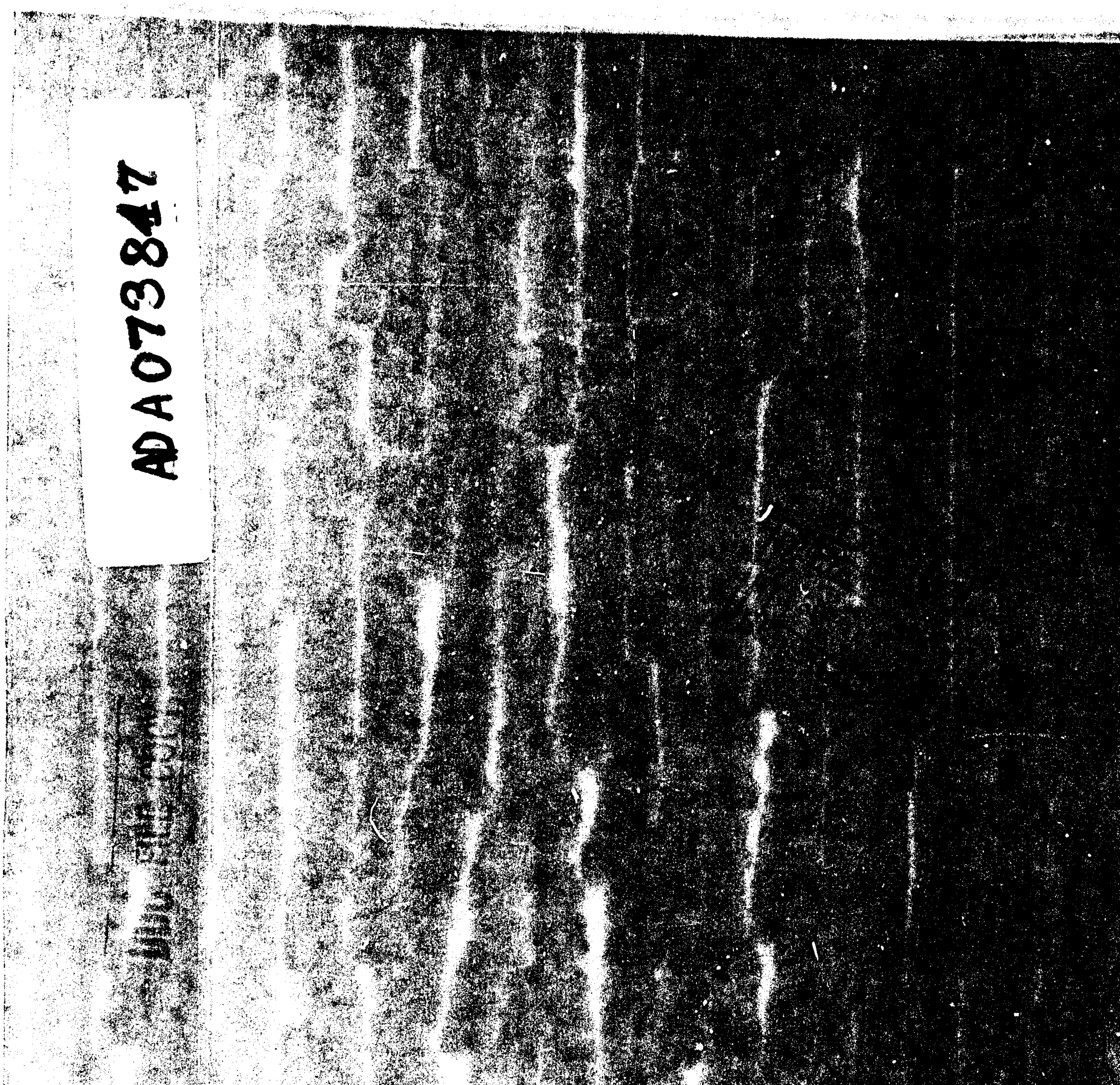

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THE CHARACTERISTICS OF FUEL MOTION WHICH AFFECT AIRPLANE DYNAMICS

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ABSTRACT

Problems in aircraft dynamics such as stability and response of the rigid airplane may be affected by fuel motion in the tanks. Such problems might also arise in connection with missiles.

In this report the response of the fuel to simple harmonic motions of a rectangular tank in translation, pitching and yawing is studied. The shape of the free surface and the values of forces and moments are obtained. Using the force and moment expressions, simple mechanical systems equivalent to the fuel are constructed. These systems respond to motions of the tank walls in the same fashion as the fuel, producing identical forces and moments. With the aid of this mechanical analogy the complete dynamic system can be handled by any desired process.

The Laplace transforms of the forces and moments are also obtained for use when the entire dynamics problem is to be solved by the transform method.

All of the above developments assume a non-viscous incompressible fluid subject to linearized boundary conditions. All tank motions (except that normal to the mean free surface) are restricted to small accelerations. In addition, the angular motions of the tank are restricted to small displacements.

INTRODUCTION

Reason for Investigation

There has been increasing interest recently in aircraft dynamics problems which are affected by the motion of fuel in partially full tanks[1,2]. The stability and response of a rigid airplane is a problem of this type. It is intuitively apparent that a large mass of fuel with a free surface may have serious dynamic effects on the airplane, particularly if there is a possibility of resonance.

Available Information

The understanding of these problems requires first an understanding of the dynamics of fuel motion. A great deal of the necessary information has been available for some time[3a], particularly the nature of the free oscillations of liquid in some tanks of simple shape. Extension of this material to include forced oscillations makes it possible to determine the forces communicated to the tank walls by an arbitrary motion of the tank (or airplane). Prior work[4] in this direction initiated by Stewart and Lorell was in part responsible for the present investigation. Similar problems have been investigated in the field of seismology. Ref. 5 gives experimental results and additional references to theoretical and experimental work in this field.

Methods Used

For convenience in solving the complete dynamics problem it may be desirable to replace the fuel by a simple mechanical system. Therefore, in addition to presenting the Laplace transforms of forces and moments, this report suggests mechanical systems which produce the same forces and moments as the oscillating fuel when the container is given an arbitrary motion. (The "arbitrary" motion must, of course, fall within the scope of the linearization processes used.) In a previous report[6] the representation of fuel by an equivalent pendulum was discussed. The characteristics of the equivalent pendulum were determined only from consideration of fluid motions in a stationary tank. This was adequate to define the frequency and the product of mass times amplitude of motion for the pendulum, so that forces corresponding to a given amplitude of the fundamental wave motion were duplicated. This provided sufficient information for the study of some stability problems as affected by fuel "sloshing." The present report offers additional information so that the more general problem of aircraft response to arbitrary disturbances can be considered, including the effect of angular tank motions.

PROCEDURE

Assumptions

In order to simplify the problem a number of restrictions are applied concerning the nature of the fluid, the motions of the fluid and the shape of the tank. The fluid is assumed incompressible (which requires no discussion), and non-viscous. Since baffles are not considered and the tanks are relatively large, viscosity has little effect on the fluid motions.

All tank motions, (except those normal to the mean free surface), are restricted to small accelerations. In addition, the angular motions of the tank are restricted to small displacements. The free surface displacement and slope must be small. These conditions insure that the equations can be linearized. This is consistent with the small perturbation methods used in airplane dynamics and does not necessarily limit the value of the results. (In this connection it should be noted that the above restrictions are sufficient for linearization, but not all necessary. The necessary and sufficient condition is probably that free surface slopes and displacements be small when measured relative to axes <u>fixed in the tank</u>.)

Only rectangular tanks are considered here in detail. The cylindrical tank of circular cross-section with free surface normal to the axis has been treated in Ref. 4.

Free Oscillations

To determine the nature of the free oscillations in a stationary tank the procedure is as follows. Since the fluid is assumed non-viscous and incompressible, the velocity potential is chosen to satisfy the Laplace equation. At the walls the normal velocity components must be zero. On the free surface the pressure must be zero, and in order to satisfy this condition with linear functions it is necessary to assume small displacements of the free surface. For rectangular tanks this problem is readily solved.

Forced Oscillations

In studying the forced oscillations of the fluid to determine an equivalent mechanical system it is sufficient to study the response of the fluid to simple harmonic motions of all frequencies. If the system responds correctly for each frequency it will respond correctly to any combination of these frequencies, since the problem has been linearized. An arbitrary forcing function can be constructed by superposition, so the mechanical system is equivalent to the fluid for any arbitrary disturbance.

The forced oscillation requires new boundary conditions corresponding to the motion of the tank walls. Comparatively simple velocity potentials can be constructed to satisfy these conditions. However, these potentials create a pressure variation at the free surface which cannot actually exist. It is then necessary to superimpose additional potentials having the time frequency of the forcing function and the space character of the free oscillations to cancel out this unwanted pressure variation. The resulting potential satisfies the boundary conditions for the moving tank and complies with the zero pressure condition on the free surface.

Tank Motions

Any tank motion of the type considered here can be constructed from motions of translation along the three axes in space and a unique set of rotations about the three axes. (One axis is taken as perpendicular to the mean free surface.) Motions normal to the free surface will be discussed later. This leaves three basic types of motion to be considered at present; translation parallel to the free surface, rotation about an axis parallel to the free surface and rotation about an axis perpendicular to the free surface. For convenience these motions are referred to as horizontal, pitching and yawing. The horizontal motion is comparatively easy to analyze, but the pitching and yawing cases are considerably more complex. The yawing case requires the use of a 3-dimensional solution of Laplace's equation and introduces a new set of natural frequencies. The pitching and horizontal motion cases require only two-dimensional solutions and possess a common set of natural frequencies.

Alternative Methods

The original method used for solving the forced oscillation problem made it possible to set up equivalent mechanical systems, and gave expressions for forces and moments only as functions of the forcing frequency. If it is not desired to use the mechanical analogy, an alternative process is to set up the entire boundary value problem in terms of the Laplace transform of the velocity potential (instead of using the velocity potential itself). This leads to expressions for the Laplace transforms of the forces and moments, which are included in this report. If Laplace transform methods are used for solving the complete dynamics problem, then these force and moment transforms enter directly as terms in the equations.

DEVELOPMENT OF EQUATIONS

Boundary Conditions

In the original development of the equations for the moving tank it was assumed that the boundary condition at any wall would be satisfied only at the mean position of the wall. In the case of horizontal tank motions such an assumption implies severe restrictions on the motions. Since these restrictions are both undesirable and unnecessary, they have been eliminated (for horizontal motion) in the following development.

Consider a rectangular tank filled to a depth h with fuel. Let the walls and bottom of the tank be given, in irrotational coordinates (x,y,z) which follow the horizontal motion of the tank, by $x = \pm \frac{a}{2}$, $y = \pm \frac{b}{2}$ and $z = -\frac{h}{2}$. If the horizontal motion of the tank center is given in inertial coordinates $(\bar{x},\bar{y},\bar{z})$ by

$$\left.\begin{aligned} \bar{x} &= X(t) \\ \bar{y} &= Y(t) \end{aligned}\right\} \qquad (1)$$

then the moving coordinates are related to the inertial coordinates by

$$\left.\begin{aligned} x &= \bar{x} - X(t) \\ y &= \bar{y} - Y(t) \\ z &= \bar{z} \end{aligned}\right\} \qquad (2)$$

Thus a function $\bar{f}(\bar{x},\bar{y},\bar{z},t)$ in the inertial coordinates would transform in the moving coordinates to

$$\bar{f}(\bar{x},\bar{y},\bar{z},t) = \bar{f}(x+X(t),y+Y(t),z,t) = f(x,y,z,t) \qquad (3)$$

The velocity potential $\bar{\phi}$ $(\bar{x},\bar{y},\bar{z},t)$ which describes motion of the fuel in response to tank motion must satisfy the conditions that at any fluid boundary the normal velocity of the fluid, $-\bar{\phi}_n$, be equal to the velocity of the boundary normal to itself. If the velocity potential is expressed in the form

$$\bar{\phi}(\bar{x},\bar{y},\bar{z},t) = -\dot{X}(t)x - \dot{Y}(t)y + \bar{\phi}'(\bar{x},\bar{y},\bar{z},t) \tag{4}$$

then $\bar{\phi}'$ must satisfy the conditions that at any fluid boundary the normal velocity of the fluid relative to the moving coordinates, $-\bar{\phi}'_n$, be equal to the normal velocity of the boundary relative to the moving coordinates. In the case of pure horizontal motion, $\bar{\phi}' = 0$ on the tank walls and bottom. If motions in pitch and yaw involve only small angular displacements of the tank from its original orientation, then the boundary conditions can be applied with sufficient accuracy at $x = \pm \frac{a}{2}$, $y = \pm \frac{b}{2}$ and $z = -\frac{h}{2}$.

If all tank motions are restricted so that they involve accelerations which are small compared to the acceleration due to gravity, and, in the case of sinusoidal motion, the forcing frequency is not too near the frequency of a mode of free oscillation of the fuel, then $\bar{\phi}'_{\bar{x}}$, $\bar{\phi}'_{\bar{y}}$, and $\bar{\phi}'_{\bar{z}}$ will be small. Also the slope of the free surface, $\bar{\eta}_{\bar{x}}$, will be small.

