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MINIMAX STOPPING RULES WHEN THE UNDERLYING
DISTRIBUTION IS UNIFORM

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STEPHEN M. SAMUELS

TECHNICAL REPORT NO. 38

OCTOBER 25, 1979

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1. INTRODUCTION

This paper deals with problems of the following kind: Suppose we wish to purchase a certain item. We will be quoted prices for the item, one at a time, and at some point we must stop listening and pay the price we have just heard. When should we stop in order to pay as little as we can?

The primary motivation for this paper was the desire to sharpen and extend some recent results of Stewart (1978), who studied certain of these problems from a Bayesian point of view. In Stewart's model the prices, X_1, X_2, \dots, X_n form, conditionally, a random sample from a uniform distribution on an interval (α, β) and the end-points have a prior density of the form

$$p_k(\alpha, \beta | \ell, u) = k(k-1)(u-\ell)^{k-1}/(\beta-\alpha)^{k+1}, \quad -\infty < \alpha < \ell < u < \beta < \infty; k > 1. \quad (1.1)$$

These densities were chosen for mathematical convenience. They are conjugate; i.e., the posterior density, say $f(\alpha, \beta | X_1, \dots, X_j)$, is of the same form as the prior density, namely

$$f(\alpha, \beta | X_1, \dots, X_j) = p_{k+j}(\alpha, \beta | \tilde{\ell}_j, \tilde{u}_j) \quad (1.2)$$

where

$$\tilde{\ell}_j = \min(\ell, X_1, \dots, X_j) \quad (1.3)$$

and

$$\tilde{u}_j = \max(u, X_1, \dots, X_j). \quad (1.4)$$

In other words the parameters ℓ and u of the prior density (1.1) play the role of the smallest and largest of k previous "pseudo-prices". (We shall regard k as an integer in (1.1) although it need not be.) This role is also reflected in the form of the Bayes rules which Stewart derives when the loss for accepting the i -th price is either of the following:

(a) the Quantile of the price: $(X_1 - \alpha) / (\beta - \alpha)$

or

(b) zero or one according as X_1 is or is not the "Best Choice" which is defined to be $\min\{\ell, X_1, \dots, X_n\}$.

(The names Quantile Problem or Best Choice Problem will be used whenever we are dealing with these losses, whether or not there is a pseudo-observation, ℓ .)

In each problem the Bayes rule -- which is the policy (i.e., stopping rule based on the prices) that minimizes the Bayes risk -- essentially depends on the sample size n and the prior density parameter k only through their sum, say $m = n + k$. It features a "learning period" of the form $\max\{i(m), k+1\} - k$ which is the subscript of the earliest price which can possibly be accepted. Beginning with that price, acceptance is based solely on the values of the current price, X_i , of $\tilde{\ell}_i$ and \tilde{u}_i , and of the number of prices remaining -- up to a maximum of $m - 2$ since the smallest integer k for which (1.1) is a probability density is $k = 2$. (Exact expressions for these Bayes rules are given at the beginning of Section 4.) Furthermore, while the risk does decrease with increasing n (or decreasing k) until $k+1 \leq i(m)$, it then remains constant for all larger n (or smaller k).

If $i(m)$ is at least 3, then setting $k = 0$ in the above policy and eliminating the pseudo-prices ℓ and u from (1.3) and (1.4) yields a perfectly legitimate stopping rule. In Section 4 we shall exploit the fact that the losses (a) and (b) are invariant under location and scale transformations to show that this policy is minimax for the family of all uniform distributions among all stopping rules based on m prices and that its risk

is a constant equal to the Bayes risk of the corresponding Bayes rule with $k=2$. Since Stewart's results shows that $i(m) \geq 3$ for $m \geq 6$ in the Quantile Problem and for $m \geq 5$ in the Best Choice Problem, this very nearly settles the minimax problem. And the minimax policy for the few remaining values of m will also be given in Section 4. The results of Section 4 utilize a general framework developed in Section 3 which applies to all bounded losses which share the invariance property of the Best Choice and Quantile Problem losses. Section 3, we think, constitutes an especially interesting "concrete" application of the invariance ideas commonly referred to as "Hunt-Stein Theorems".

Stewart's goal was to find optimal risks as well as optimal rules in order to assess the penalties which must be incurred for not knowing the actual distribution of the prices, as well as the rewards which may be received for being able to actually "measure" the prices rather than to know only the relative rank of each current price among those heard so far. (The relative rank of X_i among the first i prices is defined to be j if X_i is j -th smallest among X_1, X_2, \dots, X_i . And a relative rank rule, when there are no pseudo-prices, is a policy in which the decision of whether or not to accept a given price is based solely on how many prices have been heard so far and what is the relative rank of the current price among them.) By considering only uniformly distributed prices Stewart could invoke results of Gilbert and Mosteller (1966) and Chow et al. (1964) which give the optimal rules and risks in the two extreme cases. These are reviewed in Section 2.

Stewart showed that in the Best Choice problem there is no reward at all for being able to observe the X_i 's themselves rather than just their relative ranks. Indeed in the Bayes Rule the decision of whether or not to accept the i -th price is based solely on i and on whether or not

$X_i = \min\{\ell, X_1, X_2, \dots, X_i\}$. In the Quantile Problem, on the other hand, there is such a reward. Stewart's numerical results, together with the previously known limits reviewed in Section 2, suggested that for large n the risks of the best relative rank rule, the Bayes rule, and the optimal rule when the distribution is known, are in approximately the ratio 3.87 to 3.48 to 2. So, of the difference between the two extreme values, about 79% is due to uncertainty about which uniform distribution is being sampled from and 21% to suppression of information when we use only relative ranks rather than the prices themselves.

That exactly the same results hold for minimax risks is virtually implied by the previously remarked intimate relationship between the Bayes rules and the minimax rules. (The one small loose end is verifying that the minimax policy in the Best Choice Problem is a relative rank rule for $m \leq 5$ as well as for $m > 5$.)

For the Quantile Problem, Theorem 5.1 confirms the limit for the risks suggested by Stewart's numerical results. Further insight is provided by the monotonicity results of Section 6. And the asymptotic performance of an appealing sequence of simplified policies is also evaluated in Section 5.

2. UPPER AND LOWER BOUNDS FOR RISKS

2.1. Best Policies When the Distribution is Known

Solutions to both the Quantile Problem and the Best Choice Problem when the end-points α and β are known can be found in Gilbert and Mosteller (1966). They are as follows:

For the Quantile Problem, when the sample size is n the optimal policy is to stop as soon as

$$(X_i - \alpha) / (\beta - \alpha) \leq v_{n-i}. \quad (2.1)$$

Here X_i denotes the i -th price we will hear and the v_k 's are a single decreasing sequence of constants (the same for all sample sizes, n) which are computed recursively from the formula

$$v_0 = 1; \quad v_{k+1} = v_k(1 - v_k/2). \quad (2.2)$$

The v_k 's play a dual role in this problem: v_n is also the risk when the sample size is n and the optimal policy is used. And, as was shown by Gilbert and Mosteller,

$$nv_n \rightarrow 2 \quad \text{as } n \rightarrow \infty. \quad (2.3)$$

Thus if n is large the expected price to be paid -- expressed as a quantile of the underlying price distribution, is about $2/n$, which is just about twice what we would expect to pay if we could hear all n prices and then choose the smallest one.

For the Best Choice Problem, when the sample size is n the optimal policy is to stop as soon as $X_i = \min\{X_1, \dots, X_i\}$ and

$$(X_i - \alpha) / (\beta - \alpha) \leq d_{n-i} \quad (2.4)$$

where $d_0 = 1$ and, for $k \geq 1$, d_k satisfies

$$(1-d_k)^k = \sum_{j=1}^k j^{-1} \binom{k}{j} (1-d_k)^{k-j} d_k^j. \quad (2.5)$$

The d_k 's are decreasing in k , and writing

$$d_k = c_k / (k + c_k), \quad (2.6)$$

it follows from (2.5) and the dominated convergence theorem that

$$c_k \rightarrow c \approx .804352 \quad (2.7)$$

where c is the solution to

$$\sum_{j=1}^{\infty} c^j / j! j = 1. \quad (2.8)$$

The probability of correctly choosing the lowest price with the optimal policy was evaluated by Gilbert and Mosteller for various sample sizes. This optimal probability of best choice is a decreasing function of sample size, with limiting value

$$e^{-c} - (e^c - c - 1) \int_1^{\infty} x^{-1} e^{-cx} dx \approx .580164 \quad (2.9)$$

with c as in (2.8). The value .580164 was obtained by numerical methods by Gilbert and Mosteller. The actual algebraic expression for the limit is shown in Appendix A of this paper to be equal to

$$P(Z(1-T) < c < (Z+Z'/T)(1-TT')) \quad (2.10)$$

where Z , Z' , T , and T' are mutually independent random variables with Z and Z' each exponentially distributed with parameter one, and T and T' each uniformly distributed on the interval $(0,1)$. A proof that (2.10) is indeed the limiting optimal probability of best choice is sketched in Appendix A.

(These results for the Best Choice Problem apply not only to uniform distributions but also to any known continuous distribution F , with (2.4) replaced by " $F(X_i) \leq d_{n-i}$ ".)

