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(10) D. M. Klamer

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The problem of interpolation of bandlimited deterministic signals using Lagrange polynomials which are based upon observations taken at Poisson instants is considered. Let the signal be bandlimited to the interval $(-\pi, \pi)$ radians, and let the average sampling rate of the Poisson process be β samples per second. When the Lagrange interpolation polynomials are based upon the Poisson sampling instants $\{t_k\}$ and the observed data $\{s(t_k)\}$, then the mean-squared error converges to zero as the number of observations increases to infinity for all sampling rates $\beta > 3\pi/2$. The rate of convergence is exponential for $\beta > 3\pi/2$. The above results hold for center point interpolation (i.e., an equal number of samples on each side of the interpolation point). For the case of extrapolation, the sampling rate must be increased to 3π. (Continued)

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SUMMARY

The problem of interpolation of bandlimited deterministic signals using Lagrange polynomials which are based upon observations taken at Poisson instants is considered. Let the signal be bandlimited to the interval $(-\pi, \pi)$ radians, and let the average sampling rate of the Poisson process be β samples per second. When the Lagrange interpolation polynomials are based upon the Poisson sampling instants $\{t_k\}$ and the observed data $\{s(t_k)\}$, then the mean-squared error converges to zero as the number of observations increases to infinity for all sampling rates $\beta \geqslant 3\pi/2$. The rate of convergence is exponential for $\beta > 3\pi/2$. The above results hold for center point interpolation (i.e., an equal number of samples on each side of the interpolation point). For the case of extrapolation, the sampling rate must be increased to 3π.

The above results are compared to other sampling schemes which use Lagrange interpolation polynomials. For equally spaced data and center point interpolation, the rate must be $\rho \geqslant \pi/2$. We note that for this case the rate must be more than 1.5 (actually, $\pi/2$) times faster than the Nyquist rate of $\rho = 1$ required by the sampling theorem and that the Poisson rate is more than 4.5 $(3\pi/2)$ faster than Nyquist.

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I. INTRODUCTION

In this paper we consider the interpolation and extrapolation of bandlimited signals using Lagrange polynomials which are based upon a finite number of observations taken at Poisson sampling instants of time. We derive an upper bound on the mean-squared error based upon the bandwidth of the signal, the sampling rate of the Poisson process, and the proportion of samples to the left and right of the interpolation time of interest. This bound is then shown to go to zero as the number of observations tends toward infinity.

Let s(t) be a bandlimited signal with the representation

$$s(t) = \int_{-W}^{W} \exp(it\lambda)\, S(\lambda)\, d\lambda/2\pi, \tag{1}$$

where W is the bandwidth (in radians) of the signal and $S(\lambda)\epsilon L_1$ is its Fourier transform. Then the signal can be recovered using the well known sampling theorem [1-5]

$$s(t) = \sum_{k=-\infty}^{\infty} s(k/\rho) \frac{\sin \rho\, \pi\, (t-k/\rho)}{\rho\, \pi\, (t-k/\rho)} \tag{2}$$

where $\rho > W/\pi$ with the samples evenly spaced in the time domain. Several authors [1-5] have examined truncation type error bounds for the cardinal series (2) as a function of the number of points used in the estimate. Others have investigated the effect and necessary modifications of the cardinal series when the observations are perturbed (in a non-random manner) [6, 7]. Also, investigations using other estimation schemes, such as Lagrange polynomials [8, 9], and other polynomials [10, 11], have been reported. Finally, others have investigated estimation schemes under the premise that the observation times are random with the rate of sampling approaching infinity [12, 13] or that error-free interpolation and extrapolation schemes do exist [14, 15, 16].

The sampling instants $\{t_n, -\infty < n < \infty\}$ are assumed to be generated by a Poisson stationary point process on the real line, i.e., for $n \geqslant o$

$$t_o = \alpha_o$$

$$t_{n+1} = t_n + \alpha_{n+1}$$

and for negative indices

$$t_{-1} = -\alpha_{-1}$$

$$t_{-(n+1)} = t_{-n} - \alpha_{-(n+1)}$$

where the $\{\alpha_n\}$ are independent, identically-distributed, positive random variables with the common exponential distribution $F(x) = 1 - \exp(-\beta x)$. Note that β is the average sampling rate.

