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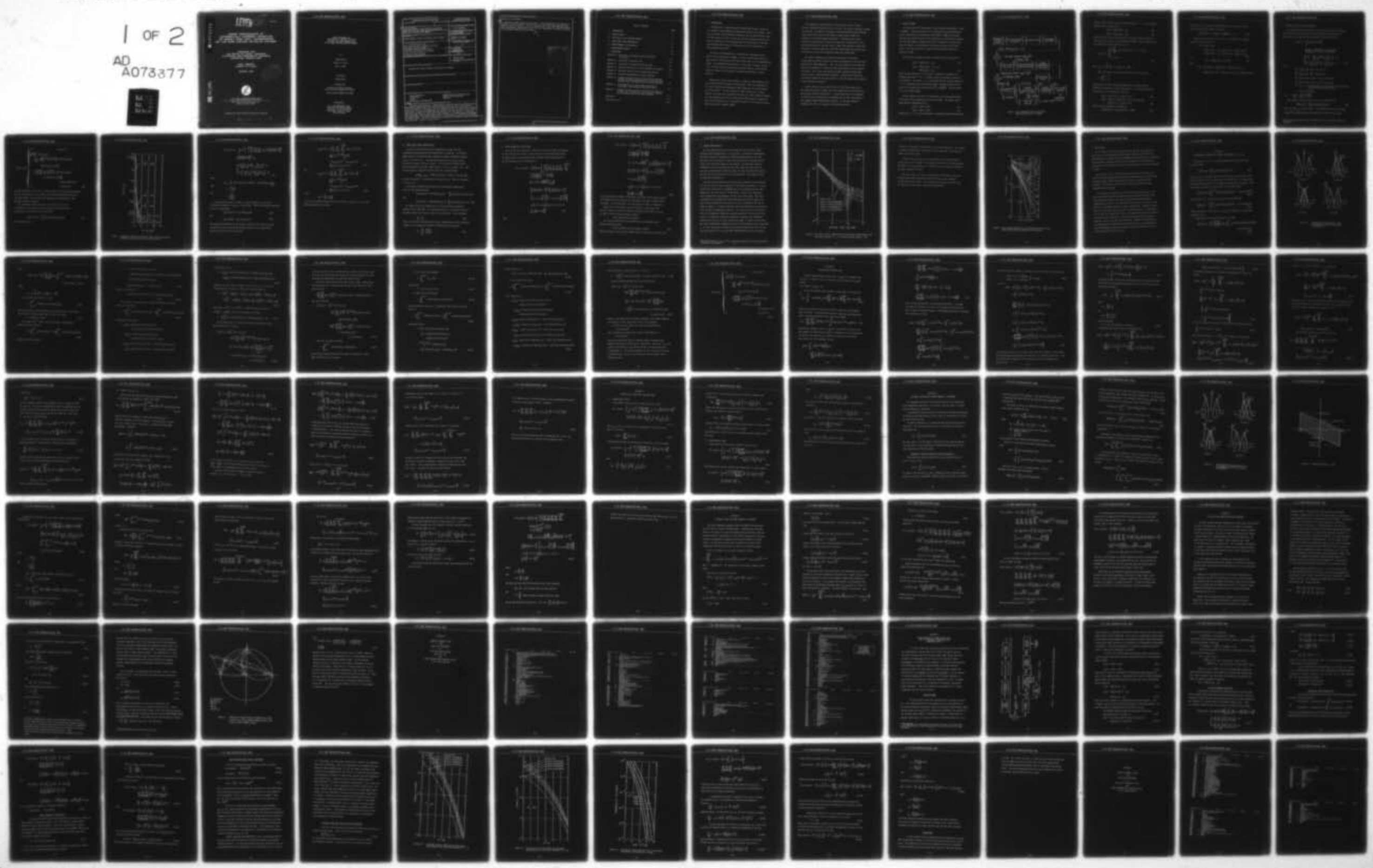

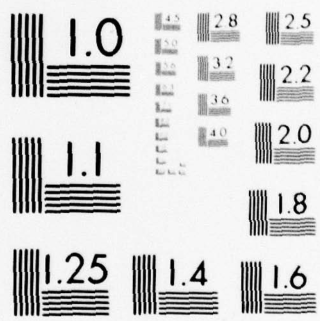


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# ERROR PERFORMANCE OF DIFFERENTIALLY COHERENT DETECTION OF BINARY DPSK DATA TRANSMISSION ON THE HARD-LIMITING SATELLITE CHANNEL

PREPARED FOR
THE OFFICE OF NAVAL RESEARCH
STATISTICS AND PROBABILITY PROGRAM
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AUGUST 1979





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# ERROR PERFORMANCE OF DIFFERENTIALLY COHERENT DETECTION OF BINARY DPSK DATA TRANSMISSION ON THE HARD-LIMITING SATELLITE CHANNEL

by

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The error performance of a differentially coherent detection method of binary differential phase-shift keying (DPSK) system operating over a hard-limiting satellite channel is derived. The main objective of our derivation was to observe the extent of error rate degradation of a DPSK system due to unequal power levels and noise correlations between the two adjacent time slot pulses. In practice, the power imbalance, or equivalently SNR imbalance, is induced by such phenomena as intersymbol interference and/or errors in phase delay in the "one-bit-delayed" reference input. The noise correlation is a practical assumption since the noise


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is necessarily band limited in the system. Error probabilities are given as a function of uplink SNR with different levels of SNR imbalances and different downlink SNR as parameters. It is discovered that, while SNR imbalance affects error performance, the probability of error is independent of noise correlation if the symbol probabilities are equal.

TABLE OF CONTENTS

## 1. INTRODUCTION

Differentially encoded phase-shift keyed (DPSK) binary signalling is a commonly used technique in digital data transmission systems. There are, of course, other modulation techniques that are more efficient than the DPSK systems in terms of power and spectral occupancy; however, the circuit simplicity and the accompanying cost-effectiveness of the DPSK system often becomes the over-riding reason for its application in a data communication system.

The choice of DPSK modulation is particularly appropriate when a small, low-cost modulator is needed. A typical application might be a buoy which has limited space and limited prime power for the communications system. An expendable platform such as a buoy must, of course, be low-cost. The data transmission from the buoy may involve a hard-limiting satellite channel for the purpose of processing the data at a shore facility.

Surprising as it may sound, however, a complete understanding of the error behavior of DPSK system has been lacking. Much less understood is the error performance mechanism of a DPSK system over the hard-limiting satellite channel. The error behavior of DPSK system over a "linear channel" has always been assessed on the basis of a most simplified fashion, e.g. the probability of error for the binary DPSK system has always been based on the expression $P_e = \frac{1}{2}\exp(-H^2)$, where $H^2$ is the carrier-to-noise power ratio at the sampling instant.

The demodulator configuration of a DPSK system requires a phase-detector (multiplier) whose inputs are "direct" and "one-bit delayed" versions of the signal. In reality, the inputs of the phase-detector have different signal power levels due to such effects as phase error, delay line attenuation, and inter-symbol interference. Furthermore, the noises between the two multiplier inputs are necessarily correlated due to the band-limited nature of any real system. When these practical aspects are considered, the error behavior of the DPSK system cannot be assessed only on the basis of the error probability expression stated above.

The lack of adequate error behavior expressions for the DPSK system under non-ideal conditions has been a consequence of the non-linear nature of the DPSK demodulator. Thus, the noise behavior at the decision time is not amenable to a relatively simple description such as coherent PSK system. As a result, the analysis of DPSK system has always been based on a most simplified ideal assumption. In particular, the noise correlation between the two phase-detector inputs has been totally ignored in the idealized assumptions.

In this report we have treated the case of the DPSK system over a hard-limiting satellite channel with power imbalance and correlated noise at the phase detector (multiplier). A surprising result of our analysis is that the error performance is independent of the noise correlation if the *a priori* symbol probabilities are equal. As has been remarked elsewhere [1], this is a departure from previous beliefs.

## 2. ANALYSIS MODEL

The binary satellite communication system under consideration is shown in Figure 1. Our main objective is to determine the error performance of the system under the assumptions that $(SNR)_1 \neq (SNR)_2$ at the phase detector input and the noises $n_1(t)$ and $n_2(t)$ are statistically dependent. The condition $(SNR)_1 \neq (SNR)_2$ arises from a difference in signal powers due, for example, to intersymbol interference or delay circuit phase error. The noise correlation reflects the band-limited nature of a practical system.

The original transmitted signals are defined as $S_1(t)$ and $S_2(t)$:

$$S_1(t) = \sqrt{2P_u}\cos(\omega t - \theta_1) \tag{1}$$

$$\begin{aligned} S_2(t) &= \sqrt{2P_u}\cos[\omega(t - T) - \theta_2] \\ &= \sqrt{2P_u}\cos(\omega t - \theta_2) \end{aligned} \tag{2}$$

where the bit duration is T; the carrier frequency is assumed to be selected such that $\omega T = 2\pi k$, k integer; and $P_u$ is the power received at the satellite. The index i = 1 is associated with the present symbol and i = 2 is associated with the previous symbol. Bandpass Gaussian noise $n_u(t)$ is also present on the uplink with $\sigma_u^2 = E\left\{n_u^2(t)\right\}$. The uplink SNR is defined as $R_u^2 \triangleq P_u/\sigma_u^2$.

