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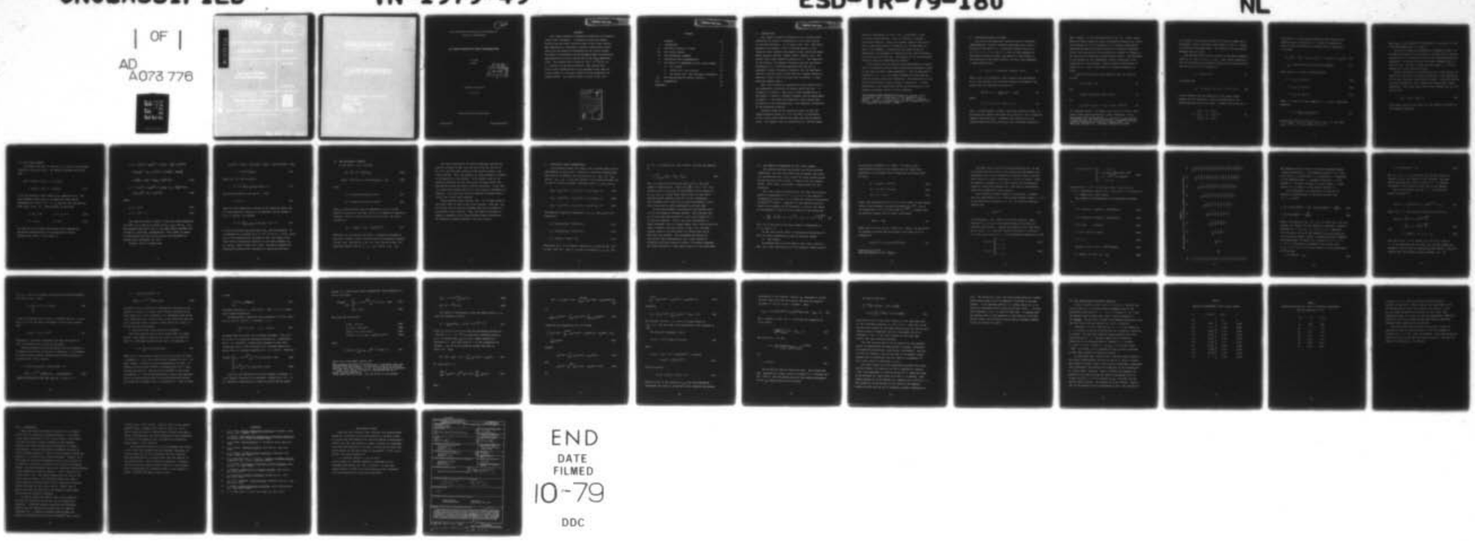

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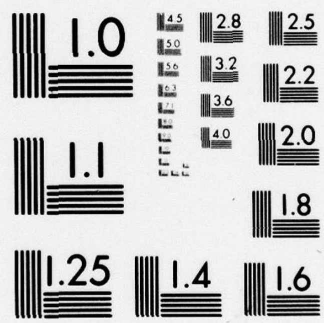


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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

# ON GAUSS'S METHOD OF ORBIT DETERMINATION

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*Group 94*



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## ABSTRACT

This report presents a statistical version of the Gaussian initial orbit technique. It neglects neither the angular velocity nor the radial velocity terms in the f and g series. More importantly it provides a rigorous, analytically simple result for the radius of convergence of the f and g series. The radius of convergence can be extremely small at periastron, approaching zero as the eccentricity of the orbit approaches unity. The leading term is given by $\sqrt{2}(1 - e)^{3/2}P/(3\pi)$ as $e \to 1$ where P is the orbital period. This implies that initial orbit determination by any procedure which uses the f and g series is a process fraught with the possibility of unknown errors. The central result is given in Eq. (29).

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# I. INTRODUCTION

This report's primary contribution is a rigorous result concerning the radius of convergence of the f and g series of celestial mechanics. It is given in Eq. (29). The result appears to be unknown in that a thorough perusal of the classical texts (Brouwer and Clemence, Brown, Brown and Shook, Danby, Escobal, Moulton, Plummer, Smart, Wintner, etc.) and several other texts reveals no mention of it. The simplicity of the formula, its apparent connection with the well-known divergence at e = 0.6627434 associated with the use of Lagrange's formula on Kepler's equation, and its importance lead me to believe that 1) there must be a simpler method of deriving it and 2) while the work here is original, it must duplicate someone else's.

The f and g series are important in orbit determination, the differential correction of orbits, and in the numerical integration of orbits. Their use in orbit determination is principally to obtain a distance estimate from the measurement of angles. In the other two cases the f and g series form the basis for the algorithms used in the numerical integration of the equations of motion.

The main thrust of the analytical result is that for highly eccentric orbits ($e \gtrsim 0.5$) the radii of convergence of the f and g series become very small near the periastron point. For example, with an eccentricity of $1/\sqrt{2}$ the common radius of convergence is $[\ell n(1 + \sqrt{2}) - 1/\sqrt{2}]P/(2\pi) \sim P/36$ (P is the orbital period). This is $\sim 20^m$ for a 2 rev/day artificial satellite. Hence, the usuable range of the series, that is when they provide 1% accuracy, is $<5^m$ (see Table 3). Thus, if we remember to include the inevitable errors of observation, the procedure of Gaussian initial orbit determination will be a difficult one. The neglect of the non-negligible velocity terms only compounds the problem*.

As we rarely ever get exactly three measurements of angles only, a statistical initial orbit determination procedure, based on the logic of Gauss, seems appropriate. This is developed in Sections II - V. It is no longer necessary to throw away any velocity effects using this formulation. The motivation for and proof of the central result is given in Section VI. A brief discussion of its implications (§VII) and alternatives to the Gaussian technique (§VIII) are also presented.

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*For a Molniya type satellite (argument of perigee = 270°, inclination = 60°, eccentricity = $1/\sqrt{2}$) observed at the equator, $\dot{\alpha} = n\sqrt{2}$, $\dot{\delta} = n\sqrt{6}$, and $\dot{r}/r = 2n$ where n is the mean motion. Clearly the angular speed and the foreshortening term are comparable and neither is small.

