AD-A073 550 POLYTECHNIC INST OF NEW YORK BROOKLYN DEPT OF ELECTR--ETC F/G 12/2
OPTIMAL REPLACEMENT OF PARTS HAVING OBSERVABLE CORRELATED STAGE--ETC(U)
JUL 79 L SHAW, C L HSU, S G TYAN N00014-75-C-0858

UNCLASSIFIED POLY-EE79-055 NL

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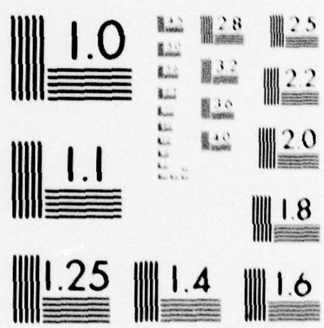

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by L. Shaw, C. L. Hsu and S. G. Tyan

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Contract No. N00014-75-6-0858
Office of Naval Research

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: POLY-EE 79-055

2. GOVT ACCESSION NO.:

3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): Optimal Replacement of Parts Having Observable Correlated Stages of Deterioration.

5. TYPE OF REPORT & PERIOD COVERED: Technical rept.

6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): L. Shaw, C. L. Hsu, [redacted] S.G. Tyan

8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0858

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Polytechnic Institute of New York
333 Jay Street Dept of Elec. Eng. + Electrophysics
Brooklyn, New York 11201

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:

11. CONTROLLING OFFICE NAME AND ADDRESS:
Chief Naval Research (Code 436)
800 N. Quincy
Arlington, Virginia 22203

12. REPORT DATE: 1 July [redacted]79

13. NUMBER OF PAGES: 25

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
Office of Naval Research
207 W. 24th Street
New York, New York 10011

29p.

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report):
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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
. . . .

18. SUPPLEMENTARY NOTES:
. . . .

19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Multivariate Exponential Distribution, Optimal Replacement, Aging, Deterioration

20. ABSTRACT (Continue on reverse side if necessary and identify by block number):
SEE NEXT PAGE

408717

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

ABSTRACT

A single component system is assumed to progress through a finite number of increasingly bad levels of deterioration. The system with level i $(0 \leq i \leq n)$ starts in state 0 when new, and is definitely replaced upon reaching the worthless state n. It is assumed that the transition times are directly monitored and the admissible class of strategies allows substitution of a new component only at such transition times. The durations in various deterioration levels are dependent random variables with exponential marginal distributions and a particularly convenient joint distribution. Strategies are chosen to maximize the average rewards per unit time. For some reward functions (with the reward rate depending on the state and the duration in this state) the knowledge of previous state duration provides useful information about the rate of deterioration.

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

OPTIMAL REPLACEMENT OF PARTS HAVING OBSERVABLE CORRELATED STAGES OF DETERIORATION[+]

by

L. Shaw[†], C-L. Hsu[*], S.G. Tyan[†]

ABSTRACT

A single component system is assumed to progress through a finite number of increasingly bad levels of deterioration. The system with level i ($0 \leq i \leq n$) starts in state 0 when new, and is definitely replaced upon reaching the worthless state n. It is assumed that the transition times are directly monitored and the admissible class of strategies allows substitution of a new component only at such transition times. The durations in various deterioration levels are dependent random variables with exponential marginal distributions and a particularly convenient joint distribution. Strategies are chosen to maximize the average rewards per unit time. For some reward functions (with the reward rate depending on the state and the duration in this state) the knowledge of previous state duration provides useful information about the rate of deterioration.

[+]This work was partially supported by Grant No. N00014-75-C-0858 from the Office of Naval Research, and it is based on a portion of the dissertation submitted by C. L. Hsu for the Ph. D. in Electrical Engineering at the Polytechnic Institute of New York, June 1977.

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Many authors have studied optimal replacement rules for parts characterized by Markovian deterioration, for example Kao [6] and Luss [9] and the many references found in those papers. Kao minimized the expected average cost per unit time for semi-Markovian deteriorating system, and considered various combinations of state and age-dependent replacement rules.

Luss examined inspection and repair models, where he assumed that the operating costs occurring during the system's life increase with the increasing deterioration. The holding times in the various states were independently, identically, and exponentially distributed. The policies examined include the scheduling of the next inspections (when an inspection reveals that the state of the system is better than certain critical state k) and preventive repairs (when an inspection reveals the state of the system being worse than or equal to k). The convenience of a Poisson-type structure for the number of events-per-unit-time made it relatively easy to allow general freedom in the selection of observation times.

The work studied here is based on a modification of the model used by Luss. Our model for deterioration is more general, but the admissible strategies used here are more restricted. Here we allow the exponentially distributed durations to have different mean values, and to be positively correlated.

The introduction here of correlation between interval durations permits the modeling of a rate of deterioration which can be estimated from a particular realization of the past durations. However, the lack of a Poisson-type of structure for the events-per-unit-time makes it much more difficult here to allow general freedom in the selection of observation times. At present, only the simple case of direct and instantaneous observation of deterioration jumps has been considered.

This model would be appropriate, for example, in a subsystem which functions, but with reduced efficiency, when some redundant components have failed; and for which failure of one component might indicate environmental stresses which increase the probability of failure for other components. In addition, deterioration in correlated stages might be used as a simple approximation for a continuously varying degradation which does not exhibit discrete stages.