Neglecting the squares and cross products of these small quantities, the linearized free surface conditions in the inertial coordinates are[3b]

$$\begin{aligned} g\bar{\eta}(\bar{x},\bar{y},t) &= \bar{\phi}_t(\bar{x},\bar{y},\tfrac{h}{2},t) \\ &\quad -\tfrac{1}{2}\left[\dot{X}^2 - 2\dot{X}\bar{\phi}'_{\bar{x}}(\bar{x},\bar{y},\tfrac{h}{2},t) + \dot{Y}^2 - 2\dot{Y}\bar{\phi}'_{\bar{y}}(\bar{x},\bar{y},\tfrac{h}{2},t)\right] + F(t) \end{aligned} \tag{5}$$

$$\bar{\eta}_t(\bar{x},\bar{y},t) = -\bar{\phi}_{\bar{z}}(\bar{x},\bar{y},\tfrac{h}{2},t) - \dot{X}\bar{\eta}_{\bar{x}}(\bar{x},\bar{y},t) - \dot{Y}\bar{\eta}_{\bar{y}}(\bar{x},\bar{y},t) \qquad (6)$$

where F(t) is an undetermined function of time. By Eqs. (2) and (3)

$$\bar{\phi}'_{\bar{x}} = \phi'_x, \bar{\phi}'_{\bar{y}} = \phi'_y, \bar{\phi}'_{\bar{z}} = \phi'_z, \bar{\eta}_{\bar{x}} = \eta_x, \bar{\eta}_{\bar{y}} = \eta_y, \bar{\phi}_t = \phi_t - \dot{X}\phi_x - \dot{Y}\phi_y, \& \ \bar{\eta}_t = \eta_t - \dot{X}\eta_x - \dot{Y}\eta_y$$

so that the free surface conditions become, in the moving coordinates:

$$g\eta = \phi_t(x,y,\tfrac{h}{2},t) + \tfrac{1}{2}\dot{X}^2 + \tfrac{1}{2}\dot{Y}^2 + F(t) \qquad (7)$$

$$\eta_t = -\phi_z(x,y,\tfrac{h}{2},t) \qquad (8)$$

The quantity $\frac{1}{2}\dot{X}^2 + \frac{1}{2}\dot{Y}^2 + F(t)$ in Eq. (7) can be set equal to zero, since F(t) is an arbitrary function of time, and $\dot{X}$ and $\dot{Y}$ are functions only of time. This quantity does not affect the fluid motions since it introduces no pressure gradients. Eliminating η between Eq. (7) and (8) the combined condition

$$\phi_{tt}(x,y,\tfrac{h}{2},t) + g\,\phi_z(x,y,\tfrac{h}{2},t) = 0 \qquad (9)$$

is obtained.

If it is assumed that the horizontally oscillating tank is equivalent to a stationary tank with oscillating horizontal force field, the preceding conclusions are reached intuitively.

To summarize, the problem now reduces to determining ϕ (x,y,z,t) to satisfy the following equations:

$$\begin{cases} \phi_{xx}(x,y,z,t) + \phi_{yy}(x,y,z,t) + \phi_{zz}(x,y,z,t) = 0 \\ -\phi_n \text{ (at tank boundary) = normal velocity of tank boundary} \\ \phi_{tt}\left(x,y,\frac{h}{2},t\right) + g\,\phi_z\left(x,y,\frac{h}{2},t\right) = 0 \end{cases}$$

In the following derivations, three of the five basic tank motions are considered. They are: horizontal motion parallel to the x-axis, pitching motion about the y-axis, and yawing motion about the z-axis. Solutions for horizontal motion parallel to the y-axis and pitching motion about the x-axis can be obtained by interchanging a and b in the first two types of motion considered.

Sinusoidal motions are considered first. Arbitrary tank motions are treated by use of Laplace transforms. In each case it is assumed that the motion involves accelerations which are small in comparison to the acceleration due to gravity and that the frequency of sinusoidal motion does not have a value in the immediate neighborhood of the frequency of a natural mode of free surface oscillation in a stationary tank. Motions in pitch and yaw are assumed to involve only small displacements while horizontal motions may involve large displacements and velocities. This is consistent with the linearized treatment of aircraft dynamic problems. Horizontal displacements of several tank widths may occur during side slip or in the case of a tank located far from the center of gravity of an aircraft undergoing small yawing motions.

Sinusoidal Horizontal Motion Parallel to the x-Axis

If the horizontal motion of the tank is given by

$$X(t) = A \sin \omega t \tag{10}$$

then the velocity potential ϕ (x,y,t) must satisfy

$$-\phi_x\left(\pm\frac{a}{2}, z, t\right) = A\omega\cos\omega t \tag{11}$$

$$\phi_z\left(x, -\frac{h}{2}, t\right) = 0 \tag{12}$$

The velocity potential which satisfies Laplace's equation and Eqs. (11) and (12) is

$$\phi = -A\omega\cos\omega t\left\{x + \sum_{n=0}^{\infty} A_n \cos\left[\frac{2n\pi}{a}x\right]\cosh\left[\frac{2n\pi}{a}\left(z+\frac{h}{2}\right)\right]\right.$$
$$\left. + \sum_{n=0}^{\infty} B_n \sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]\right\} \tag{13}$$

The term proportional to x satisfies the velocity condition on the tank walls, while the terms in the summations are needed to satisfy free surface conditions. Applying the free surface condition to Eq. (13) gives the identity

$$A\omega\cos\omega t\left\{x + \sum_{n=0}^{\infty} A_n\left(\omega^2\cosh\left[2n\pi\frac{h}{a}\right] - g\,2n\frac{\pi}{a}\sinh\left[2n\pi\frac{h}{a}\right]\right)\cos\left[2n\pi\frac{x}{a}\right]\right.$$
$$+ \sum_{n=0}^{\infty} B_n\left(\omega^2\cosh\left[(2n+1)\pi\frac{h}{a}\right] - g(2n+1)\frac{\pi}{a}\sinh\left[(2n+1)\pi\frac{h}{a}\right]\right)$$
$$\left. \times \sin\left[(2n+1)\pi\frac{x}{a}\right]\right\} \equiv 0 \tag{14}$$

Expanding x in a Fourier series and then equating coefficients to zero in the identity, one obtains

$$A_n = 0 \tag{15}$$

$$B_n = \frac{(-1)^n 4a}{\pi^2(2n+1)^2}\,\frac{\omega^2}{\left\{g(2n+1)\frac{\pi}{a}\tanh\left[(2n+1)\pi\frac{h}{a}\right] - \omega^2\right\}\cosh\left[(2n+1)\pi\frac{h}{a}\right]} \tag{16}$$

B_n becomes infinite if

$$\omega^2 = \omega_n^2 = g(2n+1)\frac{\pi}{a}\tanh\left[(2n+1)\pi\frac{h}{a}\right] \tag{17}$$

ω_n is the frequency of the nth odd harmonic of the fundamental mode of free surface oscillation in a stationary tank. The velocity potential is

$$\phi = -A\omega\cos\omega t\left\{x + \sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right.$$
$$\left.\times\frac{\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]}{\cosh\left[(2n+1)\pi\frac{h}{a}\right]}\right\} \tag{18}$$

The elevation of the free surface above $z = \frac{h}{2}$ is obtained with the aid of Eq. (7)

$$\eta = A\frac{\omega^2}{g}\sin\omega t\left\{x + \sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right.$$
$$\left.\times\sin\left[(2n+1)\frac{\pi}{a}x\right]\right\} \tag{19}$$

The horizontal force exerted by the fluid on the tank side walls is obtained by integrating the difference in pressure at equal elevations on walls

$$F_H = \int_{-\frac{h}{2}}^{\frac{h}{2}}\rho b\left\{\phi_t\left(\frac{a}{2},z,t\right) - \phi_t\left(-\frac{a}{2},z,t\right)\right\}dz \tag{20}$$

$$F_H = A\omega^2\sin\omega t\left\{\rho abh + \sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right\} \tag{21}$$

The total moment about the center of the tank is obtained from

$$M_T = \int_{\frac{h}{2}}^{-\frac{h}{2}}\rho b\left\{\phi_t\left(-\frac{a}{2},z,t\right)+g\left(\frac{h}{2}-z\right)\right\}z\,dz + \int_{-\frac{a}{2}}^{\frac{a}{2}}\rho b\left\{\phi_t\left(x,-\frac{h}{2},t\right)+gh\right\}x\,dx$$
$$+\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho b\left\{\phi_t\left(+\frac{a}{2},z,t\right)+g\left(\frac{h}{2}-z\right)\right\}z\,dz \tag{22}$$

$$M_T = A\omega^2 \sin\omega t$$

$$\times\left\{\rho\frac{a^3 b}{12}+\sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right]\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right\} \quad (23)$$

<u>Sinusoidal Pitching About The y-Axis</u>

If the pitching motion of the tank is given by

$$\theta = B\sin\omega t \quad (24)$$

where B is the maximum angular displacement of a tank wall from its mean position, then the velocity potential must satisfy the boundary conditions:

$$-\phi_x\left(\pm\frac{a}{2},z,t\right) = +B\omega z\cos\omega t \quad (25)$$

$$-\phi_z\left(x,-\frac{h}{2},t\right) = -B\omega x\cos\omega t \quad (26)$$

The velocity potential which satisfies these boundary conditions is

$$\phi = -B\omega\cos\omega t\left\{\sum_{n=0}^{\infty}(-1)^n\frac{4}{\pi^3(2n+1)^3}\left(\frac{h^2\sin\left[(2n+1)\frac{\pi}{h}z\right]\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\cosh\left[(2n+1)\frac{\pi a}{2h}\right]}\right.\right.$$

$$\left.+\frac{a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z-\frac{h}{2}\right)\right]}{\sinh\left[(2n+1)\frac{\pi}{a}h\right]}\right)$$

$$+\sum_{n=0}^{\infty}A_n'\cos\left[\frac{2n\pi}{a}x\right]\cosh\left[\frac{2n\pi}{a}\left(z+\frac{h}{2}\right)\right]$$

$$\left.+\sum_{n=0}^{\infty}B_n'\cos\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]\right\} \quad (27)$$

Applying the free surface condition, Eq. (9), the identity

$$B\omega\cos\omega t\left\{\omega^2\sum_{n=0}^{\infty}\frac{4h^2\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\pi^3(2n+1)^3\cosh\left[(2n+1)\frac{\pi a}{2h}\right]}\right.$$

$$+\omega^2\sum_{n=0}^{\infty}(-1)^n\frac{4a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]}{\pi^3(2n+1)^3\sinh\left[(2n+1)\frac{\pi}{a}h\right]}$$

$$+\sum_{n=0}^{\infty}A_n'\left(\omega^2\cosh\left[\frac{2n\pi}{a}h\right]-g2n\frac{\pi}{a}\sinh\left[2n\frac{\pi}{a}h\right]\right)\times\cos\left[2n\pi\frac{x}{a}\right]$$

$$+\sum_{n=0}^{\infty}B_n'\left(\omega^2\cosh\left[(2n+1)\frac{\pi}{a}h\right]-g(2n+1)\frac{\pi}{a}\sinh\left[(2n+1)\frac{\pi}{a}h\right]\right)$$

$$\left.\times\sin\left[(2n+1)\frac{\pi}{a}x\right]\right\}\equiv 0 \tag{28}$$

is obtained.

Expansion of the first summation in Eq. (28) in a Fourier series gives

$$\sum_{n=0}^{\infty}\frac{4h^2\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\pi^3(2n+1)^3\cosh\left[(2n+1)\frac{\pi a}{2h}\right]}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 4a}{\pi^2(2n+1)^2}\left[\frac{h}{2}-\frac{a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}\right]\sin\left[(2n+1)\frac{\pi}{a}x\right] \tag{29}$$

which upon substituting into Eq. (28) and equating coefficients to zero gives after simplification

$$A_n' = 0 \tag{30}$$

$$B_n' = (-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right] \tag{31}$$

The velocity potential is

$$\phi = -B\omega\cos\omega t\left\{\sum_{n=0}^{\infty}(-1)^n\frac{4}{\pi^3(2n+1)^3}\left[\frac{h^2\sin\left[(2n+1)\frac{\pi}{h}z\right]\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\cosh\left[(2n+1)\frac{\pi a}{2h}\right]}\right.\right.$$

$$\left.+\frac{a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z-\frac{h}{2}\right)\right]}{\sinh\left[(2n+1)\frac{\pi}{a}h\right]}\right] \tag{32}$$

$$+\sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right]$$

$$\left.\times\frac{\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]}{\cosh\left[(2n+1)\frac{\pi}{a}h\right]}\right\}$$

The elevation of the free surface is obtained by use of Eqs. (7) and (29).

$$\eta = B\frac{\omega^2}{g}\sin\omega t\left\{\sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right]\right.$$

$$\left.\times\left(\frac{\omega_n^2}{\omega_n^2-\omega^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\right\} \tag{33}$$

The horizontal force exerted by the fluid on the tank walls is obtained by use of Eq. (20)

$$F_H = B\omega^2\sin\omega t\left\{\rho\frac{a^3b}{12}+\sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3b}{\pi^4(2n+1)^4}\right.$$

$$\left.\times\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right]\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right\} \tag{34}$$

The total moment about the center of the tank is obtained from Eq. (22) where in the second integral $(g\ h)$ is replaced by $g(h-Bx\sin\omega t)$ to take into account the unequal depth due to rotation of the tank through the angle $B\sin\omega t$.

