AD-A080 354 MINNESOTA UNIV MINNEAPOLIS DEPT OF AEROSPACE ENGINE--ETC F/G 13/13
STRUCTURAL INELASTICITY XXV. COMPUTATION OF NON-UNIQUE SOLUTION--ETC(U)
NOV 79 D L WHITE, P G HODGE N00014-75-C-0177
UNCLASSIFIED AEM-H1-25 NL

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STRUCTURAL INELASTICITY XXV

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Computation of Non-unique Solutions of Elastic-Plastic Trusses

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November 1979

Technical Report

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Prepared for

OFFICE OF NAVAL RESEARCH
Arlington, VA 22217

OFFICE OF NAVAL RESEARCH
Chicago Branch Office
536 South Clark St.
Chicago, IL 60605

80 2 1 013

SECURITY CLASSIFICATION OF THIS PAGE (When

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
14 AEM-H1-25

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
6 STRUCTURAL INELASTICITY ~~XXV~~. Computation of Non-unique Solutions of Elastic-Plastic Trusses.

5. TYPE OF REPORT & PERIOD COVERED
9 Technical ~~Report~~,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
10 David L. White ~~Research Assistant~~
Philip G. Hodge, Jr ~~Professor of Mechanics~~

8. CONTRACT OR GRANT NUMBER(s)
N14-75-C-0177
15 N00014-75-C-017

9. PERFORMING ORGANIZATION NAME AND ADDRESS
University of Minnesota
Minneapolis, MN 55455

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NF 064-429

11. CONTROLLING OFFICE NAME AND ADDRESS
OFFICE OF NAVAL RESEARCH
Arlington, VA 22217

12. REPORT DATE
11 Nov[redacted]79

13. NUMBER OF PAGES
20 12 12

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)
OFFICE OF NAVAL RESEARCH
Chicago Branch Office
536 South Clark St.
Chicago, IL 60606

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Qualified requesters may obtain copies of this report from DDC

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Trusses, uniqueness, computer programs, perfect plasticity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It has recently been shown that the solution to certain well-defined boundary value problems for elastic/perfectly-plastic structures is non-unique. The present paper is concerned with the ability of finite-element computer programs to handle such solutions. Some problems arising from non-uniqueness are documented and techniques for circumventing those problems are evaluated by exploring the response of the program NonSAP to a variety of trusses with non-unique solutions.

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S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

405 395

1. Introduction. The question of uniqueness in solid mechanics problems has received a great deal of attention over the years. It is well known that the solution to a "well-posed boundary-value problem" in elasticity is unique. Hill[1]* has shown that such a problem always has a unique solution for an elastic/plastic problem when the total strain is small, the strain hardening is monotonic and not zero, and the yield function and plastic potential are identical.

However, for an elastic/perfectly-plastic material the situation is quite different. For such a material there will exist a yield-point load above which no equilibrium exists and at which the solution is non-unique. Recently, Hodge and White[2] have shown that a non-unique solution can exist for some problems with small deformations and zero strain hardening even for loads below the yield-point load. Trusses, frames, and arches have been shown to allow more than one solution.

This non-uniqueness phenomenon stems directly from the elastic/perfectly-plastic material model. The stresses are uniquely determined, as are a portion of the displacement variables. The remaining displacements can be bounded, but are not uniquely determined. In every case, the total external work is unique. In order for a structure to exhibit this lack of uniqueness, one portion must yield in such a fashion as to have more degrees of freedom than are specified by the surrounding material.

Throughout the development of the finite element method in its applications to elasticity and plasticity problems in solid mechanics, a unique solution has been assumed, at least

* Superscript numbers refer to the references listed at the end of the paper.

for loads less than the yield-point load. The purpose of this paper is to explore the response of a finite element program to a problem with a non-unique solution.

In Section 2, a simple truss problem is solved in closed form to demonstrate the origin of the phenomenon. The governing equations are written out, along with the solution in the elastic and elastic-plastic ranges. Section 3 discusses the mathematics of non-uniqueness and the computational problems to be expected.

Section 4 gives pertinent information about the finite element program NonSAP.[3] This program's response to the example truss is then carefully examined in Section 5. Section 6 comments on the generalization of these results to other programs and other classes of problems.