## II. THE MOTION OCCURS IN A PLANE

Let $\underline{r}(t)$ be the geocentric location vector in the usual (approximately) inertial reference frame where the z axis is the axis of the earth (e.g., the North Celestial Pole is at $x = 0$, $y = 0$, $z = +\infty$), the $z = 0$ plane is the extension of the earth's equatorial plane, the positive x axis points in the direction of the Vernal Equinox, and the y axis completes a right handed triple;

$$\underline{r} = (x,y,z) = r(\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta), \tag{1}$$

where $\alpha$ and $\delta$ are geocentric right ascension and declination. If G is the constant of gravitation and $M_\oplus$ is the mass of the earth, then the equations of motion are

$$d^2\underline{r}/dt^2 = \ddot{\underline{r}} = -GM_\oplus\underline{r}/|\underline{r}|^3 \equiv -\mu\underline{r}/r^3 \tag{2}$$

where

$$r^2 = x^2 + y^2 + z^2 = |\underline{r}|^2,\ r \geq 0. \tag{3}$$

For $r > 0$ there exists a unique, continuous solution of Eqs. (2) for which $\underline{r}(t)$ takes on the value $\underline{r}(t_o)$ and $\underline{v}(t) = \dot{\underline{r}}(t) = d\underline{r}(t)/dt$ takes on the value $\underline{v}(t_o)$. Moreover, the solution is a continuous function of $\underline{r}(t_o)$ and $\underline{v}(t_o)$ and a continous function of small changes in the right-hand side of Eq. (2). These statements are easily proved by using the standard existence and uniqueness theorems (§3.1ff of reference 1) for ordinary differential equations obtained from Picard's method of successive approximations. The continuity of the solution with respect to changes in the force is the foundation of perturbation theory. The dependence of the solution on six independent, initial conditions is the foundation for the statement, "Three different observations of angles (e.g., α and δ) only suffice to uniquely determine an orbit".

If we take the vector cross product of Eq. (2) with $\underline{r}(t)$ we find

$$\underline{r}(t) \times \ddot{\underline{r}}(t) = \underline{0}, \tag{4}$$

but,

$$d[\underline{r}(t) \times \underline{v}(t)]/dt = \underline{r}(t) \times \ddot{\underline{r}}(t), \tag{5}$$

so

$$\underline{L} = \underline{r}(t) \times \underline{v}(t),\ L = |\underline{L}| = [\mu a(1 - e^2)]^{1/2}, \tag{6}$$

is a constant vector. If neither $\underline{r}(t)$ nor $\underline{v}(t)$ is null*, their vector cross product determines a plane (reference 3, §31).

*The vector $\underline{r}(t)$ can vanish only if the motion is rectilinear. In that case $\underline{r}(t)$ and $\underline{v}(t)$ are collinear, $\underline{L}$ vanishes, and it's obviously impossible to determine a unique plane. If $r > 0$, then $\underline{v}(t)$ is never null. See also reference 2, §241ff.

$\underline{L}$ is normal to this plane and both $\underline{r}(t)$ and $\underline{v}(t)$ always lie in this plane. Hence, the motion takes place in this plane which is called the invariable plane. We recognize $\underline{L}$ as the angular momentum (per unit mass).

Now consider three location vectors at the three different times $t_i$, $t_j$, and $t_k$. Let $\underline{r}_n = \underline{r}(t_n)$. Then, since (reference 3, §36) the necessary and sufficient condition that three non-null vectors, $\underline{a}$, $\underline{b}$, and $\underline{c}$, be coplanar is

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{0}, \tag{7}$$

it follows that

$$\underline{r}_i \cdot (\underline{r}_j \times \underline{r}_k) = \underline{0}, \; i \neq j, \; i \neq k, \; j \neq k. \tag{8a}$$

It also follows from the properties of the triple scalar product (§37 of reference 3) that any permutation of the indices does not affect the result. Another form of Eq.(8) is

$$D = \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix} = 0. \tag{9}$$

Since this is a third order determinant there exists 6(=3!) different ways of expanding it. In order to demonstrate three of these mathematically redundant ways of expressing (9), define

$$P_{ij}(a,b) = a_i b_j - a_j b_i, \quad P_{ij}(b,a) = -P_{ij}(a,b) = P_{ji}(a,b), \qquad (10)$$

$$\varepsilon_{ijk} = \text{three dimensional Levi-Civita symbol*}, \qquad (11)$$

then, since D = 0, three of these ways are

$$D = \sum \varepsilon_{ijk} x_i P_{jk}(y,z), \qquad (12a)$$

$$= \sum \varepsilon_{ijk} y_i P_{jk}(z,x), \qquad (12b)$$

$$= \sum \varepsilon_{ijk} z_i P_{jk}(x,y), \qquad (12c)$$

where i, j, and k are each summed over 1, 2, and 3. Even more compactly,

$$D = \sum \sum E_{abc} \varepsilon_{ijk} a_i P_{jk}(b,c), \qquad (13)$$

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*$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$, all other $\varepsilon_{ijk} = 0$ for i, j, and k one of 1, 2, or 3.

where $E_{abc}$ is the alphabetical counterpart to $\varepsilon_{ijk}$ and a, b, and c are each summed over x, y, and z.

In order to remove the mathematical redundancy of Eqs. (12) we employ the physics of the problem to evaluate the P's and the observations of the problem to evaluate the remaining geocentric coordinate. To do this we need to be able to express $\underline{r}(t_n)$ in terms of $\underline{r}(t_o)$ and $\underline{v}(t_o)$ at some arbitrary epoch $t_o$. For this purpose the f and g series are used.

Before proceeding to a definition of the f and g series, we should note that there is another formulation of the coplanarity of the three location vectors which is useful. Since the vectors are three dimensional, their coplanarity implies their linear dependence. Thus, there must exist scalar constants $d_i$, $d_j$, and $d_k$ such that

$$d_i\underline{r}_i + d_j\underline{r}_j + d_k\underline{r}_k = \underline{0}. \qquad (8b)$$

This formula serves as the basis for the Gibbsian variation of the Gaussian technique.