$$M_T = B\omega^2 \sin\omega t \left\{ -\sum_{n=0}^{\infty} \frac{8\rho a^3 b}{\pi^4 (2n+1)^4} \left[\frac{h}{2} - \frac{a \tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi} - \frac{g}{\omega_n^2} \right] \right.$$

$$- \sum_{n=0}^{\infty} \frac{8\rho h^3 b}{\pi^4 (2n+1)^4} \left[\frac{a}{2} - \frac{3h \tanh\left[(2n+1)\frac{\pi a}{2h}\right]}{(2n+1)\pi} \right] + \sum_{n=0}^{\infty} \frac{\omega_n^2}{g} \frac{8\rho a^3 b}{\pi^4 (2n+1)^4} \tag{35}$$

$$\left. \times \left[\frac{h}{2} - \frac{2a \tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi} + \frac{g}{\omega_n^2} \right]^2 \left(\frac{\omega^2}{\omega_n^2 - \omega^2} \right) \right\}$$

$$+ B \sin\omega t \left\{ \rho g \frac{a^3 b}{12} \right\}$$

Sinusoidal Yawing About The z-Axis

If the yawing motion about the z-axis is given by

$$\psi = C \sin\omega t \tag{36}$$

where C is the maximum angular displacement of the tank walls from their mean position, then the velocity potential must satisfy the boundary conditions:

$$-\phi_x\left(\pm\frac{a}{2}, y, z, t\right) = -C\omega y \cos\omega t \tag{37}$$

$$-\phi_y\left(x, \pm\frac{b}{2}, z, t\right) = C\omega x \cos\omega t \tag{38}$$

$$\phi_z\left(x, y, -\frac{h}{2}, t\right) = 0 \tag{39}$$

$$M_T = B\omega^2 \sin\omega t\left\{-\sum_{n=0}^{\infty}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left[\frac{h}{2}-\frac{a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}-\frac{g}{\omega_n^2}\right]\right.$$

$$-\sum_{n=0}^{\infty}\frac{8\rho h^3 b}{\pi^4(2n+1)^4}\left[\frac{a}{2}-\frac{3h\tanh\left[(2n+1)\frac{\pi a}{2h}\right]}{(2n+1)\pi}\right]+\sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4} \tag{35}$$

$$\left.\times\left[\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right]^2\left(\frac{\omega^2}{\omega_n^2-\omega^2}\right)\right\}$$

$$+B\sin\omega t\left\{\rho g\frac{a^3 b}{12}\right\}$$

Sinusoidal Yawing About The z-Axis

If the yawing motion about the z-axis is given by

$$\psi = C\sin\omega t \tag{36}$$

where C is the maximum angular displacement of the tank walls from their mean position, then the velocity potential must satisfy the boundary conditions:

$$-\phi_x\left(\pm\frac{a}{2}, y, z, t\right) = -C\omega y\cos\omega t \tag{37}$$

$$-\phi_y\left(x, \pm\frac{b}{2}, z, t\right) = C\omega x\cos\omega t \tag{38}$$

$$\phi_z\left(x, y, -\frac{h}{2}, t\right) = 0 \tag{39}$$

and the Laplace equation in three dimensions. The velocity potential satisfying these conditions is

$$\phi = C\omega\cos\omega t\left\{\sum_{n=0}^{\infty}(-1)^n\frac{4}{\pi^3(2n+1)^3}\left[\frac{b^2\sin\left[(2n+1)\frac{\pi}{b}y\right]\sinh\left[(2n+1)\frac{\pi}{b}x\right]}{\cosh\left[(2n+1)\frac{\pi a}{2b}\right]}\right.\right.$$
$$\left.-\frac{a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]\sinh\left[(2n+1)\frac{\pi}{a}y\right]}{\cosh\left[(2n+1)\frac{\pi b}{2a}\right]}\right] \tag{40}$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{m,n}\sin\left[(2n+1)\frac{\pi}{a}x\right]\sin\left[(2m+1)\frac{\pi}{b}y\right]$$
$$\left.\times\cosh\left[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\left(z+\frac{h}{2}\right)\right]\right\}$$

where terms containing even multiples of n or m have been omitted (since they contribute nothing to the velocity potential). The term involving only the single summation is the part of the velocity potential which satisfies the velocity conditions on the tank boundaries, while the double summation term is needed to satisfy free surface conditions.

Applying the free surface condition to Eq. (40), the following identity is obtained:

$$-C\omega\cos\omega t\left\{\omega^2\sum_{m=0}^{\infty}(-1)^m\frac{4b^2\sinh\left[(2m+1)\frac{\pi}{b}x\right]}{\pi^3(2m+1)^3\cosh\left[(2m+1)\frac{\pi a}{2b}\right]}\sin\left[(2m+1)\frac{\pi}{b}y\right]\right.$$
$$-\omega^2\sum_{n=0}^{\infty}(-1)^n\frac{4a^2\sinh\left[(2n+1)\frac{\pi}{a}y\right]}{\pi^3(2n+1)^3\cosh\left[(2n+1)\frac{\pi a}{2b}\right]}\sin\left[(2n+1)\frac{\pi}{a}x\right]$$

(Equation continued on next page)

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{m,n}\Big(\omega^2\cosh\Big[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\,h\Big]-g\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}$$

$$\times\sinh\Big[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\,h\Big]\Big)\sin\Big[(2n+1)\frac{\pi}{a}x\Big]\sin\Big[(2m+1)\frac{\pi}{b}y\Big]\Big\}\equiv 0 \qquad (41)$$

Upon substitution of the Fourier expansions

$$\frac{\sinh\left[(2m+1)\frac{\pi}{b}x\right]}{(2m+1)\cosh\left[(2m+1)\frac{\pi a}{2b}\right]}=\frac{4ab}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\sin\left[(2n+1)\frac{\pi}{a}x\right]}{\left[b^2(2n+1)^2+a^2(2m+1)^2\right]} \qquad (42)$$

$$\frac{\sinh\left[(2n+1)\frac{\pi}{a}y\right]}{(2n+1)\cosh\left[(2n+1)\frac{\pi b}{2a}\right]}=\frac{4ab}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\sin\left[(2m+1)\frac{\pi}{b}y\right]}{\left[b^2(2n+1)^2+a^2(2m+1)^2\right]} \qquad (43)$$

into Eq. (41), the coefficients $A_{m,n}$ are found after simplification to be

$$A_{m,n}=B_{m,n}\left(\frac{\omega^2}{\omega_{m,n}^2-\omega^2}\right)$$

$$B_{m,n}=\frac{(-1)^{m+n}16ab\left[b^2(2n+1)^2-a^2(2m+1)^2\right]\operatorname{sech}\left[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\,h\right]}{\pi^4(2m+1)^2(2n+1)^2\left[b^2(2n+1)^2+a^2(2m+1)^2\right]} \qquad (44)$$

where

$$\omega_{m,n}=g\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\tanh\left[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\,h\right] \qquad (45)$$

The elevation of the free surface is found with the aid of Eqs. (7), (42) and (43).

$$\eta = -C\frac{\omega^2}{g}\cos\omega t\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}16ab\left[b^2(2n+1)^2-a^2(2m+1)^2\right]}{\pi^4(2m+1)^2(2n+1)^2\left[b^2(2n+1)^2+a^2(2m+1)^2\right]}\right.$$

$$\left.\times\left(\frac{\omega_{m,n}^2}{\omega_{m,n}^2-\omega^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\sin\left[(2m+1)\frac{\pi}{b}y\right]\right\} \tag{46}$$

Since the motion is purely yawing, the fuel exerts no net horizontal force or moment about the x or y-axis. The moment about the z-axis or yawing moment exerted by the fuel on the tank is obtained from the following expression

$$M_z = -\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\left\{\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\phi_t\left(\frac{a}{2},y,z,t\right)-\phi_t\left(-\frac{a}{2},y,z,t\right)\right]y\,dy\right.$$

$$\left.-\int_{-\frac{a}{2}}^{\frac{a}{2}}\left[\phi_t\left(x,\frac{b}{2},z,t\right)-\phi_t\left(x,-\frac{b}{2},z,t\right)\right]x\,dx\right\}dz \tag{47}$$

Substituting $A_{m,n}$ into Eq. (40) and applying Eq. (47) the yawing moment is found to be

$$M_z = C\omega^2\sin\omega t\left\{\frac{32\rho h}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\left[a^4\tanh\left[(2n+1)\frac{\pi b}{2a}\right]\right.\right.$$

$$\left.+b^4\tanh\left[(2n+1)\frac{\pi a}{2b}\right]\right]-\frac{1}{12}\rho abh(a^2+b^2)$$

$$\left.+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\omega_{m,n}}{g}\frac{64\rho a^3b^3\left[b^2(2n+1)^2-a^2(2m+1)^2\right]^2}{\pi^8\left[b^2(2n+1)^2+a^2(2m+1)^2\right]^2(2m+1)^4(2n+1)^4}\left(\frac{\omega^2}{\omega_{m,n}^2-\omega^2}\right)\right\} \tag{48}$$

The first summation of the right side of Eq. (48) is the effective moment of inertia of the fuel about the z-axis.

Arbitrary Horizontal Tank Motion

The Laplace transform (with respect to time) of the boundary conditions for arbitrary horizontal tank motion is

$$\mathcal{L}\left\{\phi_x\left(\pm\frac{a}{2}, z, t\right)\right\} = -\mathcal{L}\left\{\dot{X}(t)\right\} \tag{49}$$

$$\mathcal{L}\left\{\phi_z\left(x, -\frac{h}{2}, t\right)\right\} = 0 \tag{50}$$

where $X(t)$ is the displacement of the tank in inertial coordinates. Under the assumption that up until the time t = o the free surface is undisturbed, the Laplace transform of the free surface condition is

$$p^2\mathcal{L}\left\{\phi\left(x, \frac{h}{2}, t\right)\right\} + g\mathcal{L}\left\{\phi_z\left(x, \frac{h}{2}, t\right)\right\} \equiv 0 \tag{51}$$

The Laplace transform which satisfies the conditions given by Eq. (49), (50) and (51) is

$$\mathcal{L}\{\phi\} = -\mathcal{L}\{\dot{X}(t)\} \times \left[x - \sum_{n=0}^{\infty}(-1)^n \frac{4a}{\pi^2(2n+1)^2}\left(\frac{p^2}{p^2+\omega_n^2}\right)\frac{\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]}{\cosh\left[(2n+1)\frac{\pi}{a}h\right]}\right] \tag{52}$$

The Laplace transform of the elevation of the free surface above $z = \frac{h}{2}$ is given by

$$\mathcal{L}\left\{\eta(x,t)\right\} = \frac{1}{g}p\mathcal{L}\left\{\phi\left(x, \frac{h}{2}, t\right)\right\} \tag{53}$$

$$\mathcal{L}\{\eta\} = -\frac{p}{g}\mathcal{L}\{\dot{X}(t)\}\left[x - \sum_{n=0}^{\infty}(-1)^n \frac{4a}{\pi^2(2n+1)^2}\left(\frac{p^2}{p^2+\omega_n^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\right] \tag{54}$$

The Laplace transform of the horizontal force exerted by the fluid on the tank walls is given by

$$\mathcal{L}\{F_H(t)\} = \rho b p\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\mathcal{L}\{\phi(\tfrac{a}{2},z,t)\} - \mathcal{L}\{\phi(-\tfrac{a}{2},z,t)\}\right]dz \tag{55}$$

$$\mathcal{L}\{F_H\} = -p\mathcal{L}\{\dot{X}(t)\}\left[\rho abh - \sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left(\frac{p^2}{p^2+\omega_n^2}\right)\right] \tag{56}$$

The Laplace transform of the total moment exerted by the fluid on the tank is

$$\mathcal{L}\{M_T(t)\} = \rho b p\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\mathcal{L}\{\phi(\tfrac{a}{2},z,t)\} - \mathcal{L}\{\phi(-\tfrac{a}{2},z,t)\}\right]z\,dz + \rho b p\int_{-\frac{a}{2}}^{\frac{a}{2}}\mathcal{L}\{\phi(x,-\tfrac{h}{2},t)\}\,x\,dx \tag{57}$$

$$\mathcal{L}\{M_T\} = -p\mathcal{L}\{\dot{X}(t)\} \times\left[\rho\frac{a^3 b}{12} - \sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left(\frac{h}{2} - \frac{2a\tanh\left[(2n+1)\frac{\pi}{a}\frac{h}{2}\right]}{(2n+1)\pi} + \frac{g}{\omega_n^2}\right)\left(\frac{p^2}{p^2+\omega_n^2}\right)\right] \tag{58}$$

The Laplace transforms of velocity potential, free surface elevation, horizontal force and total moment representing the response of the fuel to arbitrary pitching and yawing motions of small amplitude are given below without proof.