Figure 1 shows a typical time history of deterioration and replacement. The duration in state (i-1), prior to reaching state (i), is r_{i-1}. The intervals d_i in Figure 1 represent the time required to replace a component when it has entered state i. The sequence $\{r_i\}$ will be Markov, characterized by a multi-variate exponential distribution. Reward functions will be related to the deterioration state and the time spent in each state. The decision rule specifies whether or not to replace when entering each state i, on the basis of the history of $r_{i-1}, r_{i-2}, \ldots$. The Markov property simplifies the decision rule to be a collection of $\mathcal{C}_i$ sets such that we replace on entering state i if and only if $r_{i-1} \in \mathcal{C}_i$.

The objective is to maximize the average reward per unit time:

$$L = \lim_{T \to \infty} \frac{1}{T} (\text{Total reward in } (0, T)) \tag{1}$$

$$= \frac{E[\text{Reward per renewal}]}{E[\text{Duration between renewals}]} = \frac{\mathcal{R}}{\mathcal{D}}. \tag{2}$$

(See Ross [11] page 160 for equivalence of (1) and (2).) The mean reward per renewal is defined here as:

$$\mathcal{R} = E\left[\sum_{i=0}^{N-1} \int_0^{r_i} c_i(t)dt - p_N\right], \tag{3}$$

in which:

N = state at which replacement occurs (possibly random).

p_N = replacement cost if replaced on entering state N (possibly random).

$c_i(t)$ = reward rate when in state i.

Figure 2 shows several reward rate time functions c(t) which have been considered. When one of these c(t) functions is specified for a given problem, the $c_i(t)$ in (3) are assigned values $\beta_i c(t)$ with:

$$\beta_0 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_{n-1} \geq \beta_n = 0, \tag{4}$$

to assure greater reward rates in less deteriorated states. State n corresponds to a completely failed or worthless component.

The mean duration in (2) is defined as:

$$\mathcal{D} = E\left[\sum_{i=0}^{N-1} r_i + d_N\right], \tag{5}$$

to include a possibly random time d_N for carrying out a replacement at state N.

While the ultimate objective is to choose $\mathcal{C}_i$ to maximize the L defined in (1), it is well known that a related problem of maximizing:

$$\mathcal{L}_0(\alpha) = \mathcal{R} - \alpha\,\mathcal{D}, \tag{6}$$

is simpler [1]. Indeed, the $\mathcal{C}_i$ which maximize L will be identical to those which maximize $\mathcal{L}_0(\alpha)$ for the α^* such that:

$$\mathcal{L}_0^0(\alpha^*) = 0, \quad \text{where} \quad \mathcal{L}_0^0(\alpha) \triangleq \max_{\{\mathcal{C}_i\}} \mathcal{L}_0(\alpha)$$

Section 1 considers a case in which it is found that deterioration rate information is not useful (e.g., the optimal policy is independent of the amount of correlation between successive state durations).

Sections 2 and 3 consider other reward rate structures, e.g., assuming that more deteriorated parts are rustier, hotter, or more brittle, and therefore more costly to replace. In such cases the optimal policies do make use of estimates of the deterioration rates as well as of observations of the deterioration level.

The appendix describes useful properties of the multivariate exponential $\{r_i\}$ sequence which is used to model the correlated residence times in a sequence of deterioration states.

1. Constant Reward Rate - State Independent Replacement Penalties

The constant reward rate case with $c_i(t) = \beta_i$ and with state-independent replacement penalties ($p_i = p$, $d_i = d$) is particularly simple to analyze. We will see that as long as $E[r_i | r_{i-1}, r_{i-2}, \ldots] \geq 0$ for all i, even if the r_i are not exponentially distributed, the optimal rule will be to replace the deteriorating part upon entering some critical state k*, independent of the observed durations r_i.

Based on the problem statement, the optimal decision on entering state j must maximize the mean future reward until the next renewal, $\mathcal{L}_j(\alpha)$, for a suitable α. Here:

$$\mathcal{L}_j(\alpha) = E\left[\sum_{i=j}^{N-1} \beta_i r_i \,\middle|\, r_{j-1}\right] - \alpha E\left[\sum_{i=j}^{N-1} r_i \,\middle|\, r_{j-1}\right] - p - \alpha d. \tag{8}$$

Immediately after a renewal, when $j = 0$, the expectations defining $\mathcal{L}_0(\alpha)$ are unconditional. The optimal decisions for each state will be found in terms of α, and then the proper α^* (for producing decisions which maximize L) is the one for which the maximum:

$$\max \mathcal{L}_0(\alpha^*) = \mathcal{L}_0^0(\alpha^*) = 0. \tag{9}$$

Optimization by dynamic programming begins by considering the decisions at the last step, i.e., on entering state (n-1). There are two choices, to replace (R) or not to replace ($\overline{R}$), with corresponding values:

$$\mathcal{L}_{n-1}(\alpha ;R) = -p-\alpha d, \tag{10}$$

and:

$$\mathcal{L}_{n-1}(\alpha ;\overline{R}) = E[\beta_{n-1} r_{n-1} | r_{n-2}] - \alpha E[r_{n-1} | r_{n-2}] - p - \alpha d$$

$$= E[(\beta_{n-1} - \alpha) r_{n-1} | r_{n-2}] - p - \alpha d. \tag{11}$$

Clearly, the best decision is not to replace if and only if the difference

$$\Delta_{n-1}(\alpha ; r_{n-2}) \triangleq \mathcal{L}_{n-1}(\alpha ;\overline{R}) - \mathcal{L}_{n-1}(\alpha ;R)$$

$$= (\beta_{n-1} - \alpha)\, E[r_{n-1} | r_{n-2}] \geq 0. \tag{12}$$

is non-negative. The sign of (12) will be the sign of $(\beta_{n-1} - \alpha)$, due to the non-negativity of all interval durations. Thus the best decision depends on α and the reward parameter β_{n-1}, but not on the previously observed duration. Two cases will be considered separately.

