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NONLINEAR SCHÖDINGER EVOLUTION EQUATIONS

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We consider the nonlinear Schrödinger equation
\[
\begin{cases}
\frac{i}{\alpha} \frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 & \text{in } \Omega \times [0,\infty) \\
u(x,t) = 0 & \text{in } \partial \Omega \times [0,\infty) \\
u(x,0) = u_0(x) & \text{in } \Omega
\end{cases}
\]
where $\Omega$ is a bounded domain or an exterior domain of $\mathbb{R}^2$. Such an equation has been extensively studied when $\Omega = \mathbb{R}^2$, but the methods do not apply if $\Omega \neq \mathbb{R}^2$. We prove that there exists a unique global smooth solution if $k \geq 0$ or if $k < 0$ and $|k| \int |u_0|^2 < 4$. The proof relies on a new interpolation-embedding inequality:
\[
\|u\|_{L^\infty} \leq C(1 + \log(1 + \|u\|_{H^2})) \text{ for every } u \in H^2 \text{ with } \|u\|_{H^1} \leq 1.
\]

AMS (MOS) Subject Classifications: 47H15, 46E35, 35K55, 35L60, 45E30

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Work Unit Number 1 (Applied Analysis)
SIGNIFICANCE AND EXPLANATION

The nonlinear Schrodinger equation occurs in the study of some problems in nonlinear optics (propagation of laser beams through the atmosphere or in a plasma). It has been considered by many authors. The main novelty of the present result is that it applies to the propagation of beams in channels (instead of the whole space). The proof relies on a new Sobolev-Orlicz embedding inequality which could be useful in other situations.
NONLINEAR SCHRODINGER EVOLUTION EQUATIONS

H. Brezis¹ and T. Gallouet

Let Ω be a domain in \( \mathbb{R}^2 \) with compact smooth boundary \( \Gamma (\Omega) \) (\( \Omega \) could be for example a bounded domain or an exterior domain). Consider the equation

\[
\begin{aligned}
&\frac{i}{3} \frac{\partial u}{\partial t} - \Delta u + k |u|^2 u = 0 \quad \text{in} \quad \Omega \times [0,\infty) \\
&u(x,t) = 0 \quad \text{in} \quad \Gamma \times [0,\infty) \\
&u(x,0) = u_0(x),
\end{aligned}
\]

where \( u(x,t) \) is a complex valued function and \( k \in \mathbb{R} \) is a constant.

Problem (1) which occurs in nonlinear optics when \( \Omega = \mathbb{R}^2 \) has been extensively studied in this case (see [1],[2],[3],[5],[8]), but we are not aware of any known result when \( \Omega \neq \mathbb{R}^2 \).

Our main result is the following:

Theorem 1. Let \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \). Assume that one of the following conditions holds

(a) either \( k \geq 0 \),

(b) or \( k < 0 \) and \( |k| \int |u_0(x)|^2 dx < 4 \).

Then there exists a unique solution of (1) such that

\( u \in C([0,\infty); H^2(\Omega)) \cap C^1([0,\infty); L^2(\Omega)) \).

The proof of Theorem 1 relies on several Lemmas. The first Lemma is of interest for its own sake; it is a new interpolation-embedding inequality.

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In what follows we denote by $C$ various constants depending only on $\Omega$.

**Lemma 2.** We have

\[
\|u\|_{L^\infty} \leq C(1 + \sqrt{\log(1 + \|u\|_{L^2}))}
\]

for every $u \in H^2(\Omega)$ with $\|u\|_1 \leq 1$.

**Proof.** It is well known that an $H^2$ function on $\Omega$ can be extended by an $H^2$ function on $\mathbb{R}^2$. More precisely one can construct an extension operator $P$ such that:

- $P$ is a bounded operator from $H^1(\Omega)$ into $H^1(\mathbb{R}^2)$
- $P$ is a bounded operator from $H^2(\Omega)$ into $H^2(\mathbb{R}^2)$
- $Pu|_\Omega = u$ for every $u \in H^1(\Omega)$.