2. Truss Example. Throughout this paper, we will focus on the truss shown in figure 1. All bars have the same area, A, and modulus of elasticity, E. Bars BD and CE have yield stress σ_y and all other bars yield at $3\sigma_y$. This truss is a simple example of a well-posed boundary-value problem. It is indeterminate to the first degree and its elastic solution is unique. However, as we shall show, the solution is non-unique above the elastic limit, even though the yield-point load is substantially higher.

For an elastic- perfectly plastic material model, the constitutive equations are conveniently written in rate form:

$$F_i^2 \le Y_i^2 \tag{1a}$$

$$\text{if } F_i^2 < Y_2^2 \text{ or } F_i\dot{F}_i < 0, \quad \text{then } \dot{\Delta}_i = (L_i/AE)\dot{F}_i \tag{1b}$$

$$\text{else } \dot{F}_i = 0 \text{ and } F_i\dot{\Delta}_i \ge 0 \tag{1c}$$

where Δ_i is the extension, F_i is the force, Y_i is the force at yield, and L_i is the length of bar i.

Compatibility is assured if the rates of extensions of the bars are defined in terms of the velocities of the points A, B, and C by

$$\begin{aligned} &\dot\Delta_{BD} = \dot v_B \quad \dot\Delta_{CE} = \dot v_C \quad \dot\Delta_{BC} = \dot u_C - \dot u_B \\ &\dot\Delta_{AB} = (\dot u_A - \dot u_B + \dot v_B - \dot v_A)/\sqrt{2} \\ &\dot\Delta_{AC} = (\dot u_C - \dot u_A + \dot v_C - \dot v_A)/\sqrt{2} \\ &\dot\Delta_{AD} = (\dot u_A + \dot v_A)/\sqrt{2} \quad \dot\Delta_{AE} = (\dot v_A - \dot u_A)/\sqrt{2} \end{aligned} \qquad (2)$$

Equilibrium at the joints provides six equations which may be written in rate form as

$$\begin{aligned} \dot F_{BD} - \dot F_{CE} &= \dot F_{AB} + \sqrt{2}\,\dot F_{BC} \\ &= \dot F_{AB} - \dot F_{AC} = \dot F_{AB} - \dot F_{AD} = \dot F_{AB} - \dot F_{AE} = 0 \quad (3a\text{-}e) \end{aligned}$$

$$\dot F_{BD} + \dot F_{AB}/\sqrt{2} = \dot P \qquad (3f)$$

With $\dot\Delta_i$ defined by (2), Eqs. (3) and the appropriate (1) provide thirteen equations to determine seven forces and six displacements. In the elastic range, the branch (1b) holds for all bars. Thus we have a system of thirteen equalities and thirteen unknowns. The resulting solution is

$$\dot F_{BD} = \dot P(1+4\sqrt{2})/(4\sqrt{2}+3) \quad \dot F_{BC} = -2\dot P/(4\sqrt{2}+3) \qquad (4a)$$

$$\dot F_{AB} = \dot F_{AD} = 2\sqrt{2}\dot P/(4\sqrt{2}+3) \qquad (4b)$$

$$\dot u_A = 0 \qquad \dot v_A = 2\dot F_{AD}H/AE \qquad (4c)$$

$$\dot u_B = -\dot F_{BC}H/AE \qquad \dot v_B = 2\dot F_{BD}H/AE \qquad (4d)$$

$$\dot F_{CE} = \dot F_{BD} \qquad \dot F_{AC} = \dot F_{AE} = \dot F_{AB} \qquad (4e)$$

$$\dot u_C = -\dot u_B \qquad \dot v_C = \dot v_B \qquad (4f)$$

As P is increased, (4) will hold until bars BD and CE yield for

$$P = P_e = A\sigma_y(4\sqrt{2}+3)/(1+4\sqrt{2}) = 1.30A\sigma_y \qquad (5)$$

For larger values of load, the truss is in the range of contained plastic deformation. Branch (1c) holds for bars BD and CE, whereas branch (1b) continues to hold for all other bars. Equation (1c) as applied to BD and CE implies that the equilibrium equation equation (3a) is identically satisfied. The remaining Eqs. (3) provide a unique solution for the forces:

$$\begin{aligned} &\dot F_{BD} = \dot F_{CE} = 0 \qquad \dot F_{BC} = -\dot P \\ &\dot F_{AB} = \dot F_{AC} = \dot F_{AD} = \dot F_{AE} = \sqrt{2}\dot P \end{aligned} \qquad (6)$$