If $\beta_{n-1} \geq \alpha$ then the best decision at state (n-1) is not to replace. We will now explain why, under this condition, it is best not to replace at any state less than n. Consider the situation on entering (n-2). We have already shown that it is better not to replace on entering (n-1). Thus the choice will be based on a Δ_{n-2} of the form:

$$\Delta_{n-2}(\alpha ; r_{n-3}) = E[(\beta_{n-2}-\alpha)r_{n-2} + (\beta_{n-1}-\alpha)r_{n-1} | r_{n-3}). \tag{13}$$

Here we have:

$$(\beta_{n-2}-\alpha) > (\beta_{n-1}-\alpha) > 0, \tag{14}$$

by assumption, and:

$$E[r_{n-1} | r_{n-3}] \geq 0 \quad \text{and} \quad E[r_{n-2} | r_{n-3}] \geq 0, \tag{15}$$

because all $r_i \geq 0$ with probability one. Thus $\Delta_{n-2}(\alpha ; r_{n-3}) > 0$ for all $r_{n-3} > 0$, and it is also better not to replace here. This argument can be repeated for states (n-3), (n-4), . . . , 1, 0.

The other case to consider is $\beta_{n-1} < \alpha$, which requires replacement on entering state (n-1), if the system ever reaches that state. When we consider the decision on entering (n-2), the Δ_{n-2} is:

$$\Delta_{n-2}(\alpha ; r_{n-3}) = E[(\beta_{n-2}-\alpha)r_{n-2} | r_{n-3}], \tag{16}$$

which has the sign of $(\beta_{n-2}-\alpha)$. If $(\beta_{n-2}-\alpha) < 0$, then replacement is optimal on entering (n-2) and (n-3) is considered next. This iteration may eventually reach a state (k-1) where $(\beta_{k-1}-\alpha) > 0$ and it is better not to replace. Arguments similar to those for the $\beta_{n-1}-\alpha > 0$ case show that non-replacement is the optimal decision at all state preceding the one which first arises as a non-replacement state in this backward iteration.

In summary, in the constant reward rate-constant replacement penalty case $\mathcal{L}_0(\alpha)$ is maximized by a decision rule which says replace on entering some state $k \leq n$ which depends on the reward parameters $\{\beta_i\}$ and the α:

$$k = \min\{i:(\alpha-\beta_i) > 0\} . \tag{17}$$

Finally, we must choose α^* so that $\mathcal{L}_0^0(\alpha^*) = 0$, where:

$$\mathcal{L}_0^0(\alpha) = -p-\alpha d + \sum_{0}^{k-1} (\beta_i-\alpha)E[r_i] \tag{18}$$

Figure 3 shows a typical plot of $\mathcal{L}_0^0(\alpha)$ as a continuous, piecewise linear curve whose zero crossing ($\mathcal{L}_0^0(\alpha^*) = 0$) defines α^* and the optimal replacement state k^* for maximizing L.

Example: Figure 3 shows that the optimal average reward per unit time is $L = 2\frac{5}{7}$ when $k^* = 3$, where $\beta_0 = 5$, $\beta_1 = 4$, $\beta_2 = 3$, $\beta_3 = 2$, $\beta_4 = 1$, $\beta_5 = 0$, $p = 5$, $d = 1$, $\eta_i = 2$ $(i = 0, 1, 2, 3, 4)$ and $n = 5$. From Equation (18), the optimal k is a function of α, which remains constant when α varies over each interval $\beta_{i+1} < \alpha \leq \beta_i$, as shown in the figure.

2. Increasing Replacement Penalties-Constant Reward Rate

Here we generalize the model of the previous section by allowing the replacement cost p_i and replacement duration d_i to be functions of the replacement state (i), and to be random. These parameters are assumed to have mean values $E[p_i]$ and $E[d_i]$ which are convex non-decreasing sequences in i, corresponding to the increased difficulty in replacing more deteriorated parts which may be e. g., rustier, hotter or more brittle. We also assume that the mean durations are ordered: $\eta_o \geq \eta_1 \geq \ldots \geq \eta_{n-1}$, corresponding to faster transitions of more deteriorated parts.

The foregoing assumptions, together with properties of the assumed multivariate exponential density for stage-durations (see Appendix), lead to an optimal decision policy with a nice structure. That optimal policy prescribes replacement when entering state j, if and only if $r_{j-1} < r^*_{j-1}$, where the decision thresholds are ordered: $0 \leq r^*_o/\eta_o \leq r^*_1/\eta_1 \leq \ldots \leq r^*_{n-1}/\eta_{n-1} = \infty$.

The optimal decision on entering state j must maximize the mean future reward until the next renewal, i. e., $\mathcal{L}_j(\alpha)$. For a suitable α, we have:

$$\mathcal{L}_j(\alpha) = E\left\{\sum_{i=j}^{N-1} \beta_i r_i \middle| r_{j-1}\right\} - \alpha E\left\{\sum_{i=j}^{N-1} r_i \middle| r_{j-1}\right\} - E[p_N + \alpha d_N] \quad (19)$$

For notational simplicity we define $e_i = E[p_i + \alpha d_i]$ and note that e_i is also convex and non-decreasing since we are only interested in $\alpha > 0$. The optimal decisions for each state will be found in terms of α, and then the proper α^* (for producing decisions which maximize L) is the one for which the maximum $\mathcal{L}$ vanishes:

$$\mathcal{L}_0^0(\alpha^*) = e_N(\alpha^*) + E\left[\sum_{i=0}^{N-1} \beta_i r_i - \alpha^* \sum_{i=0}^{N-1} r_i\right] = 0 . \quad (20)$$

