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AN ANALYTIC ANALYSIS OF A KALMAN FILTER AND SMOOTHER

B. Billard

S U M M A R Y

A Kalman filter and smoother which uses data from position measuring instrumentation to estimate changes in position, velocity, and acceleration is derived in detailed analytic form. The filtering and smoothing characteristics are shown to be completely determined by a single non-dimensional parameter, which itself is a simple function of the input parameters to the filter. The concept of the bandwidth of the filter and the smoother is examined, and analytic expressions are derived for the bandwidth of the smoother, and of the filter with respect to position. The analysis highlights situations in which the smoother is clearly superior to the filter.

POSTAL ADDRESS: Chief Superintendent, Electronics Research Laboratory, Box 2151, G.P.O., Adelaide, South Australia, 5001.

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1. INTRODUCTION

There are many situations in which the Kalman filter and smoother are applied, where some of the assumed input parameters (usually the plant noise are measurement noise matrices, Q and R) are unknown. The response to this problem in some cases has been to modify the filter to generate estimates of Q and R as part of the filter, or to modify the filter gain to allow for the fact that Q and R are unknown(ref.1). A summary of these various approaches has been given by Mehra (ref.2). It is noted that the incorporation of the estimation of Q, and R within the filtering process is feasible only when there is enough incoming information from the data collection system. That is, either Q and R must be varying slowly enough for them to be determined within a time scale less than that of their rate of change, or else the state vector, x_k, must be overdetermined by the measurements y_k at each data time t_k.

In applications of the Kalman filter and smoother as a general purpose tool on a testing range(ref.3), Q is normally unknown, and in many cases may be required to vary rapidly over short periods of time. This particularly occurs when the smoother is used to process data from cameras monitoring incidents such as interception, and sharp changes in acceleration. The application of the Kalman filtering and smoothing algorithms in such cases is then of necessity sub-optimum, and the practice has been to use the specification of Q as a tool to control the bandwidth of the filter and smoother. In fact this approach has been so successful in practice that it has been found unnecessary to include deterministic changes in acceleration components when monitoring, say, sharp changes in acceleration. This is despite the fact that Q does not then represent a white noise process, as required by the filter model. (This use of Q is similar to that made sometimes in overcoming filter divergence (see Jazwinski(ref.4) p305)).

The altered use of the plant noise matrix, Q, requires a different interpretation of the Kalman filter and smoother, and it is the purpose of this paper to show how, for the particular general purpose smoother of reference 3, to derive analytic expressions for the filtering gains and the smoothing frequency bandwidth as a function of Q.

As a by-product of this derivation, functional relationships which further elucidate the differences between the Kalman filter and the smoother appear. In particular, the clear superiority of the smoother over the filter becomes apparent when position measuring devices are used to estimate acceleration changes. This reinforces in a different and more specific way the existing general recognition of the comparative performance of the filter and the smoother(ref.5,6).

2. PROBLEM PRESENTATION AND ASSUMPTIONS

The model which will be utilised is basically that of the general purpose Kalman smoother(ref.3), except that the analysis will be restricted to one-dimension. This restriction should not unduly limit the applicability of the results that are obtained, since it is usually possible to find an axis system in which the filter operates essentially independently in each axis. (That is, axes are aligned with the eigenvectors of the measurement noise matrix, R). Thus the state vector, x_k, at time t_k, will be a 3-component vector of missile position, velocity, and acceleration. The equations of state are then -

$$x_k = \Phi(k,k-1)x_{k-1} + \omega_k, \tag{1}$$

where $E(\omega_k\omega_k^T) = Q_k$, $E(\omega_k) = 0$, and ω_k is the plant noise.

Here $\Phi(k,k-1)$ is the transition matrix defined according to a constant acceleration model. Thus, for $h = (t_k - t_{k-1})$,

$$\Phi(k,k-1) = \begin{bmatrix} 1, & h, & h^2/2 \\ 0, & 1, & h \\ 0, & 0, & 1 \end{bmatrix}$$

The equations describing the measurement system are

$$\begin{aligned} \delta y_k &= H_k \delta x_k + \nu_k, \\ E(\nu_k \nu_k^T) &= R_k,\ E(\nu_k) = 0 (\nu_k \text{ is the measurement noise}). \end{aligned} \tag{2}$$

We also assume that the measurement system directly (rather than indirectly) measures position, so that, if there are m measurements at time t_k, H_k is an mx3 matrix which is zero except for the first column which is all ones.