2.2 Best Relative Rank Rules

A familiar linearity property of uniform distributions -- that the mean of the i -th order statistic of a sample of size n from a uniform distribution on (α, β) is $\alpha + i(\beta - \alpha)/(n+1)$ -- yields the solution to the Quantile Problem in the restricted class of relative rank rules directly from the result of Chow et al. (1964). This linearity property implies that for any relative rank rule the expected quantile of the price accepted by the rule is just $(n+1)^{-1}$ times the expected rank of that price. Chow et al. in effect found that for any continuous distribution of prices the minimal expected rank attainable by a relative rank rule, say C_n , satisfies

$$C_n \nearrow \sum_{n=1}^{\infty} (1 + 2j^{-1})^{1/(j+1)} \approx 3.8695.$$

and is attained by a single stopping rule for each n , regardless of the distribution. Hence, for uniform distributions this same policy minimizes the expected quantile, and the minima, say r_n , satisfy

$$nr_n = nC_n / (n+1) \uparrow \approx 3.8695. \quad (2.11)$$

The Best Choice Problem with only relative rank rules permitted is an elementary one found in many places including Gilbert and Mosteller (1966). For any continuous distribution of prices the optimal policy when there are n available prices is to stop as soon as X_i is the lowest price heard so far and $i \geq i(n)$ where

$$\sum_{i=i(n)}^{n-1} i^{-1} < 1 \leq \sum_{i=i(n)-1}^{n-1} i^{-1}. \quad (2.12)$$

This policy has probability p_n of selecting the lowest of all n prices where

$$p_n = \sum_{i=i(n)-1}^{n-1} i(n)/ni \searrow e^{-1} \approx .3679. \quad (2.13)$$

2.3 Other Best Choice Problems

Recently Petruccelli (1979) has studied the Best Choice Problem where the price distribution belongs to the restricted family of just those uniform distributions with $\beta - \alpha = 1$, and found that the best relative rank rule is not minimax, in contrast to the result for the full class of all uniform distributions. By adapting the method used to get (2.10) Petruccelli found that the asymptotic minimax probability of best choice is $\approx .43517$.

Petruccelli also gives sufficient conditions under which the asymptotic minimax best choice probability achieves the value in (2.9), an example of which is the family of all normal distributions.

3. INVARIANCE AND MINIMAXITY

3.1. Invariance of the Quantile and Best Choice Problems

A property shared by both the Quantile Problem and the Best Choice Problem is that their loss functions are invariant under location and scale transformations. What this means is the following: Let $W_i^{(n)}(\underline{x}; \alpha, \gamma)$ denote the loss for accepting the i -th of the n available prices when

$$\underline{x} \equiv (x_1, \dots, x_n)$$

is the vector of prices and α and $\alpha + \gamma$ are the end-points of the underlying uniform distribution. We call the loss invariant if for each $i = 1, 2, \dots, n$

$$W_i^{(n)}(\underline{x}; \alpha, \gamma) \equiv w_i^{(n)}((\underline{x} - \alpha) / \gamma) \quad (3.1)$$

where we use the abbreviations

$$(\underline{x} - \alpha) / \gamma \equiv ((x_1 - \alpha) / \gamma, \dots, (x_n - \alpha) / \gamma)$$

and

$$a + b\underline{x} \equiv (a + bx_1, \dots, a + bx_n).$$

In the Quantile Problem

$$W_i^{(n)}(\underline{x}; \alpha, \gamma) = (x_i - \alpha) / \gamma, \quad (3.2)$$

while in the Best Choice Problem

$$\begin{aligned} W_i^{(n)}(\underline{x}; \alpha, \gamma) &= 1 \quad \text{if } x_i = \min\{x_1, \dots, x_n\} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

or, equivalently,

$$\begin{aligned}
w_i^{(n)}(\underline{x}; \alpha, \gamma) &= 1 \quad \text{if } (x_i - \alpha)/\gamma = \min_{1 \leq j \leq n} \{(x_j - \alpha)/\gamma\} \\
&= 0 \quad \text{otherwise.}
\end{aligned}
\tag{3.3}$$

Both of these loss functions are clearly invariant. (In addition, they are bounded by one and, for $i \geq 3$, symmetric in x_1 and x_2 , properties which will be needed later.)

When the loss is invariant, the risk of a stopping rule τ , say $\rho_\tau(\alpha, \gamma)$, becomes

$$\begin{aligned}
\rho_\tau(\alpha, \gamma) &\equiv E_{\alpha, \gamma} w_\tau^{(n)}((X - \alpha)/\gamma) \\
&\equiv \int_{\alpha}^{\alpha + \gamma} \dots \int_{\alpha}^{\alpha + \gamma} \sum_{i=1}^n w_i^{(n)}((\underline{x} - \alpha)/\gamma) I_{\{\underline{x}: \tau(\underline{x}) = i\}} \gamma^{-n} \prod_{i=1}^n dx_i \\
&= \sum_{i=1}^n E_{0, 1} w_i^{(n)}(\underline{X}) I_{\{\tau(\alpha + \gamma \underline{X}) = i\}}.
\end{aligned}
\tag{3.4}$$

The last equality follows from a change of variables in the integral.

(Note that although we write $\tau = \tau(x_1, \dots, x_n)$, the fact that τ is a stopping rule means that $\{\underline{x}: \tau(\underline{x}) = i\}$ depends only on the first i coordinate of \underline{x} .)

3.2. Minimaxity of Best Invariant Rules

Whenever the loss function is invariant under some group of transformations, it is natural to look for a minimax stopping rule within the class of invariant rules. In this case an invariant stopping rule τ is one which satisfies

$$\tau(a + b\underline{x}) \equiv \tau(\underline{x}) \tag{3.5}$$

for all real x_1, \dots, x_n and a and all positive b . For example, any policy of the following form is invariant:

$$\tau(X_1, \dots, X_n) = \min\{j \geq i_n : (X_j - L_j)/(U_j - L_j) \leq c_j\}, \quad (3.6)$$

where i is some integer with $2 \leq i_n \leq n$,

$$L_i = \min\{X_1, \dots, X_i\}, \quad (3.7)$$

$$U_i = \max\{X_1, \dots, X_i\}, \quad (3.8)$$

and the c_i 's are some constants with $0 \leq c_i < 1$ for $i_n \leq i < n$, and $c_n = 1$.

Also invariant are all relative rank rules, such as

$$\tau(X_1, \dots, X_n) = \min\{i \geq i_n : \sum_{j=1}^i I_{\{X_j \leq X_i\}} \leq k_i\} \quad (3.9)$$

where the k_i 's are some positive integers with $k_n = n$.

By (3.4) and (3.5), the risk of any invariant rule is constant, and can be written as

$$\rho_\tau = \int_0^1 \dots \int_0^1 \sum_{i=1}^n w_i^{(n)}(\tilde{z}) I_{\{\tau(\tilde{z}) = i\}} \prod_{i=1}^n dz_i. \quad (3.10)$$

The following Proposition can be obtained from a theorem of Kiefer (1957):

Proposition 3.1. Let X_1, X_2, \dots, X_n be IID, each uniformly distributed on the interval $(\alpha, \alpha + \gamma)$ where α is real and γ is positive. For each $i = 2, 3, \dots, n$, and $0 \leq z_i \leq 1$, let the loss function $w_i^{(n)}(z_1, \dots, z_n)$ be bounded and non-negative. Let τ be a stopping rule based on the X_i 's with $\tau \geq 2$. Finally, choose $\varepsilon > 0$. Then there is an invariant rule τ' with risk

$$\rho_{\tau'} \leq \sup_{-\infty < \alpha < \infty, \gamma > 0} \rho_{\tau}(\alpha, \gamma) + \epsilon.$$

This Proposition, proved in Appendix B, immediately yields

Theorem 3.1. Under the conditions of Proposition 3.1., if there is a best invariant rule in the class of all stopping rules based on X_1, X_2, \dots, X_n which always take at least two observations, then it is minimax.

Following Kiefer (1966) we should re-phrase Proposition 3.1 as: "For stopping rules which take at least two observations, the Hunt-Stein property holds." This is in recognition of the celebrated unpublished theorem of Hunt and Stein in the early 1940's which gave conditions under which a best invariant procedure in a hypothesis testing problem must necessarily be minimax.

3.3. Pseudo-Invariant Rules and Their Bayes Risks

Theorem 3.1 begs the question: Is there a best invariant rule and if so how do we find it? Within the restricted class of stopping rules which always take at least three observations this question turns out to have an easy answer.

Let ℓ and u be any numbers with $\ell < u$. If τ is an invariant rule with $3 \leq \tau \leq n$, then each of the following stopping rules,

$$\tilde{\tau}_1(x_1, \dots, x_{n-2}) \equiv \tau(\ell, u, x_1, \dots, x_{n-2}) - 2,$$

and

$$\tilde{\tau}_2(x_1, \dots, x_{n-2}) \equiv \tau(u, \ell, x_1, \dots, x_{n-2}) - 2,$$

will be called a pseudo-invariant rule. The name is inspired by the fact that both $\tilde{\tau}_1$ and $\tilde{\tau}_2$ differ from the invariant τ only in that they treat the constants ℓ and u as though these were the first two observations;

hence $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are based on only $n-2$ rather than n actual observations. (Later, when we restrict attention to τ 's which are symmetric in the first two variables, there will be just one pseudo-invariant $\tilde{\tau}$ corresponding to τ .)

We now consider the Bayesian problem with $n-2$ available observations in which α and β , the end-points of the underlying uniform distribution, are given the particular a priori density

$$\pi(\alpha, \beta | \ell, u) = 2(u-\ell) / (\beta-\alpha)^3, \quad -\infty < \alpha < \ell < u < \beta < \infty \quad (3.11)$$

(this is (1.1) with $k=2$) and the loss for accepting the i -th price is taken to be either $W_{i+2}^{(n)}(\ell, u, X_1, \dots, X_{n-2}; \alpha, \beta-\alpha)$ or $W_{i+2}^{(n)}(u, \ell, X_1, \dots, X_{n-2}; \alpha, \beta-\alpha)$. (The ambiguity here is only temporary.) We use the notation $\tilde{E}_{\ell, u}$ to denote expectations taken with respect to the density (3.11). This particular Bayesian problem provides the key to the problem of finding best invariant rules because the constant risk of any invariant $\tau \geq 3$ happens to be just the average of the Bayes risks of the corresponding pseudo-invariant $\tilde{\tau}_1$ and $\tilde{\tau}_2$. Using the abbreviation

$$\langle \ell, u, \tilde{X} \rangle = \left(\frac{\ell-\alpha}{\beta-\alpha}, \frac{u-\alpha}{\beta-\alpha}, \frac{X_1-\alpha}{\beta-\alpha}, \dots, \frac{X_{n-2}-\alpha}{\beta-\alpha} \right)$$

we have

Proposition 3.2. Under the same conditions as in Proposition 3.1., with $\beta = \alpha + \gamma$, if τ is invariant and $\tau \geq 3$, then for every ℓ and u with $-\infty < \ell < u < \infty$,

$$\rho_{\tau} = \frac{1}{2} \left[\tilde{E}_{\ell, u}^{w_{\tau}^{(n)}}(\ell, u, X_1, \dots, X_{n-2}) \langle \langle \ell, u, \tilde{X} \rangle \rangle + \tilde{E}_{\ell, u}^{w_{\tau}^{(n)}}(u, \ell, X_1, \dots, X_{n-2}) \langle \langle u, \ell, \tilde{X} \rangle \rangle \right] \quad (3.12)$$