## III. THE f AND g SERIES

We already know that, as long as $r > 0$, $\underline{r}(t)$ is a continuous function of $\underline{r}(t_o)$ and $\underline{v}(t_o)$. We choose to express this in the form

$$\underline{r}(t) = f[\underline{r}(t_o), \underline{v}(t_o), t - t_o]\, \underline{r}(t_o)$$
$$+ g[\underline{r}(t_o), \underline{v}(t_o), t - t_o]\underline{v}(t_o). \quad (14)$$

If we can determine f and g then we can compute the P's. The usual method of doing this is to substitute power series expansions for f and g in $T \equiv t - t_o$ into Eq. (14) and then use that result in Eq. (2). Alternatively, we note that

$$a = fa_o + g\dot{a}_o \qquad a = x, y, \text{ or } z \quad (15a)$$

$$\ddot{a} = - h(r)a \qquad h \equiv \mu/r^3 \quad (15b)$$

and that all of the higher derivatives can be computed by successive differentiation of the equations of motion. Through sixth order in T the result is

$$f = 1 - h_o T^2/2 + h_o p_o T^3/3 + h_o(3q_o - 15p_o^2 + h_o)T^4/24$$

$$+ h_o p_o(7p_o^2 - 3q_o - h_o)T^5/8 - h_o(945p_o^4 - 210h_o p_o^2$$

$$- 630p_o^2 q_o + 45q_o^2 + 24h_o q_o + h_o^2)T^6/720, \quad (16a)$$

$$g = T - h_o T^3/6 + h_o p_o T^4/4 + h_o(9q_o + h_o - 45p_o^2)T^5/120$$

$$+ h_o p_o(14p_o^2 - 6q_o - h_o)T^6/24, \quad (16b)$$

where,

$$p = \underline{r} \cdot \underline{v}/r^2, \quad (17a)$$

$$q = \underline{v} \cdot \underline{v}/r^2 - h. \quad (17b)$$

The important thing to notice is that the actual dependence of f and g on $\underline{r}_o$ and $\underline{v}_o$ is really on $|\underline{r}_o|$, $|\underline{v}_o|$, and $|\underline{r}_o \cdot \underline{v}_o|$. The important question to ask is "Do these series converge, and if they do, under what circumstances?" We'll defer an answer to this question so that we can continue the discussion of initial orbit estimation (cf. §VI).

From Eq. (15a) it follows that

$$P_{ij}(a,b) = (f_i g_j - g_i f_j)(a_o \dot{b}_o - \dot{a}_o b_o) = P_{ij}(f,g)(a_o \dot{b}_o - \dot{a}_o b_o)$$

$$= P_{ij}(f,g) E_{abc} L_c. \qquad (18)$$

Hence, Eq. (13) can be written

$$D = \sum \sum E^2_{abc} \varepsilon_{ijk} a_i P_{jk}(f,g) L_a = 0. \qquad (19)$$

This puts the physics into the P's. Since

$$\underline{r} = \underline{R} + \underline{\mathcal{R}}, \qquad (20)$$

where $\underline{R}$ is the topocentric location of the celestial object and $\underline{\mathcal{R}}$ is the geocentric location of the observer, we can replace a by $A + \mathcal{A}$ and Eq. (19) becomes

$$D = \sum \sum E^2_{abc} \varepsilon_{ijk} P_{jk}(f,g) L_a (A_i + \mathcal{A}_i) = 0. \qquad (21)$$

Of the three quantities which make up $\underline{R}_i$, two are measured. If no component of $\underline{L}$ vanishes ($i \neq 0, \pi/2$; $\Omega \neq 0, \pi/2, \pi, 3\pi/2$), since the P's are determined by the physics, Eqs. (21) appear to be three linear, homogeneous equations in the three unknowns, $R_i$, $R_j$, and $R_k$. This, though, isn't true. We need to examine the topocentric geometry more thoroughly to explicitly see this.

## IV. THE TOPOCENTRIC GEOMETRY

At any time $t = t_k$ we can write

$$r_k^2 = R_k^2 + R^2 + 2R_k R\cos Z_k, \qquad (22a)$$

$$\cos Z_k = \sin\phi'\sin\Delta_k + \cos\phi'\cos\Delta_k\cos(\tau_k - A_k), \qquad (22b)$$

where

$$\underline{R} = R(\cos\Delta\cos A, \cos\Delta\sin A, \sin\Delta), \qquad (23)$$

$$\underline{R} = R(\cos\phi'\cos\tau, \cos\phi'\sin\tau, \sin\phi'). \qquad (24)$$

Here $\tau_k$ is the sidereal time corresponding to $t_k$, $\phi'$ is the observer's geocentric latitude, and $R$ is the observer's geocentric distance. There is only one positive root of Eqs. (22) and it is given by

$$R_k = - R\cos Z_k + [r_k^2 - R^2\sin^2 Z_k]^{1/2}. \qquad (25)$$

Therefore, we can replace the object's topocentric distance at any time in terms of known quantities and its geocentric distance at that time. We replace $r_k$ by its f and g series so Eqs. (21) explicitly depend only on $r_o$, $v_o = |\underline{v}_o|$, and $V_o = |\underline{r}_o \cdot \underline{v}_o|$.

The actual application of Gauss's technique involves the explicit solution of Eqs. (21) for one of the $R_k$, the use of Eq. (25), and the total neglect of the velocity terms in the f and g series. Thus, one equation in three unknowns is reduced to one equation in one unknown. If we actually look at the equations involved, one can (for nearly circular orbits only) heuristically argue the radial velocity term away. In any case it's clear that throwing away all of the velocity information is improper. It's also clear that the Gaussian technique doesn't "solve" the initial orbit problem.

Gibbs's method starts from Eq. (8b). He includes terms of the fourth order in time in the f and g series. From Eqs. (16) we see that this is also the minimum order which includes all the effects of the velocity. Thus, the Gibbs's technique is midway in complexity and accuracy between Gauss's method and the statistical method proposed in the next section.

## V. STATISTICAL ORBIT DETERMINATION

As a practical matter one almost never acquires exactly three measurements of angles only. Therefore, instead of throwing away the velocity information, we can solve for $r_o = |\underline{r}_o|$, $v_o = |\underline{v}_o|$, and $V_o = |\underline{v}_o \cdot \underline{r}_o|$. To be more explicit let us write out Eqs. (21) in full for three, different, arbitrary times $t = t_i, t_j$, and $t_k$;

$$X_{ijk} \equiv P_{jk}(f,g)x_i + P_{ki}(f,g)x_j + P_{ij}(f,g)x_k = 0, \qquad (26a)$$

$$Y_{ijk} \equiv P_{jk}(f,g)y_i + P_{ki}(f,g)y_j + P_{ij}(f,g)y_k = 0, \qquad (26b)$$

$$Z_{ijk} \equiv P_{jk}(f,g)z_i + P_{ki}(f,g)z_j + P_{ij}(f,g)z_k = 0. \qquad (26c)$$