Optimization by dynamic programming begins by considering the decision at the last step. Since state n represents a failed component, we definitely replace the component when it enters state n. Next, we consider the decision to be made on entering state n-1. There are two choices: to replace (R) or not to replace ($\overline{R}$), with corresponding values

$$\mathcal{L}_{n-1}(\alpha;R) = -e_{n-1}, \tag{21}$$

$$\mathcal{L}_{n-1}(\alpha;\overline{R}) = E[\beta_{n-1} r_{n-1} - \alpha r_{n-1} \mid r_{n-2}] - e_n \tag{22}$$

for $\mathcal{L}_{n-1}(\alpha)$. Clearly, the best decision is not to replace if and only if

$$\Delta_{n-1}(r_{n-2}) \triangleq \mathcal{L}_{n-1}(\alpha;\overline{R}) - \mathcal{L}_{n-1}(\alpha, R)$$

is non-negative, i.e.,

$$\Delta_{n-1}(r_{n-2}) = (\beta_{n-1} - \alpha)\, E[r_{n-1} \mid r_{n-2}] + (e_{n-1} - e_n) \geq 0. \tag{23}$$

Referring to (A-6), $\Delta_{n-1}(r_{n-2})$ is a linear function of r_{n-2}, with

$$\Delta_{n-1}(0) = (\beta_{n-1} - \alpha)\eta_{n-1}(1-\rho) + (e_{n-1} - e_n).$$

Figure 4 shows the possible shapes for this function. There can be no downward zero-crossing at an $r_{n-2} > 0$.

Thus, depending on the numerical values of the parameters, there are three possible kinds of optimal decision rules when entering state (n-1):

i) replace for any r_{n-2} if $\Delta_{n-1} \geq 0$ for all $r_{n-2} \geq 0$

ii) replace for no r_{n-2} if $\Delta_{n-1} \leq 0$ for all $r_{n-2} \geq 0$

iii) replace if and only if $r^*_{n-2} > r_{n-2} \geq 0$, where $\Delta_{n-1}(r^*_{n-2}) = 0$.

In other words,

$$\mathcal{C}_{n-1}(\alpha) = \{r_{n-2} : r_{n-2} < r^*_{n-2}\}, \tag{24}$$

where r^*_{n-2} could be zero (case ii) or infinite (case i).

Next we consider the optimal decision when entering state (n-2), and assuming that the optimal decision will be made at the subsequent stage. We consider cases of $(\beta_{n-1} < \alpha)$ and $(\beta_{n-1} \geq \alpha)$ separately.

a) $(\beta_{n-1} < \alpha)$ implies replacement on entering (n-1), so

$$\Delta_{n-2}(r_{n-3}) = (\beta_{n-2}-\alpha)\, E[r_{n-2} \mid r_{n-3}] + (e_{n-2} - e_{n-1}),$$

resulting in the same three possibilities listed above for state (n-1).

b) for $(\beta_{n-1} > \alpha)$:

$$\Delta_{n-2}(r_{n-3}) = e_{n-2} + (\beta_{n-2}-\alpha)\, E[r_{n-2} \mid r_{n-3}]$$
$$+ \int_{r^*_{n-2}}^{\infty} [(\beta_{n-1}-\alpha)\, E[r_{n-1} \mid r_{n-2}] - e_n]\, f(r_{n-2} \mid r_{n-3})\, dr_{n-2}$$
$$+ \int_{0}^{r^*_{n-2}} (-e_{n-1})\, f(r_{n-2} \mid r_{n-3})\, dr_{n-2} \tag{25}$$

Equation (25) can be simplified, with the aid of the notation $(x)^+ = \max(x, 0)$, to the form

$$\Delta_{n-2}(r_{n-3}) = (e_{n-2}-e_{n-1}) + (\beta_{n-2}-\alpha)\, E[r_{n-2} \mid r_{n-3}]$$
$$+ E[(\Delta_{n-1}(r_{n-2}))^+ \mid r_{n-3}]. \tag{26}$$

Useful comparisons can be formed if normalized variables are introduced, namely

$$s_i = r_i/\eta_i; \quad \delta_i(s_{i-1}) = \Delta_i(r_{i-1})\Big|_{r_{i-1}=\eta_{i-1}s_{i-1}}$$

We now prove

A) $\delta_{n-2}(s_{n-3}) \geq \delta_{n-1}(s_{n-3})$

B) $\delta_{n-2}(s_{n-3})$ is convex with at most one upward zero crossing at an $s > 0$.

There is no harm in writing $\delta_{n-1}(s_{n-3})$ or $\delta_{n-1}(s)$ instead of $\delta_{n-1}(s_{n-2})$ for purposes of comparing functions.

To prove A), consider

$$\delta_{n-2}(s) - \delta_{n-1}(s) = [(e_{n-2}-e_{n-1}) - (e_{n-1}-e_n)] + E[(\delta_{n-1}(s))^+ | s_-]$$

$$+ [(\beta_{n-1}-\alpha)\eta_{n-2} - (\beta_{n-1}-\alpha)\eta_{n-1}] \, E[s/s_-]. \qquad (27)$$

where s_- represents the normalized duration preceding s.

The terms on the right side of (27) are non-negative due to the convexity of the e_i, $(\ )^+ \geq 0$, (A-6), and the assumed orderings of the β_i and η_i.

This completes the proof that A) is true. It follows immediately that if i) applies for state (n-1), then it is also optimal <u>not</u> to replace in state (n-2) or any earlier state. (Recall $\beta_{n-1} < \beta_{n-2} < \ldots$, and we are now considering $\alpha < \beta_{n-1}$).