Arbitrary Pitching Motion

$$\mathcal{L}\{\phi\} = -\mathcal{L}\{\dot{\theta}(t)\}\Bigg[\sum_{n=0}^{\infty}(-1)^n\frac{4}{\pi^3(2n+1)^3}\Bigg(\frac{h^2\sin\left[(2n+1)\frac{\pi}{h}z\right]\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\cosh\left[(2n+1)\frac{\pi a}{2h}\right]}$$

$$+\frac{a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z-\frac{h}{2}\right)\right]}{\sinh\left[(2n+1)\frac{\pi}{a}h\right]}\Bigg) - \sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{p^2}{p^2+\omega^2}\right) \tag{59}$$

$$\times\left(\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi}{2}\frac{h}{a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right)\frac{\sin\left[(2n+1)\frac{\pi}{a}x\right]\cosh\left[(2n+1)\frac{\pi}{a}\left(z+\frac{h}{2}\right)\right]}{\cosh\left[(2n+1)\frac{\pi}{a}h\right]}\Bigg]$$

$$\mathcal{L}\{\eta\} = -\frac{p}{g}\mathcal{L}\{\dot{\theta}(t)\} \tag{60}$$

$$\times\left[\sum_{n=0}^{\infty}(-1)^n\frac{4a}{\pi^2(2n+1)^2}\left(\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\right]$$

$$\mathcal{L}\{F_H\} = -p\mathcal{L}\{\dot{\theta}(t)\}$$

$$\times\left[\rho\frac{a^3b}{12}-\sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left(\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{a2}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right)\left(\frac{p^2}{p^2+\omega_n^2}\right)\right] \tag{61}$$

$$\mathcal{L}\{M_T\} = -p\,\mathcal{L}\{\dot{\theta}(t)\}\Bigg[-\sum_{n=0}^{\infty}\frac{8\rho a^3 b}{\pi^4(2n+1)}\left(\frac{h}{2}-\frac{a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right)$$

$$-\sum_{n=0}^{\infty}\frac{8\rho h^3 b}{\pi^4(2n+1)^4}\left(\frac{a}{2}-\frac{3h\tanh\left[(2n+1)\frac{\pi a}{2h}\right]}{(2n+1)\pi}\right) \tag{62}$$

(Equation continued on next page)

$$-\sum_{n=0}^{\infty}\frac{\omega_n^2}{g}\frac{8\rho a^3 b}{\pi^4(2n+1)^4}\left(\frac{h}{2}-\frac{2a\tanh\left[(2n+1)\frac{\pi h}{2a}\right]}{(2n+1)\pi}+\frac{g}{\omega_n^2}\right)^2\left(\frac{p^2}{p^2+\omega_n^2}\right)\Bigg]$$

$$+\mathcal{L}\{\dot\theta(t)\}\left[\rho g\frac{a^3 b}{12}\right] \qquad (62)$$

where $\theta(t)$ is the angular displacement of the tank walls in pitch.

Arbitrary Yawing Motion

$$\mathcal{L}\{\phi\}=\mathcal{L}\{\dot\psi(t)\}\Bigg[\sum_{m=0}^{\infty}(-1)^n\frac{4}{\pi^3(2m+1)^3}\left(\frac{b^2\sin\left[(2n+1)\frac{\pi}{b}y\right]\sinh\left[(2n+1)\frac{\pi}{h}x\right]}{\cosh\left[(2n+1)\frac{\pi a}{2b}\right]}\right.$$

$$\left.-\frac{a^2\sin\left[(2n+1)\frac{\pi}{a}x\right]\sinh\left[(2n+1)\frac{\pi}{a}y\right]}{\cosh\left[(2n+1)\frac{\pi a}{2b}\right]}\right)$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{m,n}\left(\frac{p^2}{p^2+\omega_{m,n}^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\sin\left[(2m+1)\frac{\pi}{b}y\right]$$

$$\times\cosh\left[\frac{\pi}{ab}\sqrt{b^2(2n+1)^2+a^2(2m+1)^2}\left(z+\frac{b}{2}\right)\right]\Bigg] \qquad (63)$$

$$\mathcal{L}\{\eta\}=\frac{p}{g}\mathcal{L}\{\dot\psi(t)\}\Bigg[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}16ab\left[b^2(2n+1)^2-a^2(2m+1)^2\right]}{\pi^4(2m+1)^2(2n+1)^2\left[b^2(2n+1)^2+a^2(2m+1)^2\right]}$$

$$\times\left(\frac{\omega_{m,n}^2}{p^2+\omega_{m,n}^2}\right)\sin\left[(2n+1)\frac{\pi}{a}x\right]\sin\left[(2m+1)\frac{\pi}{b}y\right]\Bigg] \qquad (64)$$

$$\mathcal{L}\left\{M_z\right\} = -p\mathcal{L}\left\{\dot{\psi}(t)\right\}\Bigg[\frac{32\rho h}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\Bigg(a^4\tanh\left[(2n+1)\frac{\pi b}{2a}\right]$$

$$+b^4\tanh\left[(2n+1)\frac{\pi a}{2b}\right]\Bigg) - \frac{1}{12}\rho abh(a^2+b^2) \tag{65}$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\omega_{mn}^2}{g}\,\frac{64\rho a^3 b^3\left[b^2(2n+1)^2 - a^2(2m+1)^2\right]^2}{\pi^8\left[b^2(2n+1)^2 + a^2(2m+1)^2\right](2m+1)^4(2n+1)^4}\left(\frac{p^2}{p^2+\omega_{m,n}^2}\right)\Bigg]$$

where $B_{m,n}$ is given by Eq. (44) and $\psi(t)$ is the angular displacement of the tank walls in yaw.

EQUIVALENT MECHANICAL SYSTEM

Consider a rectangular tank of length a, width b, and mean fuel depth h undergoing small amplitude sinusoidal oscillations of frequency ω about a set of fixed axes. Let the fixed axes coincide with the principal axes of the fuel when the tank is at rest in an upright position. The positive directions of these axes are shown in Fig. 1. Positive forces, moments, displacements, and rotations are defined according to the conventions of a right hand system.

The problem of replacing the fuel by an equivalent mechanical system resolves itself into finding a mechanical system for each of three types of motion:

(A) horizontal motion parallel to the x-axis and pitching about the y-axis.

(B) horizontal motion parallel to the y-axis and pitching about the x-axis.

(C) yawing motion about the z-axis.

In order for a mechanical system to be equivalent to the fuel with respect to one of the above types of tank motion, it must exert the same force and total moment on the tank as the fuel does for that type of tank motion. Motions of type (A) and (B) are of the same character so that it is only necessary to discuss one of them.

Motions of Type (A)

Consider a mechanical system composed of a fixed mass M and an infinite set of undamped spring-masses $\left\{ m_n \right\}$ so constrained as to move only parallel to the bottom of the tank and the xz-plane. Let the nth spring-mass have a spring constant k_n such that

$$k_n = \omega_n^2 m_n$$

where ω_n is the frequency of an odd harmonic of fundamental free fuel oscillation. Locate the nth spring mass at a vertical distance z_n and the fixed mass at a vertical distance Z as shown in Fig. 2. The moving masses may be thought of as point masses while the fixed mass has a distribution such that its moment of inertia about an axis parallel to the y-axis and passing through z = Z is I_y.

If the tank undergoes a horizontal oscillation of amplitude A and frequency ω parallel to the x-axis, then the equations of motion of the moving masses are given by

$$m_n \ddot{x}_n = -k_n (x_n - A \sin \omega t) \tag{66}$$

where x_n is the displacement of the moving masses parallel to the x-axis. If the tank undergoes a pitching oscillation of angular amplitude B and frequency ω, the equations of motion are

$$m_n \ddot{x}_n = -k_n (x_n - B z_n \sin \omega t) + m_n g B \sin \omega t \tag{67}$$

The horizontal force and total moment exerted by the mechanical system were obtained with the aid of the solutions of Eqs. (66) and (67). A comparison of these forces and moments (designated by primes) with the corresponding forces and moments exerted by the fuel on the tank is shown

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below. The forces and moments have been reduced to non-dimensional form in which

$M_F = \rho$ abh = total fuel mass

$W_F = g\rho$ abh = total fuel weight

g = vertical acceleration

I_{F_y} = effective moment of inertia about the y-axis of the fuel

$r_1 = \frac{h}{a}$ = tank aspect ratio

$$f_n^2 = \frac{h\omega_n^2}{g} = (2n+1)\pi r_1 \tanh[(2n+1)\pi r_1]$$

$$f^2 = \frac{h\omega^2}{g}$$

Sinusoidal horizontal motion:

$$\frac{F_H}{W_F} = \frac{A}{h} f^2 \sin\omega t \left\{ 1 + \sum_{n=0}^{\infty} \frac{8\tanh[(2n+1)\pi r_1]}{\pi^3(2n+1)^3 r_1} \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\} \tag{68}$$

$$\frac{F_H'}{W_F} = \frac{A}{h} f^2 \sin\omega t \left\{ \frac{M}{M_F} + \sum_{n=0}^{\infty} \frac{m_n}{M_F} + \sum_{n=0}^{\infty} \frac{m_n}{M_F} \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\} \tag{69}$$

$$\frac{M_T}{W_F h} = \frac{A}{h} f^2 \sin\omega t \left\{ \frac{1}{12 r_1^2} + \sum_{n=0}^{\infty} \frac{8\tanh[(2n+1)\pi r_1]}{\pi^3(2n+1)^3 r_1} \right. \tag{70}$$

$$\left. \times \left(\frac{1}{2} - \frac{\tanh\left[(2n+1)\frac{\pi}{2} r_1\right]}{(2n+1)\frac{\pi}{2} r_1} + \frac{1}{f_n^2} \right) \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\}$$

$$\frac{M_T'}{W_F h} = \frac{A}{h} f^2 \sin\omega t \left\{ \frac{M}{M_F}\frac{Z}{h} + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{Z_n}{h} + \frac{1}{f_n^2}\right) + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{Z_n}{h} + \frac{1}{f_n^2}\right) \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\} \tag{71}$$

Sinusoidal pitching motion:

$$\frac{F_H}{W_F} = B f^2 \sin\omega t \left\{ \frac{1}{12 r_1^2} + \sum_{n=0} \frac{8\tanh[(2n+1)\pi r_1]}{\pi^3(2n+1)^3 r_1} \right. \tag{72}$$

$$\left. \times \left(\frac{1}{2} - \frac{\tanh\left[(2n+1)\frac{\pi}{2} r_1\right]}{(2n+1)\frac{\pi}{2} r_1} + \frac{1}{f_n^2} \right) \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\} \tag{73}$$

$$\frac{F_H'}{W_F} = Bf^2 \sin\omega t \left\{ \frac{M}{M_F}\frac{Z}{h} + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{z_n}{h} + \frac{1}{f_n^2}\right) + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{z_n}{h} + \frac{1}{f_n^2}\right) \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \right\} \tag{73}$$

$$\begin{aligned} \frac{M_T}{W_F h} = Bf^2 \sin\omega t \Bigg\{ & \frac{I_{Fy}}{M_F h^2} + 2\sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{\pi^3 (2n+1)^3 r_1} \\ & \times \left(\frac{1}{2} - \frac{\tanh[(2n+1)\frac{\pi}{2} r_1]}{(2n+1)\frac{\pi}{2} r_1} + \frac{1}{f_n^2} \right) \frac{1}{f_n^2} + \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{\pi^3 (2n+1)^3 r_1} \\ & \times \left(\frac{1}{2} - \frac{\tanh[(2n+1)\frac{\pi}{2} r_1]}{(2n+1)\frac{\pi}{2} r_1} + \frac{1}{f_n^2} \right)^2 \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \Bigg\} + B \sin\omega t \left\{ \frac{1}{12 r_1^2} \right\} \end{aligned} \tag{74}$$

$$\begin{aligned} \frac{M_T'}{W_F h} = Bf^2 \sin\omega t \Bigg\{ & \frac{I_y}{M_F h^2} + \frac{M}{M_F}\left(\frac{Z}{h}\right)^2 + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{z_n}{h} + \frac{1}{f_n^2}\right)^2 \\ & + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{z_n}{h} + \frac{1}{f_n^2}\right)^2 \frac{1}{\left(\frac{f_n}{f}\right)^2 - 1} \Bigg\} + B \sin\omega t \left\{ \frac{M}{M_F}\frac{Z}{h} + \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{z_n}{h} + \frac{1}{f_n^2}\right) \right\} \end{aligned} \tag{75}$$

If the forces and moments exerted by the mechanical system are to be identically equal to the forces and moments exerted by the fuel, then the characteristics of the mechanical system must be given by

$$\frac{m_n}{M_F} = \frac{8 \tanh[(2n+1)\pi r_1]}{\pi^3 (2n+1)^3 r_1} \tag{76}$$

$$\frac{z_n}{h} = \frac{1}{2} - \frac{\tanh[(2n+1)\frac{\pi}{2} r_1]}{(2n+1)\frac{\pi}{2} r_1} \tag{77}$$

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$$\frac{M}{M_F} = 1 - \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{\pi^3 (2n+1)^3 r_1} \tag{78}$$

$$\frac{Z}{h} = -\frac{1}{\left(\frac{M}{M_F}\right)} \sum_{n=0}^{\infty} \frac{m_n}{M_F} \frac{Z_n}{h} \tag{79}$$

$$\frac{I_y}{M_F h^2} = \frac{I_{Fy}}{M_F h^2} - \frac{M}{M_F}\left(\frac{Z}{h}\right)^2 - \sum_{n=0}^{\infty} \frac{m_n}{M_F}\left(\frac{Z_n}{h}\right)^2 \tag{80}$$

$$\frac{h k_n}{W_F} = \frac{8 \tanh^2[(2n+1)\pi r_1]}{\pi^2 (2n+1)^2} \tag{81}$$

It is interesting to note that the rigid mechanical system has the same moment of inertia about the y-axis as the fuel if the free surface were constrained by a tank boundary. This effective moment of inertia of the fuel is found to be

$$I_{Fy} = I_{Sy} \left\{ 1 - \frac{4}{1+r_1^2} + \frac{768}{r_1(1+r_1^2)\pi^5} \sum_{n=0}^{\infty} \frac{\tanh[(2n+1)\frac{\pi}{2} r_1]}{(2n+1)^5} \right\} \tag{82}$$

where I_{Sy} is the moment of inertia about the y-axis if the fuel were solidified.

Approximate expressions, good to five significant figures, for I_{Fy}/I_{Sy}, M/M_F, and Z/h which lend themselves to easy calculation are given below.