At the receiver the signal is corrupted both by passage through the hard limiter and by additive noise on the downlink. The inputs to the phase detector (multiplier) are:

$$u_1(t) = s_1(t) + n_1(t) \tag{3}$$

$$u_2(t) = s_2(t) + n_2(t) \tag{4}$$

where $s_i(t)$, i = 1, 2, are the signals as corrupted by passage through the

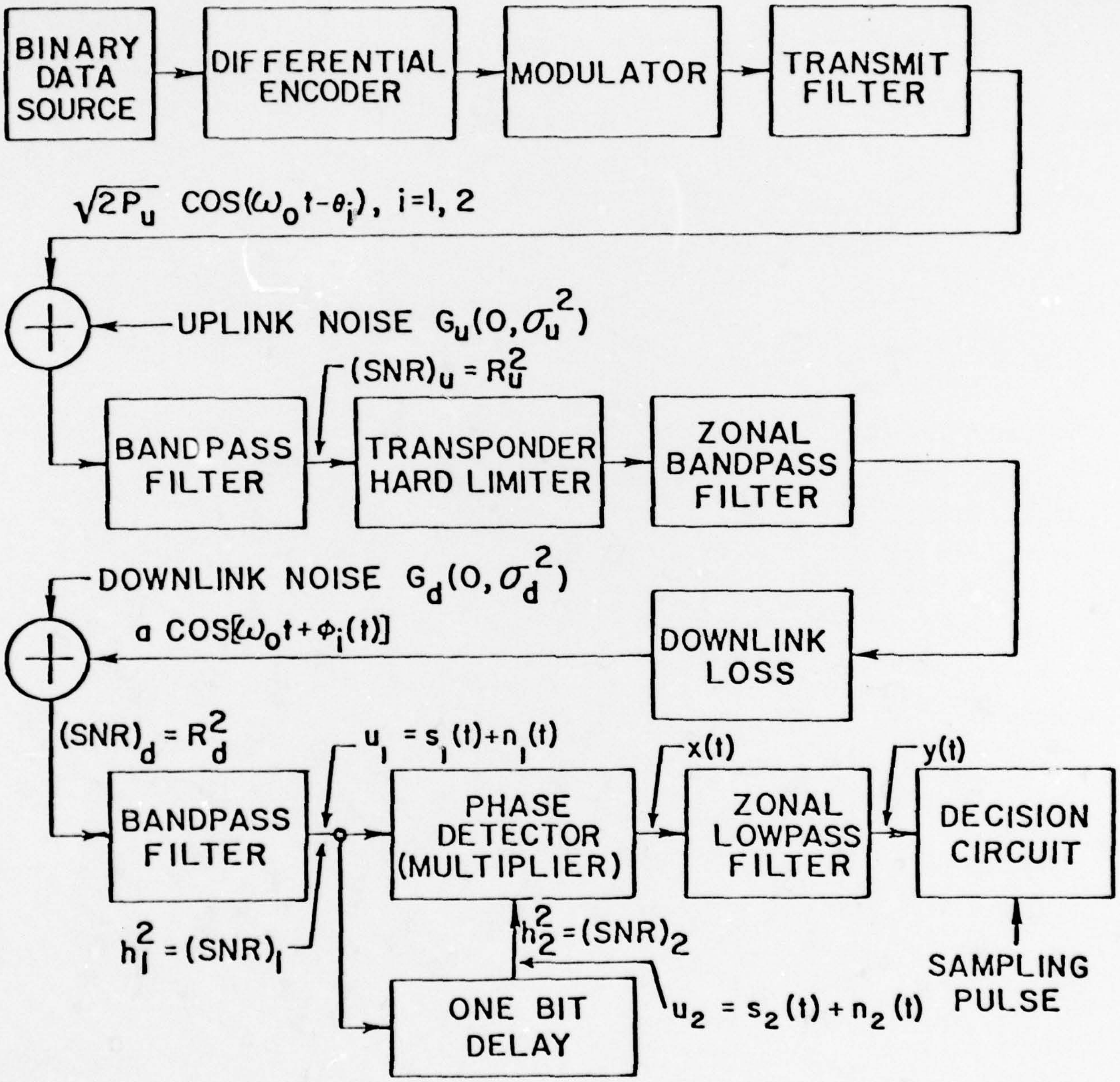


FIGURE 1 BLOCK DIAGRAM OF DPSK SYSTEM OVER A HARD-LIMITING SATELLITE CHANNEL

limiter in the presence of uplink noise and $n_i(t)$, $i = 1, 2$, are additive downlink noise. We may write

$$s_1(t) = \sqrt{2P_1}\cos(\omega t + \phi_1) \tag{5}$$

$$s_2(t) = \sqrt{2P_2}\cos(\omega t + \phi_2) \tag{6}$$

where $P_1$ and $P_2$ are the carrier powers at the phase detector and the signal phases $\phi_1$ and $\phi_2$ are identically distributed random variables with the conditional density function [2], [3]

$$f_{\phi_i}(\alpha|\theta_i) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\sum_{k=0}^{\infty} b_k \cos[k(\alpha-\theta_i)], \; i=1,2, & |\alpha-\theta_i|\le\pi \\ 0, \; |\alpha-\theta_i|>\pi \end{cases} \tag{7}$$

where $b_k = \varepsilon_k \dfrac{R_u^k}{k!}\Gamma(\frac{k}{2}+1)\ {}_1F_1(\frac{k}{2}; k+1; -R_u^2)$,

$R_u^2$ is the signal to noise ratio at the input to the limiter,

$$\varepsilon_k = \begin{cases} 1, & k = 0 \\ 2, & k > 0 \end{cases}$$

and ${}_1F_1(a;b;z)$ is the confluent hypergeometric function (Kummer's function) of parameters a and b and argument z. The bandpass Gaussian downlink noises are expressed in the form

$$n_1(t) = X_1(t)\cos\omega t + Y_1(t)\sin\omega t \tag{8}$$

$$\begin{aligned} n_2(t) &= X_1(t-T)\cos[\omega(t-T)] + Y_1(t-T)\sin[\omega(t-T)] \\ &\triangleq X_2(t)\cos\omega t + Y_2(t)\sin\omega t \end{aligned} \tag{9}$$

and the noise correlation is defined by

$$E\{n_1(t)n_2(t)\} = E\{n_1(t)n_1(t-T)\} = \sigma_d^2\rho(T) \tag{10}$$

where $\rho(T)$ is the normalized correlation with T equal to the symbol duration and the noise power is given by

$$\sigma_d^2 \triangleq E\{n_1^2(t)\} = E\{X_i^2(t)\} = E\{Y_i^2(t)\},\ i = 1,\ 2. \tag{11}$$

The decision variable y(t) at time t is the output of the zonally low-pass filtered version of

$$\begin{aligned} x(t) &= u_1(t) \times u_2(t) \\ &= [\sqrt{2P_1} \cos(\omega t - \phi_1) + X_1(t) \cos \omega t + Y_1(t) \sin \omega t] \\ &\times [\sqrt{2P_2} \cos(\omega t - \phi_2) + X_2(t) \cos \omega t + Y_2(t) \sin \omega t], \end{aligned} \tag{12}$$

i.e.

$$y(t) = \sqrt{P_1 P_2} \cos(\phi_1 - \phi_2) + N(t) \tag{13}$$

where

$$\begin{aligned} N(t) &= \tfrac{1}{2}X_1(t)X_2(t) + \tfrac{1}{2}Y_1(t)Y_2(t) + \sqrt{P_2/2}\, X_1(t) \cos \phi_2 \\ &+ \sqrt{P_2/2}\, Y_1(t) \sin \phi_2 + \sqrt{P_1/2}\, X_2(t) \cos \phi_1 + \sqrt{P_1/2}\, Y_2(t) \sin \phi_1. \end{aligned}$$

## 3. STATISTICS OF THE DECISION VARIABLE

The probability density function of the decision variable conditioned on the phase difference between the two inputs to the multiplier is a special case of the general result obtained by Miller and Lee [4]:

$$f_y(y;\rho|\phi) = \frac{1}{\sigma_d^2} \exp\left[-(h_3^2 + h_4^2)\right]$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \left[\left(\frac{1-\rho}{2}\right)h_3^2\right]^m \frac{1}{n!} \left[\left(\frac{1+\rho}{2}\right)h_4^2\right]^n$$

$$\cdot \begin{cases} \exp\left[-\dfrac{2y}{\sigma_d^2(1+\rho)}\right] G_m^n\left[\dfrac{4y}{\sigma_d^2(1-\rho^2)}\right], & y \geq 0 \\ \exp\left[\dfrac{2y}{\sigma_d^2(1-\rho)}\right] G_n^m\left[-\dfrac{4y}{\sigma_d^2(1-\rho^2)}\right], & y < 0 \end{cases} \tag{14}$$

where

$h_3^2 = \frac{1}{2(1+\rho)} (h_1^2 + h_2^2 + 2h_1h_2 \cos\phi)$,

$h_4^2 = \frac{1}{2(1-\rho)} (h_1^2 + h_2^2 - 2h_1h_2 \cos\phi)$,

$h_1^2$ = SNR at input number 1 of the multiplier,

$h_2^2$ = SNR at input number 2 of the multiplier,

$\phi = \phi_1 - \phi_2$ = the difference in phase between the narrow-band signals at the multiplier input,

$\rho$ = noise correlation,

and $$G_m^n(x) = \sum_{j=0}^{m} \binom{m+n-j}{n} \frac{x^j}{j!}.$$

The probability density function of $\phi$ can be obtained from

$$f_\phi(\beta|\theta_1,\theta_2) = \int_{-\infty}^{\infty} f_{\phi_1}(\beta+\alpha|\theta_1) f_{\phi_2}(\alpha|\theta_2)\,d\alpha \tag{15}$$

where $f_{\phi_1}$ and $f_{\phi_2}$ are given by (7). The evaluation* of (15) is straightforward but somewhat lengthy (see Appendix A), with the result being

* For an alternate approach not requiring explicit evaluation of (15), see Appendix D.

$$
f_\phi(\beta|\theta_1,\theta_2) = \begin{cases} 0, & -\infty<\beta\le\theta_1-\theta_2-2\pi \\[2mm] \left(\frac{1}{2\pi}\right)^2 \Bigg\{[2\pi-|\beta-(\theta_1-\theta_2)|] \\ \quad + \sum\limits_{k=1}^{\infty} b_k^2 \Bigg\{\left[\frac{2\pi-|\beta-(\theta_1-\theta_2)|}{2}\right] \cos[k|\beta-(\theta_1-\theta_2)|] \\ \quad\quad - \frac{1}{2k}\sin[k|\beta-(\theta_1-\theta_2)|]\Bigg\} \\ \quad -2\sum\limits_{k=1}^{\infty}\sum\limits_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2}\Big\{k\sin[k|\beta-(\theta_1-\theta_2)|] \\ \quad\quad -q\sin[q|\beta-(\theta_1-\theta_2)|]\Big\}\Bigg\}, \\ & \theta_1-\theta_2-2\pi\le\beta\le\theta_1-\theta_2+2\pi \\[2mm] 0, & \theta_1-\theta_2+2\pi\le\beta<\infty. \end{cases} \tag{16}
$$

An important aspect of the p.d.f. given in (16) is that $f_\phi(\beta|\theta_1,\theta_2)$ depends only on the difference $\theta_1-\theta_2$, which is the information-bearing parameter in a DPSK system. We can thus write $f_\phi(\beta|\theta_1,\theta_2)$ as $f_\phi(\beta|\theta_1-\theta_2)$. This density is shown in Figure 2.