Let $u \in H^2(\Omega)$ with $\|u\|_1 \leq 1$. Let $v = Pu$ and denote by $\hat{v}$ the Fourier transform of $v$. We clearly have

\[
\|1 + |\xi|^2\| \hat{v}\|_{L^2(\mathbb{R}^2)} \leq C \|u\|_{H^2(\Omega)}
\]

(3) \[
\|1 + |\xi|^2\| \hat{v}\|_{L^2(\mathbb{R}^2)} \leq C \|u\|_{H^2(\Omega)}
\]

(4) \[
\|I + |\xi|^2\| \hat{v}\|_{L^2(\mathbb{R}^2)} \leq C \|u\|_{H^2(\Omega)}
\]

(5) \[
\|u\|_{L^\infty(\Omega)} \leq \|\hat{v}\|_{L^\infty(\mathbb{R}^2)} \leq C \|v\|_{L^1(\mathbb{R}^2)}
\]

For $R > 0$ we write

\[
\|\hat{v}\|_{L^1(\mathbb{R}^2)} = \int_{|\xi| < R} |\hat{v}(\xi)|^2 d\xi + \int_{|\xi| \geq R} |\hat{v}(\xi)|^2 d\xi = \int_{|\xi| < R} \left(1 + |\xi|^2\right) |\hat{v}(\xi)|^2 \left(1 + |\xi|^2\right) \frac{1}{1 + |\xi|^2} d\xi
\]

\[
+ \int_{|\xi| \geq R} (1 + |\xi|^2) |\hat{v}(\xi)|^2 \frac{1}{1 + |\xi|^2} d\xi
\]

\[
\leq C \left[\int_{|\xi| < R} \frac{1}{1 + |\xi|^2} d\xi\right]^{1/2} + C \|u\|_{H^2} \left[\int_{|\xi| \geq R} \frac{1}{(1 + |\xi|^2)^2} d\xi\right]^{1/2}
\]

by Cauchy-Schwarz, (3) and (4). A straightforward computation leads to
\[ \|u\|_1 \leq C[\log(1 + R)]^{1/2} + C\|u\|_H^2 (1 + R)^{-1} \]

by every \( R \geq 0 \). We obtain (2) by choosing \( R = \|u\|_H^2 \).

**Lemma 3.** We have

\[ \|u\|_{H^2}^2 \leq C\|u\|_L^6 \|u\|_H^2 \] for every \( u \in H^2(\Omega) \).

**Proof of Lemma 3.** Let \( D \) denote any first order differential operator. For \( u \in H^2 \) we have

\[ |D^2(|u|^2)u| \leq C(|u|^2|D^2u| + |u| |Du|^2) \]

and so

\[ \|u\|_{H^2}^2 \leq C\|u\|_L^6 \|u\|_H^2 + C\|u\|_L^6 \|u\|_L^6 \|u\|_L^6 \|u\|_H^2 \]

On the other hand an inequality of Gagliardo-Nirenberg (see [6]) implies that

\[ \|u\|_W^{1,4} \leq C\|u\|_{L^6}^{1/2} \|u\|_H^{1/2} \]

Combining (7) and (8) we obtain (6).

Finally we recall the following well known result essentially due to Segal [7]:

**Lemma 4.** Assume \( H \) is a Hilbert space and \( A : D(A) \subset H \to H \) is an \( m \)-accretive linear operator. Assume \( F \) is a mapping from \( D(A) \) into itself which is Lipschitz on every bounded set of \( D(A) \). Then for every \( u_0 \in D(A) \), there exists a unique solution \( u \) of the equation

\[
\begin{cases}
\frac{du}{dt} + Au = Fu \\
u(0) = u_0
\end{cases}
\]

defined for \( t \in [0, T_{\max}) \) such that

\[ u \in C^1([0, T_{\max}); H) \cap C([0, T_{\max}); D(A)) \]

with the additional property that
either \( T_{\text{max}} \equiv \infty \)

or \( T_{\text{max}} < \infty \) and \( \lim_{t \to T_{\text{max}}} \| u(t) \| + \| \partial u(t) \| = \infty \).

Proof of Theorem 1. We apply Lemma 4 in \( H = L^2(\Omega) \) to \( \partial u = i\omega u, \)

\( D(A) = H^2(\Omega) \cap H_0^1(\Omega), \; Pu = ik|u|^2 u. \) We shall show that \( T_{\text{max}} = \infty \) by proving that \( \| u(t) \| \) remains bounded on every finite time interval.