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$$\frac{I_{Fy}}{I_{Sy}} = 1 - \frac{4r_1^2}{1+r_1^2} + 2.50965\left[\tanh\frac{\pi}{2r_1} + 0.004522\right]\frac{r_1^3}{1+r_1^2} \quad (83)$$

for $r_1 \leq 1$

$$\frac{M}{M_F} = 1 - \frac{0.258012}{r_1}\left[\tanh \pi r_1 + 0.051800\right] \quad (84)$$

for $r_1 \leq \frac{1}{2}$

$$\frac{M}{M_F} = 1.032049 r_1\left[\tanh\frac{\pi}{4r_1} + 0.051800\right] \quad (85)$$

for $r_1 \geq \frac{1}{2}$

$$\frac{Z}{h} = \frac{1}{2} - \frac{1}{2\left(\frac{M}{M_F}\right)} + \frac{1}{6r_1^2\left(\frac{M}{M_F}\right)} - \frac{0.164256}{r_1^2\left(\frac{M}{M_F}\right)}\left[\frac{1}{\cosh \pi r_1} + \frac{1}{3^4\cosh 3\pi r_1} + \cdots\right] \quad (86)$$

The mass ratios m_0/M_F , m_1/M_F , and M/M_F have been computed from Eqs. (76), (84) and (85) and are plotted as a function of r_1 in Fig. 3. The moving masses associated with the higher harmonics are too small to plot except very near $r_1 = 0$. The ratio of m_n to m_0 is approximately $1/(2n+1)^2$ for small r_1 and $1/(2n+1)^3$ for large r_1. These ratios are plotted on a logarithmic scale verses the ratio f_n/f_0 in Fig. 4.

The ratios of the arms z_0 , z_1 , and Z to h have been computed from Eqs. (77) and (86) and are plotted in Fig. 5. As the tank becomes shallow, the ratio $\frac{Z}{h}$ becomes infinite, but Z approaches 0.460613a as can be seen by multiplying Eq. (86) by ar_1 and letting r_1 approach zero.

The moment of inertia ratios I_{Fy}/I_{Sy} and I_y/I_{Sy} have been calculated from Eqs. (80) and (83) and are plotted in Fig. 6. It should be noted that values of I_{Fy}/I_{Sy} for $r_1 > 1$ can be obtained by replacing r_1 by $1/r_1$ in Eq. (83).

Fig. 7 is a plot of the non-dimensional form of the spring constants k_0, k_1, and k_2 as calculated from Eq. (81).

In order to get a dimensionless parameter, the horizontal force produced by the mechanical system is divided by the force which would be produced if the fuel were solidified. Fig. 8 is a plot of this ratio as a function of frequency for a tank aspect ratio of 0.25. At resonant frequencies the force ratio would not actually become infinite. In Ref. 3c, Lamb discusses a limiting value of the ratio of wave amplitude to wave length which permits compliance with free surface boundary conditions. This limiting ratio was found to be approximately 0.142 beyond which one may assume that energy dissipation through splashing would occur. Corresponding to any amplitude limitation of any particular mode of fuel oscillation there is a force limitation in that mode.

At resonant frequencies for the tank harmonics the maximum force that can be produced in the nth odd harmonic mode is $1/(2n+1)^2$ times the maximum force that can be produced in the fundamental mode (n=o). This gives some justification for discarding the moving masses associated with modes of fuel oscillation beyond the fundamental.

Motions of Type (C)

Consider a mechanical system composed of a fixed mass with moment of inertia I_z about the z-axis and an infinite set of moving masses constrained to pivot about the z-axis. If the moving masses have moments of inertia $I_{m,n}$ and are attached to torsional springs with spring constants $k_{m,n}$, then the equations of motion of the moving masses are

$$I_{m,n}\ddot{\psi}_{m,n} = -k_{m,n}\left(\psi_{m,n} - C\sin\omega t\right) \qquad (87)$$

where $C\sin\omega t$ is the angular displacement of the tank in yaw. If the spring constants are related to the natural frequencies of free fuel oscillation by

$$k_{m,n} = \omega_{m,n} I_{m,n} \qquad (88)$$

then the yawing moment exerted by the mechanical system on the tank can be found from Eq. (87) and compared to the yawing moment exerted by the fuel. This comparison is made below in which the primed moment is the moment produced by the mechanical system and where

I_{S_z} = moment of inertia about the z-axis of the solidified fuel

I_{F_z} = effective moment of inertia of the fuel

$r_1 = \dfrac{h}{a}$ = tank aspect ratio in the xz-plane

$r_2 = \dfrac{h}{b}$ = tank aspect ratio in the yz-plane

$r_3 = \dfrac{r_2}{r_1} = \dfrac{a}{b}$ = tank aspect ratio in the xy-plane

$$\frac{h\omega_{m,n}}{g} = \pi r_1\sqrt{r_3^2(2m+1)^2+(2n+1)^2}\,\tanh\left[\pi r_1\sqrt{r_3^2(2m+1)^2+(2n+1)^2}\right]$$

$$\frac{M_z}{\frac{g}{a} I_{s_z}} = \frac{C a \omega^2}{g} \sin\omega t \left\{ \frac{I_{F_z}}{I_{s_z}} \right.$$

$$\left. + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{768\left[r_3^2(2m+1)^2-(2n+1)^2\right]^2 \tanh\left[\pi r_1 \sqrt{r_3^2(2m+1)^2+(2n+1)^2}\right]}{\pi^7 r_1 (1+r_3^2)\left[r_3^2(2m+1)^2+(2n+1)^2\right]^{3/2}(2m+1)^4(2n+1)^4} \frac{1}{\left(\frac{\omega_{m,n}}{\omega}\right)^2-1} \right\} \tag{89}$$

$$\frac{M_z'}{\frac{g}{a} I_{s_z}} = \frac{C a \omega^2}{g} \sin\omega t \left\{ \frac{I_z}{I_{s_z}} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{m,n}}{I_{s_z}} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{m,n}}{I_{s_z}} \frac{1}{\left(\frac{\omega_{m,n}}{\omega}\right)^2-1} \right\} \tag{90}$$

If the two yawing moments are to be identical for all forcing frequencies, then

$$\frac{I_{m,n}}{I_{s_z}} = \frac{768\left[r_3^2(2m+1)^2-(2n+1)^2\right]^2}{\pi^6(1+r_3^2)\left[r_3^2(2m+1)^2+(2n+1)^2\right](2m+1)^4(2n+1)^4} \frac{\tanh\left[\pi r_1 \sqrt{r_3^2(2m+1)^2+(2n+1)^2}\right]}{\pi r_1 \sqrt{r_3^2(2m+1)^2+(2n+1)^2}} \tag{91}$$

$$\frac{I_z}{I_{s_z}} = \frac{I_{F_z}}{I_{s_z}} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{m,n}}{I_{s_z}} \tag{92}$$

It is interesting to note that if $r_3 = (2n+1)/(2m+1)$, then $I_{m,n} = 0$. In the case of a tank with square planform ($r_3 = 1$), the fuel associated with the frequency $\omega_{0,0}$ oscillates symmetrically about the planes $x = \pm y$ so that no net force or moment can be produced. As far as the fuel associated with the natural frequency $\omega_{m,n}$ is concerned, the planes $x = \pm(2q+1)\, a/2(2n+1)$, and $y = \pm(2q+1)\, b/2(2m+1)$ where q is an integer could be taken as tank boundaries. Thus the tank could be divided into (2m+1) times (2n+1) smaller sub-tanks. If $r_3 = (2n+1)/(2m+1)$, the smaller sub-tanks are square so that no net forces or moments are produced and $I_{m,n} = 0$.

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It is probably sufficiently accurate to consider only the moment produced by the moving masses with moments of inertia $I_{0,0}$, $I_{1,0}$, $I_{0,1}$, $I_{2,0}$, and $I_{0,2}$. The ratios of these moment of inertia to I_{S_z} have been computed from Eq. (91) as a function of r_3 for the shallow tank case, $r_1 = r_2 = 0$. They are shown in Figs. 9 and 10. To obtain the value of $I_{m,n}/I_{S_z}$ for $r_1 \neq 0$, one simply multiplies $I_{m,n}/I_{S_z}$ for $r_1 = 0$ by

$$\tanh\left[\pi r_1\sqrt{r_3^2(2m+1)^2+(2n+1)^2}\right]\Big/\left[\pi r_1\sqrt{r_3^2(2m+1)^2+(2n+1)^2}\right].$$

To obtain values for $r_3 > 1$, replace r_3 by $\frac{1}{r_3}$ and change r_1 to r_2. The ratio I_{F_z}/I_{S_z} was calculated from Eq. (83).

Mechanical Systems for Combinations of Tank Motions

A single mechanical system, that will respond the same as the fluid to all tank motions, cannot in general be constructed from the simple mechanical systems discussed without introducing negative moments of inertia. However, it is always possible to construct a single mechanical system that will represent the fluid response to both yawing motion and motion of type (A) or yawing motion and motion of type (B). For, consider the mechanical system that represents motions of type (A). The fixed mass can be distributed in the yz-plane so that the center of gravity remains fixed and its moment of inertia is I_z about the z-axis and I_y about an axis passing through $z = Z$ and parallel to the y-axis. The moment of inertia I_{F_z} can be separated into the components I_z, $I_{0,0}$, $I_{0,1}$ etc. of the mechanical system representing the response of the fluid to yawing motion.

DISCUSSION

Some problems arise concerning the proper use of the preceding material, and a few of these are discussed below. Also included are some topics which contribute only to the general description of fluid oscillations.

Mechanical Systems

For aircraft having a vertical plane of symmetry through the longitudinal axis, the longitudinal stability and the lateral-directional stability can be treated independently.

In order to handle the longitudinal stability it is necessary to know the forces and moments produced by translation of the tank and by pitching motions. A single mechanical system has been set up to represent the tank for both of these motions.

The lateral-directional case is more complicated since in general the effects of rolling motion, translation and yawing must be considered. A single mechanical system has been devised to represent the fuel tank for all of these motions, provided the translation is confined to the plane of the rolling motion. The more general case involving also longitudinal translation, (which would appear with wing tip tanks), cannot in general be represented by a single system of the present type, unless imaginary dimensions are introduced. In such a case different systems could be used to obtain forces and moments in different planes.

It is also possible to use the Laplace transforms of the forces and moments directly, without regard to mechanical systems, if the transform method is being used for the complete problem.

In some cases, particularly where tanks are located far from the C.G. of the airplane, the effect of angular motions of the tank may be negligible by comparison with the effect of translation of the tank. In such cases some simplifications may be possible.

Approximate Mechanical Systems

The forces and moments produced by the higher harmonics of the fluid motion will be negligible in many cases. This means that the equivalent mechanical system can frequently be assumed to consist of one fixed mass and one moving mass.

If the airplane has a natural frequency quite close to a higher harmonic of the fluid motion, it might be assumed that it would be necessary to retain the moving mass corresponding to this harmonic. However, consideration of the practical limitations on wave heights may alter the picture. These limitations may arise from the effects of viscosity, cavitation, or through the presence of non-linearities in free surface boundary conditions. For example, in Ref. 3c a limitation on the ratio of wave height to wave length is suggested. Such a "limit" might mark the introduction of splashing with consequent energy dissipation.

Tanks of Different Shape

Tanks of other than rectangular shape could be studied. Stewart and Lorell[4] have studied the forced oscillations in a cylindrical tank of circular cross-section with free surface normal to the axis. Some information on the free oscillations in a few other tanks of simple shape is given in the appendix to Ref. 6. Very few cases involving variable tank depth have been solved exactly. Although there are mathematical difficulties in getting exact solutions to many cases which appear physically simple, it is still possible to get approximate theoretical solutions in such cases. Also, it should be comparatively easy to determine experimentally the fundamental frequency for a tank of arbitrary shape.

Baffle Locations

For the fundamental mode in the rectangular tank the highest vertical velocities of the fluid occur at the ends of the tank in the free surface. The highest horizontal velocities occur in the free surface at the center of the tank. These results are easily verified from the expression for velocity potential.

It seems probable that the above locations are best for baffles since high local velocities are required for a high rate of energy dissipation. The practical problem is not as simple as this would suggest however, since it may be necessary for the baffles to be effective over some range of fuel heights in the tank.

Effective Value of g

If the fuel tank experiences a large constant acceleration normal to the mean free surface, then the value of g is effectively changed. If the effective value of g is quadrupled, then the natural frequencies of fluid oscillation are doubled. If the effective value of g approaches zero, as in a free fall, then the natural frequencies approach zero. Such effects would probably be of interest in connection with missile dynamics, and might be of importance in aircraft maneuvers.

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To get the correct equivalent mechanical systems and force and moment expressions it is only necessary to replace g by the effective value of g in the formulas. (This results in the introduction of an effective fuel weight also.)

Variable Normal Accelerations

It has been mentioned that a constant acceleration normal to the mean free surface changes the effective value of g and alters the natural frequencies of the fluid in the tank. It can also be shown that variable normal accelerations may alter the amplitude of the wave motion. In particular a periodic normal acceleration of the correct frequency may feed energy into an existing wave motion. This occurs through a non-linear coupling, and the equivalent mechanical system does not correctly represent the fluid to this order. Such terms are not ordinarily considered in airplane dynamics.

Non-Linearities

When the amplitude of the fuel motion is large the effect of non-linearity may become important. The non-linearity is associated with boundary conditions at the free surface, not with the Laplace equation, which is linear. The mathematical difficulties introduced by the non-linear effects are too great to permit investigation here. The region of applicability of the linearized equations can best be determined by experiment.

Viscosity

The effects of viscosity are small for unbaffled tanks of the size considered in aircraft. If baffles were present this would certainly not be the case, since the actual flow pattern can be radically different from the potential flow pattern in such cases, and heavy damping may be introduced. For this reason it is of little interest to investigate the potential flow pattern where baffles are involved. The scope of the report is therefore limited primarily to determining whether or not baffles may be necessary. It is then possible to suggest locations where baffles might be found most effective. This is based entirely on the velocity distribution in the unbaffled tank, it being assumed that the greatest energy dissipation would be produced by locating baffles in the regions of highest velocity.

