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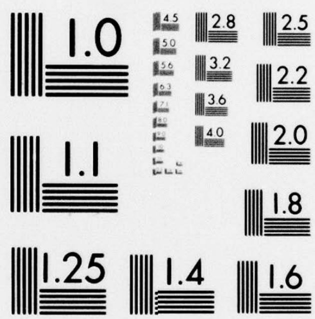
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# Approximate Confidence Intervals for an Exponential Parameter from a Sequential Life Test

by

Kenneth B. Fairbanks

Technical Report No. 88
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August 1979





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from a truncated sequential test scheme, using methods which depend heavily on numerical techniques using a computer. A more flexible approach is considered using a martingale inequality which was also given by Wald (1947) in another context. Interval estimates are found which are functions of a positive constant d which must be chosen less than an upper bound which is itself a function of the number of failures observed. It is suggested that d be chosen as a function of the sample path (i.e., after the test is complete). The validity of the confidence coefficient comes into question if this posterior selection of d is employed. Simulation studies indicate that the resulting intervals are usually conservative.

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Approximate Confidence Intervals for an Exponential Parameter from a Sequential Life Test

## Abstract

A sequential life test for the exponential location parameter was given by Epstein and Sobel (1955). This sequential test may be modified by truncating the test at r sub ∅ $r_0$ failures and/or at total test time $t'$. There may be a need or a desire to also estimate the parameter after the test decision, using the test data. Bryant and Schmee (1979) have given confidence intervals for the mean lifetime, $\theta$, Theta from a truncated sequential test scheme, using methods which depend heavily on numerical techniques using a computer. A more flexible approach is considered using a martingale inequality which was also given by Wald (1947) in another context. Interval estimates are found which are functions of a positive constant $d$ which must be chosen less than an upper bound which is itself a function of the number of failures observed. It is suggested that $d$ be chosen as a function of the sample path (i.e., after the test is complete). The validity of the confidence coefficient comes into question if this posterior selection of $d$ is employed. Simulation studies indicate that the resulting intervals are usually conservative.

# Approximate Confidence Intervals for an Exponential Parameter from a Sequential Life Test

## I. Introduction

The theory of sequential tests was developed by Abraham Wald (1947). Epstein and Sobel (1955) applied this theory directly to get a sequential life test in the exponential case. This requires the assumption that, if $\tau$ is the lifetime under consideration, the probability density function of $\tau$ is given by

$$f(\tau) = \begin{cases} \frac{1}{\theta} e^{-\tau/\theta} & \text{if } \tau \geq 0 \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta > 0$ is the average lifetime of an item. In the sequential exponential life test, the continuation region is bounded by two parallel lines inthe failure-total time plane as depicted in figure 1.1. Initially $n$ items are placed on test with or without replacement. At each failure time a decision is made to accept or reject the null hypothesis, or to continue the test. The test continues as long as the sample

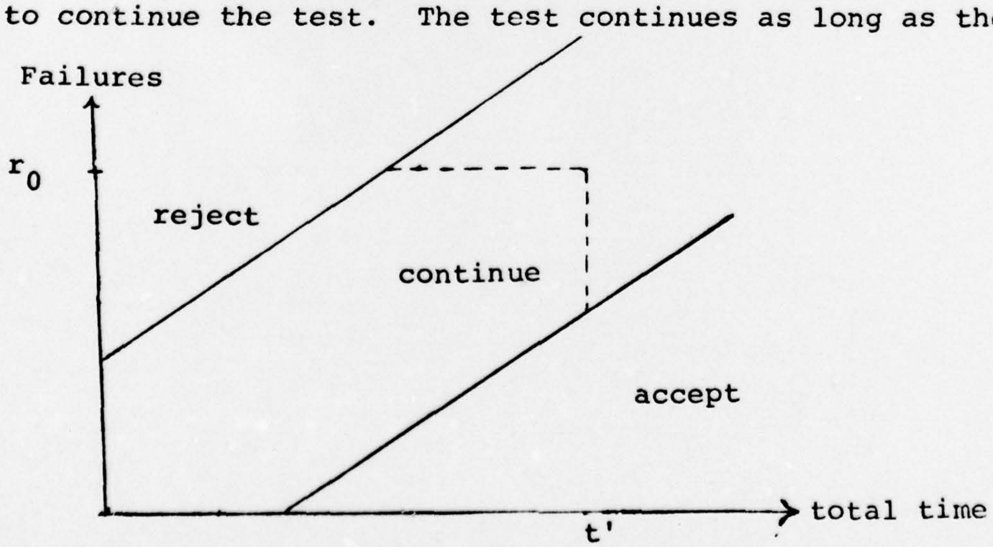


Figure 1.1. Sequential life test boundaries with time and failure truncation.

path remains inside the continuation boundaries. A modification of this test is to truncate the continuation region at $r_0$ failures and/or at total time $t'$. This truncation is also illustrated in figure 1.1. These truncated sequential tests are used by the Navy as reliability acceptance tests in MIL-STD-781C (1977). The tables given in MIL-STD-781C are for testing $H_0$: $\theta = \theta_0$ against $H_1$: $\theta = \theta_1$ where $\theta_1 < \theta_0$. Since values of $\theta$ considerably different from $\theta_0$ or $\theta_1$ could lead to the acceptance or rejection of $H_0$, it may be of interest to also estimate $\theta$ following the test decision using the test data. This report gives a method for finding approximate confidence intervals for $\theta$ from the sequential test or the truncated sequential test.