The filtering algorithms are then

$$\left.\begin{aligned} x_{k/k-1} &= \Phi(k,k-1)\ x_{k-1/k-1}, \\ P_{k/k-1} &= \Phi(k,k-1)\ P_{k-1/k-1}\ \Phi^T(k,k-1) + Q_k, \\ \Gamma_k &= H_k^T R_k^{-1} H_k, \\ \zeta_k &= H_k^T R_k^{-1} \delta y_k, \\ P_{k/k} &= P_{k/k-1}(I + \Gamma_k P_{k/k-1})^{-1}, \\ \text{and} \\ x_{k/k} &= x_{k/k-1} + P_{k/k}\ \zeta_k. \end{aligned}\right\} \tag{3}$$

Here δy_k is the vector of differences between the data and the data implied by $x_{k/k-1}$. In particular, $\delta y_k = H_k(x_k - x_{k/k-1}) + \nu_k$ since we are assuming direct measurement. Γ_k is the information matrix, and will always be zero except $(\Gamma_k)_{11} = \gamma$, say. In interpreting Γ_k and ζ_k it should be noted that if there is a single measurement of position, with standard deviation σ at each data time t_k, then $(\Gamma_k)_{11} = 1/\sigma^2$, and $(\zeta_k)_1 = (y_k - (x_{k/k-1})_1)/\sigma^2$. We may use this knowledge of the form of Γ_k to replace the fourth and sixth equations of the system of equations (3), by the single equation

$$x_{k/k} = A\ x_{k-1/k-1} + P\Gamma\ x_k \tag{4}$$

and we have found it convenient to define $A = (I - P\Gamma)\Phi$. (5)

In defining Q_k, there are two equally reasonable approaches.

If, as is usually the case, equation (1) is a discrete approximation to a continuously evolving system, then we may model the noise, Q_k, by assuming it to arise from a continuous white noise in the acceleration component. Thus $dQ/dt = \dot{q}$, say, and the discrete Q_k may be found by integrating the continuum equivalent of equation (3) from t_{k-1} to t_k. (The form of the integral becomes obvious if it is envisaged that there are a large number of "dummy" data points inserted between t_{k-1} and t_k for which there is no incoming data. i.e. $\Gamma = \zeta = 0$. Equation (3) is then recursively applied through these points). Thus we have

$$Q_k = \int_{t_{k-1}}^{t_k} \Phi(t_k - \tau) dQ/d\tau \; \Phi^T(t_k - \tau) \, d\tau$$

$$= \dot{q} \begin{bmatrix} 1/20\ h^5, & 1/8\ h^4, & 1/6\ h^3 \\ 1/8\ \ h^4, & 1/3\ h^3, & 1/2\ h^2 \\ 1/6\ \ h^3, & 1/2\ h^2, & h \end{bmatrix}$$

It is slightly less accurate to assume that noise is added to the acceleration component in "lumps" of $h\dot{q} = q$ at each data time. Then

$$Q_i = q \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{bmatrix}. \qquad (6)$$

The two definitions might be expected to give divergent results only when data is arriving at sufficiently irregular time intervals, (in which case the definition (6) would be less reasonable).

In the present analysis, however, we are looking for steady state relationships linking Q_k with other filtering parameters, and it is therefore reasonable to assume a constant data rate ($h = t_k - t_{k-1}$, all k), and to use the much simpler definition (6). This definition will in any case give answers accurate to o(h).

The final equation needed to complete the filtering system arises from the assumption of a steady state in the filtering process. Then

$$P_{k/k} = P_{k-1/k-1} = P, \text{ say}, \qquad (7)$$

and we set

$$P_{k+1/k} = P_{k/k-1} = P'.$$

The smoothing system is defined by

$$x_{k/N} = x_{k/k} + C_k(x_{k+1/N} - x_{k+1/k}), \qquad (8)$$

where

$$C_k = P_{k/k} \Phi^T(k+1,k) P^{-1}_{k+1/k} = P \, \Phi^T P'^{-1} = C.$$

The equation for P' in equation (3) and equation (5) can be used to alternatively express C as

$$C = PA^T P^{-1}. \qquad (9)$$

3. BASIC FUNCTIONS AND DEFINITIONS

3.1 Impulse response sequence

The impulse response sequence, $\underline{H}$ (n), of a digital filter is defined such that

$$f(n) = \sum_{k=-\infty}^{\infty} \underline{H}(k)\ g(n-k),$$

where g(n) is the signal sequence, and f(n) the filtered signal(ref.7).
For the filter, as defined by equation (4) we may show that

$$x_{k/k} = \sum_{N=0}^{\infty} A^N\ P\ \Gamma\ x_{k-N},$$

and hence the filter impulse response sequence is

$$\underline{H}_F(n) = \begin{cases} 0 & , n < 0, \\ A^n\ P\Gamma, & n \geqslant 0. \end{cases} \tag{10}$$

To derive the impulse response sequence for the smoother we take the basic recursive relation of equation (8) and reformulate it by repeated substitution of $x_{k+1/\infty}$, $x_{k+2/\infty}$ to give

$$x_{k/\infty} = \sum_{j=0}^{\infty} C^j\ (I-C\Phi)\ x_{k+j/k+j},$$

and hence

$$x_{k/\infty} = P \sum_{J=0}^{\infty} (A^T)^J T \sum_{L=0}^{\infty} A^L P\Gamma\ x_{k+J-L},$$

where we have set $T = P^{-1}(I-C\Phi)$.
By substituting N = J-L, and then breaking up the summation over N into 2 parts $\sum_{-\infty}^{-1} + \sum_{0}^{J}$ we have

$$x_{k/\infty} = P \sum_{J=0}^{\infty} (A^T)^J T \left[\sum_{N=1}^{\infty} A^{J+N}\ P\Gamma x_{k-N} + \sum_{N=0}^{J} A^{J-N}\ P\Gamma x_{k+N} \right].$$