Free Surface Shapes

It is of some interest that, in the free oscillations of fluid in a tank, an arbitrary shape of the free surface does not in general recur periodically. For example, in a stationary rectangular tank if the free surface is initially an inclined plane, it will never assume this shape again. This is for a non-viscous fluid, which would continue its oscillations indefinitely without loss of energy. A sinusoidal free surface whose wave length is 2/n times the tank length (where n is an integer) will repeat periodically. However, each different wave length repeats in a different period of time and these periods are not in general related to each other through rational numbers. This of course explains the non-repetition of the plane free surface since it is initially composed of these sinusoidal elements (by Fourier analysis), and after the first instant they never have simultaneously the correct relative magnitudes required to represent a plane.

Superposition Processes

Knowledge of the steady state response of the fuel to all frequencies

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of oscillation of the tank makes it possible to determine the response to an arbitrary forcing function. The complete solution of the differential equations of motion is the sum of the complementary function and the particular integral. The complementary function corresponds to the free oscillations of the fluid in a stationary tank, and the desired initial conditions can theoretically be imposed by proper choice of the infinitely many arbitrary constants associated with the fundamental and harmonics. The particular integral for an arbitrary forcing function could be obtained by superposition of solutions for different forcing frequencies through a Fourier Integral approach, for example. In practice the Laplace transform methods seem preferable.

CONCLUSIONS

In many cases it is practicable to replace the fuel by spring-mass systems for purposes of dynamic analysis. These spring-mass systems correctly represent the fuel even for angular motions of the tank. When many types of tank motion must be simultaneously considered these equivalent systems may become complicated, and possibly lose physical significance through the necessity for negative moments of inertia.

It seems probable that masses corresponding to higher harmonics of the fuel motion may often be neglected (or considered as fixed mass). This type of approximation is, of course, very important in simplifying calculations, and deserves further study.

When equivalent mechanical systems are introduced, any desired method may be used for solving the complete dynamics problem. Laplace transform methods may be used in conjunction with the mechanical systems or without reference to them. In either case terms corresponding to higher harmonics can probably be omitted.

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   c) Article 250

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FIGURE 1

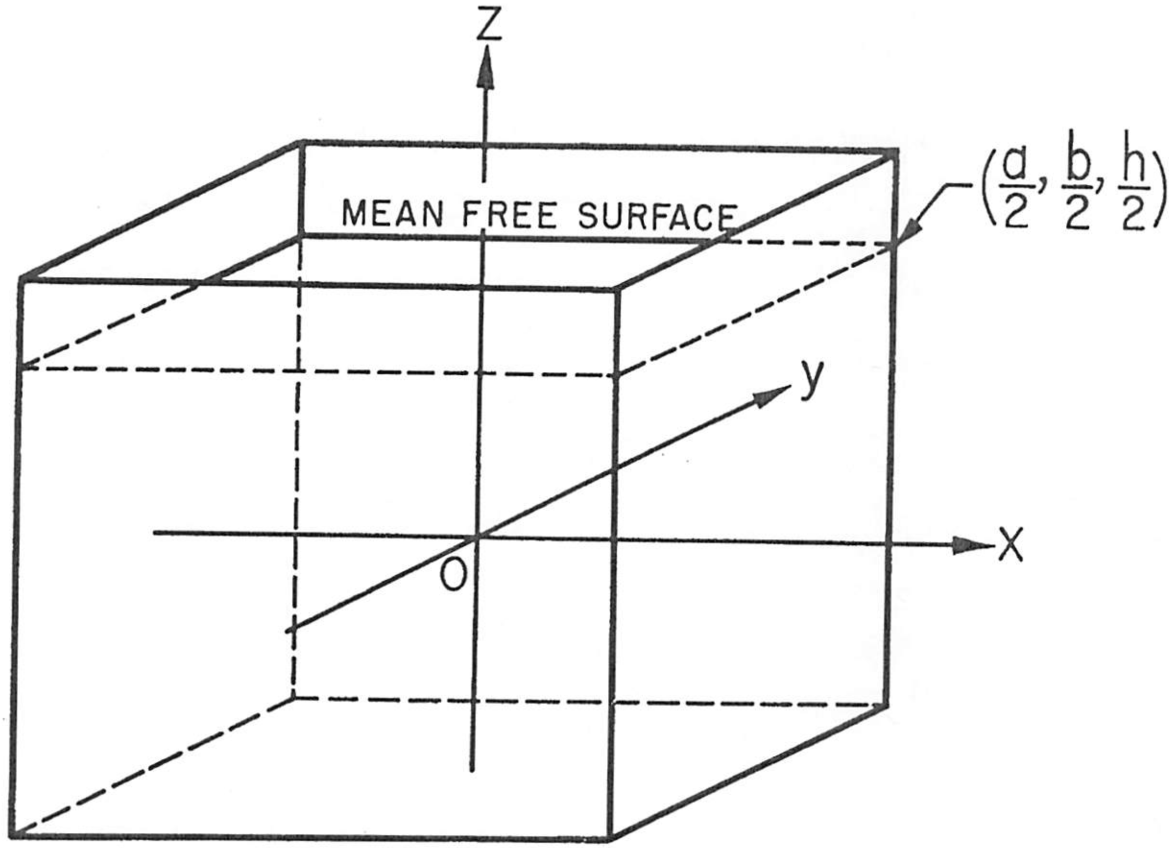


DIAGRAM OF PARTIALLY FILLED TANK
SHOWING FIXED COORDINATE SYSTEM

FIGURE 2

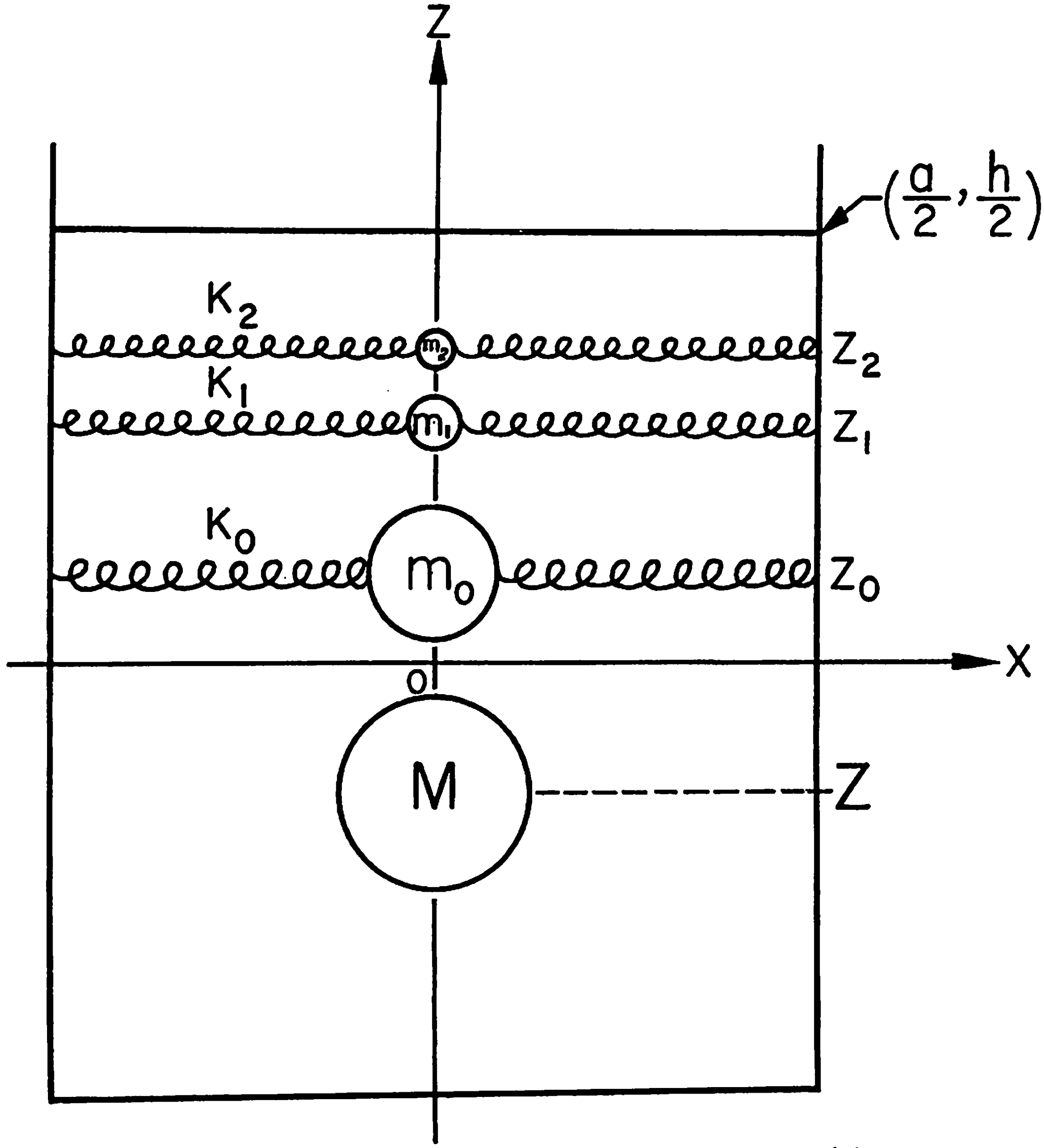
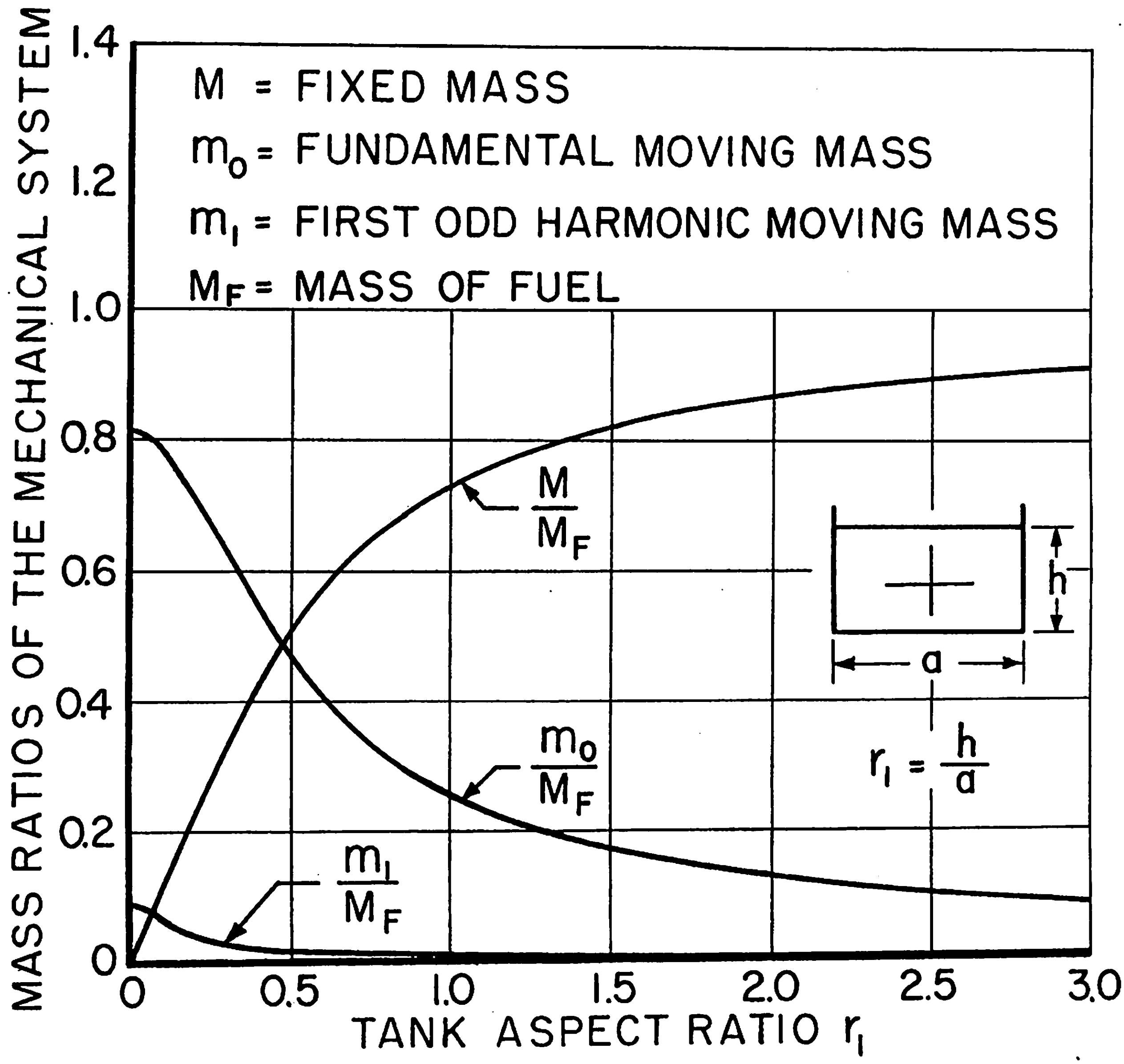