Bryant and Schmee (1979) have found confidence intervals for $\theta$ following the truncated sequential tests in MIL-STD-781C. Their results require the application of a technique developed by Aroian (1963), and Epstein, Patterson and Qualls (1963). This technique, which Aroian calls the "direct method," reduces the continuation region to a grid of discrete points, including a set of points on the boundary of the continuation region. Using an iterative procedure and the properties of the exponential distribution, a probability for each point on the boundary can be found as a function of $\theta$. Confidence intervals for $\theta$ can then be found using numerical techniques with the computer. A drawback of this method is the amount of computation required and its dependence on the location of the continuation boundary. All computations may have to be repeated if the value of a test parameter is changed.

The objective of this report is to find a method which is easier to apply computationally and perhaps is more versatile. Our approach is to use density ratios defined at each failure time and apply an inequality which was given by Wald (1947). He suggested the inequality could be applied to sequential estimation problems. O'Brien (1973) uses this inequality to find sequential confidence intervals for the shape parameter of the gamma distribution. The inequality is also known to be a special case of a well known martingale inequality. Robbins (1970) and Lai (1976) also have results in sequential estimation from this inequality.

In this report, the inequality is applied in the manner used by O'Brien, wherein the density ratio involves a positive constant which must be chosen prior to the experiment in some optimal fashion. The derivation of the intervals in this report produces a constraint on this constant which is a function of the number of failures observed. Consequently, it is suggested that the intervals may be improved if the constant is chosen as a function of the sample path generated by the failure times. Whether this posterior method of selecting the constant invalidates the desired confidence coefficient remains an open question. Computer simulations indicate that the resulting intervals remain conservative, and that, while their width is generally greater than the Bryant-Schmee intervals, the comparison is favorable in light of the utility of these intervals.

## II. Testing without Replacement

If $n$ items with exponential lifetimes are placed on test without replacement, then denote the ordered failure times as $t_1, t_2, \ldots, t_n$. Define $X_i \equiv t_i - t_{i-1}$, where $t_0 = 0$. The $X_i$, $i = 1, 2, \ldots, n$, are independent with density functions.

$$f(x_i;\theta) = \begin{cases} \frac{(n-i+1)}{\theta} e^{-\frac{(n-i+1)x_i}{\theta}}, & x_i > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Consequently we can write the joint density of $X_1, X_2, \ldots, X_k$ as

$$\prod_{i=1}^{k} f(x_i;\theta)$$

or as

$$f(x_1,x_2,\ldots,x_k) = \begin{cases} \frac{n!}{(n-k)!\,\theta^k} e^{-\frac{1}{\theta}\sum_{i=1}^{k}(n-i+1)x_i}, & x_i > 0 \\ & i=1,2,\ldots,k \\ 0 & \text{elsewhere} \end{cases}$$

Equivalently,

$$f(x_1, x_2, \ldots, x_k) = \frac{n!}{(n-k)!\,\theta^k} e^{-\frac{T_k}{\theta}},$$

where

$$T_k = \sum_{i=1}^{k} (n-i+1)x_i = \sum_{i=1}^{k} t_i + (n-k)t_k$$

$$= \text{total time on test at } t_k.$$

Also, define

$$Y_i = \begin{cases} \dfrac{f(X_1, X_2, \ldots, X_i; d_1\theta)}{f(X_1, X_2, \ldots, X_i; \theta)} & \text{if } i = 1, 2, \ldots, n \\[2ex] Y_n & \text{if } i > n. \end{cases}$$

At this point $d_1$ is an arbitrary positive constant. It can be shown quite easily that $Y_1, Y_2, \ldots$ is a martingale.

$$E[Y_{k+1} | Y_k, \ldots Y_1] = E\left[\frac{\prod_{i=1}^{k+1} f(X_i; d_1\theta)}{\prod_{i=1}^{k+1} f(X_i; \theta)} \,\Big|\, Y_k, \ldots, Y_1\right]$$

$$= \frac{\prod_{i=1}^{k} f(X_i; d_1\theta)}{\prod_{i=1}^{k} f(X_i; \theta)} E\left[\frac{f(X_{k+1}; d_1\theta)}{f(X_{k+1}; \theta)} \,\Big|\, Y_k, \ldots, Y_1\right]$$

$$= Y_k E\left[\frac{f(X_{k+1};d_1\theta)}{f(X_{k+1};\theta)}\right] = Y_k \int_0^\infty \frac{f(x_{k+1};d_1\theta)}{f(x_{k+1};\theta)} f(x_{k+1};\theta)\,dx_{k+1}$$

$$= Y_k \,.$$

Thus, $Y_1, Y_2, \ldots$ is a martingale by definition. We can now use a well known martingale inequality (see Doob [1953]) which states that, for $\varepsilon > 0$,

$$P[\max_{i \le k} Y_i > \varepsilon] \le E\frac{(Y_k)}{\varepsilon}, \text{ for all } k.$$

Since $E(Y_k) = 1$, the bound does not depend on k and the inequality becomes

$$P[\max_i Y_i > \varepsilon] \le \frac{1}{\varepsilon},$$

or

$$P[Y_i \le \varepsilon, \text{ for all } i] \ge 1 - \frac{1}{\varepsilon}. \qquad (2.1)$$

We note that the inequality ( 2.1 ) was also given by Wald (1948), without relying on the martingale property, in the following lemma.