Proof: Letting $\tilde{z}' = (z_3, \dots, z_n)$ we first re-write (3.10) as

$$\rho_{\tau} = \sum_{j=1}^2 \int_0^1 \dots \int_0^1 \int_0^{z_2} dz_1 \sum_{i=3}^n w_i^{(n)}(z_j, z_{3-j}, \tilde{z}') I_{\{\tau(z_j, z_{3-j}, \tilde{z}') = i\}} \prod_{i=2}^n dz_i. \quad (3.13)$$

Next we exploit the invariance of τ to write

$$\begin{aligned} & \tilde{E}_{\ell, u}^{w_{\tau}^{(n)}}(\ell, u, X_1, \dots, X_{n-2}) \langle \langle \ell, u, \tilde{X} \rangle \rangle \\ &= \int_u^{\infty} \int_{-\infty}^{\ell} \int_{\alpha}^{\beta} \dots \int_{\alpha}^{\beta} \sum_{i=3}^n w_i \langle \langle \ell, u, \tilde{X} \rangle \rangle I_{\{\tau(\ell, u, \tilde{X}) = i\}} [2(u-\ell)/(\beta-\alpha)]^{n+1} \left(\prod_{i=3}^n dx_i \right) d\alpha d\beta \quad (3.14) \end{aligned}$$

and make the change of variables

$$z_1 = (\ell - \alpha) / (\beta - \alpha); \quad z_2 = (u - \alpha) / (\beta - \alpha)$$

$$z_i = (x_i - \alpha) / (\beta - \alpha), \quad i = 3, \dots, n.$$

The Jacobian of this transformation is $(\beta - \alpha)^{n+1} / (u - \ell)$ so the right side of (3.14) becomes simply twice the $j=1$ term of the right side of (3.13). The $j=2$ term is derived similarly. \square

Suppose now that none of the losses $W_i^{(n)}$, $i \geq 3$, depends on the order of the first two observations. This eliminates the ambiguity in the definition of the loss function for the Bayesian problems and simplifies (3.12) for invariant rules which do not depend on the order of the first two observations.

It also enables us to ignore all other invariant rules as the following proposition shows. (The proposition is an easy consequence of (3.10) so the proof is omitted.)

Proposition 3.3. If for each $i \geq 3$, $w_i^{(n)}(z_1, z_2, \dots) \equiv w_i^{(n)}(z_2, z_1, \dots)$, then, for any invariant $\tau \geq 3$, $\rho_\tau \geq \min(\rho_{\tau'}, \rho_{\tau''})$ where τ' and τ'' are the invariant stopping rules defined by

$$\tau'(z_1, z_2, \dots) \equiv \tau(\max(z_1, z_2), \min(z_1, z_2), \dots)$$

$$\tau''(z_1, z_2, \dots) \equiv \tau(\min(z_1, z_2), \max(z_1, z_2), \dots).$$

Propositions 3.2 and 3.3 immediately yield:

Theorem 3.2. Under the conditions of Propositions 3.1 and 3.3 a sufficient condition for an invariant stopping rule $\tau^* \geq 3$ to be best among all invariant rules based on X_1, X_2, \dots, X_n which always take at least three observations is that τ^* be symmetric in the first two variables and, for each ℓ and u , the pseudo-invariant rule $\tilde{\tau}$ defined by $\tilde{\tau}(x_1, \dots, x_{n-2}) = \tau^*(\ell, u, x_1, \dots, x_{n-2}) - 2$ be a Bayes rule when there are $n-2$ available observations and the a priori density is given by (3.11).

(The phrase "...for each ℓ and u ..." in the theorem can be weakened to "...for some ℓ and u ..." but the stronger version suffices for the applications in the next section and is more relevant to the comments which follow.)

To put Theorem 3.2 in proper perspective, we need to introduce the "density" $\pi_0(\alpha, \beta) = (\beta - \alpha)^{-1}$, $-\infty < \alpha < \beta < \infty$. This is, of course, the density of an infinite measure. That measure is the so-called right invariant measure on the parameter space which is introduced in Appendix B

where it plays a key role in the proof of Proposition 3.1. If we ignore the fact that it is an "improper prior" (because it is infinite), imagine that X_1, X_2, \dots, X_n are, conditionally, IID uniform on (α, β) with (α, β) having prior "density" π_0 , and compute formally the posterior density of α and β given X_1 and X_2 , we get precisely (3.11) with l and u replaced by $\min(X_1, X_2)$ and $\max(X_1, X_2)$, respectively.

The so-called formal Bayes rule with respect to the improper prior density π_0 -- within the class of stopping rules which always take at least three observations -- is then the rule which is conditionally Bayes with respect to the proper (i.e., probability) densities $\pi(\alpha, \beta | \min(X_1, X_2), \max(X_1, X_2))$. Thus what Theorem 3.2 says is that if the formal Bayes rule is invariant (and symmetric in the first two observations) then it is best invariant among rules $\tau \geq 3$. Similar results for estimation problems are discussed in Kiefer (1966).

3.4. When to Take Less Than Three Observations

We need to know when a stopping rule which is best invariant among rules which always take at least three observations can be beaten (in the minimax sense) by a rule which may take fewer than three observations. A satisfactory result is the following:

Theorem 3.3. Under the conditions of Propositions 3.1 and 3.3., if $n = 2$ then $\tau \equiv 1$ or $\tau \equiv 2$ is minimax; if $n \geq 3$ and there exists a best invariant stopping rule τ_n^* among rules which always take at least three observations, then one of the following five invariant rules is minimax among all stopping rules:

- (a) $\tau \equiv \tau_n^*$
- (b) $\tau \equiv 1$
- (c) $\tau \equiv 2$
- (d) $\tau = 2$ if $X_2 < X_1$
 $= \tau_n^*$ otherwise
- (e) $\tau = 2$ if $X_1 < X_2$
 $= \tau_n^*$ otherwise.

Proof: There are just two key steps to the proof. The first is to observe that any τ which for some α and β has a non-zero probability of taking the first observation -- i.e., for which the Lebesgue measure, $\mu(\cdot)$, of $A \equiv \{x_1: \tau(x_1, \dots) = 1\}$ is greater than zero -- must have a maximum risk at least as great as the constant risk of the rule $\tau_1 \equiv 1$. This is obvious if A_1 contains an interval, say (c, d) , since for $\alpha = c$, $\beta = d$, the rule does always take the first observation. That it is true in general follows from the following measure theoretic result: If $\mu(A) > 0$ and if $\epsilon < 1$, then there exists a finite interval I such that $\mu(A \cap I) > \epsilon \mu(I)$ (see e.g. Halmos (1950), pp. 68). In other words, we can find parameter values α and β such that $P_{\alpha, \beta}(\tau=1)$ is as close to one as we like. Then, from the boundedness of the loss functions $\{W_i^{(n)}\}$ we conclude that $E_{\alpha, \beta}^{W_\tau^{(n)}}((X-\alpha)/(\beta-\alpha))$ will be as close as we like to $E_{\alpha, \beta}^{W_1^{(n)}}((X-\alpha)/(\beta-\alpha)) \equiv \rho_{\tau_1}$.

What we have shown so far is that $\tau_1 \equiv 1$ together with all rules which always take at least two observations forms a complete class. (This completes the proof for $n=2$.) But Proposition 3.1 implies that we still have a complete class if we consider only invariant rules which always take at least two observations. And it is easy to see that all invariant rules $\tau \geq 2$ are of the

form $\tau \equiv 2$; or $\tau = 2$ if $X_2 < X_1$, $\tau = \tau' \geq 3$ otherwise; or $\tau = 2$ if $X_2 > X_1$, $\tau = \tau' \geq 3$ otherwise; for some invariant $\tau' \geq 3$.

The second key step in the proof is to invoke the symmetry of the loss in the first two variables in dealing with the latter two cases. The same simple method which yields Proposition 3.3 also shows that we can restrict ourselves to those τ' which are symmetric in the first two variables. For such rules, the conditional risk obviously does not depend on whether or not $X_2 < X_1$. Hence conditioning on $\{\tau = 2\}$, an event of probability $1/2$, we have

$$\rho_\tau = \frac{1}{2} (E_{0,1}[w_2^{(n)}(\tilde{X}) | \tau = 2] + \rho_{\tau'}) \quad (3.15)$$

which is clearly minimized by taking $\tau' = \tau_n^*$. \square

3.5. An Algorithm for Risks of Some Invariant Rules

In Section 5 we shall want to evaluate the asymptotic risk -- in the Quantile Problem -- of some non-minimax policies of the form (3.6). Here we provide the tools for the evaluation by deriving an algorithm for computing the risk of any policy of that form.

We begin with the following

Lemma 3.1: If τ is of the form (3.6), then, for each $i \geq i_n$, $\tau I_{\{\tau \leq i\}}$ is independent of the pair (L_i, U_i) .