The probability density function of the decision variable y conditioned on the transmitted symbol $(\theta_1-\theta_2)$ with downlink noise correlation as a parameter can be found from

$$
f_y(y;\rho|\theta_1-\theta_2) = \int_{-\infty}^{\infty} f_y(y;\rho|\phi=\beta) f_\phi(\beta|\theta_1-\theta_2)\,d\beta. \tag{17}
$$

The result of (17) is

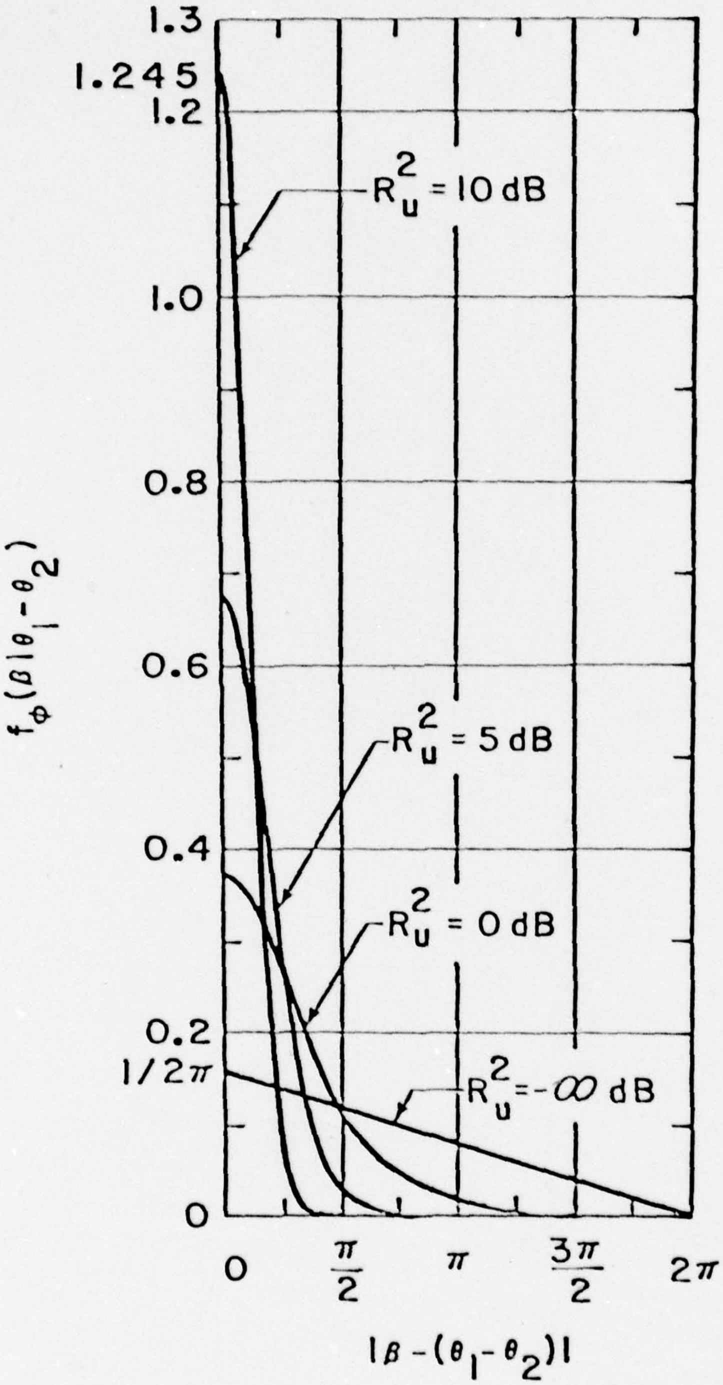


FIGURE 2 PROBABILITY DENSITY FUNCTION OF PHASE DIFFERENCE BETWEEN CONSECUTIVE SYMBOLS AT OUTPUT OF HARD LIMITER

$$f_y(y;\rho|\theta_1-\theta_2) = \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2+h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!}\frac{1}{n!}\left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)\left(h_1^2+h_2^2\right)\right]^m$$

$$\cdot\left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)\left(h_1^2+h_2^2\right)\right]^n V_{m,n}$$

$$\cdot\begin{cases} \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] G_n^m\left[-\frac{4y}{\sigma_d^2(1-\rho^2)}\right], & y<0 \\ \exp\left[-\frac{2y}{\sigma_d^2(1+\rho)}\right] G_m^n\left[\frac{4y}{\sigma_d^2(1-\rho^2)}\right], & y\geq 0 \end{cases} \tag{18}$$

where

$$V_{m,n} = \int_{-\infty}^{\infty} \exp[Y\cos\beta][1+Z\cos\beta]^m[1-Z\cos\beta]^n f_\phi(\beta|\theta_1-\theta_2)d\beta \tag{19}$$

with

$$Y = \frac{2\rho h_1 h_2}{1-\rho^2}$$

and

$$Z = \frac{2h_1 h_2}{h_1^2+h_2^2}.$$

The transmitted symbol is "mark" or "space" depending on the phase difference $|\theta_1-\theta_2|=0$ or $|\theta_1-\theta_2|=\pi$, respectively. Thus the following conditional p.d.f.'s are obtained:

$$f_y(y;\rho|\text{space}) = f_y(y;\rho||\theta_1-\theta_2|=0) \tag{20a}$$

and

$$f_y(y;\rho|\text{mark}) = f_y(y;\rho||\theta_1-\theta_2|=\pi). \tag{20b}$$

We evaluate (see Appendix B) the integral in (19) for both space ($\theta_1-\theta_2=0$) and mark ($|\theta_1-\theta_2|=\pi$) by using the binomial theorem and an expansion for powers of the cosine, with the results:

$$V_{m,n}(\text{space}) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{n}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}} \binom{s}{[s-\upsilon(s)-2\mu]/2}$$

$$\cdot\left(\frac{Z}{2}\right)^{s}(-1)^{\upsilon(s)}\,\frac{b_k^2\,\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}$$

$$\cdot\,[I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y)]$$

$$\cdot\binom{n}{s}\;{}_2F_1(-m,-s;n-s+1;-1) \tag{21a}$$

and

$$V_{m,n}(\text{mark}) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{m}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}} \binom{s}{[s-\upsilon(s)-2\mu]/2}$$

$$\cdot\left(\frac{Z}{2}\right)^{s}(-1)^{k}\,\frac{b_k^2\,\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}$$

$$\cdot\,[I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y)]$$

$$\cdot\binom{m}{s}\;{}_2F_1(-n,-s;m-s+1;-1) \tag{21b}$$

where

$$\upsilon(s) = \begin{cases} 0, & s \text{ even} \\ 1, & s \text{ odd;} \end{cases}$$

$I_n(\cdot)$ is the modified Bessel function of order n; and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function.

## 4. CONDITIONAL ERROR PROBABILITIES

In order to determine the error probability, we must know the threshold to which the decision variable $y(t)$ is compared. In a binary DPSK system the transmitted phase difference between consecutive symbols, $\theta_1-\theta_2$, is either 0 or $\pm\pi$. The threshold should be set so that correct decisions are made if noise is absent. In the absence of uplink noise the limiter does not corrupt signal phase, i.e. $\phi_1 = \theta_1$ and $\phi_2 = \theta_2$. Then in the absence of downlink noise the decision variable becomes

$$y(t)\Big|_{\text{no noise}} = \sqrt{P_1P_2}\cos(\phi_1-\phi_2) = \sqrt{P_1P_2}\cos(\theta_1-\theta_2) \quad (22)$$

and the information is contained in the sign of $y(t)$. Thus the threshold is set at zero.

From (20a) and (20b) one can obtain the conditional probabilities of error from the expressions

$$P(e;\rho|\text{space}) = \text{Prob}\{y<0|\theta_1-\theta_2=0\} = \int_{-\infty}^{0} f_y(y;\rho|\theta_1-\theta_2=0)\,dy \quad (23a)$$

and

$$P(e;\rho|\text{mark}) = \text{Prob}\{y>0||\theta_1-\theta_2|=\pi\} = \int_{0}^{\infty} f_y(y;\rho||\theta_1-\theta_2|=\pi)\,dy. \quad (23b)$$

To compute the error probabilities we need to define the downlink signal to noise ratio $R_d^2$. In a DPSK system this is conveniently taken as the SNR at the direct channel input to the multiplier. In our notation,

$$R_d^2 = h_1^2. \quad (24)$$

Also, since we are considering the case of SNR imbalance at the multiplier inputs, it is convenient to define an SNR difference measure by

$$\lambda^2 = \frac{h_2^2}{h_1^2} = \frac{(\text{SNR})_2}{(\text{SNR})_1}. \quad (25)$$

## 5. ERROR PROBABILITY EXPRESSIONS

Carrying out the integrations indicated in (23a) and (23b) (see Appendix C) and using the definitions in (24) and (25) to replace $h_1^2$ by $R_d^2$ and $h_2^2$ by $\lambda^2 R_d^2$ we obtain the following rather formidable-appearing expressions for the conditional error probabilities:

$$P(e;\rho,\lambda|\text{space}) = \pi\left(\frac{1-\rho}{4}\right) \exp\left[-\frac{2R_d^2}{1-\rho^2}\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{n} \sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}}$$

$$\frac{\varepsilon_k \varepsilon_{\mu+\upsilon(s)} (-1)^{\upsilon(s)}}{m!n!\left[\Gamma\left(\frac{k+1}{2}\right)\right]^2} \binom{n+m}{m} \binom{n}{s}$$

$$\cdot \binom{s}{[s-\upsilon(s)-2\mu]/2} \left(\frac{R_u}{2}\right)^{2k} \left(\frac{\lambda}{1+\lambda^2}\right)^s$$

$$\cdot \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right) R_d^2(1+\lambda^2)\right]^m \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right) R_d^2(1+\lambda^2)\right]^n$$

$$\cdot \left[I_{2\mu+\upsilon(s)-k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right) + I_{2\mu+\upsilon(s)+k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\right]$$

$$\cdot {}_2F_1\left(-n,1;-n-m;\frac{2}{1+\rho}\right) {}_2F_1(-m,-s;n-s+1;-1)$$

$$\cdot \left[{}_1F_1\left(\frac{k}{2};k+1;-R_u^2\right)\right]^2 \qquad (26a)$$

and

$$P(e;\rho,\lambda|\text{mark}) = \pi\left(\frac{1+\rho}{4}\right)\exp\left[-\frac{2R_d^2}{1-\rho^2}\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{m}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}}$$

$$\frac{\epsilon_k\epsilon_{\mu+\upsilon(s)}(-1)^k}{m!n!\left[\Gamma\left(\frac{k+1}{2}\right)\right]^2}\binom{m+n}{n}\binom{m}{s}$$

$$\cdot\binom{s}{[s-\upsilon(s)-2\mu]/2}\left(\frac{R_u}{2}\right)^{2k}\left(\frac{\lambda}{1+\lambda^2}\right)^s\left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)R_d^2(1+\lambda^2)\right]^m$$

$$\cdot\left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)R_d^2(1+\lambda^2)\right]^n\left[I_{2\mu+\upsilon(s)-k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\right.$$

$$\left.+\;I_{2\mu+\upsilon(s)+k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\right]$$

$$\cdot\;{}_2F_1\left(-m,1;-m-n;\frac{2}{1-\rho}\right){}_2F_1(-n,-s;m-s+1;-1)$$

$$\cdot\left[{}_1F_1\left(\frac{k}{2};k+1;-R_u^2\right)\right]^2. \qquad (26b)$$

In writing (26a) and (26b), we have made use of the relation $(2z)!=\Gamma(2z+1)=2^{2z}\Gamma(z+\frac{1}{2})\Gamma(z+1)/\sqrt{\pi}$ to simplify the coefficients arising from $b_k^2$ in (21a) and (21b). For alternate ways of writing (26a) and (26b), see Appendix E.