If we define $\underline{T} = \sum_{J=0}^{\infty} (A^T)^J T A^J$, and reverse the order of the second summation, we may show by manipulation that

$$x_{k/\infty} = P\underline{T} \sum_{N=1}^{\infty} A^N P\Gamma x_{k-N} + P \sum_{N=0}^{\infty} (A^T)^N \underline{T} P\Gamma x_{k+N}.$$

Thus, the smoother impulse response sequence is defined as

$$\underline{H}_s(N) = \begin{cases} P\,\underline{T}\,A^N\,P\Gamma \quad , N > 0, \\ P(A^T)^{-N}\,\underline{T}\,P\Gamma \ , N \leqslant 0, \end{cases} \tag{11}$$

and by manipulation of the definition of we may show that satisfies

$$\underline{T} = T + A^T \underline{T} A$$

Instead of solving this equation for $\underline{T}$, it is slightly more convenient to use an associated matrix S, defined as

$$S = \sum_{J=0}^{\infty} (A^T)^J \Gamma\Phi\, A^J.$$

S satisfies

$$S = \Gamma\Phi + A^T SA. \tag{12}$$

We may show that

$$TA = P^{-1}A - A^T P^{-1}\Phi A = P^{-1}\Phi - A^T P^{-1}\Phi A - \Gamma\Phi,$$

and hence that

$$\underline{T} = (P^{-1}\ \Phi\text{-}S)A^{-1}. \tag{13}$$

3.2 Filter system gains

If at a time t_k there is a step in $(x_k)_1$ (position) of D, then the filter will respond with a correction of $D\alpha_1$ to the predicted position coordinate. α_1 is called the filter gain with respect to position. Its inverse is the approximate number of data points required before the filtered state vector $x_{k/k}$ will fully respond to the step change D in position.

By considering step changes in velocity and acceleration in the input signal we may arrive at similar definitions of α_2 and α_3, the velocity and acceleration filter gains respectively. These definitions are in fact equivalent to the definitions of the gains α, β and γ of the α-β-γ filter.

Although a_1 represents the (non-dimensional) bandwidth of the filter with respect to position, equivalent statements cannot be made of a_2 and a_3, for reasons which will become more apparent in Section 5.

For a step change D in position at t_k, we have

$$\zeta_k^T = (\gamma D, 0, 0),$$

and hence from equations (3),

$$Da_1 = (P\zeta_k)_1 = P_{11}\,\gamma\, D.$$

(Here we have used the notation

$$P = (P_{ij});\quad i,j = 1,2,3).$$

Hence

$$a_1 = \gamma\, P_{11}. \tag{14}$$

In the same way a step change of D in velocity will lead at the next data time to an error in predicted position of hD. Then $\zeta^T = (\gamma hD, 0, 0)$, and the initial correction to velocity is an amount

$$Da_2 = (P\zeta)_2 = P_{12}\gamma hD.$$

Hence

$$a_2 = \gamma h P_{12}. \tag{15}$$

Similarly we can show that

$$a_3 = 1/2\,\gamma\, h^2\, P_{13}. \tag{16}$$

4. ANALYTIC SOLUTIONS

4.1 Filter

Letting $P' = (P'_{ij})$, and $d = (1 + \gamma P'_{11})^{-1}$, the equation for $P_{k/k}$ in equation (3) may be expanded to give

$$P = d\begin{bmatrix} P'_{11}, & P'_{12} & , & P'_{13} \\ P'_{12}, & P'_{22}/d - \gamma P'^{2}_{12}, & , & P'_{23}/d - \gamma P'_{12}\, P'_{13} \\ P'_{13}, & P'_{23}/d - \gamma P'_{12}\, P'_{13}, & & P'_{33}/d - \gamma P'^{2}_{13} \end{bmatrix}$$

The equation for $P_{k/k-1}$ in equation (3) may be similarly expanded to give P' in terms of the P_{ij}. The steady state condition (7) may then be applied to eliminate the P'_{ij}, and give 6 equations (note P and P' are symmetric) in the 7 parameters P_{11}, P_{12}, P_{13}, P_{22}, P_{23}, P_{33} and q. Thus, as expected, specification of q determines P, and in particular the filtering gains which are related to P_{11}, P_{12} and P_{13}.