Proof: It is straightforward to check that, regardless of the values of α and γ , the conditional distribution of the sequence of $(X_j - L_j)/(U_j - L_j)$'s, $j = 2, 3, \dots, i$, given L_i and U_i , is that of the sequence of $(Z_j - L_j)/(U_j - L_j)$'s where Z_1, Z_2, \dots, Z_i is a random permutation of the constants 0 and 1 to-

gether with $i-2$ IID random variables each uniform on $(0,1)$, and L'_j and U'_j are, respectively, the smallest and largest of $\{Z_1, \dots, Z_j\}$. Since this conditional distribution does not depend on L_i and U_i , the result follows immediately. \square

From the Lemma and the independence of the two vectors (X_{i+1}, \dots, X_n) and $(L_i, U_i, I_{\{\tau \leq i\}})$ we have the

Corollary: For each $i \geq i_n$ the conditional distribution of the pair (L_i, U_i) , given that $\tau > i$, is the same as its unconditional distribution. Hence so are the conditional distributions of the vector $(L_i, U_i, X_{i+1}, \dots, X_n)$ and of

$$\tau^{(i+1)} = \min\{j \geq i+1: (X_j - L_j)/(U_j - L_j) \leq c_j\}.$$

To clarify the meaning of $\tau^{(i+1)}$ in this corollary we make the following definition:

If τ is of the form (3.6), then for any i with $i_n \leq i \leq n$, we denote by $\tau^{(i)}$ the policy of the same form as τ except that i_n is replaced by i , and by $\tilde{\tau}^{(i)}$ the corresponding pseudo-invariant rule. (3.16)

In other words $\tau^{(i)}$ is τ delayed until i . And the corollary says that the conditional distribution of τ given that $\tau \geq i$ is simply that of $\tau^{(i)}$.

Suppose now that the losses do not depend on the past. That is, for each i ,

$$w_i^{(n)}((x-\alpha)/\gamma) = w_i^{(n)}((x_i-\alpha)/\gamma, (x_{i+1}-\alpha)/\gamma, \dots, (x_n-\alpha)/\gamma). \quad (3.17)$$

Then, by the corollary, for each $j > i \geq i_n$

$$E\{w_j^{(n)}((X-\alpha)/\gamma)I_{\{\tau=j\}}|\tau > i\} = E w_j^{(n)}((X-\alpha)/\gamma)I_{\{\tau^{(i+1)}=j\}}, \quad (3.18)$$

hence

$$E\{w_\tau^{(n)}((X-\alpha)/\gamma)|\tau > i\} = E w_{\tau^{(i+1)}}^{(n)}((X-\alpha)/\gamma) \equiv \rho_{\tau^{(i+1)}}. \quad (3.19)$$

If $\tau \geq 3$, then similar results hold for $\tilde{\tau}$, the pseudo-invariant policy corresponding to τ in the Bayesian problem. Using (1.2) with the prior density (3.11) we can verify immediately that the conditional distribution of $\tilde{\tau}$ given X_1, \dots, X_{i-2} is identically that of $\tilde{\tau}^{(i+1)}$ on $\{\tilde{\tau} > i-2\}$ for each $i \geq i_n$, and that, for losses satisfying (3.17), for each $j > i \geq i_n$

$$\tilde{E}\{w_j^{(n)}(\langle \ell, u, X \rangle)I_{\{\tilde{\tau}=j-2\}}|X_1, \dots, X_{i-2}\} \equiv \tilde{E}w_j^{(n)}(\langle \ell, u, X \rangle)I_{\{\tilde{\tau}^{(i+1)}=j-2\}} \text{ on } \{\tilde{\tau} > i-2\}.$$

So, corresponding to (3.18) and (3.19) we have for each $j > i \geq i_n$

$$\tilde{E}\{w_j^{(n)}(\langle \ell, u, X \rangle)I_{\{\tilde{\tau}=j-2\}}|\tilde{\tau} > i-2\} = Ew_j^{(n)}(\langle \ell, u, X \rangle)I_{\{\tilde{\tau}^{(i+1)}=j-2\}} \quad (3.20)$$

and

$$\tilde{E}\{w_{\tilde{\tau}+2}^{(n)}(\langle \ell, u, X \rangle)|\tilde{\tau} > i-2\} = Ew_{\tilde{\tau}^{(i+1)}}^{(n)}(\langle \ell, u, X \rangle). \quad (3.21)$$

Now Proposition 3.2 establishes the equality of the right sides of (3.19) and (3.21); and the proof of that proposition shows clearly that it holds "term-by-term" (i.e., for each i) so the right sides of (3.18) and (3.20) are equal. Thus, given $\tau \geq 3$ satisfying (3.6) and losses satisfying (3.17), if we define

$$\varphi_i = E(w_\tau((X-\alpha)/\gamma)|\tau > i), \quad i = i_n-1, i_n, \dots, n-1 \quad (3.22)$$

and

$$\tilde{\varphi}_i = \tilde{E}(w_{\tilde{\tau}+2}(\langle \ell, u, X \rangle)|\tilde{\tau} > i-2), \quad i = i_n-1, i_n, \dots, n-1$$

we have not only equality of φ_i and $\tilde{\varphi}_i$ for each i , but also equality of the coefficients used to compute them from the following identities:

$$\begin{aligned}\varphi_{i-1} &= E[w_i((X-\alpha)/\gamma)I_{\{\tau=i\}} | \tau > i-1] + \varphi_i P(\tau > i | \tau > i-1) \\ &= E[w_i((X-\alpha)/\gamma)I_{\{\tau^{(i)}=i\}}] + \varphi_i P(\tau^{(i)} > i)\end{aligned}\quad (3.23)$$

and

$$\tilde{\varphi}_{i-1} = \dots = \tilde{E}[w_i(\ell, u, X)I_{\{\tilde{\tau}^{(i)}=i-2\}}] + \tilde{\varphi}_i \tilde{P}(\tilde{\tau}^{(i)} > i-2).$$

So there is a choice of methods of evaluating (3.23). Either is straightforward as soon as we note that because $0 \leq c_i < 1$ for each $i < n$, we can replace (3.6) by

$$\begin{aligned}\tau(X) &= \min\{j \geq i_n : (X_j - L_{j-1}) / (U_{j-1} - L_{j-1}) \leq c_j\} \\ &= n \quad \text{if no such } j < n,\end{aligned}$$

the advantage being that X_j is independent of L_{j-1} and U_{j-1} . This implies that

$$\begin{aligned}\tilde{\tau}(X) &= \min\{j \geq i_n - 2 : (X_j - \tilde{L}_{j+1}) / (\tilde{U}_{j+1} - \tilde{L}_{j+1}) \leq c_{j+2}\} \\ &= n-2 \quad \text{if no such } j < n-2,\end{aligned}$$

where, for $i \geq 3$,

$$\tilde{L}_i = \min\{\ell, u, X_1, \dots, X_{i-2}\} \quad (3.24)$$

and

$$\tilde{U}_i = \max\{\ell, u, X_1, \dots, X_{i-2}\}. \quad (3.25)$$

We have then, by direct calculation,

$$\begin{aligned}
P(\tau^{(i)} > i) &= P(X_i > L_{i-1} + c_i (U_{i-1} - L_{i-1})) \\
&= \tilde{P}(\tilde{\tau}^{(i)} > i-2) = \tilde{P}(X_{i-2} > \tilde{L}_{i-1} + c_i (\tilde{U}_{i-1} - \tilde{L}_{i-1})) \\
&= [(i-1) - (i-2)c_i] / i, \quad i = i_n, i_n+1, \dots, n-1
\end{aligned}$$

and in the special case of the Quantile Problem,

$$\begin{aligned}
&E \left[w_i ((X-\alpha)/\gamma) I_{\{\tau^{(i)}=i\}} \right] = E \left[\frac{X_i - \alpha}{\beta - \alpha} I_{\{X_i \leq L_{i-1} + c_i (U_{i-1} - L_{i-1})\}} \right] \\
&= \tilde{E} \left[w_i (\langle \ell, u, X \rangle) I_{\{\tilde{\tau}^{(i)}=i-2\}} \right] = E \left[\frac{X_{i-2} - \alpha}{\beta - \alpha} I_{\{X_{i-2} \leq \tilde{L}_{i-1} + c_i (\tilde{U}_{i-1} - \tilde{L}_{i-1})\}} \right] \\
&= [i + (i-2)c_i + \frac{1}{2} (i-1)(i-2)c_i^2] / i(i+1), \quad i = i_n, i_n+1, \dots, n-1,
\end{aligned}$$

and

$$\varphi_{n-1} = E(X_n - \alpha) / \gamma = \frac{1}{2}.$$

Substituting back into the right side of (3.23) yields the following algorithm for the risk:

Proposition 3.4: In the Quantile Problem, if τ is of the form (3.6), then its risk, ρ_τ , equals φ_{i_n-1} , where

$$\varphi_{n-1} = \frac{1}{2}$$

$$\varphi_{i-1} = [1 + (i-2)c_i + \frac{1}{2} (i-1)(i-2)c_i^2] / i(i+1) + \varphi_i [(i-1) - (i-2)c_i] / i,$$

$$i = n-1, n-2, \dots, i_n. \quad (3.26)$$

4. MINIMAX RULES

Now we specialize to the Best Choice and Quantile Problems for which, as previously noted, the conditions of Propositions 3.1 and 3.3 are satisfied; so Theorems 3.1, 3.2, and 3.3 are at our disposal.

When $k=2$ in (1.1) -- so the a priori density is given by (3.11) -- Stewart's (1978) results are as follows:

In the Best Choice Problem the Bayes rule with $n-2$ available observations is

$$\begin{aligned}\tilde{\tau}_n &= \min\{i \geq \max\{i(n) - 2, 1\} : X_i = \min\{\ell, X_1, \dots, X_i\}\} \\ &= n-2 \quad \text{if no such } i < n-2\end{aligned}\tag{4.1}$$

where $i(n)$ satisfies (2.12).

In the Quantile Problem the Bayes rule with $n-2$ available observations is

$$\tilde{\tau}_n = \min\{i \geq i(n) - 2 : (X_i - \tilde{L}_{i+2}) / (\tilde{U}_{i+2} - \tilde{L}_{i+2}) \leq c_{i+2}(n)\}\tag{4.2}$$

where \tilde{L}_i and \tilde{U}_i are given by (3.24) and (3.25), respectively, and the $c_i(n)$'s and $i(n)$ satisfy (writing c_i for $c_i(n)$):

$$\begin{aligned}c_n &= 1 \\ c_{i-1} &= c_i(1-c_i/2) - (i+1)^{-1}c_i(1-c_i) - (i-2)^{-1}(i+1)^{-1}(1-c_i)\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}n - i(n) &= \min\{j \leq n-3 : c_{n-j-1}(n) < 0\} \\ &= n-3 \quad \text{if no such } j.\end{aligned}\tag{4.4}$$

The invariant rules corresponding to these Bayes rules are of the form (3.6), and those for the Best Choice Problem are also of the form (3.9) with $k_i \equiv 1$ for $i < n$. Thus in the Best Choice Problem the τ_n^* in Theorem 3.3 is a rule based only on relative ranks. Hence so are all five of the rules there. We therefore have

Theorem 4.1: In the Best Choice Problem, the best stopping rule based only on relative ranks is minimax.