Making use of the relation $I_j(-z)=(-1)^jI_j(z)$, j an integer, and making a change of notation in (26b) by replacing m by n and vice versa, we observe the interesting symmetry property

$$P(e;\rho,\lambda|\text{space})=P(e;-\rho,\lambda|\text{mark}). \qquad (27)$$

The total unconditional error probability is the weighted sum of (26a) and (26b) given by

$$P(e;\rho,\lambda)=P_sP(e;\rho,\lambda|\text{space})+P_mP(e;\rho,\lambda|\text{mark}) \qquad (28)$$

where $P_s$ and $P_m$ are the a priori probabilities of space ("0") and mark ("1").

## 6. COMPUTATIONAL RESULTS

We have computed the total error probability for the case of equal *a priori* symbol probabilities, i.e. $P_m=P_s=1/2$. In doing the computations, the direct numerical computation of (26a) and (26b) proved to be quite difficult due to the manner in which the multiple series converge and the effects of finite word length in the computer. We found that numerical integration of (19) in conjunction with (18), (23a) and (23b) proved to be a more computationally efficient scheme for obtaining numerical results, and this was the procedure used in our computations. The program listing is contained in Appendix G.

Our numerical evaluations of the total error probability were performed for several values of power imbalance* $\lambda^2$ and noise correlation $\rho$. We have observed the result that the computed total error probabilities are independent of $\rho$ for all values of $\lambda^2$ considered. However, the conditional error probabilities were dependent upon the noise correlation. When these conditional error probabilities were added with equal weights ($P_m=P_s=1/2$), the results coincided with the value of the error probability for $\rho=0$, as shown in Figure 3. A similar numerical result was obtained for a terrestrial link [5] (see also Appendices H and I). As in the case of the terrestrial link, the mathematical complexity prevented analytical verification of this result. It should be stated here that when one observes the conditional probabilities of error as given by (26a) and (26b), it appears certain that the total unconditional probability of error should depend on noise correlation $\rho$. In fact, it has been remarked in the previous publications [6], [7] that the error probability of a binary DPSK system depends upon the correlation

*The power imbalance can arise from intersymbol interference and other effects, as discussed in Appendix F.

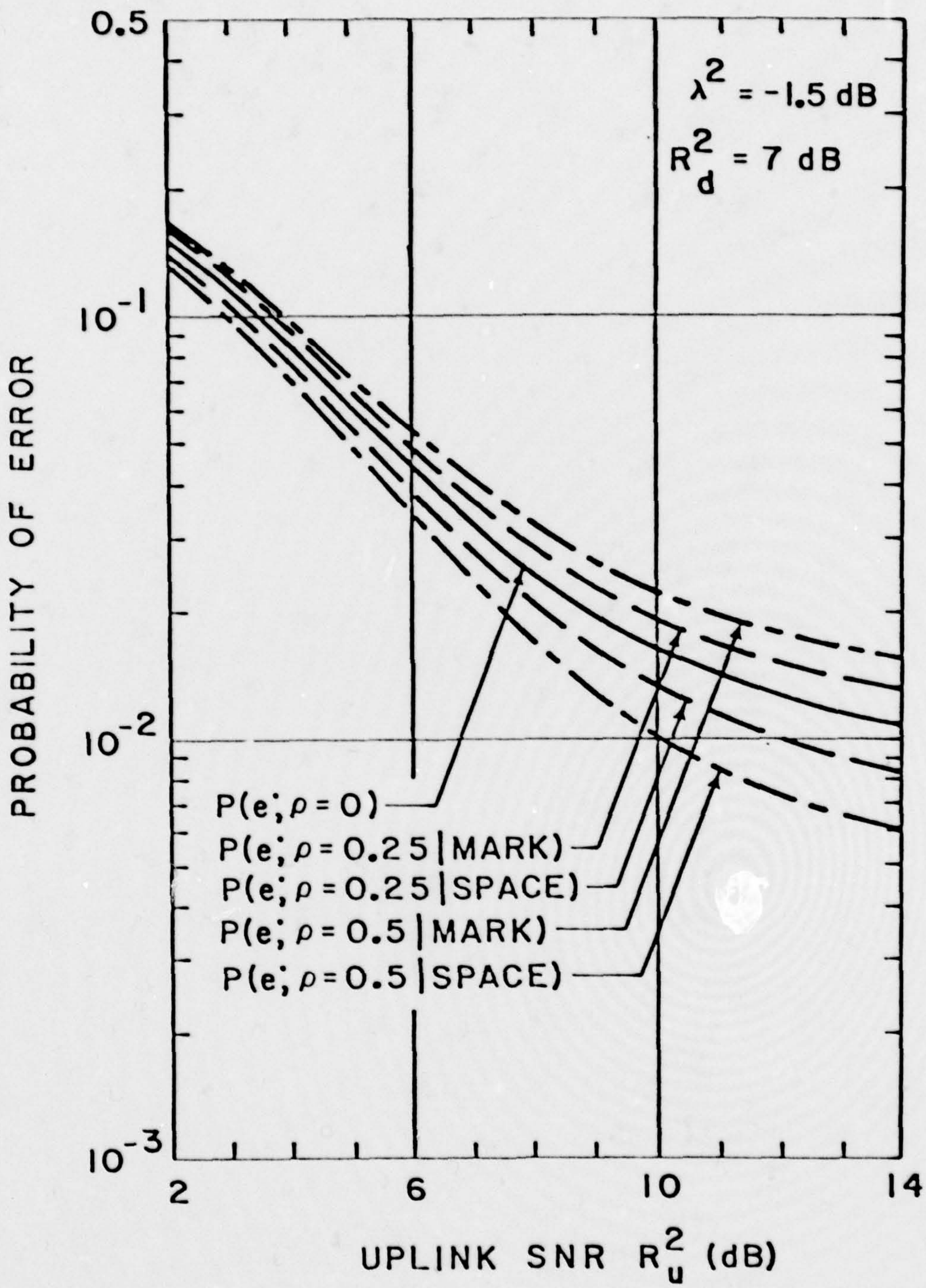


FIGURE 3 INFLUENCE OF NOISE CORRELATION ON CONDITIONAL ERROR PROBABILITIES FOR POWER IMBALANCE $\lambda^2$ = -1.5 dB AND DOWNLINK SNR $R_d^2$ = 7 dB

if there is intersymbol interference (a case of SNR imbalance). The remarks, it should be pointed out, were based on the observations of the equations rather than on the proofs.

Figure 4 shows the total error probability versus uplink SNR $R_u^2$ for several values of $\lambda^2$ and $R_d^2$. For comparison, the case of infinite downlink SNR ($R_d^2=\infty$) is also plotted; this is identical to the terrestrial link with no power imbalance ($\lambda^2$=0 dB).

The curves shown in Figure 4 corresponding to $\lambda^2$=0 dB for each case of $R_d^2$ (downlink SNR) represent the ideal case of no power imbalance at the phase detector input. Our curves for these special (ideal) cases are identical to the results reported by Weinberg [8].

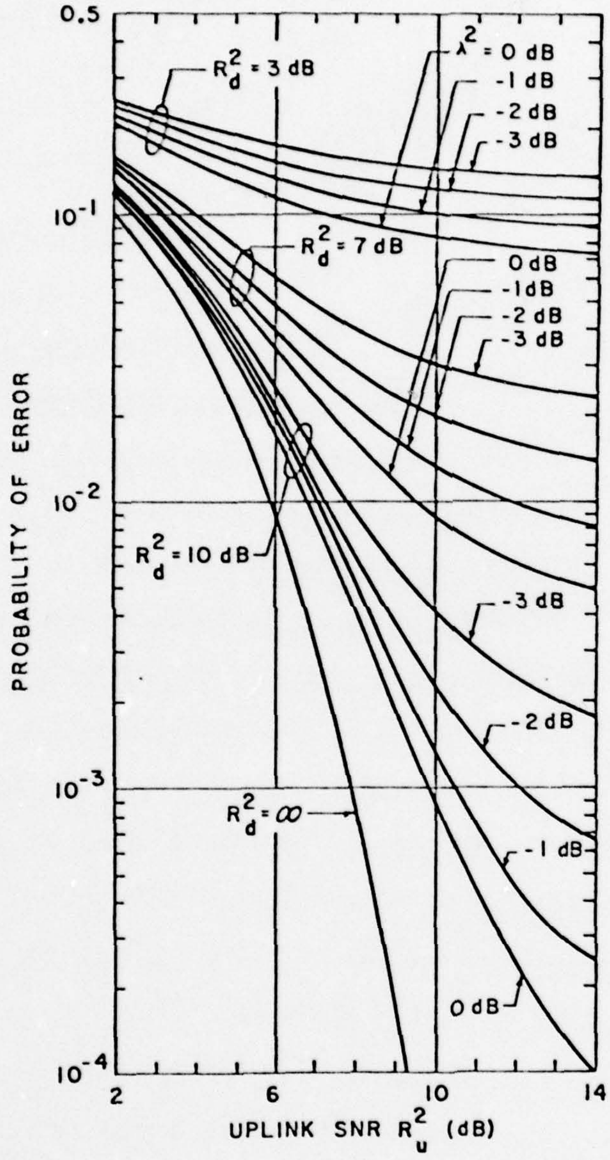


FIGURE 4 TOTAL ERROR PROBABILITY AS A FUNCTION OF UPLINK SNR WITH DOWNLINK SNR AND POWER IMBALANCE AS PARAMETERS

## 7. CONCLUSIONS

In this report we have presented the error behavior of a binary DPSK system over a hard-limiting satellite channel under the influence of an SNR imbalance at the phase detector (multiplier) inputs and noise correlation. The graphically presented curves for error probability are applicable to evaluation of system performance when the SNR imbalance at the phase detector is known.

Our numerical results show that the performance of binary DPSK over the hard-limiting satellite channel does not depend upon noise correlation when the *a priori* symbol probabilities are equal, regardless of SNR imbalance. The noise correlation has an effect only when the symbol probabilities are unequal.