Wald Lemma: Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables with joint density $f_{1n}(x_1,\ldots,x_n)$, n = 1, 2, 3, ..., under hypothesis $H_1$ and $f_{0n}(x_1,\ldots,x_n)$ under hypothesis $H_0$. Let $\alpha$ be a constant between 0 and 1. Then, under $H_0$,

$$P[(f_{1n}/f_{0n}) < \frac{1}{\alpha}, \text{ for all } n] \geq 1 - \alpha.$$

In the context of the sequential exponential life test

$$Y_i = \frac{\theta^k e^{-\frac{T_k}{d_1\theta}}}{(d_1\theta)^k e^{-\frac{T_k}{\theta}}} = d_1^{-k} e^{-\frac{T_k}{\theta}(\frac{1}{d_1}-1)}.$$

Then, applying inequality ( 2.1 ), we get

$$P[d_1^{-k} e^{-\frac{T_k}{\theta}(\frac{1}{d_1}-1)} \leq \frac{1}{\alpha}, \text{ for all } k] \geq 1 - \alpha.$$

Equivalently,

$$P\left[\frac{T_k(1-d_1)}{\theta d_1} > -k\ln d_1 + \ln \alpha, \text{ for all } k\right] \geq 1 - \alpha.$$

The right side of the inequality in brackets is positive if $d_1 < \alpha^{1/k}$. Finally we have, for $d_1 < \alpha^{1/k}$,

$$P\left[\theta < \frac{T_k(1-d_1)}{d_1 \ln\left(\frac{\alpha}{d_1^k}\right)}, \text{ for all } k\right] \geq 1 - \alpha. \qquad (2.2)$$

We will refer to this upper bound on $\theta$ as $U_k$. Inequality ( 2.2 ) can also be expressed as

$$P(\theta < \min_k U_k) \geq 1 - \alpha.$$

Likewise define

$$Z_i = \begin{cases} \dfrac{f(X_1,X_2,\ldots,X_i;\theta/d_2)}{f(X_1,X_2,\ldots,X_i;\theta)} & \text{if } i = 1,2,\ldots,n \\ \\ Z_n & \text{if } i > n, \end{cases}$$

where $d_2$ is an arbitrary positive constant. From inequality ( 2.1 ) we have

$$P[Z_k < \frac{1}{\alpha}, \text{ for all } k] \geq 1 - \alpha,$$

where

$$Z_k = \left(\frac{d_2\theta}{\theta}\right)^k \frac{e^{-T_k d_2/\theta}}{e^{-T_k/\theta}} = d_2^k \, e^{-\frac{T_k}{\theta}(d_2-1)} .$$

This yields

$$P[d_2^k \, e^{-\frac{T_k}{\theta}(d_2-1)} < \frac{1}{\alpha}, \text{ for all } k] \geq 1 - \alpha,$$

or

$$P\left[\frac{T_k(1-d_2)}{\theta} < -\ln(\alpha d_2^k), \text{ for all } k\right] \geq 1-\alpha.$$

If $-\ln(\alpha d_2^k) > 0$, i.e., $d_2 < 1/\alpha^{1/k}$, then

$$P\left[\frac{T_k(1-d_2)}{-\ln(\alpha d_2^k)} < \theta, \text{ for all } k\right] \geq 1-\alpha. \qquad (2.3)$$

We shall refer to this lower bound on $\theta$ as $L_k$. If $L_k$ is to be positive it must be true that $d_2 < 1$ and $d_2 < 1/\alpha^{1/k}$. Since $\alpha^{1/k} < 1$, we have the single constraint $d_2 < 1$. Here also, it is true that

$$P[\max_k L_k < \theta] \geq 1-\alpha,$$

since ( 2.3 ) is valid for all k. Applying the Bonferroni inequality gives

$$P(\max_k L_k<\theta, \min_k U_k>\theta) \geq P(\max_k L_k<\theta) + P(\min_k U_k>\theta) - 1,$$

or

$$P[\max_k L_k < \theta < \min_k U_k] \geq 1-\alpha+1-\alpha-1 = 1-2\alpha.$$

This establishes the following theorem.

<u>Theorem 2.1</u> If a sequential exponential life test is conducted without replacement and terminates with r failures, then a two-sided confidence interval for the

mean lifetime $\theta$, having confidence coefficient at least $1 - 2\alpha$, is given by

$$\left[\max_{k \le r} \frac{T_k(1-d_2)}{-\ln(\alpha d_2^k)}, \min_{k \le r} \frac{T_k(1-d_1)}{d_1 \ln(\frac{\alpha}{d_1^k})}\right]$$

where $0 < d_1 < \alpha^{1/k}$ and $0 < d_2 < 1$.

Due to the nature of the interval, it may happen that

$$\max_{k \le r} L_k \ge \min_{k \le r} U_k.$$

This leaves us with a confidence interval whose lower limit exceeds its upper limit. Computer simulation studies have shown that this inversion of the endpoints occurs infrequently, often as a consequence of an unusually early failure. To avoid this predicament, the following iterative scheme is proposed whenever

$$\max_{k \le r} L_k \ge \min_{k \le r} U_k.$$

(1) The upper limit of the confidence interval, say U, is assigned the value of the next larger member in the set $\{U_k : k = 1,2,\ldots,n\}$.

(2) If U is greater than the lower limit, say L, the endpoints are ordered correctly and the procedure stops.

(3) If U is smaller than the lower limit, assign to L the value of the next smaller number of the set $\{L_k : k = 1,2,\ldots,n\}$.

(4) If $L \geq u$ return to step (1).

Since inequalities ( 2.2 ) and ( 2.3 ) are valid for all k, the interval that results from applying this procedure will still have a confidence coefficient at least $1-2\alpha$.