From (2.12) it is easy to check that in the Best Choice Problem $i(2) = 1$, $i(3) = i(4) = 2$ and $i(n) \geq 3$ for $n \geq 5$.

In the Quantile Problem the risks of the five stopping rules in Theorem 3.3 are $\rho_{\tau_n^*}$, $1/2$, $1/2$, $\frac{1}{2}(1/3 + \rho_{\tau_n^*})$, and $\frac{1}{2}(2/3 + \rho_{\tau_n^*})$, respectively. (This follows from invariance plus the fact that if Z_1 and Z_2 are IID uniform on $(0,1)$, then $EZ_1 = EZ_2 = 1/2$, $E(Z_2|Z_2 < Z_1) = 1/3$, and $E(Z_2|Z_2 > Z_1) = 2/3$.) Since $\rho_{\tau_n^*}$ must be less than $1/2$ (the risk of the rule $\tau \equiv 3$) only the first and fourth of these five stopping rules are admissible. Now $\rho_{\tau_n^*}$ is decreasing in n simply because the class $\{\tau: 3 \leq \tau \leq n\}$ is increasing in n . Thus the minimax stopping rule will be completely determined as soon as we find the smallest n for which $\rho_{\tau_n^*} \leq 1/3$.

This is easily obtained using the fact that, by Theorem 3.2, $\rho_{\tau_n^*}$ is also the Bayes risk of its pseudo-invariant counterpart $\tilde{\tau}_n$ given by (4.2). That is, for each $n \geq 3$,

$$\rho_{\tau_n^*} = b_{n-2} \tag{4.5}$$

and

$$m_n = \min\{b_{n-2}, \frac{1}{2}(\frac{1}{3} + b_{n-2})\} \tag{4.6}$$

where

$$b_n = \text{minimum Bayes risk in Quantile Problem} \\ \text{with } n \text{ available observations and} \\ \text{a priori density given by (3.11)} \quad (4.7)$$

and

$$m_n = \text{minimax risk in Quantile Problem with} \\ n \text{ available observations.} \quad (4.8)$$

Stewart's proof, based on backward induction, provides what we need. Let

$$\psi_i^{(n)} = \inf_{\{\tau: i-2 < \tau \leq n-2\}} \tilde{E}[(X_\tau - \alpha) / (\beta - \alpha) | X_1, \dots, X_{i-2}], \quad i = 2, 3, \dots, n-1 \quad (4.9)$$

so

$$\psi_2^{(n)} = b_{n-2}. \quad (4.10)$$

As Stewart showed, the $\psi_i^{(n)}$'s and the $c_i(n)$'s are related by

$$\psi_i^{(n)} = (i+1)^{-1} \{(i-1)c_i(n) + 1\} \quad i(n) - 1 \leq i \leq n-1 \quad (4.11)$$

and

$$\psi_2^{(n)} = \psi_3^{(n)} = \dots = \psi_{i(n)-1}^{(n)} \quad (4.12)$$

[PUT TABLE 1 HERE]

Table 1 uses (4.3), (4.6), (4.10), and (4.11) to get the explicit solution to the Quantile Problem for $n \leq 6$. We can see from the table that the smallest n for which $\rho_{\tau_n^*} = b_{n-2} \leq 1/3$ is $n = 6$. With these numerical results we can now state

Theorem 4.2: In the Quantile Problem, for all $n \geq 6$, $m_n = b_{n-2}$ and the minimax stopping rule based on n available observations is

$$\tau_n = \min\{i \geq i(n) : (X_i - L_i) / (U_i - L_i) \leq c_i(n)\}$$

where $L_i = \min\{X_1, \dots, X_i\}$, $U_i = \max\{X_1, \dots, X_i\}$ and the $c_i(n)$'s and $i(n)$ are given by (4.3) and (4.4), respectively. For $n = 3, 4$, or 5 , the minimax rule is $\tau = 2$ or τ_n according as $X_2 < X_1$ or $X_2 > X_1$. And for $n = 2$, $\tau \equiv 1$ is minimax.

5. ASYMPTOTIC MINIMAX RISKS

Since the minimax rule in the Best Choice Problem is just the best relative rank rule, its asymptotic risk is $1 - e^{-1} \approx .6321$, from (2.13). (This limit should be compared with the value $\approx 1 - .5802 = .4198$ from (2.9) which is the asymptotic risk when the distribution is known.)

For the Quantile Problem, we shall prove the following theorem:

Theorem 5.1: $\lim_{n \rightarrow \infty} nm_n = (3 + 2\sqrt{2})^{1/\sqrt{2}} \approx 3.4780$.

(This limit should be compared with $\lim_{n \rightarrow \infty} nv_n = 2$ and $\lim_{n \rightarrow \infty} nr_n \approx 3.8695$, from (2.3) and (2.11), which are the asymptotic risks when the distribution is known, and when the best relative rank rule is used, respectively.)

5.1. Preliminary Results

We collect here results which will be needed for the proofs of Theorems 5.1 and 6.1.

Taking $\psi_n^{(n)} \equiv n/(n+1)$ for convenience, and re-writing (4.11) as

$$c_i^{(n)} = (i-1)^{-1} \{((i+1)\psi_i^{(n)} - 1)\}, \quad i(n) - 1 \leq i \leq n \quad (5.1)$$

we can then re-write (4.3) in either of the following forms:

$$\psi_{i-1}^{(n)} = \frac{1}{2(i-1)(i+1)} + \left(1 - \frac{1}{i(i-1)}\right) \psi_i^{(n)} - \frac{1}{2} \left(1 - \frac{2}{i(i-1)}\right) (\psi_i^{(n)})^2, \quad i(n) \leq i \leq n \quad (5.2)$$

or

$$\psi_i^{(n)} - \psi_{i-1}^{(n)} = \frac{(i-2)(i+1)}{2i(i-1)} \left\{ \left(\psi_i^{(n)} - \frac{1}{i+1}\right) \left(\psi_i^{(n)} + \frac{i}{(i-2)(i+1)}\right) \right\}, \quad i(n) \leq i \leq n \quad (5.3)$$

Also (4.4) becomes

$$\begin{aligned}
n - i(n) &= \min\{j \leq n-3: \psi_{n-j-1}^{(n)} < (n-j)^{-1}\} \\
&= n-3 \quad \text{if no such } j.
\end{aligned} \tag{5.4}$$

From (5.2)-(5.4) we can show that

$$\begin{aligned}
i(n) &= [(\psi_{i(n)-1}^{(n)})^{-1}] \\
&= [m_n^{-1}] \quad \text{for } n \geq 6.
\end{aligned} \tag{5.5}$$

(Here $[x]$ denotes the integer part of x .) Hence $i(n)$ is non-decreasing in n , a fact which can be strengthened as follows: Since the right side of (4.3) is increasing in both i and c_i it follows that

$$c_{n-j}^{(n)} \nearrow \text{ in } n \text{ for each } j=1,2,\dots \text{ and } n \geq i(n)+j. \tag{5.6}$$

Combining this with (4.4) we conclude that $i(n+1) \leq i(n)+1$. Thus, for each n ,

$$i(n+1) = i(n) \quad \text{or} \quad i(n)+1. \tag{5.7}$$

We also need to know that

$$t < 1 \Rightarrow \sup_n \sup_{i \leq nt} n\psi_i^{(n)} < \infty. \tag{5.8}$$

Here a probabilistic argument is simpler than an analytic one. Using (2.11) and the fact that $\psi_i^{(n)}$ is no greater than the risk of any rule which totally ignores the first $n-i$ observations, we have

$$(n-i)\psi_i^{(n)} \leq (n-i)m_{n-i} < (n-i)r_{n-i} < 4$$

for which (5.8) follows immediately.

5.2. Proof of Theorem 5.1

By (5.8) we can multiply both sides of (5.3) by n^2 , let $n \rightarrow \infty$, and conclude that for $i > i(n)$

$$i/n \rightarrow t \Rightarrow n\psi_i^{(n)} \rightarrow f(t) \quad (5.9)$$

where f satisfies the Riccati equation

$$f'(t) = (1/2)(f^2(t) - t^2) \quad (5.10)$$

on $(0,1)$ with the boundary condition $f(1^-) = \infty$. And from (5.5) we can also conclude that

$$i(n)/n \rightarrow t^* \quad (5.11)$$

and

$$nm_n \rightarrow f(t^*), \quad (5.12)$$

t^* being the t for which $f(t)$ is minimized and where $f(t) = t^{-1}$.

The change of variables

$$f(t) = -2g'(t)/g(t) \quad (5.13)$$

in (5.10) yields the linear equation

$$4t^2 g''(t) = g(t)$$

which has the general solution

$$g(t) = at^{(1+\sqrt{2})/2} - bt^{(1-\sqrt{2})/2}.$$

The boundary condition $f(1) = \infty$ implies $a = b$ which uniquely determines f .

The result, from (5.13), is

$$f(t) = \frac{(1+\sqrt{2})(3-2\sqrt{2}+t^{\sqrt{2}})}{t(1-t^{\sqrt{2}})} \quad (5.14)$$

which attains its minimum at

$$t^* = (3-2\sqrt{2})^{1/\sqrt{2}} = 1/(3+2\sqrt{2})^{1/\sqrt{2}} = 1/f(t^*) \approx .2875.$$

Applying (5.12) completes the proof. \square

From (5.1), (5.9), and (5.14) we obtain the

Corollary: If $i/n \rightarrow t > t^*$, then

$$(n-i)c_i(n) \rightarrow (1-t)(f(t) - t^{-1}) = \frac{(2+\sqrt{2})(1-t)(t^{\sqrt{2}} - (3-2\sqrt{2}))}{t(1-t^{\sqrt{2}})}. \quad (5.15)$$

The right side of (5.15) increases to 2 as t increases to 1. Table 2 gives some values.