II. THE ESTIMATE AND ITS BOUND

The estimation scheme is based upon the classical Lagrange polynomial interpolator. Using the N observations $\{s(t_k), N_\ell \leq k \leq N_u\}$ and the sampling instants $\{t_k\}$ we form the estimate at time t as the Lagrange polynomial interpolator

$$\hat{s}(t) = \sum_{j=N_\ell}^{N_u} s(t_j) \prod_{\substack{k=N_\ell \\ k \neq j}}^{N_u} \frac{t - t_k}{t_j - t_k}.$$

We assume that the N = m + n sample times are ordered such that m sample times occur before the time at which the estimate is desired and that n sample times occur after the time of interest. That is, for some appropriate k

$$t_{k+1} < \ldots < t_{k+m} < t < t_{k+m+1} < \ldots < t_{k+m+n},$$

where the estimate is done for time t. The estimate will be designated $\hat{s}_{m/n}(t)$ for interpolation or simply as $\hat{s}_m(t)$ in the case of extrapolation when n = 0. Note that in the case of extrapolation the inequalities

$$t_{k+m} < t < t_{k+m+1}$$

continue to hold so that the extrapolation does not take place in the "too distant future."

Writing the sampling times in terms of the recurrence times (Figure 1)

$$t_{k+i} = t - L_{-(m+1-i)}(t), \quad 1 \leq i \leq m,$$

$$t_{k+m+i} = t + L_i(t), \quad 1 \leq i \leq n \quad ,$$

the Lagrange interpolation polynomial estimate becomes

$$\hat{s}_{m/n}(t) = \sum_{j=1}^{m} s\left(t - L_{-j}(t)\right) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m} \frac{L_{-\ell}(t)}{L_{-\ell}(t) - L_{-j}(t)} \prod_{\ell=1}^{n} \frac{L_\ell(t)}{L_\ell(t) + L_{-j}(t)}$$

$$+ \sum_{j=1}^{n} s\left(t + L_j(t)\right) \prod_{\ell=1}^{m} \frac{L_{-\ell}(t)}{L_j(t) + L_{-\ell}(t)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} \frac{L_\ell(t)}{L_\ell(t) - L_j(t)}$$

and the extrapolation polynomial estimate is

$$\hat{s}_m(t) = \sum_{j=1}^{m} s\left(t - L_{-j}(t)\right) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m} \frac{L_{-\ell}(t)}{L_{-\ell}(t) - L_{-j}(t)}$$

where $L_j(t)$ is the j^{th} forward recurrence time and $L_{-j}(t)$ is the j^{th} backward recurrence time.

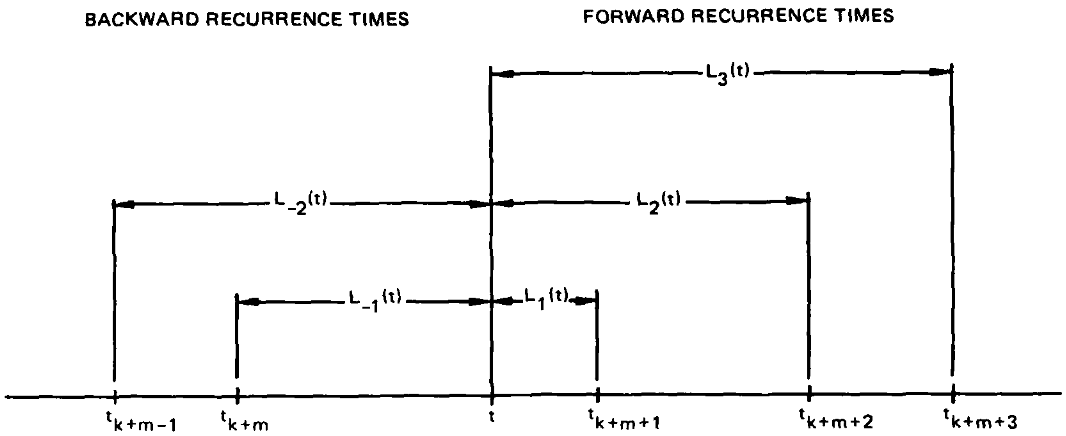

Figure 1. The first backward recurrence time $L_{-1}(t)$ at time t is defined as the distance between t and the first previous sampling time. The second backward recurrence time $L_{-2}(t)$ is the distance between t and the second previous sampling instant. Likewise, for the forward recurrence times, $L_j(t)$ is the distance between t and the j^{th} sampling point after t.