Accept decisions in the sequential test occur between failure times. Consequently, if an accept decision occurs and j failures have been observed, the total time on test when the decision is made to accept $H_0$, denoted by $T_{a,j}$ say, will exceed $T_j$, the total time on test at the $j^{th}$ failure. Furthermore

$$\frac{T_{a,j}(1-d_2)}{-\ln(\alpha d_2^k)} > \frac{T_j(1-d_2)}{-\ln(\alpha d_2^k)}$$

and it may also be true that

$$\frac{T_{a,j}(1-d_2)}{-\ln(\alpha d_2^k)} > \max_{k\leq j} \frac{T_k(1-d_2)}{-\ln(\alpha d_2^k)} .$$

This suggests that we might shorten the confidence interval for $\theta$ by increasing the lower limit utilizing this total test time at the time of an accept decision. Note that the upper limit cannot be improved in the same manner since there we seek the minimum of a set, and $T_{a,j} > T_j$ implies the minimum will occur at a failure time. The lower limit of the interval in Theorem 2.1 was chosen as the maximum of the set $\{L_k, k = 1,2,\ldots,r\}$. The following lemma admits the use of

$$\frac{T_{a,j}(1-d_2)}{-\ln(\alpha d_2^k)}$$

as the lower limit when an accept decision occurs.

Lemma 2.2 If the sequential life test terminates in an accept decision with j failures and total test time $T_{a,j}$, then

$$P\left[\theta > \frac{T_{a,j}(1-d_2)}{-\ln(\alpha d_2^j)}\right] \geq 1 - \alpha .$$

Proof: We note that this is an exponential life test, without replacement, which is observed for a total test time t'. This test can be shown to be equivalent to a Poisson process with parameter $\lambda = 1/\theta$ which is observed for actual time t'. We will apply that result here with $t' = T_{a,j}$. A well known result for a Poisson process (see Barlow and

Proschan (1975), Theorem 3.7) may be adapted as follows:

Given j failures in a Poisson process in $[0, T_{a,j}]$, the distribution of the failure times (here interpreted as actual times) $T_1, T_2, \ldots, T_j$ is the distribution of the order statistics from the uniform distribution over the interval $[0, T_{a,j}]$.

As a consequence, if $y_i$ represents the observed value of $T_i$, then

$$f(y_1, y_2, \ldots, y_j \mid j) = \frac{j!}{T_{a,j}^{\,j}} ,$$

and

$$P(\text{j Failures}) = \frac{e^{-\frac{T_{a,j}}{\theta}} (T_{a,j}/\theta)^j}{j!} .$$

Thus

$$f(y_1, y_2, \ldots, y_j, j) = \frac{e^{-\frac{T_{a,j}}{\theta}}}{\theta^j} .$$

Now if we apply Wald's lemma in the manner leading to Theorem 2.1 letting

$$f_{1j} = \frac{d_2^j \, e^{-d_2 T_{a,j}/\theta}}{\theta^j} \text{ and } f_{0j} = e^{-T_{a,j}/\theta} \frac{}{\theta^j} \ .$$

the conclusion of the lemma follows by the same algebraic manipulations.

There remains the problem of selecting the values of $d_1$ and $d_2$ which are in some sense optimal, or near optimal, while satisfying the given constraints. We shall say the values of $d_1$ and $d_2$ are optimal if they minimize the length of the confidence interval in Theorem 2.1 . The constraint $d_1 < \alpha^{1/k}$, $k = 1, 2, \ldots, r$, poses a dilemma. If $d_1$ is a fixed constant as the theory behind Theorem 2.1 requires, then it must be true that $d_1 < \alpha$. It is not likely that a value of $d_1$ in this range will be "optimal" in any sense of the word. On the other hand if a value of $d_1$ is chosen greater than $\alpha$, and the test terminates with one failure, then there is no upper bound (other than $+ \infty$) for the confidence interval.

The martingale inequality on which Theorem 2.1 is based has wide applicability. It is reasonable to suspect that in a specific application the bound may be quite conservative. Specifically the probability in Theorem 2.1 may considerably exceed $1-2\alpha$. This suggests that some liberties might be taken in selecting $d_1$ and $d_2$ without violating the probability inequalities in ( 2.2 ) and ( 2.3 ). Therefore, we propose to choose $d_1$ and $d_2$ as functions of the sample path generated by the sequential test. The following scheme implements this proposal.

Treat $U_k$, $k = 1, 2, \ldots, n$, as a function of $d_1$ and for each failure time, $t_k$, find the value of $d_1$ which minimizes $U_k$. This optimal value is found to be the root of the equation

$$g(d) = 1 - (\frac{1}{k}) \ln \alpha - d + \ln d.$$

Note that the solution is not dependent on $T_k$ and so, for a given $\alpha$, a set of optimal values of $d_1$, say $\{d_{11}, d_{12}, d_{13}, \ldots, d_{1n}\}$ can be computed by numerical methods. A table of these values, for several values of $\alpha$, is given in Appendix 1. At each failure time, $t_k$, compute $U_k$ using $d_{1k}$. Then if the minimum value of $U_k$ occurs at $t_m$ say, we set $d_1 \equiv d_{1m}$. The following lemma shows that for each k, the relationship $d_{1k} < \alpha^{1/k}$ will hold.

<u>Lemma 2.3</u>. If $d_{1k}$ is the root of the equation $g(d) = 1 - (1/k) \ln \alpha - d + \ln d$, then $d_{1k} < \alpha^{1/k}$.

Proof: First, $g'(d) = 1/d - 1 > 0$, which implies that $g(d)$ is increasing on the interval $(0, 1)$. Also $g(0) = -\infty$ and $g(1) = -(1/k) \ln \alpha > 0$. It follows that $g(d)$ crosses the axis once from below in $(0, 1)$. But $g(\alpha^{1/k}) = 1-\alpha^{1/k} > 0$. Thus $d_{1k} < \alpha^{1/k}$.

In the same way, consider $L_k$ as a function of $d_2$ and find the set $\{d_{21}, d_{22}, \ldots, d_{2n}\}$ of values which maximizes

$L_k$ for $k = 1, 2, \ldots, n$. These values are the roots of the equations $h(d) = 1 - (1/k) \ln \alpha - \ln d - d^{-1}$ and are also tabled for some specific values of $\alpha$ in Appendix 1. If the maximum value of $L_k$ occurs at $t_\ell$, set $d_2 \equiv d_{2\ell}$. Using the same steps as in lemma 2.3 , it is seen that $d_{2k} < 1$ for each k.