The explicitly appearing components of $\underline{r}_i$, $\underline{r}_j$, and $\underline{r}_k$ are to be replaced by

$$x_i = R_i \cos\Delta_i \cos A_i + R\cos\phi' \cos\tau_i, \qquad (27a)$$

$$y_i = R_i \cos\Delta_i \sin A_i + R\cos\phi' \sin\tau_i, \qquad (27b)$$

$$z_i = R_i \sin\Delta_i + R\sin\phi', \text{ etc.} \qquad (27c)$$

Each value of $R_i$ is an implicit function of $r_i$ given by Eqs. (22) or Eqs. (22b, 25). Each $r_i$ is given in terms of $r_o$, $v_o$, $V_o$, and

$T_i = t_i - t_o$ through the f and g series. We form the quantity S,

$$S = \sum_{i,j,k=1}^{N} (X_{ijk}^2 + Y_{ijk}^2 + Z_{ijk}^2). \qquad (28)$$

The i,j,k sums are unrestricted from 1 to N (= the total number of observations). There are $3\binom{N}{3} = N(N-1)(N-2)/2$ non-zero terms in the sum. We now minimize S with respect to $r_o$, $v_o$, and $V_o$. Having found these values, we then solve for the orbital elements using all of this information. This procedure still rests on the rapid convergence of the f and g series but no other approximation has been made.

An efficient method of searching for the minimum value of S is the method of steepest descent. This requires that all of the second derivatives of f and g with respect to h, p, and q be computed. Since f and g are polynomials in these variables, this is a simple matter once f and g series are known. However, from the pattern in Eqs. (16), one must include terms of the tenth order in T before this can be done with any accuracy. The programming and testing of statistical orbit determination, for a wide variety of different artificial satellite orbits, is currently underway. All terms inclusive of those twelth order in T are being used.

## VI. THE RADIUS OF CONVERGENCE OF THE f AND g SERIES

This section has been partitioned into three different subsections. We first show (§VIA) why we suspect that there is a problem with the f and g series for large eccentricities. We then sum the f and g series in terms of the true anomaly (§VIB). Next (§VIC), we provide a rigorous proof for the following:

The f and g series defined by Eq. (14) have the following radius of convergence in $T = t - t_o$; if $e = 0$ the radius of convergence is infinite, if e is unity the radius of convergence is $(8Q^3/9\mu)^{1/2}$ where $\mu = GM_\oplus$ and Q is the distance from the focus of the parabola to its directrix, if $e \varepsilon (0,1)$ the radius of convergence is given by $P\lambda/(2\pi)$ where P is the period and

$$\lambda = \left[M_o^2 + \left\{\ell n\left[1 + (1 - e^2)^{1/2}\right] - \ell n e - (1 - e^2)^{1/2}\right\}^2\right]^{1/2} . \quad (29)$$

Here $M_o$ is the value of the mean anomaly corresponding to $t = t_o$, $M_o \varepsilon [-\pi,\pi]$.

In the next section (§VII) we discuss the implications of this for orbit determination using the Gaussian method.

### A. The Problem

We obtained the first few terms of the f and g series in Eqs. (16). The coefficients of T are algebraic combinations of the auxiliary variables h, p, and q. In order to get a quantitative feeling for the relative size of these three quantities let us compute their average over an entire orbit*. We find,

$$\langle h\rangle = \mu/[a^3(1 - e^2)^{3/2}], \tag{30a}$$

$$\langle p\rangle = 0, \ \langle p^2\rangle = e^2\langle h\rangle/2, \tag{30b}$$

$$\langle q\rangle = (1 - e^2)\langle h\rangle. \tag{30c}$$

Hence, the coefficient of $T^k$ in f is of the order, on the average (given a very heuristic averaging procedure) of $h^{k/2}$. For g, the coefficient of $T^k$ is of the order of $h^{\frac{k-1}{2}}$. A simple proof by induction coupled with the exact relationship

$$dh/dt = -3hp \tag{31}$$

shows that it's true for all values of k. Hence, for the series to converge, we would need (by Cauchy's root rest, §17-4 of reference 4)

$$T(\langle h\rangle)^{1/2}<1 \text{ or } T/P<(1-e^2)^{3/4}/(2\pi). \tag{32}$$

Clearly as $e\to 1$ $T/P\to 0$.

*For any quantity u, $\langle u\rangle = \int_0^P u\,dt/P$.

The above can be easily criticized especially for the use of averaging and the limit as the eccentricity approaches unity. The proper limit of the formulas of elliptical motion involves both letting $e \to 1$ and $a \to \infty$ such that $Q = a(1-e)$ is finite. Hence, an asymptotic expansion in 1-e that starts from the parabolic formulas is much more appropriate. This will be investigated after we've summed the f and g series.

More generally though, despite our averaging, it appears that the convergence of the f and g series depends (since $h = \mu/r^3$), on the satellite's distance, e.g., Eq. (32) without averaging implies that

$$r > (T^2 \mu)^{1/3} \tag{33}$$

for convergence. This leads us to ask the question, "What fraction of an orbit is the distance greater than some pre-selected lower bound?". Suppose we parametrize the lower bound by $a\rho$ with $\rho \geq 0$. Then the resulting fraction is a function of $\rho$ and e only, say $H(\rho,e)$. A straightforward computation yields

$$H(\rho,0) = \begin{cases} 1 & \rho \leq 1, \\ 0 & \rho > 1. \end{cases} \tag{34a}$$

$$H(\rho,1) = \begin{cases} 0 & \rho < Q, \\ 1 & \rho \geq Q. \end{cases} \tag{34b}$$

$$H(\rho,e) \text{ for } 0<e<1 = \begin{cases} 1 & \rho \leq 1 - e. \\ 1 - (1/\pi)\left\{\cos^{-1}\left[(1-\rho)/e\right] - \left[e^2 - (1-\rho)^2\right]^{1/2}\right\} & 1-e \leq \rho \leq 1+e, \\ 0 & \rho \geq 1 + e. \end{cases} \tag{34c}$$

We note $H(1,e) = (2e + \pi)/(2\pi)$ and $\partial H/\partial\rho < 0$ for $e \varepsilon (0,1)$. $H(\rho,e)$ is tabulated in Table 1 for $e = 0(0.1)0.9$, $\rho = 1-e(0.1)1+e$.