The mathematical complexity of the error rate expressions has, thus far, prevented an analytical verification of the numerical results that the noise correlation has no effect on performance when the symbol probabilities are equal. The mathematical formulation for the error rate, however, is possibly amenable to further investigation. Appendix E points out a few first steps which may lead to further analytical investigation of the error rate properties; however, the mathematical relations involved are in an area of mathematics, e.g. generalized multiple hypergeometric functions, which is not yet fully developed. Further investigations may be mathematically interesting, but the directions of such investigations and the immediacy of practical results of such investigations are by no means clearly evident at this time.

## APPENDIX A

## DERIVATION OF PROBABILITY DENSITY FUNCTION OF $\phi = \phi_1 - \phi_2$

The pdf of $\phi_i$, $i=1,2$, is given by equation (7) of the main text. We must evaluate

$$f_\phi(\beta|\theta_1,\theta_2) = \int_{-\infty}^{\infty} f_{\phi_1}(\beta+\alpha|\theta_1) f_{\phi_2}(\alpha|\theta_2)d\alpha. \tag{A-1}$$

In view of the restricted range over which the $f_{\phi_i}$'s are non-zero, (A-1) must be treated as four separate cases (see Figure A-1). If $-\beta+\theta_1+\pi<\theta_2-\pi$, or $\theta_1-\theta_2+2\pi<\beta$, then there is no overlap of the two $f_{\phi_i}$'s and $f_\phi \equiv 0$. In the second case, we have $\theta_2-\pi<-\beta+\theta_1+\pi<\theta_2+\pi$, which implies that

$$f_\phi(\beta|\theta_1,\theta_2) = \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} f_{\phi_1}(\beta+\alpha|\theta_1) f_{\phi_2}(\alpha|\theta_2)d\alpha, \quad \theta_1-\theta_2 \le \beta \le \theta_1-\theta_2+2\pi. \tag{A-2}$$

In the third case, $-\beta+\theta_1-\pi \le \theta_2+\pi \le -\beta+\theta_1+\pi$, which implies that

$$f_\phi(\beta|\theta_1,\theta_2) = \int_{\theta_1-\pi-\beta}^{\theta_2+\pi} f_{\phi_1}(\beta+\alpha|\theta_1) f_{\phi_2}(\alpha|\theta_2)d\alpha, \quad \theta_1-\theta_2-2\pi \le \beta \le \theta_1-\theta_2 . \tag{A-3}$$

In the fourth case $-\beta+\theta_1-\pi>\theta_2+\pi$, or $\beta<\theta_1-\theta_2-2\pi$, there is no overlap and $f_\phi(\beta|\theta_1,\theta_2) \equiv 0$.

Substituting (7) into (A-3) and (A-2), we find

$$f_\phi(\beta|\theta_1,\theta_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{k=0}^{\infty}\sum_{q=0}^{\infty} b_k b_q \int_{\theta_1-\pi-\beta}^{\theta_2+\pi} \cos[k(\beta+\alpha-\theta_1)]\cos[q(\alpha-\theta_2)]d\alpha,$$

$$\theta_1-\theta_2-2\pi \le \beta \le \theta_1-\theta_2 \tag{A-4}$$

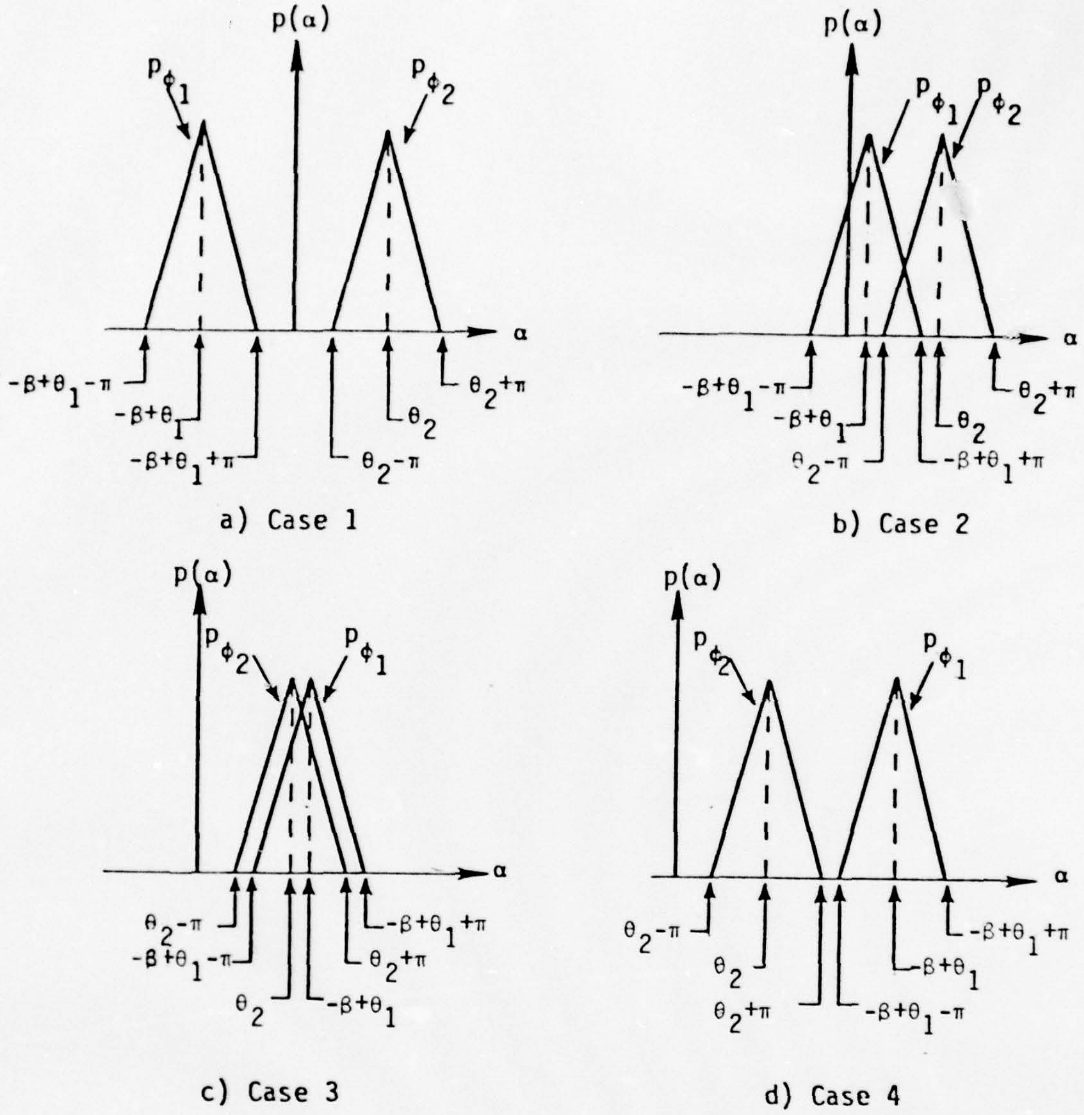


FIGURE A-1 DETERMINATION OF INTEGRATION LIMITS FOR PROBABILITY DENSITY OF $\phi_1-\phi_2$ AS A FUNCTION OF $\beta$

and

$$f_\phi(\beta|\theta_1,\theta_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{k=0}^{\infty}\sum_{q=0}^{\infty} b_k b_q \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos[k(\beta+\alpha-\theta_1)]\cos[q(\alpha-\theta_2)]\,d\alpha,$$

$$\theta_1-\theta_2\le\beta\le\theta_1-\theta_2+2\pi \qquad \text{(A-5)}$$

where

$$b_k = \varepsilon_k \frac{R_u^k}{k!}\,\Gamma\left(\frac{k}{2}+1\right)\ {}_1F_1\left(\frac{k}{2};\ k+1;\ -R_u^2\right).$$

Now consider the integral in (A-4):

$$I = \int_{\theta_1-\pi-\beta}^{\theta_2+\pi} \cos[k(\beta+\alpha-\theta_1)]\ \cos[q(\alpha-\theta_2)]\ d\alpha. \qquad \text{(A-6)}$$

We have three cases, determined by the parameters k and q. When $k=q=0$, the integrand of (A-6) becomes equal to one, and we have

$$I = 2\pi+\beta+\theta_2-\theta_1,\quad k=q=0. \qquad \text{(A-7)}$$

The second case is $k=q\neq0$, for which we apply a trigonometric identity to write (A-6) as

$$I = \frac{1}{2}\int_{\theta_1-\pi-\beta}^{\theta_2+\pi} \cos(k\beta+k\theta_2-k\theta_1)\ d\alpha + \frac{1}{2}\int_{\theta_1-\pi-\beta}^{\theta_2+\pi} \cos(2k\alpha+k\beta-k\theta_1-k\theta_2)\ d\alpha. \qquad \text{(A-8)}$$

Integrating (A-8) we obtain

$$I = \frac{1}{2}\cos(k\beta+k\theta_2-k\theta_1)\ (\theta_2+\pi-\theta_1+\pi+\beta)$$

$$+\frac{1}{4k}[\sin(2k\theta_2+2k\pi+k\beta-k\theta_1-k\theta_2) - \sin(2k\theta_1-2k\pi-2k\beta+k\beta-k\theta_1-k\theta_2)]$$

$$= \frac{1}{2}\cos\,[k\beta+k(\theta_2-\theta_1)]\ (2\pi+\beta+\theta_2-\theta_1)$$

$$+\frac{1}{4k}[\sin(k\beta+k\theta_2-k\theta_1) + \sin(k\beta+k\theta_2-k\theta_1)] \tag{A-9a}$$

which can be written as

$$I = \frac{1}{2}(2\pi+\beta+\theta_2-\theta_1)\cos[k\beta+k(\theta_2-\theta_1)] + \frac{1}{2k}\sin[k\beta+k(\theta_2-\theta_1)],$$

$$k=q\neq 0. \tag{A-9b}$$

In the third case, we have $k\neq q$. Using a trigonometric identity, (A-6) can be written as

$$I = \frac{1}{2}\int_{\theta_1-\pi-\beta}^{\theta_2+\pi}\cos[(k-q)\alpha+k\beta+q\theta_2-k\theta_1]d\alpha + \frac{1}{2}\int_{\theta_1-\pi-\beta}^{\theta_2+\pi}\cos[(k+q)\alpha+k\beta-k\theta_1-q\theta_2]d\alpha. \tag{A-10}$$