Combination of Eqs. (6) and (2), together with Eqs. (1b) for the five elastic bars provides five equations whose solution may be written

$$\dot u_A = 0 \qquad \dot v_A = 2\sqrt{2}(H/AE)\dot P \qquad (7a)$$

$$\dot u_B = -(H/AE)\dot P + \dot\theta \qquad \dot v_B = (4\sqrt{2}+1)(H/AE)\dot P + \dot\theta \qquad (7b)$$

$$\dot u_C = (H/AE)\dot P + \dot\theta \qquad \dot v_C = (4\sqrt{2}+1)(H/AE)\dot P - \dot\theta \qquad (7c)$$

where

$$\dot\theta = (\dot v_B - \dot v_C)/2 \qquad (7d)$$

Evidently θ represents a rigid-body rotation of ABC about point A.

Equation (1c) for the plastic bars BD and CE provides only inequalities which may be written

$$-(4\sqrt{2}+1)(H/AE)\dot P \le \dot\theta \le (4\sqrt{2}+1)(H/AE)\dot P \qquad (7e)$$

Therefore, the solution is not unique since any $\dot{\theta}$ satisfying (7e) provides displacement rates which satisfy all of the governing equations.

Physically, the $\dot{P}$ terms in Eqs. (7) define a symmetric motion in which equal upward motions of B and C extend the plastic bars BD and CE, and the $\dot{\theta}$ terms define a rigid-body rotation of triangle ABC about point A which extends one plastic bar and compresses the other. The requirement that the superposition of these two motions results in extension of both BD and CE is expressed by (7e).

This solution is valid up to the yielding of the four diagonal bars at

$$P = P_y \equiv \sigma_y A(1+3/\sqrt{2}) = 3.12A\sigma_y \tag{8}$$

At this load the truss collapses with multiple degrees of freedom.

3. The Mathematics Of Non-Uniqueness. This non-uniqueness phenomenon can be discussed on two levels, the physical and the mathematical. On the one hand, the actual physical structure can assume a range of displaced shapes. This can be explained by examining the structure and qualitatively discussing its behavior. On the other hand, our mathematical model also exhibits a non-unique solution. The origin of this phenomenon can be traced to certain features of the model. Naturally, a correspondence between the physical and mathematical view points of the phenomenon can be made.

The first hint of non-uniqueness lies in the constitutive equations (1c). The fact that (1c) is an inequality implies

that the relationship between stress and strain is not unique, even for a continuously loaded bar. If this relationship were unique, as in the case of a strain hardening model, then the solution for the structure would be unique. This inequality also accounts for non-uniqueness at collapse.

In order for the solution to be non-unique, the problem must have a homogeneous equilibrium equation which degenerates into an identity as the bars yield. In this example, Eq. (3a) degenerates to 0 = 0 when bars BD and CE yield simultaneously. In the general case of an indeterminate truss with a single parameter loading system and more than one degree of freedom, it is always possible to write all but one of the equilibrium equations in homogeneous rate form. If all of the bars appearing in a homogeneous equation except bar k yield, the equation predicts $\dot{F}_k = 0$ regardless of the value of $\dot{\delta}_k$. If bar k is elastic this implies that $\dot{\delta}_k = 0$. However, if bar k yields at the same load as the last other bar in the equation, then $\dot{F}_k$ is already known to be zero and the equilibrium equation is edundant. The loss of an equilibrium equation is a loss of one piece of information and corresponds to the loss of uniqueness.

The redundant equilibrium equation causes the stiffness matrix to be singular. Since virtually all finite element programs are based on the inversion of a stiffness matrix, this is a major obstacle. The difficulty can be surmounted by making small changes in the geometry, in the loading, or in the yield stresses, or by the introduction of strain hardening. These changes will result in a modified problem whose unique solution will be in the range of permissible solutions for the original non-unique problem. By carefully selecting the changes, the analyst can define the range of possible solutions.

Consider the introduction of strain hardening to the example truss. If all bars are assumed to strain harden with the same slope E_t, the solution will be unique, and will be symmetric. If strain hardening is added only to bar BD, then BD will not extend after yielding. Bar CE will extend, and this solution will represent one extreme of the range of permissible displacements. If strain hardening is added only to bar CE, the solution will represent the opposite extreme. By choosing different non-zero values of E_t for the two bars we can obtain any intermediate solution. Further, if the value (or values) of E_t are extremely small compared to the elastic modulus E, the velocity at A and all of the bar forces will be only insignificantly different from the unique values determined by the perfectly-plastic model in Eqs. (7a) and (6).