The equations (after some manipulation) are

$$\left.\begin{aligned} dq &= \gamma P_{13}^2, \\ dhP_{33} &= \gamma P_{13}\ P_{12}, \\ (1-d)P_{13} &= d(hP_{23} + 1/2\ h^2\ P_{33}), \\ 1/2\ \gamma P_{12}^2 &= d(hP_{23} + 1/2\ h^2\ P_{33}), \\ (1-d)P_{12} &= d(hP_{22} + hP_{13} + 3/2\ h^2 P_{23} + 1/2\ h^3 P_{33}), \\ \text{and} \qquad 0 &= (2hP_{12} + h^2 P_{13} + h^2 P_{22} + h^3 P_{23} + 1/4\ h^4 P_{33}). \end{aligned}\right\} \qquad (17)$$

Here we have chosen to carry d separately, since we may use equation (14) directly to show that $d = 1-\alpha_1$.

By substituting from equations (14) to (16) we may solve the equations (17) to give the functional relationships

$$\left.\begin{aligned} 4\alpha_1\alpha_3 &= \alpha_2^2, \\ \text{and} \qquad \alpha_2^2 - 4(2-\alpha_1)\alpha_2 + 4\alpha_1^2 &= 0. \end{aligned}\right\} \qquad (18)$$

If we define $g = 2-\alpha_1 + 2(1-\alpha_1)$, we may show by manipulation that

$$\alpha_2 = 2\alpha_1^2/g, \text{ and } \alpha_3 = \alpha_1^2/g^2. \qquad (19)$$

Note in particular that the relationships (18) and (19) are dependent only on the structure of the filtering system (the constant acceleration model for missile dynamics, the plant noise model, and the position measuring instrumentation), and not on the parameters of that system.

An alternative form of equation (19) which may be useful is found by setting $r^2 = (1-\alpha_1)$. Then

$$\alpha_1 = 1-r^2,\ \alpha_2 = 2(1-r)^2, \text{ and } \alpha_3 = (2-r)^3/(1+r). \qquad (20)$$

The equation for the noise increment, q, may be found by substituting for P_{13} in equation (17):

$$q = 4\alpha_3^2/(d\gamma h^4).$$

This equation may alternatively be reconstructed so as to define a non-dimensional parameter of the filtering system,

$$\epsilon^2 = \alpha_3^2/(1-\alpha_1) = \gamma q h^4/4, \qquad (21)$$

and ϵ will then determine all other variables in the system.

In terms of r, equation (21) becomes

$$\epsilon = (1-r)^3/(r(1+r)).$$

The remaining terms in P may also be found from equation (17):

$$\begin{aligned} P_{22} &= \alpha_2^2(12-10\alpha_1-\alpha_2)/(8h^2\gamma\alpha_1(1-\alpha_1)), \\ P_{23} &= \alpha_3(2\alpha_1-\alpha_2)/(h^3\ \gamma(1-\alpha_1)), \text{ and} \\ P_{33} &= 2\alpha_2\ \alpha_3/(h^4\ \gamma(1-\alpha_1)). \end{aligned}$$

4.2 Smoother

Having obtained P, Φ and Γ in analytic form, it is obviously a straightforward step, though perhaps tedious, to substitute in the equation (5) and (9) to find the smoothing matrix, C, in analytic form. However, rather than follow this path, we choose to seek an analytic form of S, which plays a key role in the impulse response form of the smoother (equation (11)), and which will also be important in determining the step function response in Section 5.

We may immediately note from the definitions of A and S, and our knowledge that $P \propto 1/\gamma$, that A is independent of γ, and $S \propto \gamma$.

To examine the dependence of these functions on $h = (t_k - t_{k-1})$ we note that

$$\Phi_{ij} \propto h^{j-i},$$

and

$$P_{ij} \propto h^{2-i-j}.$$

It follows from the definition of A that

$$A_{ij} \propto h^{j-i},$$

and hence we can show that equation (12) is satisfied if

$$S_{ij} \propto h^{i+j-2}.$$

The same technique may also be applied to equation (11) to show that

$$(\underline{H}_s(N))_{ij} \propto h^{j-i}, \text{ all } N.$$

Having determined the dependence of all smoothing matrices on γ and h, we may without loss of generality assume $\gamma = h = 1$ in subsequent calculations.

By direct calculation we may show

$$A = \begin{bmatrix} 1-\alpha_1, & 1-\alpha_1, & 1/2(1-\alpha_1) \\ -\alpha_2, & 1-\alpha_2, & 1/2(1-\alpha_2) \\ -2\alpha_3, & -2\alpha_3, & 1-\alpha_3 \end{bmatrix},$$

and

$$A^{-1} = \begin{bmatrix} \beta_1, & -1, & 1/2 \\ \beta_2, & 1, & -1 \\ 2\beta_3, & 0, & 1 \end{bmatrix},$$

where

$$\begin{aligned} \beta_1 &= (1-\alpha_2+\alpha_3)/\beta_4, \\ \beta_2 &= (\alpha_2-2\alpha_3)/\beta_4, \\ \beta_3 &= \alpha_3/\beta_4, \end{aligned}$$