To clarify this procedure, consider the following example: Test plan I of MIL-STD-781C is used to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_0/\theta_1 = 1.5$, $\alpha = \beta = .1$. The test ends with an accept decision based on six observed failures. At each failure time, the total time on test was computed as were the upper and lower 95% confidence limits using ( 2.2 ) and ( 2.3 ) and the optimal values of $d_1$ and $d_2$ given in Appendix 1. These computations are summarized in Table 2.1 . A lower confidence limit was also computed at the acceptance boundary as described by lemma 2.2, using $T_{a,6} = 13.91$. Failure times were simulated using $\theta = \theta_0 = 1.5$. According to our procedures based on Theorem 2.1 , an approximate 90% confidence interval for $\theta$ is [.984, 4.101] with $d_1 = .2628$ and $d_2 = .4244$. The Bryant-Schmee 90% confidence interval for this example is [1.108, 4.681].

The second method is really a simplification of this first method. During computer simulations it was noticed that, for a fixed $d_1$ or $d_2$, $\max_{k \le r} L_k$ and $\min_{k \le r} U_k$

TABLE 2.1

Computation Results for Confidence Interval Example

| Failure $k$ | Total Test Time $T_k$ | $d_{1k}$ | $U_k$ | $d_{2k}$ | $L_k$ |
|---|---|---|---|---|---|
| 1 | 1.362 | .0187 | 72.649 | .1741 | .2371 |
| 2 | 2.623 | .0900 | 14.572 | .2601 | .3411 |
| 3 | 4.618 | .1589 | 9.690 | .3181 | .4896 |
| 4 | 4.815 | .2159 | 5.576 | .3615 | .4351 |
| 5 | 5.389 | .2628 | 4.101 | .3959 | .4267 |
| 6 | 12.555 | .3020 | 6.928 | .4244 | .8880 |
| $T_{a,6}$ | 13.91 | -- | -- | .4244 | .9838 |

usually occurred at or near $t_r$, the last observed failure time. Hence, if we set $d_1 = d_{1r}$ and $d_2 = d_{2r}$ we should get results nearly the same as with the first method. It was also felt that this second method would be easier to justify analytically. Unfortunately, an analytic justification of either of these methods remains an unsolved problem. It is not known if these posterior methods of selecting $d_1$ and $d_2$ invalidate the probability inequalities (2.2) and (2.3).

To obtain some empirical conclusions, we conducted computer simulations of two truncated sequential test plans from MIL-STD-781C. At the time of test decision 60%, 80%, and 90% confidence intervals for the mean failure time $\theta$ were formed using both methods discussed here for selecting $d_1$ and $d_2$. These confidence coefficients were chosen so that our results could be compared with those obtained by Bryant and Schmee (1979). Each simulation used 500 trials. Estimates of the confidence coefficient, and the average value of the endpoints were computed for our methods and for the Bryant and Schmee intervals. Partial results are given in Tables 2.2 through 2.5. It was noted that, when the test ended with an accept decision and one failure, the upper endpoint on the confidence interval was often very large, which tended to inflate the average endpoints. Because of this, the average endpoints were also computed excluding those trials which terminated in an accept

TABLE 2.2

Simulation Results for 90% Confidence Interval from MIL-STD-781C, Test Plan I; $\theta_0/\theta = 1.5$
$\alpha = \beta = .1$

| $\theta$ | Estimated Coefficient | Ave. Endpoints, at Least 1 Failure | | Ave. Endpoints, at Least 2 Failures | |
|---|---|---|---|---|---|
| | .884 | .307 , | .917 | .307 , | .917 |
| .5 | .898 | .259 , | 1.059 | .259 , | 1.059 |
| | .946 | .256 , | 1.102 | .256 , | 1.102 |
| | .912 | .714 , | 2.399 | .708 , | 1.637 |
| 1 | .917 | .655 , | 3.208 | .650 , | 1.760 |
| | .964 | .647 , | 3.228 | .642 , | 1.804 |
| | .916 | 1.020 , | 7.038 | 1.003 , | 3.743 |
| 1.5 | .912 | .942 , | 8.487 | .930 , | 4.192 |
| | .962 | .941 , | 8.553 | .930 , | 4.367 |
| | .912 | 1.137 , | 15.331 | 1.096 , | 5.800 |
| 2.0 | .870 | 1.018 , | 20.113 | .989 , | 6.734 |
| | .906 | 1.017 , | 20.265 | .988 , | 7.081 |

Note: In each of Tables 2.2 through 2.5 ,

(1) The first entry in each cell corresponds to Bryant-Schmee results.

(2) The second entry corresponds to the first method: using the optimal $d_1$ and $d_2$ at each failure.

(3) The third entry corresponds to the second method: using the optimal $d_1$ and $d_2$ from the final failure time.