Alternatively, we could modify the problem by making a small change in the position of C, leaving B exactly where it is in Fig. 1. Other possibilities would be to slightly change the value of A or E, to change the yield stress in one of the bars, to introduce a small horizontal load, or to slightly change one of the loads P.

All of the above-mentioned techniques yield an invertable stiffness matrix. However, if the changes are sufficiently small, the matrix will probably be ill-conditioned. An ill-conditioned matrix can lead to high computer time requirements, poor accuracy, or no solution at all depending on the method used to solve the simultaneous equations.

An analysis of the practical significance of non-uniqueness is further complicated by the fact that round-off error may mask the phenomenon. Although the actual elastic solution precisely satisfies Eqs. (4e) and (4f), round-off error in the computer

program may produce results which are slightly different. If, say, $\dot{F}_{CE}-\dot{F}_{BD} \neq 0$ by however small an amount, the program will predict yielding of only one of bars CE or BD and a unique solution will be obtained at one of the extremes allowed by (7e).

In Sec. 6 we will comment on these phenomena in relation to a particular computer program solution.

4. Non-SAP. In order to understand the results NonSAP[3] gives for a non-unique problem, we need to understand the iteration scheme it uses for non-linear problems. It will also be helpful to discuss the yielding of a single truss element.

NonSAP follows the usual practice of solving the problem for some initial load, then solving the incremental problem for larger loads. The user can specify both the initial load and the increment. For linear problems, the program solves:

$$ {}^{i}K u = {}^{i+1}R - {}^{i}F \qquad (9)$$

where

${}^{i}K$ is the tangent stiffness at load increment i,

u is the vector of nodal displacement increment from i to i+1, i.e., $u = {}^{i+1}u - {}^{i}u$,

${}^{i}F$ is the nodal point force vector equivalent to the element stress at increment i,

${}^{i+1}R$ is the external load vector applied at step i+1.

For non-linear or piecewise linear problems, the program first uses Eqn. (9) for j=1 and then solves

$$ {}^{i}K\, \Delta u^{(j)} = {}^{i+1}R - {}^{i+1}F^{(j-1)} \qquad (10)$$

repeatedly, iterating until the correct solution has been reached. This method corresponds to a modified Newton's iteration.

The user has two controls on this process. The size of the load increment has a large influence on the number of iterations required to reach a solution. Secondly, the user is asked to set a limit on the number of iterations.

The yielding of a single truss element is fundamental to the analysis of an elastic- plastic truss. For the perfectly plastic model, Non-SAP gives the correct linear solution up to yield. The solution for the load increment spanning yield cannot be computed. The program gives one of two messages, depending on the limit on iterations. For a high limit, the message is "stop because out-of-balance loads larger than incremental loads." For a low iteration limit, the message is "iteration limit reached."

For a linear strain-hardening material model, the ability of the program to run past yield depends on the amount of strain hardening, the size of the load increment, and the iteration limit. For large values of strain hardening the problem is handled with one iteration per load step both before and after yield, but the load step spanning yield requires several iterations. For small values of strain hardening and a large load increment, the program will stop at the last load before yield just as in the perfectly-plastic case. If the load increment is sufficiently small, the program will run past yield. However, at each successive step past yield the number of iterations required will increase until the limit is reached or the program stops for some other reason.

Thus we see that the occurence of yield must be inferred

from the program messages. For the perfectly plastic model, the program cannot run past yield. For the strain hardening model, the program may run past yield, in which case yield is recognized by examining the stresses and displacements.

The collapse of a structure composed of truss elements is indicated in a similar fashion. Prior to collapse, the load step spanning yielding of one or more bars will require several iterations. The program cannot compute a solution for the load step spanning collapse of a perfectly plastic structure. Neither can it compute the solution for the load step spanning yield of the structure as a whole for a strain-hardening model with small strain hardening and reasonable load increments. In both cases the message will be either "iteration limit reached" or "out-of-balance loads larger than incremental loads," depending on the iteration limit.

5. <u>NonSAP and Non-Unique Structures</u>. A large number of computer experiments were performed on the truss in Fig. 1 in order to explore how the program NonSAP[3] responds to the phenomenon of non-uniqueness. We considered the perfectly-plastic truss with a variety of numbering schemes for the bars and elements and with various load increments. We also introduced a small amount of strain-hardening or slight increase in yield stress in one or both of the bars BD and CE. Tables 1 and 2 summarize the results and show that a great variety of solutions could be obtained by making apparently small changes in the problem.