and

$$\beta_4 = (1-\alpha_1).$$

Because of the relative simplicity of A^{-1}, it is simpler to reform equation (12) in terms of A^{-1} rather than A. Then

$$(A^{-1})^T S A^{-1} - S = (A^{-1})^T \Gamma\Phi A^{-1}.$$

The solution of this set of 9 equations in the 9 unknowns S_{ij}, i, j = 1,2,3 was in the first instance attempted by hand. A range of relatively simple subrelations could be derived, but they were still too cumbersome to solve in this way. The relations were

$$\begin{aligned}
&S_{11} = S_{12} + S_{21},\\
&S_{12} = 2(S_{13} + S_{22}),\\
&S_{21} = 2(S_{31} + S_{22}),\\
&8\beta_3 S_{33} = 1+2(\beta_4-1)(S_{13}+S_{31}),\\
&S_{31} = S_{32} = -S_{23},\\
&2(1+\beta_4)(S_{13}+S_{31}) + 2(2\beta_1+\beta_2+2\beta_4) S_{22} = 1,\\
&2c(S_{13}-S_{31}) = \beta_2+\beta_3, \text{ and}\\
&2c\, S_{31} = -\beta_4, \text{ where}\\
&c = \beta_2+\beta_3-\beta_4(\beta_2+3\beta_3).
\end{aligned}$$

These subrelations were then solved using routines developed in APL to manipulate polynomials of several variables(ref.8). The parameters β_1, β_2, β_3, β_4 were represented initially as polynomials in r and s = 1/(1+r), using the definitions of equations (20). Subsequently the solutions S_{ij} could be represented as rational polynomials in r. The solutions which were found using this technique give

$$S = \begin{bmatrix} 4(1-r)^2\ (1+2r-r^2), & 2(1-r)^2\ (2+5r-r^2), & 1+4-6r^2 + 2r^3 \\ 2r(1-r)^2\ (1+r) & , -r(1+r)\ (1-4r+r^2), & 2r^2(1+r) \\ -2r^2(1+r) & , -2r^2(1+r) & , 4r^3 \end{bmatrix} /\Delta,$$

where $\Delta = 4(1-r)^3(1+4r+r^2)$, and we recall that $a_1 = 1-r^2$.

5. RESPONSE TO STEP FUNCTION

5.1 General step function

In this section we shall consider how the impulse response sequences derived in Section 3 may be used to find the filter and smoother responses to step changes in the input signal. We are particularly interested in the three separate cases of step changes in position, velocity, and acceleration. However they may be treated together by using a general input signal of

$$x_k = \begin{cases} \Phi^k x_-, & k < 0, \\ x_0, & k = 0, \\ 0, & k > 0, \end{cases} \tag{22}$$

where x_- and x_0 are constant vectors to be defined later.

The restriction for $k > 0$ is not severe since we may easily recover the complementary set of step functions (with $x_k = 0$, $k < 0$) by using the linearity of the filter and smoother.

The equation (1) of Section 3.1 may be applied to the signal x_k to give

$$x_{k/k} = \sum_{N=k}^{\infty} A^N P\Gamma x_{k-N}, \; k \geqslant 0$$

and hence

$$x_{k/k} = \begin{cases} A^k \; x_{o/o}, & k \geqslant 0, \\ \Phi^k \; x_- , & k < 0. \end{cases} \tag{23}$$

The equations (11) for the smoother may similarly be applied and simplified for $k \geqslant 0$ to show that

$$x_{k/\infty} = (\Phi\text{-PS})A^{k-1} \; x_{o/o} = P \; \underline{T} \; x_{k/k}, \; k \geqslant 0. \tag{24}$$

Consider a signal X_k such that

$$X_k = \Phi^k \; x_- - x_k .$$

That is, X_k is the step function complementary to x_k, for which $X_k = 0$, $k < 0$. Because of the linearity of the smoother we know that $X_{k/\infty} = \Phi^k x_- - x_{k/\infty}$.
If we apply the impulse function (11) to X_k we have, for $k < 0$

$$X_{k/\infty} = \sum_{N=k}^{-\infty} P(A^T)^{-N} \; \underline{T} \; P\Gamma \; X_{k-N}$$

and hence

$$X_{k/\infty} = P(A^T)^{|k|} \; P^{-1} \; X_{o/\infty}.$$

Substituting for $X_{k/\infty}$ and $X_{o/\infty}$ we have finally

$$x_{k/\infty} = P(A^T)^{|k|} \underline{T} \; x_{o/o} + (\Phi^k - P(A^T)^{|k|} \; P^{-1})x_- , \; k < 0 \tag{25}$$

Equations (23) to (25) thus completely define in terms of analytically known matrices the filter and smoother response to the general step function signal defined by equation (24).

5.2 Numerical solutions

We consider the input signal defined by equation (22), for which $x_o = x_-$, and hence $x_{o/o} = x_-$.

In particular we are interested in determining the ability of the filter and the smoother to estimate step changes in position, velocity, and acceleration respectively.