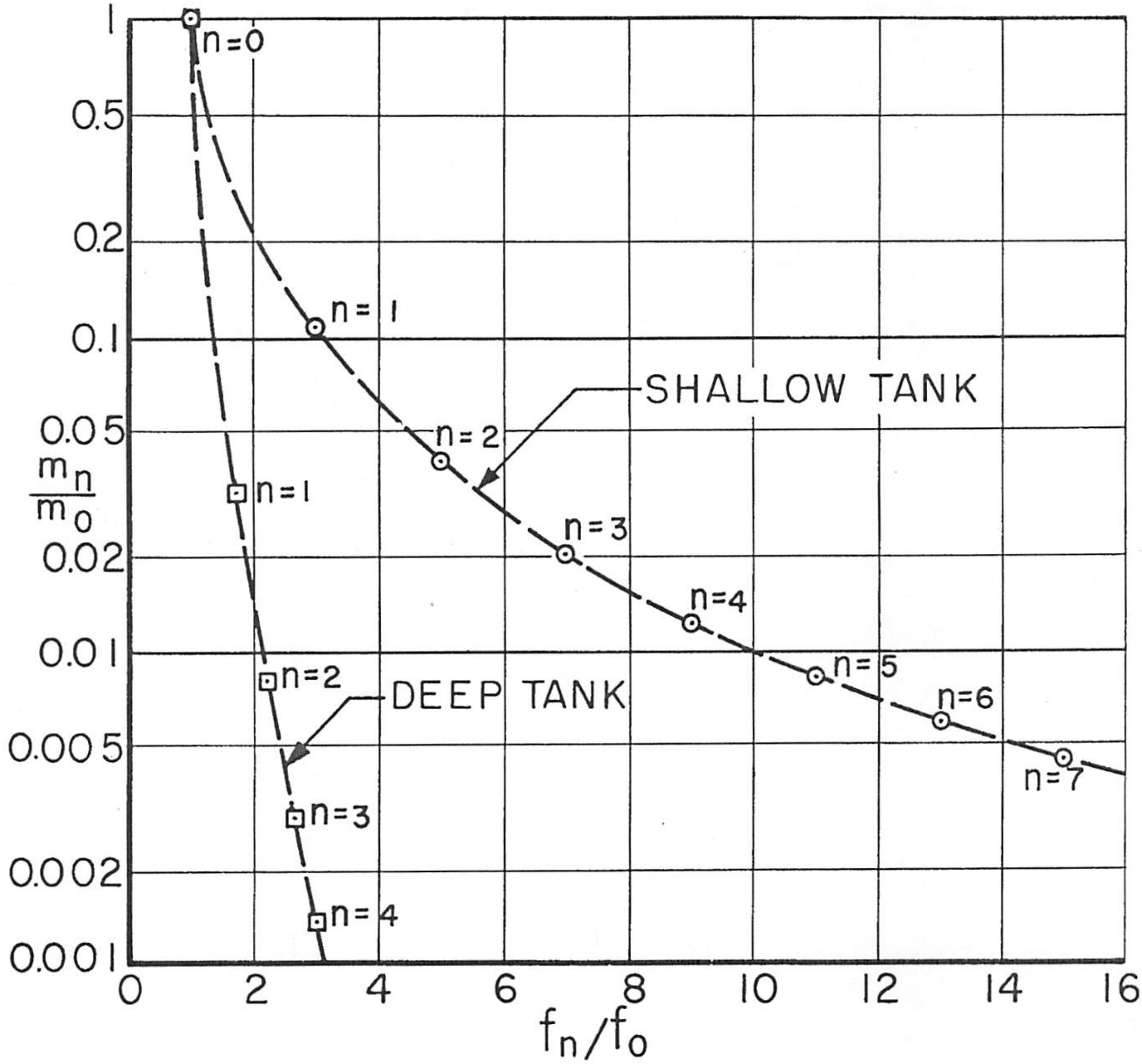
DIAGRAM OF MECHANICAL SYSTEM REPRESENTING RESPONSE OF FUEL TO HORIZONTAL AND PITCHING MOTIONS OF THE TANK

FIGURE 3



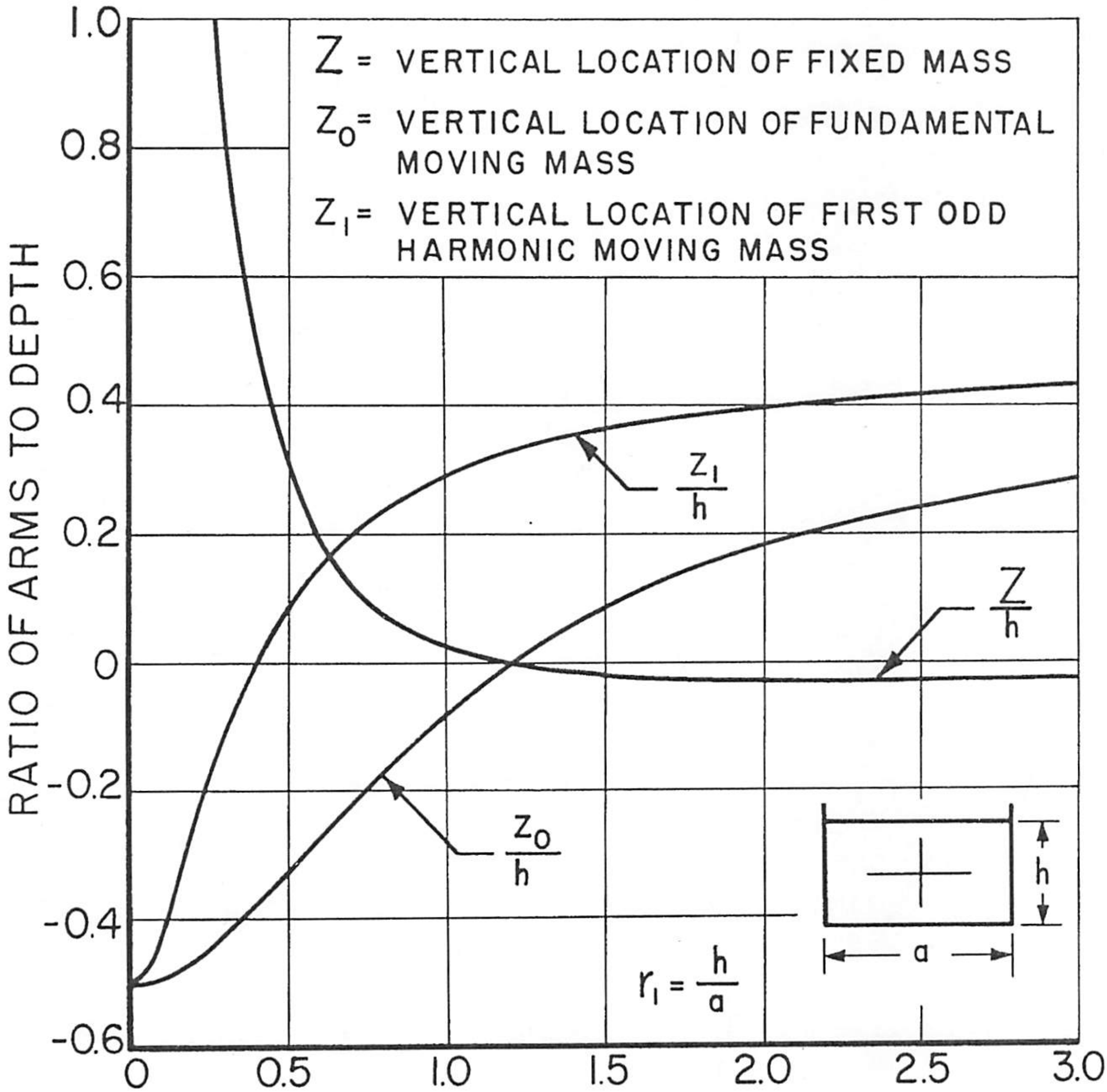
RATIO OF FIXED MASS AND FIRST TWO MOVING MASSES TO FUEL MASS vs TANK ASPECT RATIO

FIGURE 4



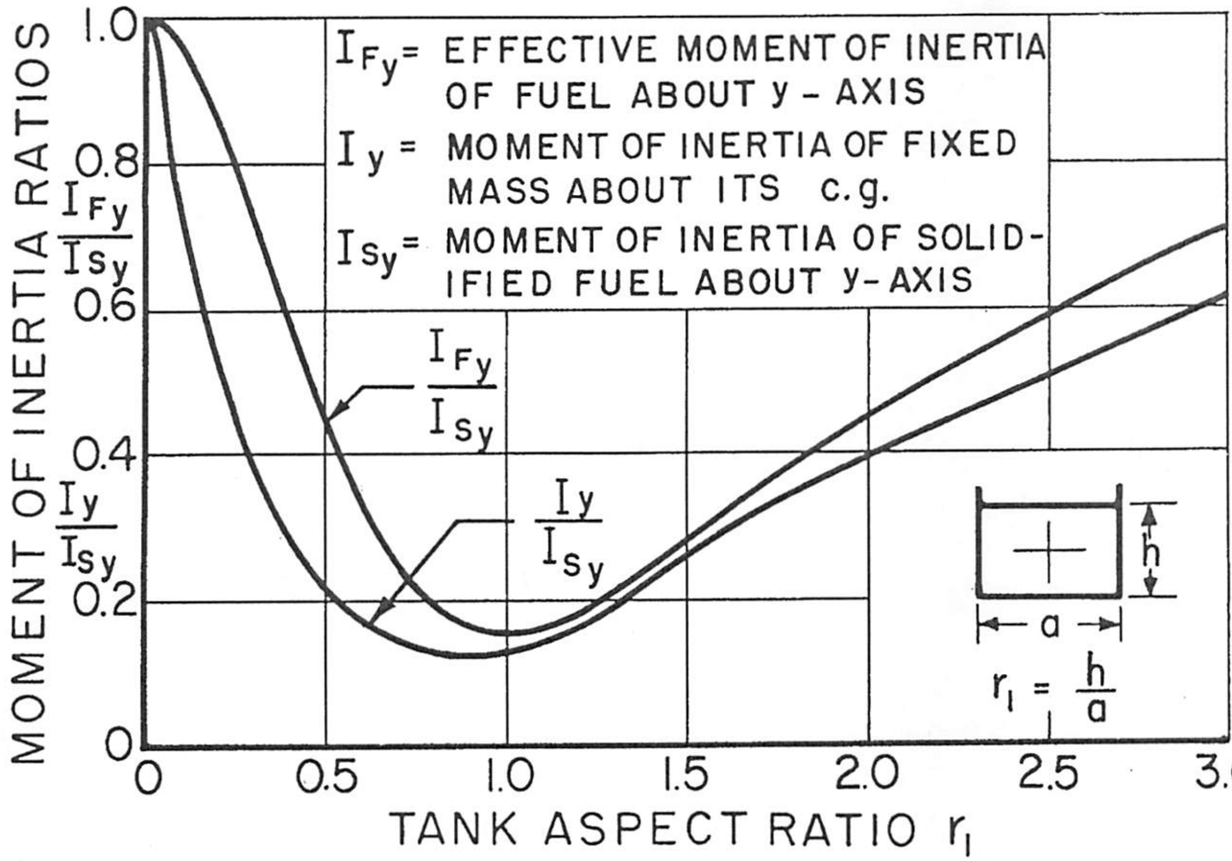
RATIOS OF MOVING MASSES
TO FUNDAMENTAL MOVING MASS
vs RATIO OF THEIR RESPECTIVE NATURAL
FREQUENCIES TO FUNDAMENTAL FREQUENCY

FIGURE 5



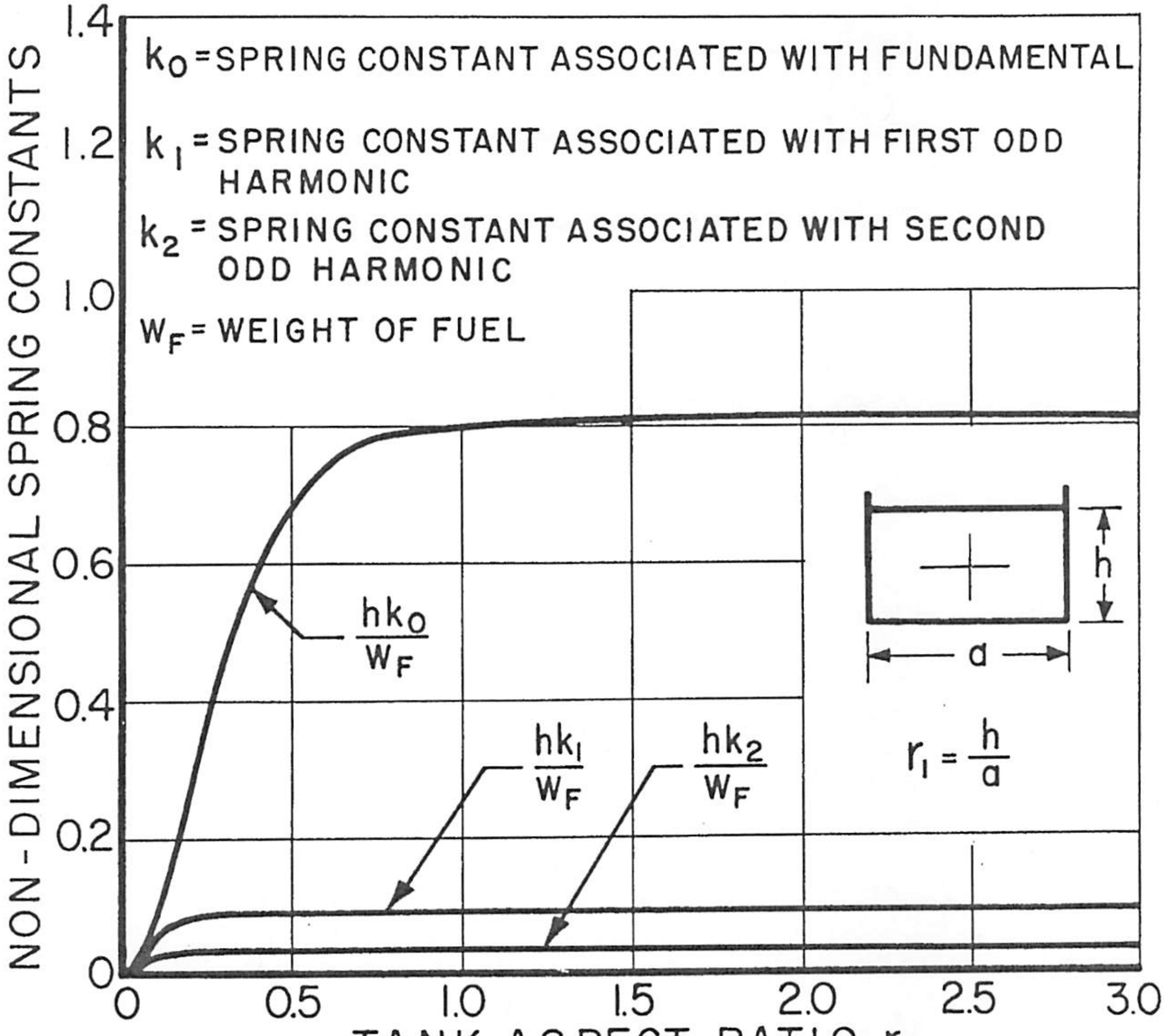
RATIOS OF THE VERTICAL LOCATIONS OR ARMS OF THE FIXED MASS AND FIRST TWO MOVING MASSES TO DEPTH vs TANK ASPECT RATIO