Integrating (A-10) yields

$$I = \frac{1}{2(k-q)}\{\sin[k\theta_2-q\theta_2+(k-q)\pi+k\beta+q\theta_2-k\theta_1]$$

$$-\sin[k\theta_1-q\theta_1-(k-q)\pi-k\beta+q\beta+k\beta+q\theta_2-k\theta_1]\}$$

$$+\frac{1}{2(k+q)}\{\sin[k\theta_2+q\theta_2+(k+q)\pi+k\beta-k\theta_1-q\theta_2]$$

$$-\sin[k\theta_1+q\theta_1-(k+q)\pi-k\beta-q\beta+k\beta-k\theta_1-q\theta_2]\}$$

$$= \frac{1}{2(k-q)}\{\sin[(k-q)\pi+k\beta+k(\theta_2-\theta_1)] - \sin[-(k-q)\pi+q\beta+q(\theta_2-\theta_1)]\}$$

$$+\frac{1}{2(k+q)}\{\sin[(k+q)\pi+k\beta+k(\theta_2-\theta_1)] - \sin[-(k+q)\pi-q\beta-q(\theta_2-\theta_1)]\}$$

which simplifies to

$$I = \frac{1}{2(k-q)}\{\sin[(k-q)\pi+k\beta+k(\theta_2-\theta_1)] + \sin[(k-q)\pi-q\beta-q(\theta_2-\theta_1)]\}$$

$$+ \frac{1}{2(k+q)}\{\sin[(k+q)\pi+k\beta+k(\theta_2-\theta_1)] + \sin[(k+q)\pi+q\beta+q(\theta_2-\theta_1)]\}. \tag{A-11}$$

Making use of the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$, and recognizing that $\cos(K\pi) = (-1)^K$, (A-11) can be written as

$$I = \frac{(-1)^{k-q}}{k-q} \sin\left[\left(\frac{k-q}{2}\right)\beta + \left(\frac{k-q}{2}\right)(\theta_2-\theta_1)\right] \cos\left[\left(\frac{k+q}{2}\right)\beta + \left(\frac{k+q}{2}\right)(\theta_2-\theta_1)\right]$$

$$+ \frac{(-1)^{k+q}}{k+q} \sin\left[\left(\frac{k+q}{2}\right)\beta + \left(\frac{k+q}{2}\right)(\theta_2-\theta_1)\right] \cos\left[\left(\frac{k-q}{2}\right)\beta + \left(\frac{k-q}{2}\right)(\theta_2-\theta_1)\right]. \tag{A-12}$$

Noting that $(-1)^{k-q} = (-1)^{k-q+2q} = (-1)^{k+q}$ and that $2\sin x \cos y = \sin\left(\frac{x+y}{2}\right) + \sin\left(\frac{x-y}{2}\right)$, (A-12) can be written in the form

$$I = \frac{(-1)^{k+q}}{k^2-q^2}\{k\sin[k\beta+k(\theta_2-\theta_1)] - q\sin[q\beta+q(\theta_2-\theta_1)]\}, \quad k\neq q. \tag{A-13}$$

We now substitute (A-7), (A-9), and (A-13) into (A-4) to obtain, replacing $(\theta_2-\theta_1)$ by $-(\theta_1-\theta_2)$,

$$f_\phi(\beta|\theta_1,\theta_2) = \left(\frac{1}{2\pi}\right)^2 [2\pi+\beta-(\theta_1-\theta_2)]$$

$$+ \left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{\infty} b_k^2 \left\{\left[\frac{2\pi+\beta-(\theta_1-\theta_2)}{2}\right]\cos[k\beta-k(\theta_1-\theta_2)]\right.$$

$$\left.+ \frac{1}{2k}\sin[k\beta-k(\theta_1-\theta_2)]\right\} + \left(\frac{1}{2\pi}\right)^2 \sum_{\substack{k=0 \\ k\neq q}}^{\infty}\sum_{q=0}^{\infty} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2}$$

$$\cdot \{k\sin[k\beta-k(\theta_1-\theta_2)] - q\sin[q\beta-q(\theta_1-\theta_2)]\},$$

$$\theta_1-\theta_2-2\pi \le \beta \le \theta_1-\theta_2. \tag{A-14}$$

In (A-14) the first term in square brackets accounts for the k=q=0 term of (A-4), the single summation accounts for the main diagonal k=q, and the double sum encompasses the remaining terms of (A-4).

We observe from (A-13) that for $q \neq k$, $I(k=k_0, q=q_0) = I(k=q_0, q=k_0)$. Thus the double sum in (A-14) can be reduced to one infinite sum and one finite sum:

$$2 \sum_{k=1}^{\infty} \sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2} \{k \sin [k\beta-k(\theta_1-\theta_2)] - q \sin [q\beta-q(\theta_1-\theta_2)]\}.$$

Thus (A-14) becomes

$$\begin{aligned} f_\phi(\beta|\theta_1,\theta_2) = &\left(\frac{1}{2\pi}\right)^2 [2\pi+\beta-(\theta_1-\theta_2)] \\ &+ \left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{\infty} b_k^2 \left\{\left[\frac{2\pi+\beta-(\theta_1-\theta_2)}{2}\right] \cos[k\beta-k(\theta_1-\theta_2)] \right. \\ &\left. + \frac{1}{2k} \sin[k\beta-k(\theta_1-\theta_2)]\right\} \\ &+ 2\left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{\infty} \sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2} \{k \sin [k\beta-k(\theta_1-\theta_2)] \\ &- q \sin [q\beta-q(\theta_1-\theta_2)]\}, \\ &\theta_1-\theta_2-2\pi \le \beta \le \theta_1-\theta_2. \end{aligned} \tag{A-15}$$

From (A-5), we need to evaluate

$$J = \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos(k\alpha+k\beta-k\theta_1) \cos(q\alpha-q\theta_2)\, d\alpha \tag{A-16}$$

which differs from (A-6) only in the limits of integration. Again there are three cases to consider.

If k=q=0, then (A-16) becomes

$$J = \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} d\alpha, \quad k=q=0 \tag{A-17}$$

which yields

$$J = 2\pi-\beta-(\theta_2 - \theta_1), \quad k=q=0. \tag{A-18}$$

If $k=q\neq 0$, then (A-16) becomes

$$J = \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos(k\alpha+k\beta-k\theta_1)\cos(k\alpha-k\theta_2)\, d\alpha. \tag{A-19}$$

Using the identity $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$ we can write (A-19) as

$$J = \frac{1}{2}\int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos[k\beta+k(\theta_1-\theta_2)]\, d\alpha + \frac{1}{2}\int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos[2k\alpha+k\beta-k\theta_2-k\theta_1]\, d\alpha. \tag{A-20}$$

Integrating (A-20),

$$\begin{aligned} J &= \frac{1}{2}[2\pi-\beta-(\theta_2-\theta_1)]\cos[k\beta+k(\theta_2-\theta_1)] \\ &\quad + \frac{1}{4k}\{\sin[2k\theta_1+2k\pi-2k\beta+k\beta-k\theta_2-k\theta_1] \\ &\qquad - \sin[2k\theta_2-2k\pi+k\beta-k\theta_2-k\theta_1]\} \\ &= \frac{2\pi-\beta-(\theta_2-\theta_1)}{2}\cos[k\beta+k(\theta_2-\theta_1)] \\ &\quad + \frac{1}{4k}\{\sin[-k\beta-k(\theta_2-\theta_1)] - \sin[k\beta+k(\theta_2-\theta_1)]\} \end{aligned} \tag{A-21}$$

which simplifies to

$$J = \frac{1}{2}\,[2\pi-\beta+(\theta_1-\theta_2)]\,\cos[k\beta-k(\theta_1-\theta_2)] - \frac{1}{2k}\,\sin[k\beta-k(\theta_1-\theta_2)],\ k=q\neq 0. \tag{A-22}$$

If $k\neq q$, then (A-16) becomes

$$J = \frac{1}{2}\int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos[(k-q)\alpha+k\beta+q\theta_2-k\theta_1]\,d\alpha + \frac{1}{2}\int_{\theta_2-\pi}^{\theta_1+\pi-\beta} \cos[(k+q)\alpha+k\beta-k\theta_1-q\theta_2]\,d\alpha \tag{A-23}$$

which integrates to

$$\begin{aligned}
J = {} & \frac{1}{2(k-q)}\{\sin[k\theta_1-q\theta_1+(k-q)\pi-k\beta+q\beta+k\beta+q\theta_2-k\theta_1] \\
& \quad - \sin[k\theta_2-q\theta_2-(k-q)\pi+k\beta+q\theta_2-k\theta_1]\} \\
& + \frac{1}{2(k+q)}\{\sin[k\theta_1+q\theta_1+(k+q)\pi-k\beta-q\beta+k\beta-k\theta_1-q\theta_2] \\
& \quad - \sin[k\theta_2+q\theta_2-(k+q)\pi+k\beta-k\theta_1-q\theta_2]\} \\
= {} & \frac{1}{2(k-q)}\{\sin[(k-q)\pi+q\beta+q(\theta_2-\theta_1)] - \sin[-(k-q)\pi+k\beta+k(\theta_2-\theta_1)]\} \\
& + \frac{1}{2(k+q)}\{\sin[(k+q)\pi-q\beta-q(\theta_2-\theta_1)] - \sin[-(k+q)\pi+k\beta+k(\theta_2-\theta_1)]\} \\
= {} & \frac{1}{2(k-q)}\{\sin[(k-q)\pi+q\beta+q(\theta_2-\theta_1)] + \sin[(k-q)\pi-k\beta-k(\theta_2-\theta_1)]\} \\
& + \frac{1}{2(k+q)}\{\sin[(k+q)\pi-q\beta-q(\theta_2-\theta_1)] + \sin[(k+q)\pi-k\beta-k(\theta_2-\theta_1)]\} \\
= {} & \frac{1}{2(k-q)}\{\cos[(k-q)\pi]\,\sin[q\beta+q(\theta_2-\theta_1)] - \cos[(k-q)\pi]\,\sin[k\beta+k(\theta_2-\theta_1)]\} \\
& + \frac{1}{2(k+q)}\{-\cos[(k+q)\pi]\,\sin[q\beta+q(\theta_2-\theta_1)] - \cos[(k+q)\pi]\,\sin[k\beta+k(\theta_2-\theta_1)]\}
\end{aligned} \tag{A-24}$$

which simplifies, recognizing $\cos n\pi = (-1)^n$, to

$$J = -\frac{(-1)^{k+q}}{k^2-q^2}\{k\sin[k\beta-k(\theta_1-\theta_2)] - q\sin[q\beta-q(\theta_1-\theta_2)]\},\ k\neq q. \quad \text{(A-25)}$$

Putting (A-18), (A-22), and (A-25) into (A-5) yields

$$f_\phi(\beta|\theta_1,\theta_2) = \left(\frac{1}{2\pi}\right)^2 \{2\pi - [\beta-(\theta_1-\theta_2)]\}$$

$$+\left(\frac{1}{2\pi}\right)^2 \sum_{k=1}^{\infty} b_k^2 \left\{\left[\frac{2\pi-[\beta-(\theta_1-\theta_2)]}{2}\right] \cos[k\beta-k(\theta_1-\theta_2)]\right.$$

$$\left. -\left(\frac{1}{2k}\right)\sin[k\beta-k(\theta_1-\theta_2)]\right\} - \left(\frac{1}{2\pi}\right)^2 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q$$

$$\cdot \frac{(-1)^{k+q}}{k^2-q^2}\{k\sin[k\beta-k(\theta_1-\theta_2)] - q\sin[q\beta-q(\theta_1-\theta_2)]\},$$

$$\theta_1-\theta_2 \le \beta \le \theta_1-\theta_2+2\pi \quad \text{(A-26)}$$

where we have again used the symmetry property of the double summation to introduce a finite upper limit on the inner summation.