Thus, for step in position, $x_{1-} = (1,0,0)^T$;

for step in velocity, $x_{2-} = (0,1,0)^T$;

and for a step in acceleration, $x_{3-} = (0,0,1)^T$.

The filter and smoother responses comparable to each of these steps will simply be the appropriate diagonal term of the matrix coefficients of x_- in equations (23), (24) and (25).

Figures 1 and 2 give the results of numerical calculation of these responses for two separate values of α_1. Each figure shows the filtered position component when there is an input step in position ($x_- = x_{1-}$); the filtered velocity component when $x_- = x_{2-}$; the filtered acceleration component when $x_- = x_{3-}$; and the smoothed position component when $x_- = x_{1-}$. The other smoothed curves (for velocity and acceleration) were not included, as they were found to have exactly the same shape as the smoothed position component, except that they were displaced by half a data point and one data point respectively to the left.

It is quite easy to show analytically that an appropriate choice of x_o as $\frac{1}{2}x_-$ or 0 will simply displace the smoothed position component half or one data point respectively to the left. However the problem of showing analytically that the respective smoothed position and velocity, or position and acceleration components are identical is not readily soluble (though it should be possible). (It is of course quite straightforward to prove the necessary identities for $k = 0,1,-1$).

There are several features which are immediately apparent from the figures 1 and 2.

(i) The fact that the smoother curves are identical means that the smoother is able to accurately estimate changes in the derivatives of the measured components of the state vector. At DRCS this has been utilized extensively to allow camera instrumentation to estimate sharp changes in acceleration. It is clear that the filter does not perform nearly so well in this area. That is, its performance deteriorates rapidly as it is called on to estimate higher derivatives of the measured component.

(ii) The filter velocity and acceleration curves show a clear lag before there is a response to the step change in velocity and acceleration respectively. Since we may show from equation (23) that the position, velocity, and acceleration curves at $k = 1$ are simply $(1-\alpha_1)$, $(1-\alpha_2)$ and $(1-\alpha_3)$, then it is evident that α_1 is a reasonable measure of bandwidth, whereas α_2 and α_3 are not, because of the lag.

(iii) Overshoot with the smoother is very much less than for any of the filter curves.

5.3 Smoother response

Rather than rely on numerical results, it is possible to derive an analytical expression which gives a reasonable measure of the bandwidth of the smoother.
From equation (24) we have

$$x_{1/\infty} - x_{o/\infty} = (\Phi\text{-PS})\ (I-A^{-1})\ x_{o/o}.$$

Hence the magnitudes of the slopes of each of the smoothed curves at k = 0 are simply the diagonal elements of

$$W = -(\Phi\text{-PS})(I-A^{-1}). \tag{26}$$

We choose to define the smoother bandwidth, α_s, in terms of the smoothed velocity response to a step in velocity, since this curve passes through the midpoint at k = 0,

$$(\text{i.e. } (x_{o/\infty})_2 = 1/2(x_-)_2).$$

All the matrices on the right hand side of equation (26) are known analytically, and hence we may use the APL polynomial manipulation routines to show that

$$\alpha_s = W_{22} = (1-r^2)/(1+4r+r^2), \tag{27}$$

where $r^2 = (1-\alpha_1)$. (Note that all diagonal terms of W will be independent of γ and h).
We may also note from this equation that

$$\alpha_1/\alpha_s = 1+4r+r^2 > 1,$$

that is, for a given set of input parameters to the filter and the smoother, the smoother will exhibit a narrower bandwidth than will the filter with respect to position. This might be expected from intuition, since the smoother is influenced by twice as much data as the filter.
The relationship between α_1 and α_s is plotted in figure 3.

6. CONCLUSION

This paper has presented derivations of the detailed analytic form of a Kalman filter and smoother when position-measuring instrumentation is used to estimate position, velocity, and acceleration changes in a point under observation. In the course of an analysis of this system, a non-dimensional parameter, ϵ, has been defined (equation (21)) which is shown to completely determine the filtering and smoothing characteristics of the system.

Through analysis of the response of the system to various input step functions, a relationship is derived connecting the bandwidth of the smoother to the bandwidth of the filter with respect to position. This relationship shows that for given input parameters, the filter will have greater bandwidth (with respect to position) than will the smoother.

The analysis also shows the clear superiority of the smoother over the filter when changes in velocity and acceleration are being estimated with position-measuring instrumentation.

In two specific respects the analysis is incomplete. Firstly, it has not been possible to derive a simple analytic representation of the bandwidth of the filter with respect to velocity and acceleration, though it is clear from figures 1 and 2 that they would differ markedly from that for position. Secondly, it has not been possible to show analytically that there is an identically shaped smoother output of position, velocity, and acceleration when there is a step in the input signal of the same component.