B. Summing the f and g Series

The parametric representation of the geocentric position is

$$x/r = \cos\delta\cos\alpha = \cos\Omega\cos u - \sin\Omega\cos i\sin u, \tag{35a}$$

$$y/r = \cos\delta\sin\alpha = \sin\Omega\cos u + \cos\Omega\cos i\sin u, \tag{35b}$$

$$z/r = \sin\delta \quad = \sin i\sin u, \tag{35c}$$

$$r = a(1 - e^2)/(1 + e\cos v), \tag{35d}$$

$$u = v + \omega, \tag{35e}$$

$$\tan(v/2) = \left[(1 + e)/(1 - e)\right]^{1/2}\tan(E/2), \tag{35f}$$

$$E - e\sin E = M = n(t - t_o) + M_o. \tag{35g}$$

**TABLE 1**

**The Percentage of An Orbit the Distance Exceeds ρ·(semi-major axis)**

| ρ/e | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 0.1 | | | | | | | | | | 100 |
| 0.2 | | | | | | | | | 100 | 98.0 |
| 0.3 | | | | | | | | 100 | 96.2 | 96.4 |
| 0.4 | | | | | | | 100 | 94.3 | 93.8 | 94.6 |
| 0.5 | | | | | | 100 | 91.9 | 90.9 | 91.4 | 92.6 |
| 0.6 | | | | | 100 | 89.1 | 87.5 | 87.6 | 88.7 | 90.3 |
| 0.7 | | | | 100 | 85.4 | 83.2 | 83.2 | 84.2 | 85.8 | 87.8 |
| 0.8 | | | 100 | 80.3 | 77.7 | 77.7 | 78.8 | 80.6 | 82.7 | 85.1 |
| 0.9 | | 100 | 72.2 | 69.8 | 70.4 | 72.0 | 74.2 | 76.6 | 79.3 | 82.0 |
| 1.0 | 100 | 53.2 | 56.4 | 59.5 | 62.7 | 65.9 | 69.1 | 72.3 | 75.5 | 78.6 |
| 1.1 | | 0 | 36.8 | 48.2 | 54.3 | 59.2 | 63.5 | 67.5 | 71.3 | 74.9 |
| 1.2 | | | 0 | 33.9 | 44.4 | 51.5 | 57.2 | 62.1 | 66.6 | 70.8 |
| 1.3 | | | | 0 | 31.4 | 42.2 | 49.9 | 56.0 | 61.4 | 66.2 |
| 1.4 | | | | | 0 | 30.0 | 41.0 | 48.9 | 55.4 | 61.0 |
| 1.5 | | | | | | 0 | 29.2 | 40.3 | 48.4 | 55.1 |
| 1.6 | | | | | | | 0 | 28.7 | 39.8 | 48.1 |
| 1.7 | | | | | | | | 0 | 28.4 | 39.6 |
| 1.8 | | | | | | | | | 0 | 28.3 |
| 1.9 | | | | | | | | | | 0 |
| 2.0 | | | | | | | | | | |

The standard origin of time is the time of perigee passage (usually denoted by T). The remaining notation is a = semi-major axis, e = eccentricity, i = inclination, $\Omega$ = longitude of the ascending node, $\omega$ = argument of perigee, v = true anomaly, E = eccentric anomaly, M = mean anomaly, n = mean motion, nP = $2\pi$ where P is the period, and $n^2a^3 = \mu$.

If we express Eq. (14) explicitly in terms of the parametric representation of the motion then we find ($T = t - t_o$ here)

$$f = \left\{- r(T)\dot{r}_o \sin\left[v(T) - v_o\right] + r(T) r_o \dot{v}_o \cos\left[v(T) - v_o\right]\right\}/L, \tag{36a}$$

$$g = \left\{r(T) r_o \sin\left[v(T) - v_o\right]\right\}/L, \tag{36b}$$

where L was defined in Eq. (6). All v's refer now to the true anomaly, not $|d\underline{r}/dt|$ and the subscript o indicates a quantity evaluated at $t = t_o$ (e.g., $v_o = 0$ if $t_o = T$). Equations (36) are exact as long as the Jacobian $\partial(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)/\partial(a, e, i, \Omega, \omega, M_o)$ doesn't vanish. Although we haven't computed this determinant, we expect it to vanish for e = 0, i = 0, $i = \pi/2$. If the motion is assumed to be circular then $p = q = 0$, $h = \mu/a^3 = n^2$ and by induction on the coefficients of $T^n$ we find

$$f = \cos[v(T) - v_o], \tag{37a}$$

$$g = (1/n)\sin[v(T) - v_o]. \quad (37b)$$

But if e = 0, v = E = M and since the sine and cosine are analytic for all real values of their argument, the power series expansions of f and g in terms of $t - t_o$ converge everywhere. This proves the first part of the theorem.

Let us now consider the case e = 1. The analog of Kepler's equation is

$$\tan(v/2) + (1/3)\tan^3(v/2) = (\mu/2Q^3)^{1/2}(t - t_o) \quad (38)$$

where $v(t_o) = 0$. The explicit solution for tan(v/2) is

$$\tan(v/2) = \left[cT + (1 + c^2T^2)^{1/2}\right]^{1/3} + \left[cT - (1 + c^2T^2)^{1/2}\right]^{1/3}, \quad (39a)$$

with

$$c^2 = 9\mu/8Q^3, c > 0, T = t - t_o. \quad (39b)$$

From this result, we can compute sinv and cosv in terms of T. If we substitute this into Eqs. (36) and then expand the binomials, we will have the appropriate power series. They will converge when the binomial series converges, e.g., for

$c|T| \leq 1$. Now we can address the earlier criticism concerning the limit as $e \to 1$. Since

$$Q = \lim_{\substack{a\to\infty \\ e\to 1}} a(1 - e), \quad Q \text{ finite,} \tag{40}$$

it will be asymptotically correct to replace Q by $a(1 - e)$ in c. Thus, as $e \to 1$ we can expect convergence of the f and g series when

$$3\pi|T|/P < \sqrt{2}(1 - e)^{3/2}. \tag{41}$$

Although of a different functional form than the result in (32), the problem with $e \to 1$ does not go away.*

Having provided this much motivation for the existence of a problem, we now turn to solving it rigorously. It is simpler to work with the eccentric anaomaly than the mean anomaly so we rewrite the f and g sums as

$$f = (a/Lr_o)(r_o\dot{r}_o \sin v_o + L\cos v_o)(\cos E - e)$$

$$+ \left[a(1 - e^2)^{1/2}/r_o L\right]\left(L\sin v_o - r_o\dot{r}_o \cos v_o\right)\sin E, \tag{42a}$$

*This is the limit of Eq. (29) when $M_o = 0$ and $e \to 1^-$.

$$g = -(ar_o/L)\sin v_o(\cos E - e)$$

$$+ \left[ar_o(1 - e^2)^{1/2}/L\right]\cos v_o \sin E. \tag{42b}$$

We need not only to expand the trigonometric functions of the eccentric anomaly in a power series (which converges for all real values of E), but to expand E in a convergent power series in M, substitute this, term by term, in the series for cosE and sinE, rearrange terms to obtain a power series for f and g in M, and then show convergence.