In (A-15), we can rewrite the limits of applicability as

$$-2\pi \le \beta-(\theta_1-\theta_2) \le 0$$

and in (A-26) we can rewrite the limits of applicability as

$$0 \le \beta-(\theta_1-\theta_2) \le 2\pi.$$

Thus (A-15) and (A-26) apply to adjacent regions, depending upon whether $\beta-(\theta_1-\theta_2)<0$ or $\beta-(\theta_1-\theta_2)>0$, respectively. Letting $\psi = \beta-(\theta_1-\theta_2)$ in (A-15) and (A-26), we can see that (A-15) is merely (A-26) with $\psi$ replaced by $-\psi$. Since (A-15) applies for $\psi<0$, $-\psi=|\psi|$ over the range of applicability. For $\psi>0$, $|\psi|=\psi$ and thus (A-15) and (A-26) can be combined to give

$$f(\beta|\theta_1,\theta_2) = \begin{cases} 0, & -\infty<\beta\le\theta_1-\theta_2-2\pi \\ \left(\frac{1}{2\pi}\right)^2 \Bigg\{[2\pi-|\beta-(\theta_1-\theta_2)|] & \\ \quad + \sum_{k=1}^{\infty} b_k^2 \Bigg\{\left[\frac{2\pi-|\beta-(\theta_1-\theta_2)|}{2}\right] \cos[k|\beta-(\theta_1-\theta_2)|] & \\ \quad\quad - \frac{1}{2k} \sin[k|\beta-(\theta_1-\theta_2)|]\Bigg\} & \\ \quad -2 \sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2} \Bigg\{k \sin[k|\beta-(\theta_1-\theta_2)|] & \\ \quad\quad -q \sin[q|\beta-(\theta_1-\theta_2)|]\Bigg\}\Bigg\}, & \\ & \theta_1-\theta_2-2\pi\le\beta\le\theta_1-\theta_2+2\pi \\ 0, & \theta_1-\theta_2+2\pi\le\beta<\infty. \end{cases} \tag{A-27}$$

# APPENDIX B

## EVALUATION OF EQUATION (19)

We will evaluate (19) for two cases: $|\theta_1-\theta_2| = 0$ ("space") and $|\theta_1-\theta_2| = \pi$ ("mark"). First we treat the case of "space" being transmitted.

### B.1 "SPACE" ($|\theta_1-\theta_2| = 0$)

We use the binomial theorem twice to write (19) of the main text as

$$V_{m,n} = \int_{-2\pi}^{2\pi} \exp[\gamma\cos\beta]\, f_\phi(\beta|0) \left[\sum_{\ell=0}^{m} \binom{m}{\ell} Z^\ell \cos^\ell\beta\right]\left[\sum_{r=0}^{n} \binom{n}{r}(-Z)^r \cos^r\beta\right] d\beta \tag{B-1}$$

where we have taken into account the finite range over which $f_\phi(\beta|0)$ does not vanish. Interchanging the order of summation and integration, which is clearly permissable due to the finite limits,

$$V_{m,n} = \sum_{\ell=0}^{m}\sum_{r=0}^{n} \binom{m}{\ell}\binom{n}{r}(-1)^r Z^{\ell+r} \int_{-2\pi}^{2\pi} \exp[\gamma\cos\beta] \cos^{\ell+r}\beta\, f_\phi(\beta|0)\,d\beta. \tag{B-2}$$

Denoting the integral in (B-2) as $W_S(\ell+r)$, using (16) of the main text for $f_\phi(\beta|0)$, and splitting the range of integration into two subintervals for positive and negative $\beta$ to enable us to eliminate the absolute value signs from the integrands, we have

$$W_S(M) = \int_{-2\pi}^{0} \cos^M\beta\, e^{\gamma\cos\beta}\left(\frac{1}{2\pi}\right)^2 \Bigg\{(2\pi+\beta)\cdot$$

$$+\sum_{k=1}^{\infty} b_k^2 \left[\left(\frac{2\pi+\beta}{2}\right)\cos(k\beta) + \frac{1}{2k}\sin(k\beta)\right]$$

$$+ 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2 - q^2}\left[k \sin(k\beta) - q \sin(q\beta)\right]\Bigg\} d\beta$$

$$+ \int_0^{2\pi} \cos^M\beta e^{Y\cos\beta}\left(\frac{1}{2\pi}\right)^2\Bigg\{(2\pi-\beta)$$

$$+\sum_{k=1}^{\infty} b_k^2\left[\left(\frac{2\pi-\beta}{2}\right)\cos(k\beta) - \frac{1}{2k}\sin(k\beta)\right]$$

$$-2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2 - q^2}\left[k \sin(k\beta) - q \sin(q\beta)\right]\Bigg\} d\beta \qquad \text{(B-3)}$$

where we have defined $M=\ell+r$ for notational simplicity. If we make the change of variable $\gamma=-\beta$ in the first integral of (B-3) we find that it equals the second integral. Interchanging the order of summation and integration, we have:

$$W_S(M) = 2\left(\frac{1}{2\pi}\right)^2\Bigg\{\int_0^{2\pi} 2\pi \cos^M\beta\, e^{Y\cos\beta} d\beta - \int_0^{2\pi} \beta \cos^M\beta\, e^{Y\cos\beta} d\beta$$

$$+\sum_{k=1}^{\infty} b_k^2\left[\pi\int_0^{2\pi} \cos k\beta \cos^M\beta\, e^{Y\cos\beta} d\beta - \frac{1}{2}\int_0^{2\pi} \beta\cos k\beta \cos^M\beta\, e^{Y\cos\beta} d\beta\right.$$

$$\left.- \frac{1}{2k}\int_0^{2\pi} \sin k\beta \cos^M\beta\, e^{Y\cos\beta} d\beta\right]$$

$$- 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2}\left[k\int_0^{2\pi} \sin k\beta \cos^M\beta\, e^{Y\cos\beta} d\beta\right.$$

$$\left.- q\int_0^{2\pi} \sin q\beta \cos^M\beta\, e^{Y\cos\beta} d\beta\right]\Bigg\}. \qquad \text{(B-4)}$$

In (B-4) we make the change of variable $\alpha=\beta-\pi$ and make use of the identities

$$\left.\begin{aligned} \cos(Q\alpha + Q\pi) &= (-1)^Q \cos Q\alpha \\ \sin(Q\alpha + Q\pi) &= (-1)^Q \sin Q\alpha \end{aligned}\right\} \quad Q \text{ integer} \tag{B-5}$$

to write

$$\begin{aligned} W_S(M) = 2\left(\frac{1}{2\pi}\right)^2 \Bigg\{ & 2\pi(-1)^M \int_{-\pi}^{\pi} \cos^M \alpha e^{-Y\cos\alpha} d\alpha - \int_{-\pi}^{\pi} \alpha(-1)^M \cos^M \alpha \, e^{-Y\cos\alpha} d\alpha \\ & - \pi \int_{-\pi}^{\pi} (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \\ & + \sum_{k=1}^{\infty} b_k^2 \Bigg[ \pi \int_{-\pi}^{\pi} (-1)^k \cos k\alpha (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \\ & - \frac{1}{2} \int_{-\pi}^{\pi} \alpha(-1)^k \cos k\alpha (-1)^M \cos^M \alpha \, e^{-Y\cos\alpha} d\alpha \\ & - \frac{1}{2} \int_{-\pi}^{\pi} \pi(-1)^k \cos k\alpha (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \\ & - \frac{1}{2k} \int_{-\pi}^{\pi} (-1)^k \sin k\alpha (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \Bigg] \\ & - 2 \sum_{k=1}^{\infty} \sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2 - q^2} \Bigg[ k \int_{-\pi}^{\pi} (-1)^k \sin k\alpha (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \\ & - q \int_{-\pi}^{\pi} (-1)^q \sin q\alpha (-1)^M \cos^M \alpha e^{-Y\cos\alpha} d\alpha \Bigg] \Bigg\}. \end{aligned} \tag{B-6}$$

In (B-6) the second, fifth, seventh, eighth, and ninth integrals vanish because the integrands are odd functions and the limits are symmetric about zero. Also, we may write the integrals of even functions as twice the integrals over $[0,\pi]$ to give

$$W_S(M) = 4\left(\frac{1}{2\pi}\right)^2 (-1)^M \left\{\pi \int_0^\pi e^{-Y\cos\alpha} \cos^M \alpha d\alpha + \sum_{k=1}^{\infty} b_k^2 (-1)^k \right.$$

$$\left. \cdot \frac{\pi}{2} \int_0^\pi e^{-Y\cos\alpha} \cos^M \alpha \cos k\alpha d\alpha \right\}. \qquad \text{(B-7)}$$

Combining formulas for the even and odd powers of the cosine [9,1.320.5 and 1.320.7] we can expand the power of the cosine in terms of cosines of multiple arguments:

$$\cos^M \zeta = \frac{1}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2} \cos\{[2\mu+\upsilon(M)]\zeta\} \qquad \text{(B-8)}$$

where

$$\varepsilon_\mu = \begin{cases} 1, & \mu=0 \\ 2, & \mu>0 \end{cases}$$

and

$$\upsilon(M) = \begin{cases} 0, & M \text{ even} \\ 1, & M \text{ odd} . \end{cases}$$

Using the identity

$$\cos z \cos \zeta = \frac{1}{2} [\cos(z+\zeta) + \cos(z-\zeta)] \qquad \text{(B-9)}$$

in connection with (B-8) allows us to write (B-7) as

$$W_S(M) = 4\left(\frac{1}{2\pi}\right)^2 (-1)^M \left\{\pi \int_0^\pi e^{-Y\cos\alpha} \frac{1}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_\mu \binom{M}{[M-\upsilon(M)-2\mu]/2} \cos\{[2\mu+\upsilon(M)]\alpha\} d\alpha \right.$$

$$+ \sum_{k=1}^{\infty} b_k^2 (-1)^k \frac{\pi}{2} \int_0^\pi e^{-Y\cos\alpha} \frac{1}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot\left[\frac{1}{2}\cos\{[2\mu+\upsilon(M)+k]\alpha\} + \frac{1}{2}\cos\{[2\mu+\upsilon(M)-k]\alpha\}\right] d\alpha\Bigg\}. \tag{B-10}$$