To prove B), which is only of interest when an $r^*_{n-2} > 0$ exists, we refer to the theorem in the appendix. The test difference $\delta_{n-2}(s)$ can be written as

$$\delta_{n-2}(s) = E[e_{n-2} - e_{n-1} + (\beta_{n-2}-\alpha)\eta_{n-2}s + (\delta_{n-1}(s))^+ | s_-] \qquad (28)$$

in which the integrand has the properties required by h(s) in the theorem. To see this, we note that $r^*_{n-2} > 0$ implies that $(\delta_{n-1}(0))^+ = 0$, so the integrand is non-positive at $s = 0$. Thus, $s_{n-2}(s)$ has the shape stated in B), implying that

$$\mathcal{C}_{n-3} = \{r_{n-3}: r_{n-3} \leq r^*_{n-3}\} \tag{29}$$

where r^*_{n-3} may be zero, infinity, or the non-negative value defined by $\delta_{n-3}(r^*_{n-3}/\eta_{n-3}) = 0$.

The foregoing arguments can be repeated for r_{n-4}, $r_{n-5} \ldots r_0$ to prove that the optimal replacement policy has the form:

Replace on entering state i if and only if $r_i \leq r^*_i$

where

$$0 \leq r^*_o/\eta_o \leq r^*_1/\eta_1 \leq \ldots \leq r^*_{n-1}/\eta_{n-1} = \infty$$

When repeating the proof for earlier stages, the $(\ )^+$ term in (28) is modified to the form, e.g., $[(\delta_{n-2}(s))^+ - (\delta_{n-1}(s))^+]$. This term is zero for $x = 0$, and generally non-negative, due to A), so the basic theorem is still applicable.

3. Computational Procedure

The preceding section derived the structure of the optimal decision rule for the case where replacement is more difficult and more expensive when the part is more deteriorated. The corresponding optimal decision thresholds can be formed as follows:

a) choose an initial α.

b) Find the $r^*_i(\alpha)$ $(i = n-2, n-1, \ldots, 0)$ recursively, via numerical integration of expressions like (26) (where $r^*_{n-3}(\alpha)$ is defined by the condition $\Delta_{n-2}(r^*_{n-3}) = 0$).

c) Compute

$$\mathcal{L}_0^0(\alpha) = e_0 - e_1 + \int_0^\infty [(\beta_0 - \alpha) r_0 + (\Delta_1(r_0))^+] f(r_0) dr_0.$$

d) If $|\mathcal{L}_0^0(\alpha)| < \epsilon$, for sufficiently small ϵ, say $L_{max} = \alpha^* = \alpha$: otherwise repeat the computational cycle starting with a new α.

The following properties of $\mathcal{L}_0^0(\alpha)$ can be used to generate an α-sequence which converges to α^*.

1. $\mathcal{L}_0^0(\alpha)$ is monotone decreasing, since $\mathcal{L}_0(\alpha)$ has this property for a fixed policy (see Eq. (19)); and if $\mathcal{L}_0^0(\alpha_2) \geq \mathcal{L}_0^0(\alpha_1)$ for $\alpha_2 > \alpha_1$, then the policy used to achieve $\mathcal{L}_0^0(\alpha_2)$ could be used to achieve an $\mathcal{L}_0(\alpha_1) > \mathcal{L}_0^0(\alpha_1)$ -- a contradiction.

2. When $\rho = 0$, all r_i^* are zero or infinite: replacement always occurs on arrival at a critical state i^*. Use of that policy will achieve the same average reward for durations having any value of ρ. Thus, a useful bound on $\alpha^*(\rho)$ is $\alpha^*(0) \leq \alpha^*(\rho)$; $0 \leq \rho \leq 1$.

3. When $\rho = 1$, future r_i are completely predictable ($Var(r_i | r_{i-1}) = 0$ in (A-7)), so $\alpha^*(1) \geq \alpha^*(\rho)$. In this case there is essentially a single random variable r_0, and the r_i^* can be calculated without the need for numerical integration of Bessel functions.

4. Numerical Example

Table I lists parameter values for a replacement problem which falls under the assumptions of Section 2.

i	0	1	2	3	4	5
β_i	5	4	3	2	1	
η_i	1	0.9	0.8	0.7	0.6	
$E[p_i]$		2	2.2	2.4	2.6	2.8
$E[a_i]$		1	1.1	1.2	1.3	1.4

TABLE I

Case 1 ($\rho = 0$)

Since future durations are independent of past ones, the optimal policy replaces when a critical state i^* is reached. The general optimal reward expression

$$\alpha^*(\rho) = \frac{E\left[\sum_0^{N-1} \beta_i r_i - p_N\right]}{E\left[\sum_0^{N-1} r_i - d_N\right]} ,$$

becomes, in this case

$$\alpha^*(0) = \max_j \left[\frac{\sum_0^{j-1} \beta_i \eta_i - E[p_i]}{\sum_0^{j-1} \eta_i + E[d_i]}\right] = \max_j A(j)$$

Direct evaluation shows

j	1	2	3	4	5
A(j)	1.5	2.13	2.205	2.085	1.89

with $j^* = 3$ and $\alpha^*(0) = 2.205$.

Case 2 ($\rho = 1$)

Since $r_i = r_0 \eta_i / \eta_0$ in this case, the optimal rule specifies a replacement state $j(r_0)$ as a function for r_0.

For any such policy

$$\mathcal{L}_0(\alpha, j(r_0)) = E_{r_0}\left[-p_j - \alpha d_j + \frac{r_0}{\eta_0} \sum_0^{j-1} \eta_i(\beta_i - \alpha)\right] .$$

This expectation will be maximized if $j(r_0)$ maximizes the bracketed term for each r_0. Making the necessary comparisons for a sequence of α-values leads to the policy

$$j^* = 1, \text{ if } r_0 \le 0.2698$$
$$= 2, \text{ if } 0.2698 < r_0 \le 0.7083$$
$$= 3, \text{ if } 0.7083 < r_0.$$

for which $|\mathcal{L}_0| < 0.003$ and $\alpha^*(1) = 2.25$.