C. The Series f(M), g(M) and Their Convergence

The first thing we need to do is obtain E(M) in a power series. From Kepler's equation, Eq. (35g), we can solve for E(M) in a convergent Fourier series (§17.2 of reference 5),

$$E = M + \sum_{\nu=1}^{\infty} (2/\nu)J_\nu(\nu e)\sin(\nu M), \tag{43}$$

where $J_\mu(z)$ is the Bessel function of the first kind, of order $\mu$, argument z. For e real, this series converges on [0,1]. We want, however, a Taylor's series representation for E(M). To compute the derivatives $d^\ell E/dM^\ell$ we differentiate (43) $\ell$ times term by term and sum. The sum will be $d^\ell E/dM^\ell$ if the series obtained by $(\ell - 1)$-fold differentiation term by term converges and if the series obtained by $\ell$-fold differentiation term by term converges uniformly (§18.5 of reference 4). Thus, we need to show

$$\sum_{\nu=1}^{\infty} \nu^{\ell-1} J_\nu(\nu e) \begin{matrix}\sin\\ \cos\end{matrix}(\nu M) \tag{44}$$

converges uniformly $\forall \ell \geq 1$ and $e \varepsilon (0,1)$. The $e = 0$, $e = 1$ cases we've already disposed of.

By Weierstrass's comparison test (reference 6, §3.34), since for all real M, $|\sin \nu M| \leq 1$, $|\cos \nu M| \leq 1$, if

$$\sum_{\nu=1}^{\infty} |\nu^{\ell-1} J_\nu(\nu e)| \quad \ell \geq 1, \quad e \varepsilon (0,1), \tag{45}$$

converges then the series (44) converges uniformly. But $J_\nu(\nu e)$ for $e \varepsilon (0,1]$ is a positive decreasing function of $\nu$ (reference 5 §8.5). It, therefore, follows from D'Alembert's ratio test (reference 4 §17.4) that the series (45) converges whence the series (44) all converge uniformly for $\ell \geq 1$, $e \varepsilon (0,1]$. Therefore,

$$d^\ell E/dM^\ell = \begin{cases} \delta_{\ell 1} + 2(-1)^{\frac{\ell-1}{2}} \sum_{\nu=1}^{\infty} \nu^{\ell-1} J_\nu(\nu e) \cos(\nu M), \ell \text{ odd}, & (46a) \\ 2(-1)^{\frac{\ell}{2}} \sum_{\nu=1}^{\infty} \nu^{\ell-1} J_\nu(\nu e) \sin(\nu M), \ell \text{ even}. & (46b) \end{cases}$$

Since all the derivatives of E with respect to M exist, E is an analytic function of M (reference 7 §169H of Vol. II). It is, therefore, expressible in a Taylor's series and the Taylor series, in a sufficiently small neighborhood, does converge to E. At M = 0* we have,

$$d^{\ell}E/dM^{\ell}\Big|_{M=0} = \begin{cases} \delta_{\ell 1} + 2(-1)^{\frac{\ell-1}{2}} \sum_{\nu=1}^{\infty} \nu^{\ell-1} J_{\nu}(\nu e), \ \ell \text{ odd}, & (47a) \\ 0, \ \ell \text{ even}. & (47b) \end{cases}$$

The first few values are**

$$E'(0) = 1/(1-e), \quad (48a)$$

$$E'''(0) = -e/(1-e)^4, \quad (48b)$$

$$E^{v}(0) = e(1 + 9e)/(1-e)^7, \quad (48c)$$

$$E^{vii}(0) = -e(1 + 54e + 225e^2)/(1-e)^{10}. \quad (48d)$$

Thus,

$$E = M/(1-e) + \sum_{k=1}^{\infty} S_{2k+1} M^{2k+1}/\Gamma(2k+2), \quad (49)$$

where $\Gamma(z)$ is the gamma function and

*The expansion about $M_o = 0$ is equivalent to expanding about the instant of perigee passage. For any other value of $M_o$ we would find that the f and g series in $(M - M_o)$ has radius of convergence $r$. Hence, once the $M_o = 0$ result is established the $M_o \neq 0$ result follows immediately.

**The reader can see in Eqs. (48) the kernel of the problem.

$$S_{2k+1} = 2(-1)^k \sum_{\nu=1}^{\infty} \gamma_{2k}(\nu, e), \tag{50a}$$

$$\gamma_{2k}(\nu, e) \equiv \nu^{2k} J_\nu(\nu e). \tag{50b}$$

The radius of convergence of the E(M) power series, $r$, is given by (reference 4 §19.5).

$$1/r = \lim_{k\to\infty} \sup |S_{2k+1}/\Gamma(2k+2)|^{1/(2k+1)}. \tag{51}$$

Since for $e\varepsilon(0,1]$ $J_\nu(\nu e)$ is a positive decreasing function of $\nu$ and for $k > 0$, $\nu \geq 1$ $\nu^{2k}$ is a positive increasing function of $\nu$, it follows that $\gamma_{2k}(\nu,e)$ has a single maximum as a function of $\nu$. Call this value of $\nu$, N (not necessarily an integer). Let [N] be the greatest integer less than or equal to N. Then,

$$\forall m \leq [N],\ \gamma_{2k}(m-1,e) < \int_{m-1}^{m} \gamma_{2k}(\nu,e)d\nu < \gamma_{2k}(m,e), \tag{52}$$

So, since $J_o(0) = 1$,

$$\sum_{m=0}^{[N]-1} \gamma_{2k}(m,e) < \int_0^{[N]} \gamma_{2k}(\nu,e)d\nu < \sum_{m=0}^{[N]} \gamma_{2k}(m,e). \tag{53}$$

Also,

$$\forall m \geq 1,\ \gamma_{2k}([N] + m,e) > \int_{[N]+m}^{[N]+m+1} \gamma_{2k}(\nu,e)d\nu > \gamma_{2k}([N]+m+1,e), \tag{54}$$

so

$$\sum_{m=[N]+1}^{\infty} \gamma_{2k}(m,e) > \int_{[N]+1}^{\infty} \gamma_{2k}(\nu,e)d\nu > \sum_{m=2+[N]}^{\infty} \gamma_{2k}(m,e). \tag{55}$$