The standard Lagrange polynomial interpolation error for a function f which is at least N times differentiable is given by [16, page 56]

$$\frac{(t - t_1)(t - t_2) \dots (t - t_N)}{N!} f^{(N)}(x)$$

where $\min(t, t_1) < x < \max(t, t_N)$. The point x depends upon $\{t_k\}$, t, and f. Since s(t) has a Fourier transform which is bandlimited, s(t) is an entire function and thus possesses derivatives of all orders. The above Lagrange interpolation error formula therefore applies for any N and gives the error as

$$e_{m/n}(t) = s^{(m+n)}(x) \prod_1^m L_{-j}(t) \prod_1^n L_j(t)/(m+n)!$$

for some x. The mean-squared error is given by

$$\epsilon^2_{m/n}(t) = E\,[e_{m/n}(t)]^2. \qquad (3)$$

By applying Bernsteins inequality [17, page 138], equation (1) gives

$$|s^{(m+n)}(x)| \leq W^{(m+n)} B \qquad (4)$$

where B is the maximum value of s(t), i.e.,

$$B = \sup_t |s(t)|.$$

That B exists and is finite is guaranteed by the fact that $S(\lambda) \in L_1$.

We now consider the joint second moments of the backward and forward recurrence times. Since the Poisson point process is stationary, the joint density of the recurrence times $L_{-1}(t), \ldots, L_{-m}(t), L_1(t), \ldots, L_n(t)$ is independent of t [18] and the joint density for the interpolation case is

$$f_{L_{-1}, \ldots, L_{-m}, L_1, \ldots, L_n}(x_1, \ldots, x_{m+n}) = \beta^{m+n} e^{-\beta(x_m + x_{m+n})}$$

and for the extrapolation case

$$f_{L_{-1}, \ldots, L_{-m}}(x_1, \ldots, x_m) = \beta e^{-\beta x_m}$$

We have

$$E\left\{\prod_1^m L_{-j}^2(t) \prod_1^n L_j^2(t)\right\}$$

$$= \beta^{m+n} \int_0^\infty dx_m \int_0^{x_m} dx_{m-1} \cdots \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \, e^{-\beta x_m} \prod_{i=1}^m x_i^2$$

$$\int_0^\infty dx_{m+n} \int_0^{x_{m+n}} dx_{m+n-1} \cdots \int_0^{x_{m+2}} dx_{m+1} \, e^{-\beta x_{m+n}} \prod_{i=m+1}^{m+n} x_i^2 . \tag{5}$$

Since both integrals have the same form we need only to evaluate one of them; now

$$\beta^m \int_0^\infty dx_m \, x_m^2 e^{-\beta x_m} \int_0^{x_m} dx_{m-1} \, x_{m-1}^2 \cdots \int_0^{x_3} dx_2 \, x_2^2 \int_0^{x_2} dx_1 \, x_1^2$$

$$= \beta^m \int_0^\infty dx_m \, e^{-\beta x_m} x_m^2 \int_0^{x_m} dx_{m-1} \, x_{m-1}^2 \cdots \int_0^{x_3} dx_2 \frac{1}{3} x_2^5$$

$$= \beta^m \int_0^\infty dx_m \, e^{-\beta x_m} x_m^2 \left(\frac{1}{3} \cdot \frac{1}{6} \cdots \frac{1}{3(m-1)}\right) x_m^{3m-1}$$

$$= \Gamma(3m)/3^{m-1} \Gamma(m) \beta^{2m}. \tag{6}$$

Combining (4), (5) and (6) into (3) we have that the mean squared error is bounded by

$$\epsilon^2_{m/n}(t) \leqslant \left(\frac{W}{\beta}\right)^{2(m+n)} \frac{9B^2\,\Gamma(3m)\,\Gamma(3n)}{3^{m+n}\,[\Gamma(m+n+1)]^2\,\Gamma(m)\,\Gamma(n)}\,. \tag{7}$$

A similar expression holds for $\epsilon^2_m(t)$.