Case 3 $(\rho = \frac{1}{2})$

We know that $2.205 < \alpha^*(\frac{1}{2}) < 2.25$. A pilot calculation along the lines indicated in the previous section shows that $r_0^*(\frac{1}{2}) = 0$, $r_j^*(\frac{1}{2}) = \infty$ for $j \ge 2$, and

$$r_1^* = \frac{9(\alpha^* - 2)}{8(3 - \alpha^*)},$$

where α^* is chosen to make the following $\mathcal{L}_0(\alpha)$ vanish.

$$\mathcal{L}_0(\alpha) = 6.4 - 3\alpha + \int_0^\infty \int_{r_1^*}^\infty \left[(\frac{\alpha}{2} - 1) + \frac{4}{9}(3 - \alpha) r_1 \right] \frac{e^{-(2r_o + \frac{r_1}{0.225})}}{0.45} I_0(2.981 \sqrt{r_0 r_1}) \, dr_1 dr_0$$

While the method for finding $\alpha^*(\frac{1}{2})$ is clear, further numerical work seems unwarranted. The precision needed to get a meaningful answer is not justified by the minimal improvement that the optimal policy will have over the suboptimal use, for all ρ, of the optimal $\rho = 0$ policy (replace when reaching state 3).

5. Conclusions

A multivariate exponential distribution has been used to describe successive stages of deterioration. Optimal replacement strategies have been found for the class of decision rules which can continuously observe the deterioration state, and which may make replacements only at the times of state transitions. Similar results have been found for the other reward rates shown in Figure 2 (linear; and constant after an initial set-up interval for readjustment to the new state) [5].

The optimal replacement policy derived in Section 2 makes use of observations which allow estimation of the current rate of deterioration for the correlated stages of deterioration. The numerical example demonstrated a situation where a much simpler policy, which ignores the correlation, can be almost as good. The relatively easy calculation of the optimal rewards for $\rho = 0$ and $\rho \doteq 1$ provides information about the necessity for use of the more complex optimal solution. It is worth noting that we have been unable to find other combinations of values for the parameters in Table I such that there is a substantial relative increase in the maximum reward as ρ increases from zero to infinity.

The ordering of state dependent rewards, mean durations, etc. assumed here are physically reasonable, and lead to nice ordering of the decision regions. However, other β_i, n_i, p_i, d_i orderings might be more appropriate in other situations. The model introduced here for dependent stage durations could be used in those cases, together with dynamic programming optimization, although the solutions may not have comparably neat structures.

One reasonable generalization would allow transitions from state i to any state $j > i$. This would not change the form of the solution in the case of constant replacement penalties. However, the possibility of these additional transitions does ruin the structure when replacement penalties increase with the deterioration state. (The $\delta_{n-2}(s) > \delta_{n-1}(s)$ argument is no longer valid.)

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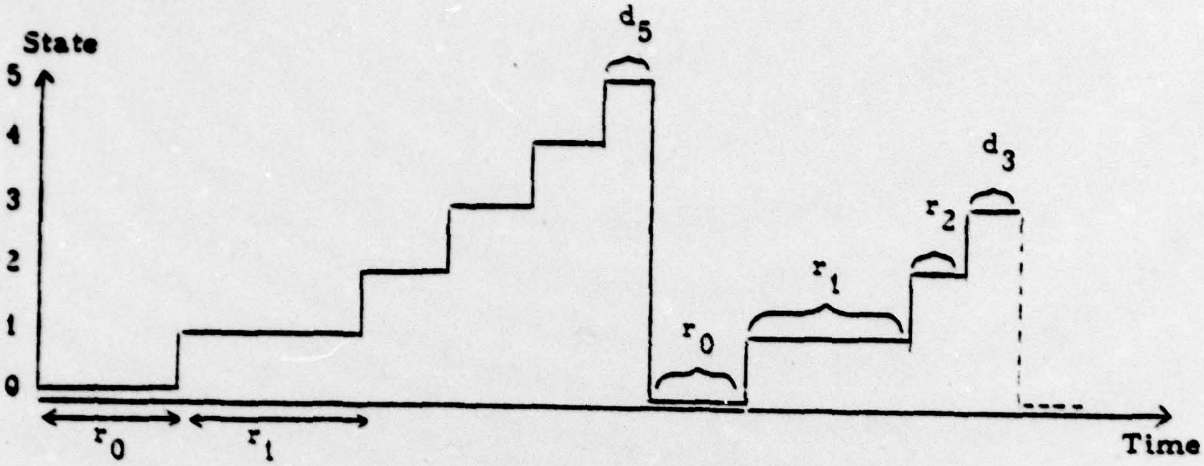

Figure 1:

History of Deterioration and Replacement (n=5) .