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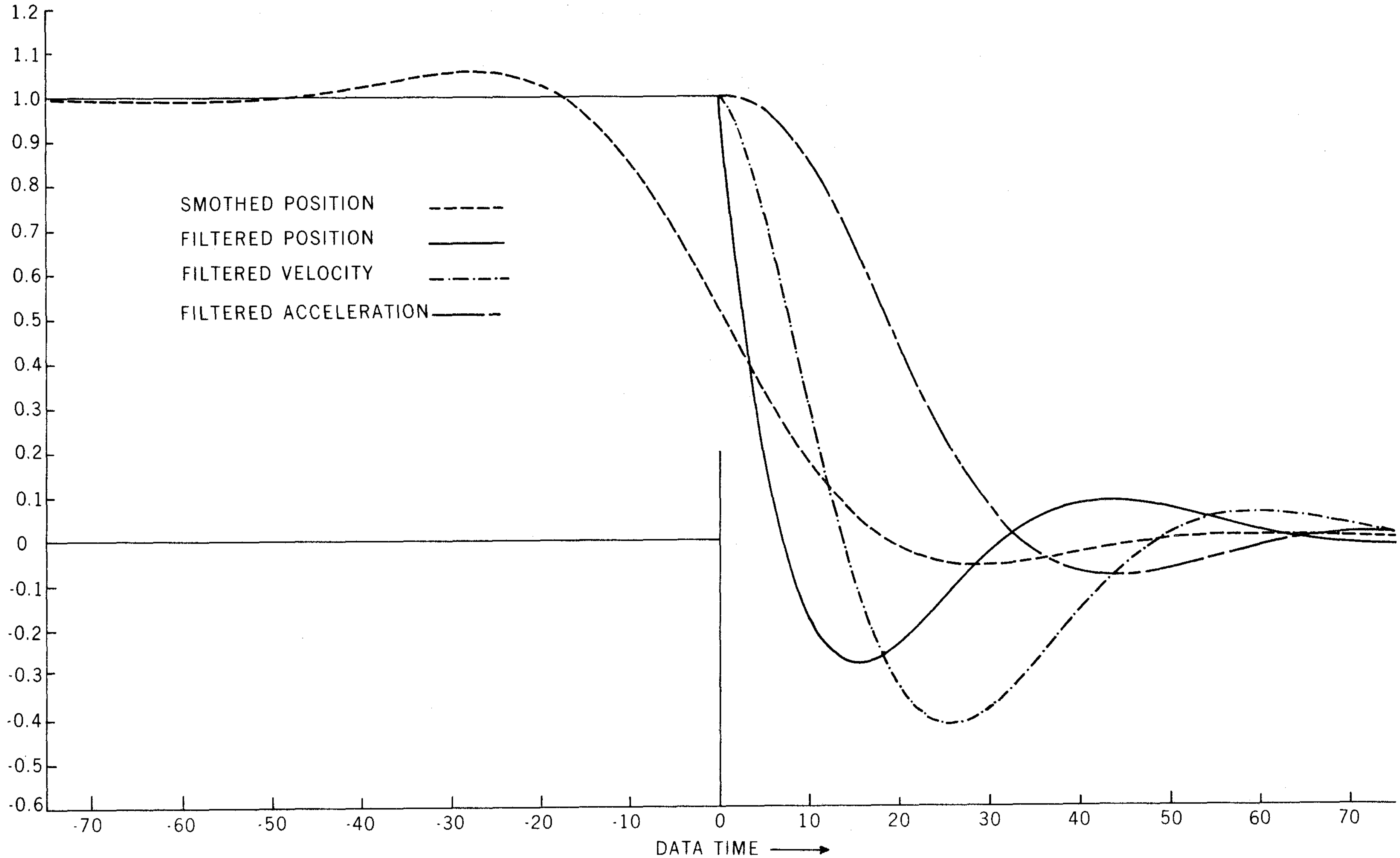

Figure 1. Response to a step in the input signal of the corresponding component
$a_1 = 0.2$