TABLE 2.3

Simulation Results for 90% Confidence Interval from MIL-STD-781C, Test Plan III; $\theta_0/\theta_1 = 2$, $\alpha = \beta = .1$

| $\theta$ | Estimated Coefficient | Ave. Endpoints, at Least 1 Failure | | Ave. Endpoints at Least 2 Failures | |
|---|---|---|---|---|---|
| | .886 | .259 , | 1.398 | .259 , | 1.398 |
| .5 | .944 | .195 , | 1.733 | .195 , | 1.733 |
| | .974 | .193 , | 1.803 | .193 , | 1.803 |
| | .912 | .868 , | 3.746 | .558 , | 2.396 |
| 1 | .931 | .472 , | 4.344 | .463 , | 2.715 |
| | .962 | .469 , | 4.401 | .460 , | 2.797 |
| | .938 | .792 , | 8.729 | .768 , | 3.891 |
| 1.5 | .944 | .707 , | 11.239 | .688 , | 4.272 |
| | .970 | .706 , | 11.430 | .687 , | 4.521 |
| | .956 | .940 , | 18.898 | .895 , | 5.838 |
| 2.0 | .962 | .855 , | 25.942 | .825 , | 7.232 |
| | .986 | .855 , | 26.140 | .825 , | 7.616 |

TABLE 2.4

Simulation Results for 80% Confidence Interval from MIL-STD-781C, Test Plan I

| θ | Estimated Coefficient | Ave. Endpoints, at Least 1 Failure | | Ave. Endpoints, at Least 2 Failures | |
|---|---|---|---|---|---|
| | .784 | .342 , | .798 | .342 , | .798 |
| .5 | .816 | .286 , | .907 | .286 , | .907 |
| | .898 | .281 , | .964 | .281 , | .964 |
| | .810 | .760 , | 1.871 | .752 , | 1.504 |
| 1 | .845 | .696 , | 2.434 | .691 , | 1.611 |
| | .920 | .680 , | 2.492 | .674 , | 1.696 |
| | .808 | 1.125 , | 4.485 | 1.103 , | 3.022 |
| 1.5 | .790 | 1.018 , | 5.323 | 1.003 , | 3.456 |
| | .892 | 1.011 , | 5.385 | .996 , | 3.609 |
| | .867 | 1.290 , | 8.433 | 1.247 , | 4.661 |
| 2.0 | .802 | 1.120 , | 10.425 | 1.091 , | 5.298 |
| | .867 | 1.118 , | 10.661 | 1.089 , | 5.657 |

TABLE 2.5

Simulation Results from 80% Confidence Interval from MIL-STD-781C, Test Plan III

| θ | Estimated Coefficient | Ave. Endpoints, at Least 1 Failure | | Ave. Endpoints at Least 2 Failures | |
|---|---|---|---|---|---|
| | .788 | .294 , | 1.085 | .294 , | 1.085 |
| .5 | .872 | .215 , | 1.346 | .215 , | 1.346 |
| | .922 | .212 , | 1.437 | .212 , | 1.437 |
| | .830 | .601 , | 1.857 | .601 , | 1.857 |
| 1.0 | .887 | .489 , | 2.098 | .489 , | 2.098 |
| | .920 | .484 , | 2.198 | .484 , | 2.198 |
| | .856 | .916 , | 5.356 | .882 , | 2.978 |
| 1.5 | .849 | .798 , | 7.064 | .772 , | 3.527 |
| | .900 | .794 , | 7.188 | .769 , | 3.731 |
| | .892 | 1.081 , | 9.882 | 1.020 , | 4.311 |
| 2.0 | .912 | .946 , | 12.656 | .903 , | 5.043 |
| | .956 | .945 , | 12.693 | .903 , | 5.332 |

decision with one failure. As expected, the Bryant-Schmee intervals, on the average, were somewhat shorter than the intervals from both of our methods. However, we also noted that for some sample paths, our methods gave shorter intervals than the corresponding Bryant-Schmee intervals. The simulation results also indicate that the intervals generated from our posterior methods of selecting $d_1$ and $d_2$ remain conservative. We also note that our methods may be just as easily applied to an untruncated sequential test, while this is not so with the Bryant-Schmee approach. Their method requires non-trivial computation of probabilities at every discrete acceptance point up to that point where the sample path ended. In the untruncated sequential test the sample path may continue for a long time, resulting in a rather formidable computation problem to get the Bryant-Schmee intervals. We believe that, considering the relative ease of computation and application, these approximate confidence intervals provide a competitive alternative to those of Bryant and Schmee.

## III. A Confidence Interval with No Observed Failures

The sequential life test may end in acceptance with no failures observed. In this situation the confidence interval should be one sided since there is no information on which to base an upper limit. If no failures are observed by total time $T_0$, then we have observed a Poisson process with parameter $\lambda = n/\theta$ for a total time $T_0$, or actual time $T_0/n$. This is equivalent to type I, or time, censoring and Epstein (1960) showed that a one-sided $100(1-\alpha)$ percent confidence interval for $\theta$ is

$$\left[\frac{2T_0}{\chi^2_{\alpha,2}}, \infty\right]. \tag{3.1}$$

When no failures are observed, the two-sided confidence interval of Theorem 2.1 reduces to

$$\left[\frac{T_0(1-d_2)}{-\ln \alpha}, \frac{T_0(1-d_1)}{d_1 \ln \alpha}\right] \tag{3.2}$$

If $k = 0$, the constraint on $d_1$ becomes $0 < d_1 < 0$, leaving the upper endpoint undefined. Since no failure times are known, we define the upper endpoint to be infinity. The lower endpoint decreases as $d_2$ decreases and approaches $T_0/(-\ln \alpha)$ as $d_2$ goes to zero in the limit. The resulting interval is

$$\left[\frac{T_0}{-\ln \alpha}, \infty\right]. \tag{3.3}$$

But from the relationship between the chi-square and Poisson distributions we have

$$P(\chi^2_2 > \chi^2_{\alpha,2}) = e^{-\chi^2_{\alpha,2}/2} = \alpha.$$

Hence,

$$- \ln \alpha = \chi^2_{\alpha,2}/2 ,$$

and ( 3.1 ) is equivalent to ( 3.3 ), and either provides a one-sided 100(1-α) percent confidence interval for θ when no failures have been observed in the life test.