First we multiply (1) by \( \overline{u} \) and consider the imaginary part. This leads to

\[
\| u(t) \|_{L^2} = \| u_0 \|_{L^2}.
\]

Next we multiply (1) by \( \frac{\partial u}{\partial t} \) and consider the real part. This leads to

\[
\frac{1}{2} \int |\nabla u(x,t)|^2 dx + \frac{k}{4} \int |u(x,t)|^4 dx = E_0
\]

where

\[
E_0 = \frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 dx + \frac{k}{4} \int_\Omega |u_0(x)|^4 dx.
\]

We claim that \( \| u(t) \| \) remains bounded for \( t > 0 \). Indeed, this is clear when \( k \geq 0 \). While if \( k < 0 \) we have

\[
\int |\nabla u(x,t)|^2 \leq \frac{|k|}{2} \int |u(x,t)|^4 dx + 2 E_0.
\]

On the other hand an inequality of Gagliardo and Nirenberg ([6]) shows that (1)

\[
(1) \quad \text{In order to obtain the constant } 1/2 \text{ one proceeds as follows. For } \varphi \in C_0^\infty(\mathbb{R}^2) \text{ we have } |\varphi(x_1,x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_1}(t,x_2)| dt, \]

\[
|\varphi(x_1,x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_2}(x_1,s)| ds. \text{ Thus } \int |\varphi|^2 dx \leq \frac{1}{4} \int_{\mathbb{R}^2} |\varphi_{x_1}| dx \int_{\mathbb{R}^2} |\varphi_{x_2}| dx.
\]

Choosing \( \varphi = |u|^2 \) leads to

\[
\int |u|^4 dx \leq \int |u|^2 dx (\int |u_{x_1}|^2 dx)^{1/2} (\int |u_{x_2}|^2 dx)^{1/2} \leq \frac{1}{2} \int |u|^2 dx + \int |\nabla u|^2 dx.
\]
(12) \[
\int |u|^4 dx \leq \frac{1}{2} \int |u|^2 dx \int |\nabla u|^2 dx
\]
\[
= \frac{1}{2} \int |u_0|^2 dx \int |\nabla u|^2 dx.
\]
Combining (11), (12) and assumption (b) in Theorem 1 we see that
(13) \[
\|u(t)\|_{H^1} \leq C
\]
where \(C\) is independent of \(t\).

We now denote by \(S(t)\) the \(L^2\) isometry group generated by \(-\Delta\). From (1) we have
\[
u(t) = S(t)u_0 + ik \int_0^t S(t-s)|u(s)|^2 u(s) ds
\]
and so
\[
\Delta u(t) = S(t)\Delta u_0 + ik \int_0^t S(t-s)\Delta [|u(s)|^2 u(s)] ds.
\]
Thus
(14) \[
\|\Delta u(t)\|_{L^2} \leq \|\Delta u_0\|_{L^2} + |k| \int_0^t \|\Delta [|u(s)|^2 u(s)]\|_{L^2} ds.
\]
Lemma 3 implies that
\[
\|\Delta [|u(s)|^2 u(s)]\|_{L^2} \leq C\|u(s)\|^2 \|u(s)\|_{H^2}.
\]
From Lemma 2 and estimate (13) we deduce that
\[
\|u(s)\|_{L^2} \leq C(1 + \sqrt{\log(1 + \|u(s)\|_{H^2}^2)}).
\]
Hence (14) leads to
(15) \[
\|u(t)\|_{H^2} \leq C + C \int_0^t \|u(s)\|_{H^2} \left[1 + \log(1 + \|u(s)\|_{H^2}^2)\right] ds.
\]
We denote by $G(t)$ the RHS in (15); thus

$$G'(t) = C \frac{\|u(t)\|}{H^2} [1 + \log(1 + \|u(t)\|)] \leq CG(t)[1 + \log(1 + G(t))].$$

Consequently

$$\frac{d}{dt} \log[1 + \log(1 + G(t))] \leq C$$

and we find an estimate for $\|u(t)\|_2$ of the form

$$\|u(t)\|_2 \leq e^{a e^{\beta t}}$$

for some constants $a$ and $\beta$. Therefore $\|u(t)\|_2$ remains bounded on every finite time interval and so we must have $T_{\max} = \infty$.

Remarks. 1) The proof of Theorem 1 leads to an estimate of the form

$$\|u(t)\|_\infty \leq a e^{\beta t}.$$ We do not know whether $\|u(t)\|_\infty$ remains actually bounded as $t \to \infty$.

2) When $k < 0$ and $|k| \int |u_0|^2 > 4$, it is known (see [4],[2]) if $\Omega = \mathbb{R}^2$ that the solution of (1) corresponding to some initial conditions may blow up in finite time. A similar phenomenon presumably occurs when $\Omega \neq \mathbb{R}^2$. 

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\]