Combining the inequalities (53, 55) yields

$$\int_{0}^{\infty} \gamma_{2k}(\nu,e)d\nu - \int_{[N]}^{[N]+1} \gamma_{2k}(\nu,e)d\nu + \gamma_{2k}([N],e) + \gamma_{2k}([N] + 1,e)$$

$$> \sum_{m=0}^{\infty} \gamma_{2k}(m,e). \tag{56}$$

However,

$$\gamma_{2k}(N,e) > \int_{[N]}^{N} \gamma_{2k}(\nu,e)d\nu > \gamma_{2k}([N],e), \tag{57a}$$

and

$$\gamma_{2k}(N,e) > \int_{N}^{[N]+1} \gamma_{2k}(\nu,e)d\nu > \gamma_{2k}([N] + 1,e), \tag{57b}$$

so

$$\int_{[N]}^{[N]+1} \gamma_{2k}(\nu,e)d\nu > \gamma_{2k}([N],e) + \gamma_{2k}([N]+1,e). \tag{57c}$$

Therefore,

$$I_{2k} = \int_0^\infty \gamma_{2k}(\nu,e)d\nu > \sum_{m=0}^{\infty} \gamma_{2k}(m,e) = |S_{2k} + 1|/2. \tag{58}$$

We now have, for all $k > 0$, $e\varepsilon(0,1]$ an upper bound for $|S_{2k} + 1|$. The next step is the evaluation of the integral in (58).

We can write (reference 5 §8.5)

$$J_\nu(\nu e) = (1/\pi) \int_0^\pi \exp[-\nu F(\theta,e)]d\theta, \tag{59}$$

where

$$F(\theta,x) = \ell n[\theta + (\theta^2 - x^2\sin^2\theta)^{1/2}] - \ell n(x\sin\theta) - \cot\theta(\theta^2 - x^2\sin^2\theta)^{1/2}, \tag{60}$$

and we can prove,

$$F(\theta,x) \geq F(0,x) \geq F(0,1) = 0. \tag{61}$$

Replace $J_\nu(\nu e)$ in the integral of $I_{2k}$ with this expression, interchange the order of integration (this requires the uniform convergence of the improper integral $I_{2k}$ [reference 6, §4.44]; we shall explicitly show this below) and make the change of variable $\mu = \nu F(\theta,x)$ in the $\nu$ integral. Then,

$$2I_{2k} = (2/\pi)\int_0^{\pi} [d\theta/F^{2k+1}(\theta,e)]\int_0^{\infty} \mu^{2k}\exp(-\mu)d\mu > |S_{2k} + 1|. \quad (62)$$

The $\mu$ integral is just $\Gamma(2k + 1)$ and from the inequality on $F(\theta,e)$ above,

$$\frac{2\Gamma(2k + 1)}{\pi F^{2k+1}(0,e)}\int_0^{\pi} d\theta > |S_{2k} + 1|. \quad (63)$$

The result for $\kappa$ is, then

$$1/\kappa = \lim_{k\to\infty} \left|\frac{2\Gamma(2k + 1)}{\Gamma(2k + 2)F^{2k+1}(0,e)}\right|^{1/(2k+1)}. \quad (64)$$

Or,

$$\kappa = F(0,e). \quad (65)$$

Let us now look back on what we've done. We've shown that $E(M)$, expressed as a power series in $M$ about $M = 0$, converges for $|M| < F(0,e)$. The one missing point is the uniform convergence of the $I_{2k}$ integral which we now provide.

We need to show that

$$\int_0^\infty \nu^{2k}\left\{(1/\pi)\int_0^\pi \exp[\,-\,\nu F(\theta,e)]d\theta\right\}d\nu$$

$$= (1/\pi)\int_0^\pi \left\{\int_0^\infty \nu^{2k}\exp[\,-\nu F(\theta,e)]d\nu\right\}d\theta. \qquad (66)$$

This will be true if the inner integral on the right hand side of (66) converges uniformly in $\theta$ for $\theta\varepsilon[0,\pi]$. From (61) we can majorize the integral by the gamma function again so that by de la Vallee Poussin's comparison test for convergence (§6.12 of reference 7, Vol. I) the inner integral on the right hand side of (66) does converge uniformly.

The last thing we need to do is inquire into the permissibility of substituting one power series in another, performing the Cauchy multiplications, rearranging, and summing. From Vol. II, §161 of reference 7 we can see that in the present circumstances this is permissible and the radius of convergence of the f and g series is precisely $\kappa$. Q.E.D.

One might wonder as to the relationship between this result and the results, for instance in §100 of reference 9, wherein r and v are expressed in a Fourier series of M. These are known to be divergent for some values of M once $e > 0.6627434$. This number happens to be the modulus of a complex root of $F^2(0,e) = 1$. The connection arises because of the nature of the Kapteyn series for E(M) and the use of Lagrange's formula (reference 10

§56). The series for r and v are really power series in e whose coefficients happen to be trigonometric functions of the mean anomaly. If we expanded sinM etc. in a power series in M, its radius of convergence would clearly be at most 0.66---. In fact, from reference 11 §4.3, it would be much less. It appears that the theorem given in the beginning of this Section is the most general result one can obtain without special arguments depending on the values of e and M.

## VII. THE IMPLICATIONS FOR ORBITAL ANALYSIS

$F(0,e)$ is given in Table 2 for $e = 0(0.1)1.0$. We note that $F(0,0.1)/F(0,0.7) = 11$ which shows the dramatic drop in the permissible time span as $e \to 1$. If $n = 2$ rev/day, $e = 0.7$, then the maximum time for which the f and g series converge is <21 minutes. We also give in Table 2 the maximum eccentricity such that the f and g series converge for $|T|/P \leq 0(0.05)0.5$.

Although it is clear that for $M_o = 0$, $\kappa \to 0$ very rapidly as $e \to 1$, we shall rarely be so unfortunate as to observe a satellite at perigee. The two most promising search patterns will find satellites near $\delta = 0$. The most common high eccentricity satellites can be characterized by (roughly) $n = 2$ rev/day, $e = 1/\sqrt{2}$, $\omega = 3\pi/2$. Hence, $E = \pi/4$, $M_o = (\pi-2)/4$ and $P\kappa/(2\pi) = 38^m.3$. Of course, $V_o$ is a maximum now and $v_o$ nearly a maximum so that their neglect is especially serious.