## IV. Testing with Replacement

In this section we will show that the derivations in Section II are also valid if the sequential test is *conducted with replacement*. Testing with replacement is equivalent to observing a Poisson process with parameter $\lambda = n/\theta$, where again, n is the number of items placed on test. As in Section II , define $X_i \equiv t_i - t_{i-1}$. The $X_i$, $i = 1, 2, \ldots, n$, are independent and are exponentially distributed with mean lifetime $\theta/n$. Hence,

$$f(x_1,\ldots,x_k;\theta) = \frac{n^k}{\theta^k} e^{-\frac{n}{\theta}\sum_{i=1}^{k} x_i} = \frac{n^k}{\theta^k} e^{-\frac{nt_k}{\theta}} = \frac{n^k}{\theta^k} e^{-\frac{T_k}{\theta}} ,$$

where $T_k = nt_k$ = total time on test at $t_k$. We see that the joint densities of $X_1, X_2, \ldots, X_k$ when testing with or without replacement differ only by a constant involving n and k. Thus, the density ratios in Section II will be the same as here in the replacement case. It follows that

Theorem 2.1 also holds when testing with replacement.

Now, to establish the analogous result to Lemma 2.2, we note that a total test time of $T_{a,j}$ is equal to actual time $t_{a,j} = T_{a,j}/n$ if failed items are replaced. If $t_i$ is the $i^{th}$ ordered failure time and $y_i$ represents the observed value of $t_i$, then

$$f(y_1, y_2, \ldots, y_j | j) = \frac{j!}{t_{a,j}^{\;j}} \, ,$$

and

$$P(j \text{ Failures}) = \frac{e^{-\frac{nt_{a,j}}{\theta}} \left(\frac{nt_{a,j}}{\theta}\right)^j}{j!} \, .$$

Thus

$$f(y_1, y_2, \ldots, y_j, j) = \frac{n^j}{\theta^j} e^{-\frac{nt_{a,j}}{\theta}} = \frac{n^j}{\theta^j} e^{-\frac{T_{a,j}}{\theta}} \, .$$

Except for the constant $n^j$, this is the same result obtained in the proof of Lemma 2.2. The conclusion of Lemma 2.2, for testing with replacement, follows in the same manner as in that proof.

It follows that all results and conclusions of Section II, where testing is without replacement, also apply to the sequential test scheme where failed items are replaced.

# APPENDIX 1

Tabled here are the optimal $d_2$ values for determining the lower limit of the 100(1-$\alpha$) percent two-sided confidence interval.

| Failures | Confidence Coefficient | | | | |
|---|---|---|---|---|---|
| k | 99% | 98% | 95% | 90% | 80% |
| 1 | 0.1186 | 0.1309 | 0.1522 | 0.1741 | 0.2045 |
| 2 | 0.1879 | 0.2045 | 0.2323 | 0.2602 | 0.2973 |
| 3 | 0.2380 | 0.2568 | 0.2876 | 0.3180 | 0.3575 |
| 4 | 0.2771 | 0.2973 | 0.3298 | 0.3615 | 0.4023 |
| 5 | 0.3092 | 0.3301 | 0.3637 | 0.3962 | 0.4371 |
| 6 | 0.3364 | 0.3575 | 0.3918 | 0.4243 | 0.4656 |
| 7 | 0.3597 | 0.3812 | 0.4160 | 0.4485 | 0.4894 |
| 8 | 0.3804 | 0.4023 | 0.4366 | 0.4691 | 0.5096 |
| 9 | 0.3988 | 0.4208 | 0.4551 | 0.4876 | 0.5271 |
| 10 | 0.4151 | 0.4371 | 0.4718 | 0.5034 | 0.5430 |
| 11 | 0.4300 | 0.4516 | 0.4867 | 0.5184 | 0.5570 |
| 12 | 0.4437 | 0.4656 | 0.4999 | 0.5315 | 0.5702 |
| 13 | 0.4560 | 0.4779 | 0.5122 | 0.5430 | 0.5816 |
| 14 | 0.4674 | 0.4894 | 0.5236 | 0.5544 | 0.5922 |
| 15 | 0.4779 | 0.4999 | 0.5342 | 0.4641 | 0.6027 |
| 16 | 0.4885 | 0.5096 | 0.5430 | 0.5746 | 0.6115 |
| 17 | 0.4973 | 0.5192 | 0.5518 | 0.5825 | 0.6203 |
| 18 | 0.5061 | 0.5271 | 0.5605 | 0.5904 | 0.6273 |
| 19 | 0.5148 | 0.5359 | 0.5685 | 0.5992 | 0.6344 |
| 20 | 0.5219 | 0.5430 | 0.5764 | 0.6062 | 0.6414 |
| 21 | 0.5298 | 0.5509 | 0.5834 | 0.6115 | 0.6484 |
| 22 | 0.5368 | 0.5570 | 0.5904 | 0.6186 | 0.6537 |
| 23 | 0.5430 | 0.5641 | 0.5957 | 0.6238 | 0.6590 |
| 24 | 0.5500 | 0.5702 | 0.6027 | 0.6309 | 0.6660 |
| 25 | 0.5553 | 0.5764 | 0.6080 | 0.6361 | 0.6695 |
| 26 | 0.5614 | 0.5816 | 0.6133 | 0.6414 | 0.6748 |
| 27 | 0.5676 | 0.5869 | 0.6186 | 0.6449 | 0.6801 |
| 28 | 0.5729 | 0.5922 | 0.6238 | 0.6520 | 0.6836 |
| 29 | 0.5781 | 0.5975 | 0.6273 | 0.6555 | 0.6871 |
| 30 | 0.5816 | 0.6027 | 0.6326 | 0.6590 | 0.6924 |
| 31 | 0.5869 | 0.6062 | 0.6379 | 0.6643 | 0.6977 |
| 32 | 0.5922 | 0.6115 | 0.6414 | 0.6678 | 0.7012 |
| 33 | 0.5957 | 0.6150 | 0.6449 | 0.6730 | 0.7047 |
| 34 | 0.5992 | 0.6203 | 0.6484 | 0.6766 | 0.7082 |
| 35 | 0.6045 | 0.6238 | 0.6520 | 0.6801 | 0.7117 |