FIGURE 6



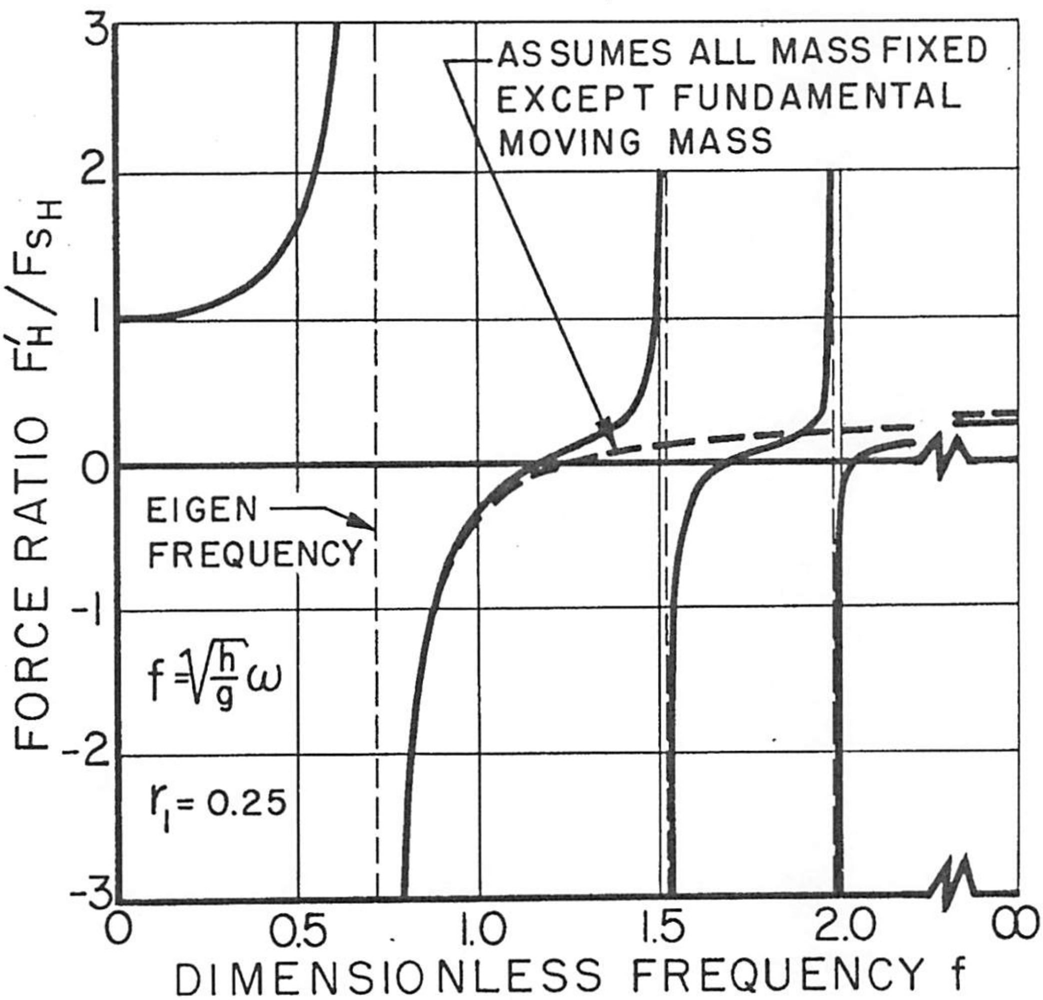
RATIOS OF MOMENTS OF INERTIA OF FUEL AND FIXED MASS TO THE MOMENT OF INERTIA OF THE SOLIDIFIED FUEL vs TANK ASPECT RATIO

FIGURE 7



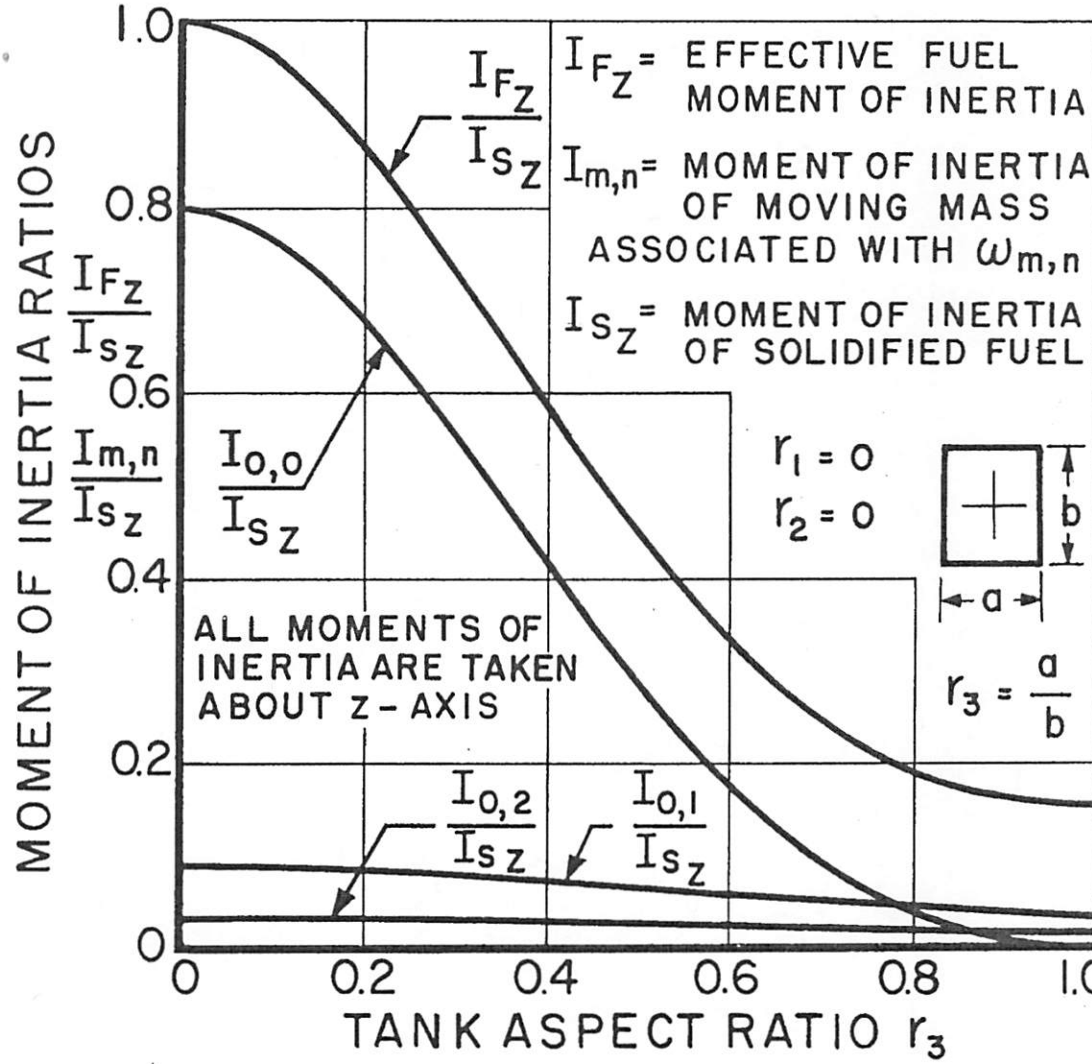
NON-DIMENSIONAL SPRING CONSTANTS OF MECHANICAL SYSTEM vs TANK ASPECT RATIO

FIGURE 8



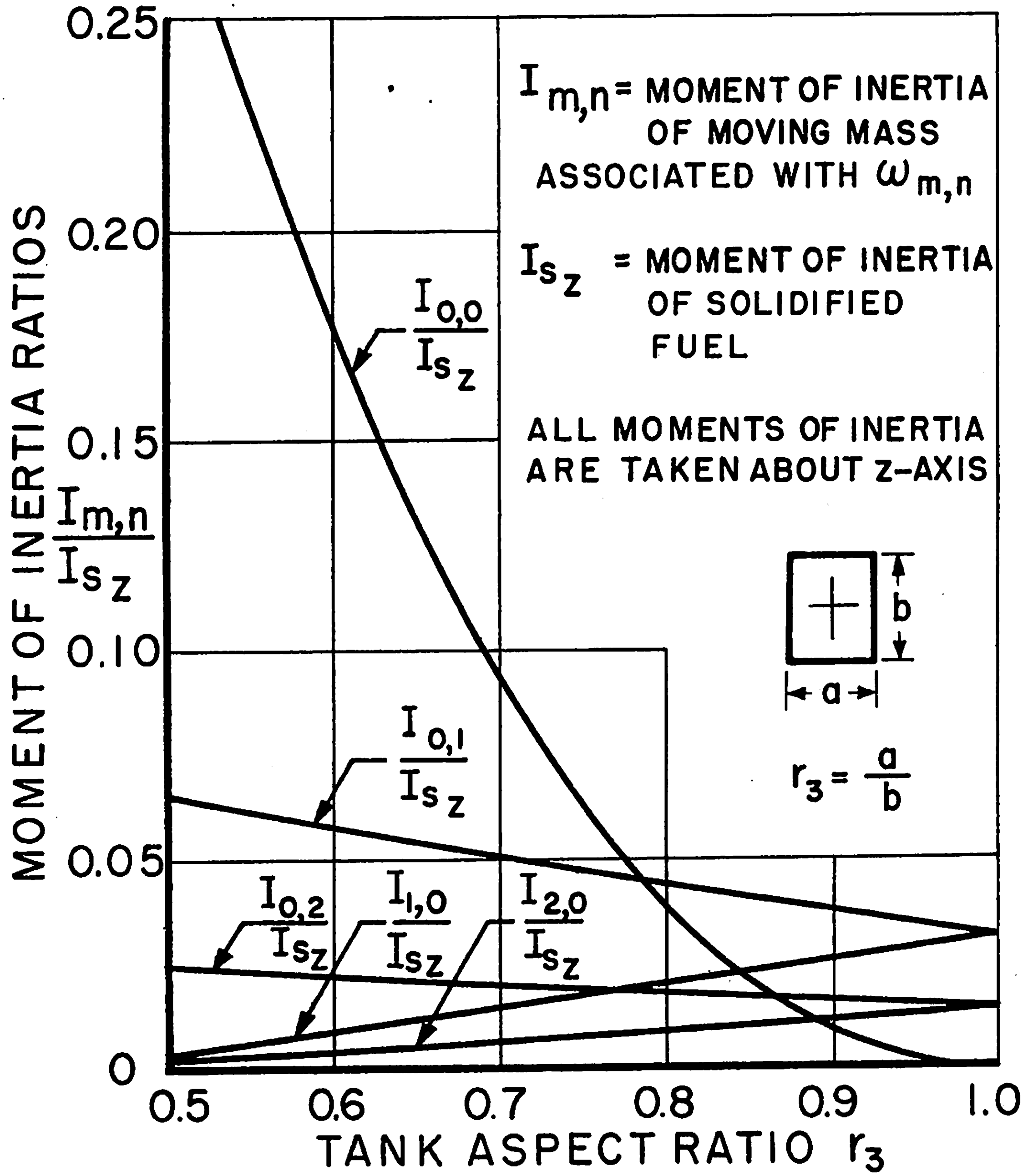
RATIO OF FORCE EXERTED BY MECHANICAL SYSTEM TO FORCE EXERTED BY SOLIDIFIED FUEL vs DIMENSIONLESS FREQUENCY FOR HORIZONTAL MOTION

FIGURE 9



YAWING MOMENT OF INERTIA RATIOS
FOR SHALLOW TANK

FIGURE 10



MOMENT OF INERTIA RATIOS $I_{m,n}/I_{s_z}$
FOR THE SHALLOW TANK
vs TANK ASPECT RATIO r_3

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