It would seem that, since we can't know the orbital phase of the initial observations, the best we can do is retain a reasonable number of terms in the f and g series, numerically investigate their divergence, and restrict the time span of the observations to a safe, small, duration. Table 3 contains the maximum time duration we can use and keep the relative percentage error at 1% and 0.1% for $e = 0.7$ as a function of $M_o = 0(10)60°$ for the twelfth order f series. The results for g are similar. When we turn to the rapidity of the convergence of the f and g series as

TABLE 2

RADIUS OF CONVERGENCE OF THE f AND g SERIES

| e | $r = F(0,e)$ | $\|T\|/P$ | e |
|---|---|---|---|
| 0.0 | ∞ | 0 | $1^-$ |
| 0.1 | 2.00 | 0.05 | 0.589 |
| 0.2 | 1.31 | 0.10 | 0.410 |
| 0.3 | 0.920 | 0.15 | 0.293 |
| 0.4 | 0.650 | 0.20 | 0.212 |
| 0.5 | 0.451 | 0.25 | 0.154 |
| 0.6 | 0.299 | 0.30 | 0.112 |
| 0.7 | 0.181 | 0135 | 0.082 |
| 0.8 | 0.0931 | 0.40 | 0.060 |
| 0.9 | 0.0313 | 0.45 | 0.044 |
| 1.0 | 0 | 0.50 | 0.032 |

TABLE 3

Maximum Durations for 1 and 0.1% Relative Convergence: Twelfth Order and e = 0.7

| $M_o$ | $T_1$ | $T_{0.1}$ |
|---|---|---|
| 0° | $5^m.7$ | $1^m.3$ |
| 10 | 6.7 | 1.4 |
| 20 | 11.7 | 3.1 |
| 30 | 17.5 | 5.6 |
| 40 | 23.6 | 7.3 |
| 50 | 29.8 | 10.2 |
| 60 | >30.0 | 15.0 |

a function of e, T, and $M_o$ we discover that the series converge or they don't. What I mean by this tautology is that the partial sums of the f and g series, for fixed e, T, and $M_o$, are either constant as a function of their order (and essentially equal to f and g) or they are meaningless. Numerical values for the third through twelfth order partial sums were computed before reaching this conclusion.

The other side of the problem is the desire to have the observations span as large an extent of the true argument of latitude [e.g., u of Eq. (35e)] as possible. For highly elliptical orbits this will be the case only near periastron, just when the radius of convergence is its smallest.

## VIII. ALTERNATIVES

While the statistical method outlined in §V is clearly better than the traditional Gaussian method, it too relies on the rapid convergence of the f and g series. With angles only data the other classical methods are the Gibbsian variation of Gauss's method and the procedures of Laplace and Lagrange. I briefly mentioned the Gibbsian variation of Gauss's method on page 12. Since it does minimally include the velocity, it is better than Gauss's procedure. However, as it too relies on the rapid convergence of the f and g series, it is subject to the above criticism of the statistical Gaussian method. The Laplace method requires two numerical differentiations of the topocentric direction cosines. I don't think this is a good thing to do. The Laplace method does not rely on the f and g series though. The Lagrangian method only uses a single numerical differentiation of the topocentric direction cosines but does use the f and g series. Hence, this is open to the same difficulties as the Gaussian, Gauss-Gibbs, and statistical Gaussian techniques.

If one has angles and angular rates, which should be the case for artificial satellites, one can improve the situation. A modified Laplace method has been developed wherein only one numerical differentiation is required (reference 12). A modified Lagrange method without any numerical differentiations has been developed[12] but it still relies on the f and g series. Finally, there is yet another method[12] which is exact (given that the force is $-\mu \underline{r}/r^3$) and only uses two sets of observations of angles and angular rates. Unfortunately, the exact technique is very susceptable to errors in the angular rates. [To make this statistical would require f and g series.]

If we step back from all of this and remember the history of the first three hundred years of dynamical astronomy, we realize almost all of the development has been concerned with perturbations of a circular orbit with the observer in the orbital plane. As the practical problem is to compute initial orbits for highly eccentric motion, it would appear that a perturbation of a parabolic orbit is the analysis to pursue. Moulton[9] has made a start on this.

# REFERENCES


1. E. L. Ince, Ordinary Differential Equations (Longmans, Green and Co. Ltd., London, 1927).

2. A. Wintner, The Analytical Foundations of Celestial Mechanics (Princeton Univ. Press, Princeton, 1947).

3. J. W. Gibbs, Vector Analysis (C. Scribner's Sons, New York, 1909).

4. A. E. Taylor, Advanced Calculus (Ginn and Co., New York, 1955).

5. G. N. Watson, Theory of Bessel Functions (Cambridge Univ. Press, Cambridge, 1922).

6. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge Univ. Press, Cambridge, 1902).

7. J. Pierpont, The Theory of Functions of Real Variables (Ginn and Co., New York, 1905).

8. J. Pierpont, Functions of a Complex Variable (Ginn and Co., New York, 1914).

9. F. R. Moulton, Celestial Mechanics (MacMillan Co., Ltd., New York, 1902).

10. T. J. I'a. Bromwich, Infinite Series (MacMillan and Co., Ltd., New York, 1908).

11. K. Knopp, Infinite Sequences and Series (Dover Publications Inc., New York, 1956).

12. L. G. Taff and D. L. Hall, Cel. Mech. 16, 481 (1977).

## NOTE ADDED IN PROOF

After the full review of this technical note Donald Batman showed me a reference in the second edition of Moulton's book. It occurs under the heading "The Laplacian Method of Determining Orbits" (pg. 202) and refers to a paper of Moulton's (published after the first edition of the text) in which one can find "the determination of the exact realm of convergence" of the f and g series. The full reference is

F. R. Moulton, Astron. J. 23, 93 (1903).

In this paper, by a method completely independent of the technique used herein, Eq. (29) is derived. He also gave tables similar to Table 2 but was discussing orbit determination for asteroids, not artificial satellites.

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This report presents a statistical version of the Gaussian initial orbit technique. It neglects neither the angular velocity nor the radial velocity terms in the f and g series. More importantly it provides a rigorous, analytically simple result for the radius of convergence of the f and g series. The radius of convergence can be extremely small at periastron, approaching zero as the eccentricity of the orbit approaches unity. The leading term is given by $\sqrt{2}(1-e)^{3/2}P/(3\pi)$ as $e \rightarrow 1$ where P is the orbital period. This implies that initial orbit determination by any procedure which uses the f and g series is a process fraught with the possibility of unknown errors. The central result is given in Eq. (29).


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