[PUT TABLE 2 HERE]

5.3. A Simplified Sequence of Policies

The corollary to Theorem 5.1 shows that when n is large the minimax policy is to let approximately $t^*n (\approx .2875n)$ observations go by, then stop as soon as

$$(X_i - L_i)/(U_i - L_i) \leq a_n((n-i)/n)/(n-i) \quad (5.16)$$

where $a_n(t)$ is approximately the right side of (5.15), so $a_n(t) \nearrow 2$ as t increases from t^* to 1. Not surprisingly, this is similar to what happens when the distribution is known. By (2.3) v_{n-i} is approximately $2/(n-i)$ for large n . Indeed it can easily be checked that replacing the right side of (2.1) by $2/(n-i)$ does not increase the asymptotic risk. It is therefore of interest to see what happens if we correspondingly alter the minimax policies by replacing $c_i(n)$ by $2/(n-i)$ and $i(n)$ by a free variable chosen so as to minimize the risk.

Substituting $c_i = 2(n-i)^{-1}$ into (3.24), writing $\varphi_i^{(n)}$ for φ_i , and re-arranging terms, we can show that if $i/n \rightarrow t$ as $n \rightarrow \infty$, then $n\varphi_i^{(n)} \rightarrow f(t)$ when

$$f'(t) = \frac{1+t}{t(1-t)} \left(f(t) - \frac{1+t^2}{t(1-t^2)} \right),$$

and f satisfies the boundary condition $(1-t)^2 f(t) \rightarrow 0$ as $t \rightarrow 1$. The unique solution is

$$f(t) = \frac{(1-t^2) + 2t^2 |\ln t|}{2t(1-t)^2}. \quad (5.17)$$

Thus if we take $i/n \rightarrow t$, the risk is asymptotically $n^{-1}f(t)$. The best choice of t is the one for which $f(t)$ is minimized. By numerically

evaluating f we find that its minimum value is about 3.8215, attained at $t \approx .3225$.

Comparing this value of 3.8215 with the corresponding values of 3.4780 for the minimax policies and 3.8695 for the best relative rank rules, we see that replacing $c_i(n)$ by $2(n-i)^{-1}$ yields policies which are sub-optimal but still manage to outperform the best relative rank rules by a whisker.

6. MONOTONICITY PROPERTIES OF THE MINIMAL RISK

(This section is devoted entirely to the Quantile Problem.)

All of the minimal risk sequences which have been introduced -- v_n and r_n in Section 2 and m_n and b_n in Section 4 -- are decreasing in n simply because the bigger n is the larger is the class of available stopping rules. Multiplying the minimal risk by n , however, produces an increasing sequence. This was in effect shown for nr_n by Chow et al. (1964), as noted in (2.11). The method used there was to condition on the arrival time of the worst of $n+1$ arrivals, which leads to a randomized relative-rank-based rule for n arrivals such that -- in the notation of this paper -- $(n+1)$ times its risk is less than $(n+2)r_{n+1}$.

It is quite easy to show that

$$nv_n \uparrow \text{ in } n, \quad (6.1)$$

and two proofs will be given in Section 6.1: a purely analytic one and one using the method of Chow et al. The latter proof will then be generalized in Section 6.2 to yield the main result of this section:

Theorem 6.1: $nm_n \uparrow$ in n .

(We should remark that nb_n is also increasing in n since $b_n = m_{n+2}$ for $n \geq 4$ and monotonicity for smaller n can be checked using Table 1.)

6.1. Monotonicity of nv_n

The analytic proof starts with the recursion (2.2). (Note that if we let $R_n = 1 - v_n$ in (2.2), then

$$R_{n+1} = \frac{1}{2} (1 + R_n^2); \quad R_1 = \frac{1}{2}$$

which is Gilbert and Mosteller's formula (5a-1.) Now $x(1-x/2)$ is increasing on $(0,1)$ so

$$v_n \leq \frac{2}{n+1} \Rightarrow v_{n+1} \leq \frac{2}{n+1} \left(1 - \frac{1}{n+1}\right) = \frac{2n}{(n+1)^2} \leq \frac{2}{n+2}.$$

Since $v_1 \leq 2/(1+1)$ we have by induction

$$v_n \leq \frac{2}{n+1} \text{ for all } n;$$

hence, from (2.2) again,

$$v_{n+1} \geq v_n \left(1 - \frac{1}{n+1}\right) \tag{6.2}$$

which is (6.1). \square

For the second proof we first take $\alpha=0$, $\beta=1$, purely for notational convenience. Now suppose we are told both the value, U_{n+1} , and the arrival time of the maximum (i.e., worst) of X_1, \dots, X_{n+1} . Then clearly we can improve on v_{n+1} by using an optimal policy on the other n X_i 's, which are conditionally IID, uniform on $(0, U_{n+1})$. The optimal risk is then conditionally $v_n U_{n+1}$; hence unconditionally $v_n E U_{n+1} = v_n (n+1)/(n+2)$. Thus,

$$\frac{n+1}{n+2} v_n \leq v_{n+1}$$

which is slightly stronger than (6.2). \square

6.2 Monotonicity of nm_n

Now let us "tie one hand behind our back" by insisting that only invariant rules be allowed on the other n X 's. It is not hard to see that the best such rule does not depend on U_{n+1} and its conditional risk is

just $m_n U_{n+1}$. But the very fact that the rule does not depend on U_{n+1} prevents us from asserting at once that it is an improvement on m_{n+1} . (If so, then we are done, as before.) Here is where the work begins.

First we check directly from Table 1 that nm_n is increasing for $n \leq 6$. So henceforth we will take $n \geq 6$. Since the minimax rules have constant risk we can simplify the notation by taking $\alpha = 0, \beta = 1$. Let us also abbreviate τ_{n+1} , the minimax rule with $n+1$ observations, by simply τ .

Now we define the random variable Z by $X_Z = U_{n+1}$, and let

$$U'_i = U_i \quad \text{if } i < Z$$

$$= \max\{X_j : j \leq i, j \neq Z\} \quad \text{if } i \geq Z.$$

Then, noting that $\tau = Z$ only if $Z = n+1$, we let

$$\sigma = \tau I_{\{\tau < Z\}} + n I_{\{\tau = Z = n+1\}} + \sigma'_Z I_{\{\tau > Z; Z \leq n\}}$$

where for $1 \leq z \leq n$,

$$\sigma'_z = \min\{i > \max\{z, i(n)\} : (X_i - L_{i-1}) / (U'_{i-1} - L_{i-1}) \leq c_{i-1}(n)\}.$$

Thus, on $\{\tau > Z; Z \leq n\}$, σ is essentially the minimax rule with n observations except that it ignores X_Z entirely and treats X_{Z+1}, \dots, X_{n+1} as though they were X_Z, \dots, X_n . Conditional on U_{n+1} , σ is in effect a randomized invariant rule applied to n IID random variables, uniform on $(0, U_{n+1})$. Hence

$$E(X_\sigma | U_{n+1}) = m U_{n+1}$$

where the constant m is at least as large as m_n . Thus

$$EX_{\sigma} \geq m_n EU_{n+1} = m_n (n+1)/(n+2).$$

Since $EX_{\tau} = m_{n+1}$ we see that to prove Theorem 6.1 it suffices to show that

$$EX_{\sigma} \leq EX_{\tau}.$$

Clearly

$$X_{\sigma} I_{\{\tau < Z\}} = X_{\tau} I_{\{\tau < Z\}}$$

and, letting $A = \{\tau = Z = n+1\}$,

$$X_{\sigma} I_A = X_n I_A < X_{n+1} I_A = X_{\tau} I_A.$$

So, what remains to be shown is

$$EX_{\sigma} I_{\{\tau > Z, Z \leq n\}} \leq EX_{\tau} I_{\{\tau > Z, Z \leq n\}}. \quad (6.3)$$

We do so by showing that the corresponding inequality holds identically on $\{Z \leq n\}$ for the conditional expectations given Z and U_{n+1} . And, to simplify matters, we divide both conditional expectations by U_{n+1} which eliminates the dependence on U_{n+1} . Specifically (6.3) is implied by

$$E(X_{\sigma} U_{n+1}^{-1} I_{\{\tau > Z\}} | Z, U_{n+1}) \leq E(X_{\tau} U_{n+1}^{-1} I_{\{\tau > Z\}} | Z, U_{n+1})$$

identically on $\{Z \leq n\}$ (6.4)

where both sides of (6.4) are functions of Z only.

We can describe these two functions of Z as follows: Let Y_1, Y_2, \dots, Y_n be IID random variables, each uniform on $(0,1)$; let L'_i and U'_i be the smallest and largest, respectively of Y_1, \dots, Y_i ; also define

$$V_i = L'_{i-1} + c_i(n)(U'_{i-1} - L'_{i-1})$$

and

$$V'_i = L'_{i-1} + c_{i+1}(n+1)(1 - L'_{i-1}).$$

For each $z \leq n$ let

$$\sigma(z) = \min\{i \geq \max\{z, i(n)\} : Y_i \leq V_i\}$$

and

$$\tau(z) = \min\{i \geq \max\{z, i(n+1)-1\} : Y_i \leq V'_i\}.$$

Then, on $\{Z=z\}$, the left and right sides of (6.4) are $EY_{\sigma(z)}$ and $EY_{\tau(z)}$ respectively.