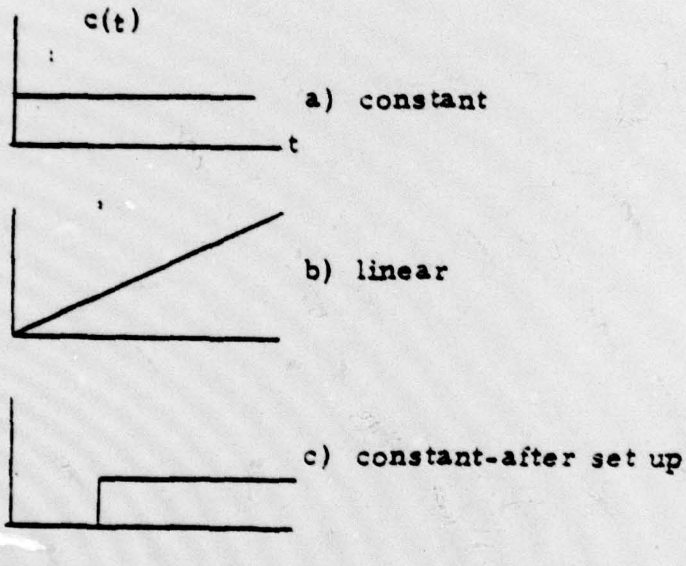

Figure 2:

Reward Rate Time Functions .

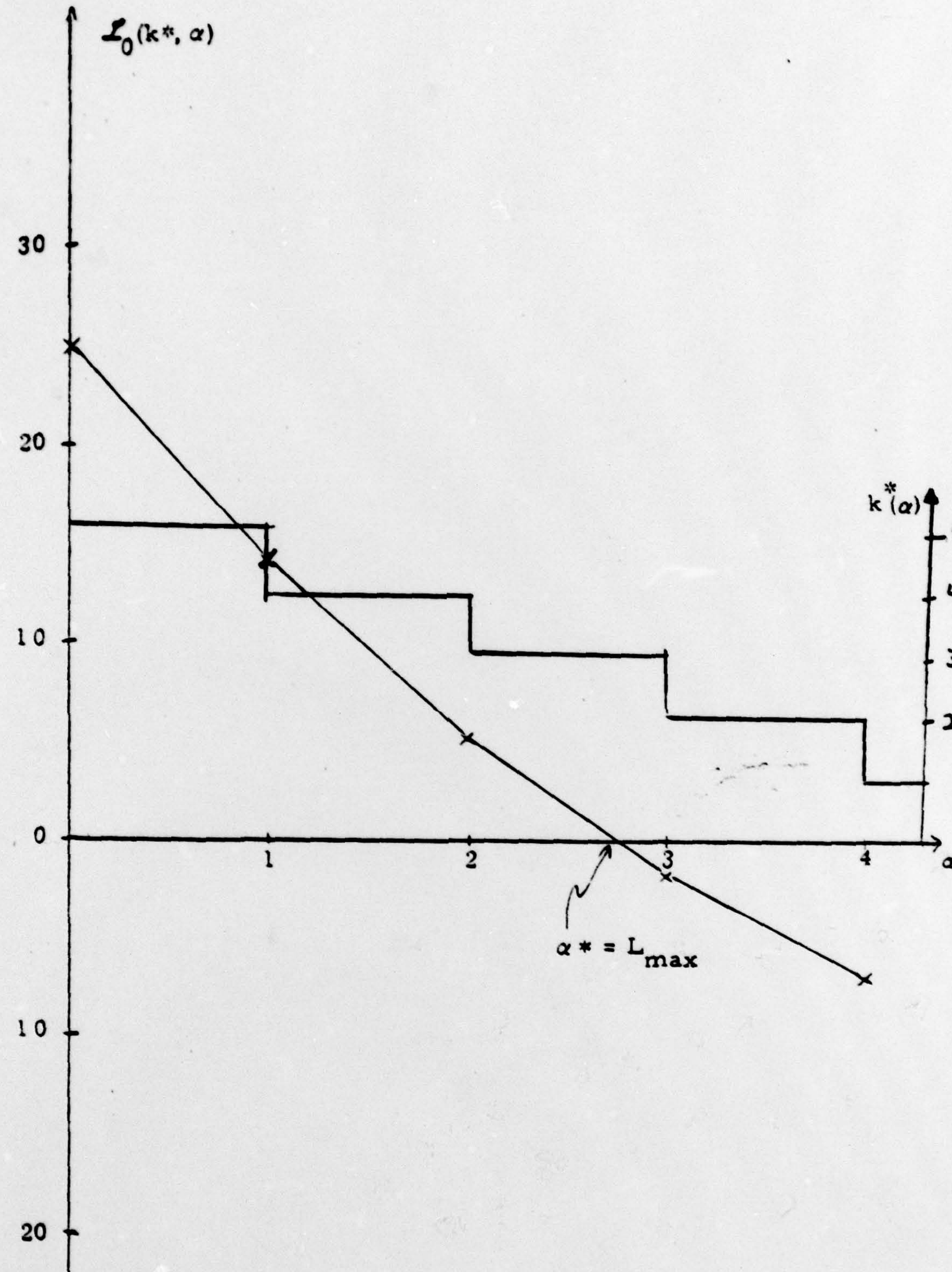

Figure 3:
Optimal Reward Search:
Constant Reward Rate Case.

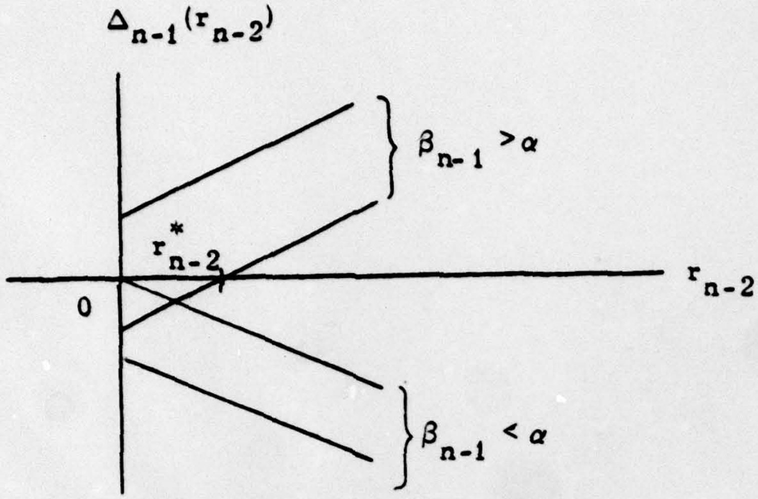

Figure 4:

Possible Shapes for $\Delta_{n-1}(r_{n-2})$.