Table 1 shows the results for the perfectly-plastic material model. Table 2 considers the strain-hardening models, all of which have the numbering scheme used for case A in Table 1.

In interpreting these tables, we recall that

$$P_e/P_y = 0.417 \qquad (11)$$

P_{max}/P_y is the last printed load value. For a complete solution it should be between $1 - \Delta P/P_y$ and 1. The value of $\theta(AE/HP_y)$ was computed from the output for P_{max} and integrals of (7d). According to the integral of (7e), it should satisfy

$$-6.656(P_{max}-P_e)/P_y \leq \theta AE/HP_y \leq 6.656(P_{max}-P_e)/P_y \qquad (12)$$

The value of $\theta = 0$ corresponds to the symmetric solution.

Each of the three possible NonSAP responses to a nonunique truss are represented in the tables. For a low iteration limit, as in case E2, the program will not be able to solve the load increment spanning yield of the first bar to go plastic. Note that the message is one of the two messages which indicate collapse. In cases A-D, NonSAP recognizes the singular stiffness matrix. Whether or not the program recognizes the singularity depends on the node numbering scheme, whereas the load at which this recognition takes place depends on the element numbering. For some node numbering schemes, for instance E and F, NonSAP will run to collapse without detecting the singular stiffness matrix.

Each variation on the perfectly-plastic model gives a distinct value for the displacements. The values of θ always fall within the bounds given by (12), but are different for each numbering scheme and load increment.

Computer experiments were also carried out for smaller and larger perfectly-plastic trusses, and the results further emphasize the importance of round-off error. For the three bar truss in Fig. 2, NonSAP recognizes the singularity immediately past

P_e, regardless of the numbering scheme, presumably because the small number of bars gives less opportunity for round-off error. As a result, the program terminates at P_e and does not carry on to the yield-point load regardless of the numbering scheme used.

On the other hand, the truss in Fig. 3 has many more bars with correspondingly greater opportunities for round-off error. NonSAP will compute the complete solution for this truss for all of the numbering systems tried. However, the number of iterations per step after yield and the displacement solution will depend on the numbering of the nodes and elements.

The program is more likely to correctly diagnose the singular stiffness matrix for a small truss. For some trusses, including the truss in Fig. 1, the numbering of the nodes will determine whether or not the singularity is recognized and the element numbering will determine at what load this takes place. For large trusses, the program will generally run to collapse.

The most straightforward technique for forcing uniqueness is the introduction of a small strain-hardening coefficient, E_t. Ideally, this would allow the analyst to explore the permissible range of the non-unique variables as well as give a valid solution up to collapse. Unfortunately, Table 2 shows that NonSAP does not always respond well to small amounts of strain-hardening.

Strain-hardening is only important in the bars BD and CE. The remaining bars do not yield before the structure collapses, therefore we have no interest in their post-yield behavior. If the same value of E_t is used for both bars, NonSAP will run to collapse regardless of the size of E_t. The symmetric solution is obtained.

If a small amount of strain-hardening is introduced to one of the bars BD or CE, the program will not run to collapse, and may not run past P_e. For $E_t/E = 10^{-5}$, cases J and K, the program runs through the load step spanning yield, but no further. For somewhat larger E_t/E NonSAP will not run past P_e. For sufficiently large E_t/E in conjunction with a sufficiently small load step, as in cases N and O, the program will run to collapse or until it is stopped by the user. The solution does represent one extreme of the permissible range for θ. The other extreme can be established by introducing strain-hardening only in bar CE.

By introducing a different E_t for each of the two critical bars it should be possible to achieve any θ within the acceptable range. For example, in case P, $\theta AE/HP_y$ is approximately one third of its maximum permissible value. If the difference between the two coefficients is sufficiently large as in case Q, then the program will respond as if the smaller coefficient were zero and stop at approximately $P = P_e$.

A second technique for forcing uniqueness is the alteration of yield stresses. This leads to one bar yielding prior to the other, which prevents the degeneration of the homogeneous equilibrium equation into an identity. Cases S and T show that this technique will only work for a sufficiently large yield stress differential used with a sufficiently small load increment. This trade-off is similar to the trade-off when strain-hardening is introduced in only one bar.