| Failures k | Confidence Coefficient 99% | 98% | 95% | 90% | 80% |
|---|---|---|---|---|---|
| 36 | 0.6080 | 0.6273 | 0.6555 | 0.6836 | 0.7152 |
| 37 | 0.6115 | 0.6309 | 0.6590 | 0.6871 | 0.7152 |
| 38 | 0.6150 | 0.6344 | 0.6625 | 0.6889 | 0.7187 |
| 39 | 0.6186 | 0.6379 | 0.6660 | 0.6906 | 0.7223 |
| 40 | 0.6221 | 0.6414 | 0.6695 | 0.6941 | 0.7258 |
| 41 | 0.6256 | 0.6449 | 0.6730 | 0.6977 | 0.7293 |

Tabled here are the optimal $d_1$ values for determining the upper limit of the 100(1-α) percent two-sided confidence interval.

| Failures k | Confidence Coefficient 99% | 98% | 95% | 90% | 80% |
|---|---|---|---|---|---|
| 1 | 0.0018 | 0.0037 | 0.0093 | 0.0187 | 0.0382 |
| 2 | 0.0267 | 0.0382 | 0.0619 | 0.0900 | 0.1329 |
| 3 | 0.0673 | 0.0864 | 0.1215 | 0.1590 | 0.2108 |
| 4 | 0.1090 | 0.1329 | 0.1742 | 0.2156 | 0.2712 |
| 5 | 0.1478 | 0.1742 | 0.2190 | 0.2625 | 0.3190 |
| 6 | 0.1825 | 0.2108 | 0.2576 | 0.3020 | 0.3590 |
| 7 | 0.2137 | 0.2429 | 0.2907 | 0.3356 | 0.3912 |
| 8 | 0.2415 | 0.2712 | 0.3190 | 0.3639 | 0.4195 |
| 9 | 0.2664 | 0.2966 | 0.3444 | 0.3893 | 0.4439 |
| 10 | 0.2888 | 0.3190 | 0.3678 | 0.4107 | 0.4654 |
| 11 | 0.3098 | 0.3405 | 0.3873 | 0.4312 | 0.4849 |
| 12 | 0.3288 | 0.3590 | 0.4059 | 0.4498 | 0.5005 |
| 13 | 0.3464 | 0.3756 | 0.4225 | 0.4654 | 0.5161 |
| 14 | 0.3620 | 0.3912 | 0.4381 | 0.4800 | 0.5298 |
| 15 | 0.3766 | 0.4068 | 0.4517 | 0.4927 | 0.5434 |
| 16 | 0.3903 | 0.4195 | 0.4654 | 0.5064 | 0.5551 |
| 17 | 0.4029 | 0.4322 | 0.4771 | 0.5181 | 0.5668 |
| 18 | 0.4146 | 0.4439 | 0.4888 | 0.5278 | 0.5746 |
| 19 | 0.4264 | 0.4556 | 0.4985 | 0.5376 | 0.5844 |
| 20 | 0.4371 | 0.4654 | 0.5083 | 0.5473 | 0.5942 |
| 21 | 0.4458 | 0.4751 | 0.5181 | 0.5571 | 0.6020 |
| 22 | 0.4667 | 0.4849 | 0.5278 | 0.5649 | 0.6098 |
| 23 | 0.4654 | 0.4927 | 0.5356 | 0.5727 | 0.6176 |
| 24 | 0.4732 | 0.5005 | 0.5434 | 0.5785 | 0.6234 |
| 25 | 0.4810 | 0.5083 | 0.5512 | 0.5864 | 0.6293 |
| 26 | 0.4888 | 0.5161 | 0.5571 | 0.5942 | 0.6371 |
| 27 | 0.4966 | 0.5239 | 0.5629 | 0.6000 | 0.6410 |
| 28 | 0.5044 | 0.5298 | 0.5707 | 0.6059 | 0.6488 |
| 29 | 0.5109 | 0.5376 | 0.5766 | 0.6117 | 0.6527 |
| 30 | 0.5161 | 0.5434 | 0.5824 | 0.6176 | 0.6566 |
| 31 | 0.5239 | 0.5493 | 0.5883 | 0.6215 | 0.6605 |
| 32 | 0.5298 | 0.5551 | 0.5942 | 0.6273 | 0.6683 |
| 33 | 0.5356 | 0.5610 | 0.5981 | 0.6332 | 0.6722 |
| 34 | 0.5395 | 0.5668 | 0.6039 | 0.6371 | 0.6761 |
| 35 | 0.5454 | 0.5707 | 0.6098 | 0.6410 | 0.6800 |
| 36 | 0.5512 | 0.5746 | 0.6137 | 0.6449 | 0.6839 |
| 37 | 0.5551 | 0.5805 | 0.6176 | 0.6488 | 0.6878 |
| 38 | 0.5610 | 0.5844 | 0.6215 | 0.6527 | 0.6917 |
| 39 | 0.5649 | 0.5903 | 0.6254 | 0.6566 | 0.6956 |
| 40 | 0.5707 | 0.5942 | 0.6293 | 0.6605 | 0.6995 |
| 41 | 0.5746 | 0.5981 | 0.6332 | 0.6644 | 0.7034 |

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