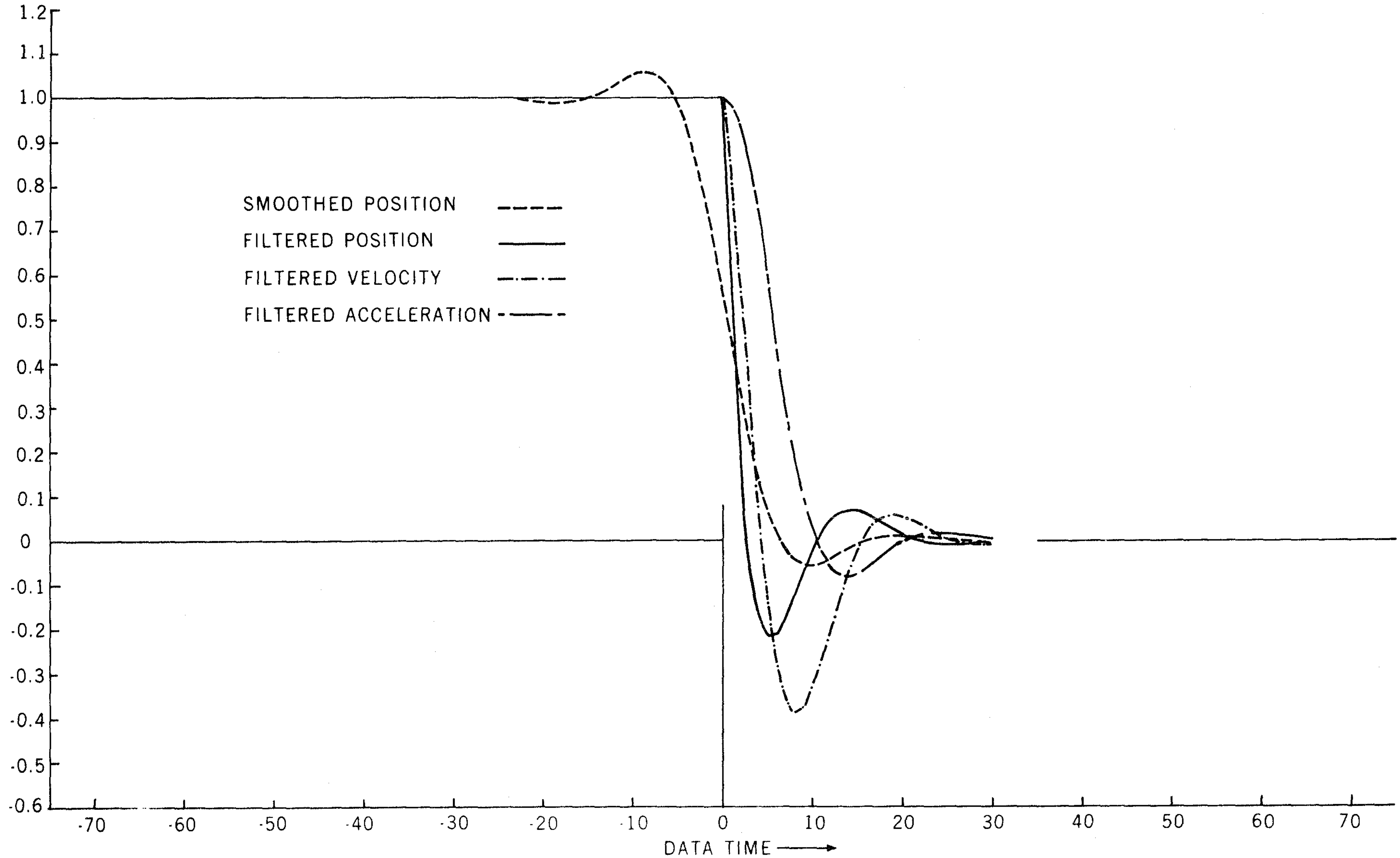

Figure 2. Response to a step in the input signal of the corresponding component $a_1 = 0.5$

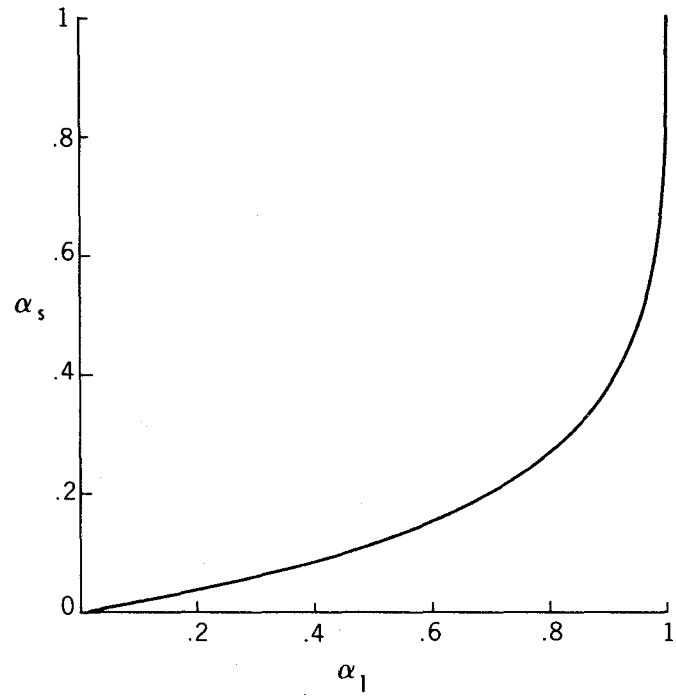

Figure 3. Smoothing response (α_s) vs filtering response w.r.t. position (α_1)

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A Kalman filter and smoother which uses data from position measuring instrumentation to estimate changes in position, velocity, and acceleration is derived in detailed analytic form. The filtering and smoothing characteristics are shown to be completely determined by a single non-dimensional parameter, which itself is a simple function of the input parameters to the filter. The concept of the bandwidth of the filter and the smoother is examined, and analytic expressions are derived for the bandwidth of the smoother, and of the filter with respect to position. The analysis highlights situations in which the smoother is clearly superior to the filter.

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