Now we are in a position to establish (6.4) by backward induction. Let

$$\hat{\psi}_i = \hat{\psi}_i(z) = E(Y_{\sigma(z)} | \sigma(z) > i)$$

$$\delta_i = \delta_i(z) = E(Y_{\tau(z)} | \tau(z) > i).$$

Then

$$\hat{\psi}_{n-1} = \delta_{n-1} = \frac{1}{2} \tag{6.5}$$

and

$$\hat{\psi}_{z-1}(z) = EY_{\sigma(z)}; \quad \delta_{z-1}(z) = EY_{\tau(z)}. \tag{6.6}$$

Moreover

$$\begin{aligned}\hat{\psi}_{i-1} &= \psi_{i-1}^{(n)} \quad \text{if } \max\{z, i(n)\} \leq i \leq n-1 \\ &= \hat{\psi}_{z-1} \quad \text{if } z \leq i < i(n),\end{aligned}\tag{6.7}$$

where $\psi_i^{(n)}$ is defined by (4.9); also

$$\begin{aligned}\delta_{i-1} &= EY_i I\{Y_i \leq V_i\} + \delta_i P(Y_i > V_i) \quad \text{if } \max\{z, i(n+1)-1\} \leq i \leq n-1 \\ &= \delta_{z-1} \quad \text{if } z \leq i < i(n+1) - 1.\end{aligned}\tag{6.8}$$

By (6.6) the proof will be complete when we show that

$$\delta_i \geq \hat{\psi}_i \quad \text{for } i = n-1, n-2, \dots, z-1.\tag{6.9}$$

Equality holds for $i = n-1$ by (6.5) so we can proceed by induction. First we need to evaluate (6.8). Independence of Y_i and V_i , together with the moments $EL'_{i-1} = i^{-1}$, $E(L'_{i-1})^2 = 2i^{-1}(i+1)^{-1}$, gives

$$\begin{aligned}\delta_{i-1} &= \frac{1}{2} EV_i'^2 + \delta_i E(1-V_i') \\ &= \frac{1 + (i-1)c_{i+1}^{(n+1)} + \frac{1}{2} i(i-1)(c_{i+1}^{(n+1)})^2}{i(i+1)} + (1-c_{i+1}^{(n+1)})\left(\frac{i-1}{i}\right) \delta_i \\ &\quad \text{if } \max\{z, i(n+1)-1\} \leq i \leq n-1 \\ &= \delta_{z-1} \quad \text{if } z \leq i \leq i(n+1) - 1.\end{aligned}$$

For $0 \leq c, x \leq 1$, let

$$d_{i-1}(c, x) = \frac{1 + (i-1)c + \frac{1}{2} i(i-1)c^2}{i(i+1)} + (1-c) \frac{i-1}{i} x$$

and

$$g_{i-1}(c, x) = \frac{1 + (i-2)c + \frac{1}{2} (i-1)(i-2)c^2}{i(i+1)} + \left[\frac{(i-1) - (i-2)c}{i} \right] x.$$

Then

$$\delta_{i-1} = d_{i-1}(c_{i+1}(n+1), \delta_i) \quad \text{if } \max\{z, i(n+1) - 1\} \leq i \leq n-1$$

and, by (3.26) and (6.7),

$$\hat{\psi}_{i-1} = g_{i-1}(c_i(n), \hat{\psi}_i) \quad \text{if } \max\{z, i(n)\} \leq i \leq n-1,$$

Also, by direct calculation,

$$d_{i-1}(c, x) - g_{i-1}(c, x) = \frac{i-1}{i(i+1)} c \left(c - \frac{(i+1)x - 1}{i-1} \right), \quad (6.10)$$

which is clearly non-negative if and only if $c \geq [(i+1)x - 1]/(i-1)$.

We now assert that, for $i \geq \max\{z, i(n+1) - 1\}$, $\delta_i \geq \hat{\psi}_i$ implies

$$\delta_{i-1} \geq d_{i-1}(c_{i+1}(n+1), \hat{\psi}_i) \geq g_{i-1}(c_{i+1}(n+1), \hat{\psi}_i) \geq \hat{\psi}_{i-1}. \quad (6.11)$$

The first inequality holds by the induction hypothesis because d_{i-1} is increasing in x . The second inequality holds by (6.10), (5.1), and (5.6) if $i \geq i(n)$; but by (5.7) the only remaining case is $z \leq i(n) = i(n+1)$; $i = i(n+1) - 1$, in which case the inequality holds by just (6.10) because then $[(i+1)\hat{\psi}_i - 1]/(i-1) < 0$. Finally the last inequality of (6.11) holds because

$$\begin{aligned} \min_{0 \leq c \leq 1} g_{i-1}(c, \hat{\psi}_i) &= \hat{\psi}_{i-1} && \text{if } i \geq i(n) \\ &> \hat{\psi}_i = \hat{\psi}_{i-1} && \text{if } i = i(n) - 1. \end{aligned}$$

Thus (6.11) shows that $\delta_j \geq \hat{\psi}_j$ where

$$j = \max\{z, i(n+1) - 1\} \leq \max\{z; i(n)\}.$$

Referring to (6.7) and (6.8) we see that $\delta_{z-1} = \delta_j$ and $\hat{\psi}_{z-1} = \hat{\psi}_j$. Thus (6.9) holds and the proof of Theorem 6.1 is complete. \square

APPENDIX A

Derivation of (2.10): Without loss of generality we may assume that the underlying distribution is uniform on $(0,1)$; and for compatibility with the work of others we shall pose the equivalent problem of maximizing the probability of selecting the largest of X_1, X_2, \dots, X_n . So the optimal policy is to stop as soon as $X_i = \max(X_1, X_2, \dots, X_i)$ and $X_i \geq 1 - d_{n-i}$ where $d_0 = 1$ and, for $i < n$, d_{n-i} is defined by (2.5). For convenience we let

$$b_k = 1 - d_k$$

and

$$M_k = \max(X_1, \dots, X_k).$$

It should be noted that Gilbert and Mosteller's argument for the optimality of this policy was heuristic. A rigorous argument can be constructed either by dynamic programming or by a Markov chain argument. The latter is provided by Bojdecki (1978) who, in effect, uses the easily-verifiable identity

$$\sum_{j=1}^i \frac{\binom{i}{j} (1-x)^j}{j x^j} = \sum_{j=1}^i \frac{(x^{-j} - 1)}{j} \quad (\text{A.1})$$

to re-write (2.5).

We now let X_1, X_2, \dots be an infinite sequence of IID random variables each uniform on $(0,1)$, let

$$\tau_n = \min\{i: X_i = M_i \geq b_{n-i}\},$$

so τ_n is the optimal policy based on n prices, and let σ_n and σ_n' be defined by

$$X_{\sigma_n} = M_n \quad \text{and} \quad X_{\sigma'_n} = M_{\sigma_n - 1}$$

so σ_n is the "arrival time" of the largest of the first n prices and σ'_n is the arrival time of the largest price prior to the σ_n -th. Then, because $b_{n-i} \searrow$ in i while $M_i \nearrow$ in i , we can write

$$P(X_{\tau_n} = M_n) = P(M_n > b_{n-\sigma_n} \quad \text{and} \quad M_{\sigma_n - 1} < b_{n-\sigma'_n}) \quad (\text{A.2})$$

Now we make the change of variables:

$$Z_n = n(1-M_n); \quad T_n = \sigma_n/n$$

$$Z'_n = (\sigma_n - 1)(1 - M_{\sigma_n - 1}/M_n); \quad T'_n = \sigma'_n/(\sigma_n - 1)$$

and use (2.6) so that (A.2) becomes

$$P(X_{\tau_n} = M_n) = P(A_n \cap B_n) \quad (\text{A.3})$$

where

$$A_n = \{Z_n(1 - T_n + n^{-1}c_{n(1-T_n)}) < c_{n(1-T_n)}\}$$

and, letting

$$K_n = n(1 - T_n T'_n) + T'_n,$$

$$B_n = \{c_{K_n} < [Z_n + (T_n - \frac{1}{n})^{-1} (1 - n^{-1}Z_n)Z'_n][1 - T_n T'_n + n^{-1}(c_{K_n} + T'_n)]\}.$$

Using familiar properties of the uniform distribution, one can verify the weak convergence result

$$(Z_n, Z'_n, T_n, T'_n) \xrightarrow{\mathcal{D}} (Z, Z', T, T')$$

where $Z, Z', T,$ and T' are as described following (2.10). This in turn implies that (A.3) converges to (2.10) as $n \rightarrow \infty$.

To get from (2.10) to (2.9) we first condition on $Z = z, T = t,$ and $T = t'$; the conditional probability is

$$e^{-t(c/(1-tt') - z)^+} I_{\{z < c/(1-t)\}}.$$

Integrating this multiplied by the exponential density of Z yields the conditional probability given $T = t$ and $T = t'$ which is

$$(1-t)^{-1} e^{-ct/(1-tt')} (1 - e^{-c(1-t)/(1-tt')}) + e^{-c/(1-tt')} - e^{-c/(1-t)}.$$

The final step of integrating this expression over the unit square requires the change of variables

$$u = (1-t)/(1-tt'), \quad v = 1/(1-tt')$$

on all but the last term. This, with the help of (2.8) and the identity

$$\int_0^1 u^{-1} (e^{cu} - 1) = \sum_{k=1}^{\infty} c^k / k! k,$$

yields the expression

$$\int_0^{\infty} \int_0^1 v^{-2} (v-u)^{-1} e^{-c(v-u)} du dv.$$

Letting $w = v - u$ and interchanging the order of integration then leads directly to (2.9) which can easily be numerically evaluated from the identity

$$\int_1^{\infty} x^{-1} e^{-cx} dx = |\log c| - \gamma - \sum_{j=1}^{\infty} (-c)^j / j! j$$

where γ is Euler's constant $\approx .577216$.

APPENDIX B

Proof of Proposition 3.1: (This proof is nothing but Kiefer's proof (1957; p. 588-589) of a much more general theorem, written out in longhand for this special case, and deleting those aspects of the general proof which become extraneous in this case. The proof uses the notation introduced in Section 3.)

The key step in the proof is this: Given any stopping rule $\tau \geq 2$, we use it to define a large collection of corresponding invariant rules as follows: For each real α and positive γ let $\tau_{\alpha,\gamma}$ be the stopping rule:

$$\tau_{\alpha,\gamma}(\underline{x}) = \tau(\alpha + \gamma(\underline{x} - x_1) / |\underline{x} - x_1|).$$

(Note that if $\{\underline{x}: \tau(\underline{x}) = 1\}$ were non-empty, then $\tau_{\alpha,\gamma}(\underline{x})$ would fail to be a stopping rule.)