Using Sterling's approximation

$$\Gamma(z) = z^{z-1/2}\, e^{-z} \sqrt{2\pi}$$

we have that

$$\frac{\Gamma(3m)\,\Gamma(3n)}{\Gamma^2(m+n+1)\,\Gamma(m)\,\Gamma(n)}$$

$$\simeq \frac{e^2}{6\pi}\,\frac{3^{3(m+n)}}{(m+n+1)}\left(\frac{m}{m+n+1}\right)^{2m}\left(\frac{n}{m+n+1}\right)^{2n}.$$

Hence, the mean-squared interpolation error is asymptotically bounded by

$$\epsilon^2_{m/n}(t) \leqslant \frac{K}{m+n+1}\left(\frac{3W}{\beta}\right)^{2(m+n)}\left(\frac{m}{m+n+1}\right)^{2m}\left(\frac{n}{m+n+1}\right)^{2n}. \tag{8}$$

We note that for extrapolation the mean-squared error is asymptotically bounded by

$$\epsilon^2_m(t) \leqslant \frac{K}{m+1}\left(\frac{3W}{\beta}\right)^{2m}. \tag{9}$$

From (8) we evaluate the mean-squared error for the case when m = an and obtain

$$\epsilon^2_{m/n}(t) \leqslant \frac{K}{m(1+1/a)}\left(\frac{3\,\frac{a}{1+a}\,W}{\beta}\right)^{2m}\left(\frac{3\,\frac{1}{1+a}\,W}{\beta}\right)^{2m/a}.$$

A sufficient condition for $\epsilon^2_{m/n}(t)$ to converge to zero is that

$$\frac{3aW}{(1+a)\beta} \leqslant 1 \qquad \text{and} \qquad \frac{3W}{(1+a)\beta} \leqslant 1$$

or

$$3W/\beta \leqslant \min\left\{(1+a)/a,\ 1+a\right\}.$$

Then the mean-squared error goes to zero as the number of samples goes to infinity. We note that for $a \geqslant 1$, $(1+a)/a$ is the minimum and for $o < a \leqslant 1$, $1+a$ is the minimum. In either case, the maximum occurs at a = 1, which allows for the minimum sampling rate of β under which $\epsilon^2_{m/n}(t)$ goes to zero. This case has an equal number of samples on either side of the point of interpolation and corresponds to the general behavior of classical Lagrange polynomial interpolation of functions. Also note that for this case (a = 1) the average sampling rate of the Poisson process must equal or exceed 3W/2 (i.e., 1.5 π above the Nyquist rate) for asymptotic error free recovery by the above analysis. For the extrapolation case, if the Poisson sampling rate exceeds 3W (see (9)) then asymptotic error free recovery of the signal takes place.

III. DISCUSSION

In [14] Beutler established the existence of error free recovery schemes for a bandlimited signal process when the Poisson point process has a rate β that exceeds the Nyquist rate of the signal process, i.e., $\beta > W/\pi$. However, no reconstruction scheme was given only its existence was proved. As pointed out above, when Poisson sample instants are used in the Lagrange interpolation polynomial with an equal number of observations to the left and right of the interpolation point, then error free reconstruction is possible for a deterministic signal if the Poisson sampling rate is $\beta \geqslant 3W/2$, or in the case of extrapolation, $\beta \geqslant 3W$. We also point out that for the case of a random signal process no results are known. The problem centers around finding a bound for the conditional expectation $E\left\{ s(x) \mid \{t_n\} \right\}$, since the point x is not only a function of the sampling times $\{t_n\}$ but also depends upon the signal process s(t) itself.

For the case of observations spaced at equal intervals, say h, with an equal number of samples on each side of the interpolation point, Radzyner and Bason [8] showed that the Lagrange interpolation error will asymptotically go to zero if $\rho \geqslant W/2$ where ρ is the sampling rate and $h = 1/\rho$ is the spacing between samples. As pointed out above, the Poisson average sampling rate for the analogous case is required to be $\beta \geqslant 3W/2$, or three times faster than the rate for equally spaced samples.

Finally, we mention the results of Leneman-Lewis [12] and Beutler [13] in which various estimation schemes employing randomly sampled observations are investigated. The performance bound, for a fixed interpolation scheme, was expressed in terms of the sampling rate as it asymptotically approaches infinity (with a fixed number of observations), whereas the results above are obtained for a fixed sampling rate as the number of observations tend toward infinity.

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