6. Conclusions. When the simple truss problem in Fig. 1 is analyzed by writing out and solving the equations, it is perfectly

clear that the solution exists and is unique for all variables up to P_e, that it is unique for some variables but non-unique for others between P_e and P_y, and that no solution exists above P_y. Further, it is not difficult to obtain all of the determinable information about the solution. However, the possibility of non-uniqueness was clearly not considered by the creators of NonSAP[3], nor, preseumably, by the authors of other multi-purpose computer programs. Therefore, the user of these programs must be prepared to interpret various program messages in relation to non-uniqueness, even though they were not written with this phenomenon in mind.

In this paper we have endeavored to investigate the seriousness of the problem of non-uniqueness and the effectiveness of various means of circumventing it. The phenomenon is of more than academic interest because small changes in the statement of the original program lead to well-posed problems with different solutions. The stresses are uniquely determined in the original problem and are essentially the same for each of the associated unique problems. Therefore, if only the stresses are of interest, any one of the possible solutions will be satisfactory. If the non-unique displacement variables of are of interest, predictions of the program may be in gross error.

In most cases, the program gives no indication that the problem has a non-unique solution. For a relatively small truss with a perfectly-plastic material model, the program may discover and report a singular stiffness matrix. In other cases, for instance K, Q, and S, the program stops well short of the yield point load and prints the same message it prints at yield point load. Misinterpretation of these results can be prevented by a check to see that a mechanism does in fact exist.

Fortunately, NonSAP appears to be able to compute a solution up to the yield point load for large trusses in spite of non-uniqueness. We suspect that round-off error masks the phenomenon and prevents the recognition of the singularity. More examples need to be tested to verify this conclusion. Once again, it is important to point out that the program gives no hint that additional solutions with different displacements also exist.

The introduction of strain-hardening will usually allow the program to compute a solution up to the yield point load. The alteration of yield stresses, small changes in geometry, and certain applications of strain-hardening will be less successful. In general, NonSAP appears to respond well to unique problems whose solutions are close to the center of the permissible range given by the non-unique solution. In cases of symmetric or slightly asymmetric strain-hardening, the program runs to the yield point load and gives the symmetric or a nearly symmetric solution. On the other hand, NonSAP responds poorly to unique problems whose solution lies at or near one of the extremes of the permissible range.

This latter difficulty can be traced to the iterative nature of the solution process through the observation that one of the bars will alternate between a yielded and an elastic state during the iteration process. In cases J, K, and S the process does not converge. In cases L, M, and Q the process does not converge rapidly enough to give a solution within the maximum allowed iterations. In each of these cases, the solution lies very near the yield extension of one of the bars. It is no surprise

that an iterative process faces difficulties in the neighborhood of such a sharp change in the stiffness matrix. Thus we see that in some cases the program is unable to find a solution even through the problem is well posed and has a unique solution.

Some structural analysis programs designed to deal only with perfectly plastic truss problems speed computations by yielding more than one bar at a time. When bar A reaches yield at $|F_A| = F_{A0}$, the program will check all other bars and will call all bars B for which $|F_B| \geq 0.98\ F_{B0}$, say, plastic also. This may artificially introduce non-uniqueness into a problem whose solution is actually unique.

This paper has begun the study of the ability of finite element programs to handle non-unique problems by considering one program, NonSAP, and one class of structures, trusses. This study needs to be expanded in two directions to include both more programs and more classes of structures. To achieve this end, it will be necessary to discover additional structures which have non-unique solutions.

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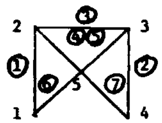
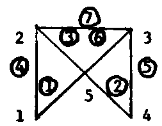
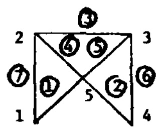
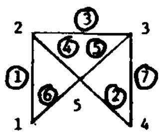
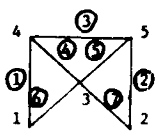
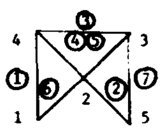
Table 1

$E_t = 0$ for all bars

Case	Model Numbering	Iteration Limit	$\Delta P/P_y$	P_{max}/P_y	$\frac{\theta AE}{HP_y}$	$\frac{6.656(P_{max}-P_e)}{P_y}$	Message
A		75	.075	.67	-.213	1.68	ZP
B		75	.075	.75	-1.90	2.22	ZP
C		75	.075	.90	-.61	3.21	ZP
D_1		75	.0375	.49	.167	.49	ZP
D_2		75	.075	.67	1.17	1.68	ZP
E_1		75	.075	.97	-.21	3.68	IL
E_2		15	.075	.38	0	-0	IL
F		75	.075	.97	1.21	3.68	IL