If we now let $\Theta = \mathbb{R} \times \mathbb{R}^+$ denote the parameter space and μ be any probability measure on Borel sets of Θ , then -- simply because a maximum is at least as large as an average, which in turn is at least as large as a minimum -- we have

$$\sup_{\Theta} \rho_{\tau}(\alpha, \gamma) \geq \int_{\Theta} \rho_{\tau}(\alpha, \gamma) \mu(d\alpha, d\gamma) \quad (\text{B.1})$$

and

$$\int_{\Theta} \rho_{\tau_{\alpha,\gamma}} \mu(d\alpha, d\gamma) \geq \inf_{\Theta} \rho_{\tau_{\alpha,\gamma}}. \quad (\text{B.2})$$

Hence, if we could find a μ for which the right side of (B.1) is equal to the left side of (B.2), the proof would be complete. This is not unreasonable to hope for, because, as will be seen below, the requirement is that μ be right invariant, i.e., that for each $G \subset \Theta$ and $(\alpha, \gamma) \in \Theta$, $\mu(G \circ (\alpha, \gamma)) = \mu(G)$, where

$$(\alpha', \gamma') \circ (\alpha, \gamma) = (\alpha' + \alpha\gamma', \gamma'\gamma) \quad (\text{B.3})$$

and

$$G \circ (\alpha, \gamma) = \{(\alpha', \gamma') \circ (\alpha, \gamma) : (\alpha', \gamma') \in G\}. \quad (\text{B.4})$$

A right invariant measure does indeed exist, but it is, unfortunately, necessarily an infinite measure; e.g., $\pi_0(d\alpha, d\gamma) = \gamma^{-1}d\alpha d\gamma$ is one such, as can easily be checked (see Zacks (1971), pp. 334-335). Hence we must resort to showing that

$$\lim_{m \rightarrow \infty} \int_{\Theta} (\rho_{\tau}(\alpha, \gamma) - \rho_{\tau\alpha, \gamma}) \pi_m(d\alpha, d\gamma) = 0. \quad (\text{B.5})$$

where

$$\pi_m(G) = \pi_0(GG_m) / \pi_0(G_m)$$

and the G_m 's are successively larger subsets of Θ with finite π_0 measure.

Let $\mathcal{U}_n = [0, 1]^n$ and, for each $\tilde{z} \in \mathcal{U}_n$, let

$$G_i(\tilde{z}) = \{(\alpha, \gamma) \in \Theta : \tau(\alpha + \gamma\tilde{z}) = i\}; \quad (\text{B.6})$$

so $G_2(\tilde{z}), G_3(\tilde{z}), \dots, G_n(\tilde{z})$ is a measurable partition of Θ . It follows from (B.3) and (B.4) that for any $(a, b) \in \Theta$,

$$G_i(\tilde{z}) \circ (a, b) = \{(\alpha, \gamma) \in \Theta : \tau(\alpha + \gamma(z-a)/b) = i\}. \quad (\text{B.7})$$

Now, using the right side of (3.4), then interchanging the order of integration, the right side of (B.1) becomes

$$\int_{\Theta} \mu(d\alpha, d\gamma) \int_{\mathcal{U}_n} \sum_{i=2}^n w_i(\tilde{z}) I_{\{\tilde{z} : \tau(\alpha + \gamma\tilde{z}) = i\}} d\tilde{z} = \sum_{i=2}^n \int_{\mathcal{U}_n} w_i(\tilde{z}) \mu(G_i(\tilde{z})) d\tilde{z}. \quad (\text{B.8})$$

And, from (3.10) -- with $\tau = \tau_{\alpha, \gamma}$ -- followed by an interchange of the order of integration and (B.7), the left side of (B.2) becomes

$$\begin{aligned} & \int_{\Theta} \mu(d\alpha, d\gamma) \int_{\mathcal{U}_n} \sum_{i=2}^n w_i(z) \mathbb{I}_{\{z: \tau(\alpha + \gamma(z-z_1))/|z_2-z_1|=i\}} dz \\ &= \sum_{i=2}^n \int_{\mathcal{U}_n} w_i(z) \mu\{G_i(z) \circ (z_1, |z_2-z_1|)\} dz. \end{aligned} \quad (\text{B.9})$$

Comparing (B.8) and (B.9) we see that right invariance of μ does indeed imply equality of the right side of (B.1) and the left side of (B.2). Take $\mu = \pi_0 = \gamma^{-1} d\alpha d\gamma$, and

$$G_m = \{(\alpha, \gamma) \in \Theta: |\alpha| \leq A_m, \quad 0 < b_m \leq \gamma \leq B_m < \infty\}$$

where $A_m \uparrow \infty$, $b_m \downarrow 0$, $B_m \uparrow \infty$ and $B_m / \ln B_m = o(A_m)$. Define π_m by

$$\pi_m(G) = \pi_0(GG_m) / \pi_0(G_m).$$

Then, from (B.8) and (B.9),

$$\begin{aligned} & \int_{\Theta} |\rho_{\tau}(\alpha, \gamma) - \rho_{\tau_{\alpha, \gamma}}| \pi_m(d\alpha, d\gamma) \\ &= \sum_{i=2}^n \int_{\mathcal{U}_n} w_i(z) |\pi_m\{G_i(z) - \pi_m\{G_i(z) \circ (z_1, |z_2-z_1|)\}\}| dz. \end{aligned} \quad (\text{B.10})$$

Now, denoting $(z_1, |z_2-z_1|)$ by $g(z)$, we have

$$\begin{aligned} & \pi_0(\{G_i(z) \circ g(z)\} \cap G_m) = \pi_0(G_i(z)G_m \circ g(z)) \\ & + \pi_0(\{(G_i(z) - G_m) \circ g(z)\} \cap G_m) \\ & - \pi_0(G_i(z)G_m \circ g(z) - G_m) \end{aligned} \quad (\text{B.11})$$

The first term on the right side of (B.11) equals $\pi_0(G_i(z)G_m)$ because π_0 is invariant. The second and third terms are at most $\pi_0(\{\tilde{G}_m \circ g(z)\} \cap G_m) = \pi_0(G_m - G_m \circ g(z))$, and $\pi_0(\{G_m \circ g(z)\} - G_m)$ respectively. Thus the right side of (B.10) is at most

$$\sum_{i=2}^n \|w_i(\cdot)\| \int_{\mathcal{U}_n} \frac{\pi_0(G_m \Delta \{G_m \circ g(z)\})}{\pi_0(G_m)} dz,$$

where $\|w_i(\cdot)\| = \sup_{\mathcal{U}_n} w_i(z)$ which is finite by assumption. So to establish (B.5) all we need to show is that, for each $z \in \mathcal{U}_n$ with $|z_2 - z_1| > 0$,

$$\frac{\pi_0(G_m \Delta \{G_m \circ g(z)\})}{\pi_0(G_m)} \rightarrow 0.$$

Using (B.3) with $(\alpha, \gamma) = (z_1, |z_2 - z_1|)$ we can verify that

$$\begin{aligned} G_m \Delta \{G_m \circ g(z)\} &\subset \{|\alpha| \leq A_m, b_m \leq \gamma \leq b_m/|z_2 - z_1|\} \\ &\cup \\ &\{|\alpha - A_m| \leq |z_1|\gamma, b_m \leq \gamma \leq B_m\} \\ &\cup \\ &\{|\alpha + A_m| \leq |z_1|\gamma, b_m \leq \gamma \leq B_m\} \\ &\cup \\ &\{-A_m - z_1\gamma \leq \alpha < A_m - z_1\gamma, B_m \leq \gamma < B_m/|z_2 - z_1|\}. \end{aligned}$$

Hence

$$\frac{\pi_0(G_m \Delta \{G_m \circ g(z)\})}{\pi_0(G_m)} \leq \frac{2A_m \ell_n |z_2 - z_1| + 4|z_1|(B_m - b_m) + 2A_m \ell_n |z_2 - z_1|}{2A_m \ell_n (B_m/b_m)},$$

which goes to zero provided $B_m / \ln B_m = o(A_m)$.

Thus (B.5) holds, which completes the proof of the proposition. \square

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2. Asymptotic Form of Minimax Rule in Quantile Problem

\underline{t}	$\lim_{n \rightarrow \infty} n(1-t)c_{[nt]}(n)$
.288	.0041
.289	.0127
.29	.0212
.30	.1035
.40	.7199
.50	1.1128
.60	1.3894
.70	1.5967
.80	1.7593
.90	1.8908
.95	1.9478
.99	1.9899
.995	1.9950
.999	1.9990

1. Solution to Quantile Problem for $n = 3, 4, 5,$ and 6

	n			
	<u>3</u>	<u>4</u>	<u>5</u>	<u>6</u>
$c_5(n)$	—	—	1.0000	.5000
$\psi_5(n)$	—	—	—	.5000
$c_4(n)$	—	1.0000	.5000	.3056
$\psi_4(n)$	—	—	.5000	.3833
$c_3(n)$	1.0000	.5000	.2750	.1470
$\psi_3(n)$	—	.5000	.3875	.3235
$c_2(n)$.5000	.1875	.0609	-.1084
$\psi_2(n) = b_{n-2}$.5000	.3958	.3354	.2972
m_n	.4167	.3646	.3344	.2972

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MINIMAX STOPPING RULES WHEN THE UNDERLYING
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An invariance-based method of obtaining the minimax stopping rule when sampling from an unknown uniform distribution is presented and applied to two problems, maximizing the probability of selecting the smallest observation and minimizing the expected quantile of the observation selected. In the first problem the minimax rules use only the relative ranks of the observations; in the second they are shown to achieve asymptotic risk $(3+2\sqrt{2})^{1/\sqrt{2}}/n \approx 3.4780/n$ which is intermediate between the values $\approx 3.8695/n$ for the best rules based on relative ranks and $2/n$ when the distribution is known. Except for a few small values of the sample size, n , the minimax rules are the formal Bayes rules with respect to an improper a priori "density" whose a posteriori density given the first two observations is proper.

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