APPENDIX

Dependence Relationships Among Multivariate Exponential Variables

Many multivariate distributions have been described and applied to reliability problems [4, 8, 10]. In each case the marginal univariate distributions are of the negative exponential form. Properties of the distribution used here are most easily derived by exploiting its relationship to multivariate normal distributions [3, 5].

The multivariate exponential variables $r_1, r_2, \ldots, r_n$ can be viewed as sums of squares:

$$r_i = w_i^2 + z_i^2, \qquad \text{(A-1)}$$

where $\underline{w}$ and $\underline{z}$ are independent, zero mean, identically distributed normal vectors, each with covariance matrix Γ. It follows that the r_i have exponential marginal distributions with

$$E[r_i] = 2\,\gamma_{ii} \qquad \text{(A-2)}$$

$$\rho_{r_i r_j} = [\rho_{w_i w_j}]^2.$$

We specialize to the case where the underlying normal sequences $\{w_i\}$ and $\{z_i\}$ are Markovian

$$\gamma_{ij} = \sqrt{\gamma_{ii}\,\gamma_{jj}}\;\rho^{|i-j|} \qquad \text{(A-3)}$$

and find that $\{r_i\}$ is also Markov with the joint density

$$f(r_0, r_1, r_2, \ldots, r_{n-1}) = \left[(1-\rho)^{n-1} \prod_{i=0}^{n-1} \eta_i\right]^{-1} \cdot \prod_{i=0}^{n-2} I_0\left[\frac{2}{1-\rho}\sqrt{\frac{\rho}{\eta_i \eta_{i+1}}}\sqrt{r_i r_{i+1}}\right]$$

$$\cdot \exp\left[-\frac{1}{1-\rho}\left(\frac{r_0}{\eta_0} + \frac{r_{n-1}}{\eta_{n-1}} + \sum_{i=1}^{n-2} \frac{r_i(1+\rho)}{\eta_i}\right)\right]; \; n > 2, \tag{A-4}$$

Equation (11) uses the modified Bessel function $I_0(\ )$ and the notations $E[r_i] = \eta_i$ and $\rho_{r_i, r_{i+1}} = \rho$. (When n = 2, the summation in exp ( ) vanishes.)

The conditional density is easily shown to satisfy the Markov property and [5]

$$f(r_i | r_{i-1}) = [\eta_i(1-\rho)]^{-1} \exp\left[-\frac{1}{(1-\rho)} \frac{r_i}{\eta_i} + \frac{\rho r_{i-1}}{\eta_{i-1}}\right] \cdot I_0\left(\frac{2}{1-\rho}\sqrt{\frac{\rho r_i r_{i-1}}{\eta_i \eta_{i-1}}}\right) \tag{A-5}$$

with

$$E[r_i | r_{i-1}] = \eta_i + (r_{i-1} - \eta_{i-1})\rho \;\; \eta_i / \eta_{i-1}. \tag{A-6}$$

$$Var[r_i | r_{i-1}] = \eta_i^2[(1-\rho)^2 + 2\rho(1-\rho) r_{i-1}/\eta_{i-1}] . \tag{A-7}$$

These conditional moments show, e.g., that the conditional mean of r_i exceeds its mean in proportion to the amount by which r_{i-1} exceeds its mean, and that conditional mean is a linearly increasing function of r_{n-1}.

The Dynamic Programming arguments used here required calculations of conditional expectations based on (A-5). As is often the case [2], the total positivity properties of $f(r_i | r_{i-1})$ are very useful for determining structural properties of the optimal policy.

It is straightforward to show that both $f(r_i, r_{i-1})$ and $f(r_i | r_{i-1})$ are totally positive of all orders (TP_∞), [5, 7]. This means, for $f(r_i, r_{i-1})$, that the following determinants are non-negative for any N and any

$\alpha_1 < \alpha_2 \ldots < \alpha_N;\ \beta_1 < \beta_2 \ldots < \beta_N.$

$$\begin{vmatrix} f(\alpha_1, \beta_1) & f(\alpha_1, \beta_2) \ldots & f(\alpha_1, \beta_N) \\ \vdots & \vdots & \vdots \\ f(\alpha_N, \beta_1) \ldots & \ldots\ldots\ldots & f(\alpha_N, \beta_N) \end{vmatrix} \geq 0$$

THEOREM: If $h(y)$ is continuous and convex, and satisfies the bounds

i) $h(0) \leq 0$

ii) $|h(y)| \leq a + b\,y^{2m}$; $a > 0$, $b > 0$, $y > 0$, m = positive integer.

Then $g(x)$ is continuous, convex, bounded in the sense

$|g(x)| \leq a' + b'\,x^{2m}$; $a' > 0$, $b' > 0$, $x > 0$;

and belongs to one of the three following categories:

(I) $g(x) \geq 0$ for all $x \geq 0$,

(II) $g(x) \leq 0$ for all $x \geq 0$ except with a possible zero at $x = 0$,

(III) there exists a unique x^*, $0 < x^* < \infty$, such that $g(x) > 0$ for all $x > x^*$; and $g(x) < 0$ for $x < x^*$ except for a possible zero at $x = 0$.

This theorem is used to define optimal decision regions according to the sign of a function like $g(x)$, with x^* corresponding to a decision threshold.