Message : ZP: stop because of zero pivot IL: iteration limit reached

OB: stop because out-of-balance loads are larger than incremental loads

Table 2

All runs use numbering scheme A from Table 1

Case	Strain Hardening (E_t/E) BD	CE	Iteration Limit	$\Delta P/P_y$	P_{max}/P_y	$\frac{\theta AE}{HP_y}$	$\frac{6.656(P_{max}-P_e)}{P_y}$	Message
G	10^{-3}	10^{-3}	75	.075	.97	0	3.68	IL
H	10^{-6}	10^{-6}	75	.075	.97	0	3.68	IL
I	10^{-10}	10^{-10}	75	.075	.97	0	3.68	IL
J	0	10^{-5}	75	.075	.45	0	.21	OB
K	10^{-5}	0	75	.075	.45	0	.21	OB
L	10^{-2}	0	75	.075	.37	0	-0	IL
M	10^{-2}	0	75	.038	.41	0	-0	IL
N	10^{-2}	0	75	.019	.58*	-1.08	1.08	
O	10^{-1}	0	75	.038	.75*	-2.19	2.22	
P	10^{-5}	$2x10^{-5}$	75	.038	.97	1.23	3.68	IL
Q	10^{-5}	10^{-2}	75	.075	.37	0	0	IL
R	10^{-5}	10^{-3}	75	.075	.97	3.6	3.68	IL
	Yield Stress (σ_y/30KSI) BD	CE						
S	1.001	1	75	.075	.45	.005	.21	OB
T	1.01	1	75	.075	.97	-3.68	3.68	IL

*Solution was stopped by user at this load

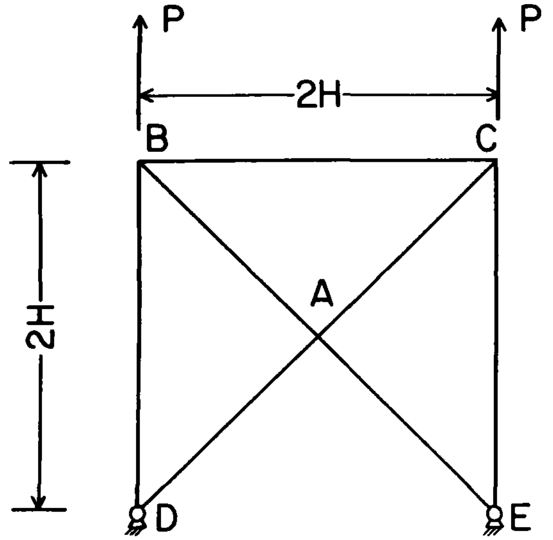


Figure 1

Seven-bar Truss

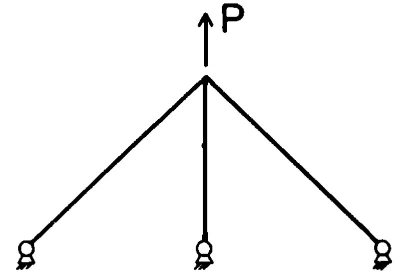


Figure 2

Three-bar Truss

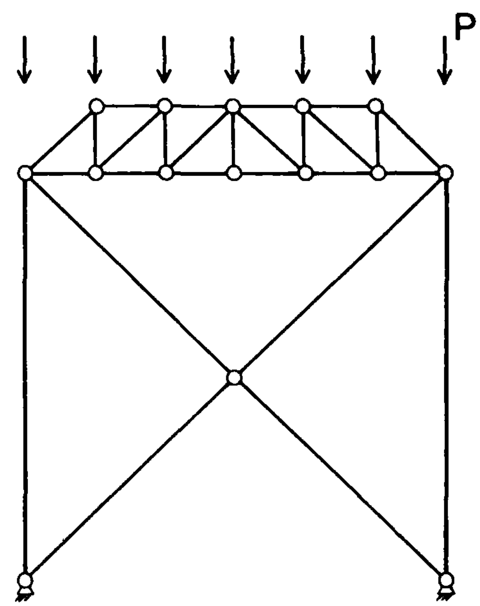


Figure 3

Large Truss