Interchanging the order of summation and integration in (B-10), we obtain

$$W_S(M) = 4\left(\frac{1}{2\pi}\right)^2 (-1)^M \Bigg\{\frac{\pi}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot\int_0^{\pi} e^{-Y\cos\alpha}\cos\{[2\mu+\upsilon(M)]\alpha\}d\alpha$$

$$+\sum_{k=1}^{\infty} b_k^2 (-1)^k \frac{\pi}{2} \frac{1}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot\left[\frac{1}{2}\int_0^{\pi} e^{-Y\cos\alpha}\cos\{[2\mu+\upsilon(M)+k]\alpha\}d\alpha\right.$$

$$\left.+\frac{1}{2}\int_0^{\pi} e^{-Y\cos\alpha}\cos\{[2\mu+\upsilon(M)-k]\alpha\}d\alpha\right]\Bigg\}. \tag{B-11}$$

All of the integrals remaining in (B-11) are of the form [10, 9.16.9]

$$\int_0^{\pi} e^{-Y\cos\alpha} \cos N\alpha \, d\alpha = \pi I_N(-Y). \tag{B-12}$$

Also noting that $I_N(-Y) = (-1)^N I_N(Y)$ for N integer, (B-11) becomes

$$W_S(M) = 4\left(\frac{1}{2\pi}\right)^2 (-1)^M \Bigg\{\frac{\pi^2}{2^M} (-1)^{\upsilon(M)} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2} I_{2\mu+\upsilon(M)}(Y)$$

$$+\sum_{k=1}^{\infty} b_k^2(-1)^k \frac{\pi}{2} \frac{\pi}{2^{M+1}} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot\left[(-1)^{2\mu+\upsilon(M)+k} I_{2\mu+\upsilon(M)+k}(Y) + (-1)^{2\mu+\upsilon(M)-k} I_{2\mu+\upsilon(M)-k}(Y)\right]\Bigg\} \tag{B-13}$$

Since $(-1)^k = (-1)^{-k}$ and $(-1)^{2\mu} = 1$, (B-13) becomes

$$W_S(M) = 4\left(\frac{1}{2\pi}\right)^2 (-1)^M \left\{ \frac{\pi^2(-1)^{\upsilon(M)}}{2^M} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2} I_{2\mu+\upsilon(M)}(Y) \right.$$

$$+ \sum_{k=1}^{\infty} b_k^2 \frac{\pi^2(-1)^{\upsilon(M)}}{2^{M+2}} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\left. \cdot \left[ I_{2\mu+\upsilon(M)+k}(Y) + I_{2\mu+\upsilon(M)-k}(Y) \right] \right\}. \qquad \text{(B-14)}$$

Noting that $b_0 = 1$, the single summation in (B-14) can be brought inside the double summation by introducing the factor $1/\varepsilon_k$ since for k=0 the two modified Bessel functions are identical. Thus,

$$W_S(M) = \frac{(-1)^{M+\upsilon(M)}}{2^{M+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \frac{1}{\varepsilon_k} \varepsilon_{\mu+\upsilon(M)} b_k^2 \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot \left[ I_{2\mu+\upsilon(M)-k}(Y) + I_{2\mu+\upsilon(M)+k}(Y) \right]. \qquad \text{(B-15)}$$

Putting (B-15) into (B-2) we arrive at the form

$$V_{m,n} = \sum_{\ell=0}^{m} \sum_{r=0}^{n} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{\ell+r-\upsilon(\ell+r)}{2}} \binom{m}{\ell} \binom{n}{r} Z^{\ell+r} (-1)^{\ell+\upsilon(\ell+r)}$$

$$\cdot \frac{b_k^2 \, \varepsilon_{\mu+\upsilon(\ell+r)}}{2^{\ell+r+1} \varepsilon_k} \binom{\ell+r}{[\ell+r-\upsilon(\ell+r)-2\mu]/2}$$

$$\cdot \left[ I_{2\mu+\upsilon(\ell+r)-k}(Y) + I_{2\mu+\upsilon(\ell+r)+k}(Y) \right]. \qquad \text{(B-16)}$$

Noting that

$$\binom{a}{b} = 0 \text{ if } b<0 \text{ or } b>a, \qquad \text{(B-17)}$$

we can extend the limits of the summations over $\ell$, r, and $\mu$ in (B-16) to infinity. Then we can interchange the order of summations and use Bailey's theorem [11,pp. 58-59] to replace the doubly infinite sums over $\ell$ and r with one infinite sum and one finite sum. Thus, with $s = \ell+r$ and $t=\ell$, (B-16) becomes

$$V_{m,n} = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{s=0}^{\infty} \binom{s}{[s-\upsilon(s)-2\mu]/2} \left(\frac{Z}{2}\right)^s (-1)^{\upsilon(s)}\, b_k^2\, \frac{\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}$$

$$\cdot \left[ I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y) \right] \sum_{t=0}^{s} \binom{m}{t} \binom{n}{s-t} (-1)^t \quad . \qquad \text{(B-18)}$$

It can be shown [12,p. 17] that the finite sum over t in (B-18) is a Gauss hypergeometric function with a negative numerator parameter,

$$\sum_{t=0}^{s} \binom{m}{t} \binom{n}{s-t} (-1)^t = \binom{n}{s}\, {}_2F_1(-m,-s;n-s+1;-1). \qquad \text{(B-19)}$$

Putting (B-19) into (B-18) and again using (B-17) to introduce finite limits in the transformed summations, we have finally that

$$V_{m,n}(\text{space}) = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{s=0}^{n} \sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}} \binom{s}{[s-\upsilon(s)-2\mu]/2} \left(\frac{Z}{2}\right)^s (-1)^{\upsilon(s)}\, \frac{b_k^2\, \varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}$$

$$\cdot \left[ I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y) \right] \binom{n}{s}\, {}_2F_1(-m,-s;n-s+1;-1) \qquad \text{(3-20)}$$

which is (21a) of the main text.

B.2 "MARK" ($|\theta_1 - \theta_2| = \pi$)

Again we use the binomial theorem twice and interchange the order of integration and summation to obtain, for "mark",

$$V_{m,n} = \sum_{\ell=0}^{m} \sum_{r=0}^{n} \binom{m}{\ell}\binom{n}{r}(-1)^r Z^{\ell+r} \int_{\theta_1-\theta_2-2\pi}^{\theta_1-\theta_2+2\pi} \exp\left[\gamma\cos\beta\right] \cos^{\ell+r}\beta \; f_\phi(\beta \,|\, |\theta_1-\theta_2| = \pi)\,d\beta. \tag{B-21}$$

Equation (B-21) is identical to (B-2) except that $|\theta_1 - \theta_2| = \pi$. Denote the integral in (B-21) as $W_M(\ell+r)$. Since $|\theta_1-\theta_2| = \pi$ we have two cases, namely $\theta_1-\theta_2 = -\pi$ and $\theta_1-\theta_2 = \pi$, which occur with equal probability. Therefore,

$$W_M(\ell+r) = \frac{1}{2}\int_{-3\pi}^{\pi} \exp\left[\gamma\cos\beta\right] \cos^{\ell+r}\beta \;\; f_\phi(\beta|\theta_1-\theta_2 = -\pi)\,d\beta$$

$$+ \frac{1}{2}\int_{-\pi}^{3\pi} \exp\left[\gamma\cos\beta\right] \cos^{\ell+r}\beta \;\; f_\phi(\beta|\theta_1-\theta_2 = \pi)\,d\beta. \tag{B-22}$$

Using (16) of the main text for $f_\phi(\beta|\theta_1 - \theta_2)$, letting $M = \ell + r$, we can write for the first integral in (B-22)

$$W_M^-(M) = \left(\frac{1}{2\pi}\right)^2 \int_{-3\pi}^{-\pi} e^{\gamma\cos\beta}\cos^M\beta \left\{(3\pi+\beta) + \sum_{k=1}^{\infty} b_k^2 \left\{\frac{3\pi+\beta}{2} \cos\left[k(\beta+\pi)\right]\right.\right.$$

$$\left.\left. + \frac{1}{2k}\sin\left[k(\beta+\pi)\right]\right\} + 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2 - q^2}\right.$$

$$\left.\cdot \left\{k\sin\left[k(\beta+\pi)\right] - q\sin\left[q(\beta+\pi)\right]\right\}\right\}d\beta \; + \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} e^{\gamma\cos\beta}\cos^M\beta$$

$$\cdot\left\{(\pi-\beta)+\sum_{k=1}^{\infty} b_k^2\left\{\left(\frac{\pi-\beta}{2}\right)\cos\left[k(\beta+\pi)\right]-\frac{1}{2k}\sin\left[k(\beta+\pi)\right]\right\}\right.$$

$$\left.-2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q\frac{(-1)^{k+q}}{k^2-q^2}\left\{k\sin\left[k(\beta+\pi)\right]-q\sin\left[q(\beta+\pi)\right]\right\}\right\}d\beta \tag{B-23}$$

and for the second integral in (B-22)

$$W_M^+(M)=\left(\frac{1}{2\pi}\right)^2\int_{-\pi}^{\pi} e^{\gamma\cos\beta}\cos^M\beta\left\{(\pi+\beta)+\sum_{k=1}^{\infty} b_k^2\left\{\left(\frac{\pi+\beta}{2}\right)\cos\left[k(\beta-\pi)\right]+\frac{1}{2k}\sin\left[k(\beta-\pi)\right]\right\}\right.$$

$$\left.+2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q\frac{(-1)^{k+q}}{k^2-q^2}\left\{k\sin\left[k(\beta-\pi)\right]-q\sin\left[q(\beta-\pi)\right]\right\}\right\}d\beta$$

$$+\left(\frac{1}{2\pi}\right)^2\int_{\pi}^{3\pi} e^{\gamma\cos\beta}\cos^M\beta\left\{(3\pi-\beta)+\sum_{k=1}^{\infty} b_k^2\left\{\left(\frac{3\pi-\beta}{2}\right)\cos\left[k(\beta-\pi)\right]\right.\right.$$

$$\left.-\frac{1}{2k}\sin\left[k(\beta-\pi)\right]\right\}-2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q\frac{(-1)^{k+q}}{k^2-q^2}$$

$$\left.\cdot\left\{k\sin\left[k(\beta-\pi)\right]-q\sin\left[q(\beta-\pi)\right]\right\}\right\}d\beta. \tag{B-24}$$

If the change of variable $\gamma=-\beta$ is made in (B-23) we find that $W_M^-(M) = W_M^+(M)$. Thus the equally weighted sum of the two integrals in (B-22) equals the unweighted value of either integral.

Using the form in (B-24) for $W_M(M)$, we make the change of variable $\gamma=\beta-\pi$. Since $\cos(\gamma+\pi) = -\cos\gamma$, we have

$$W_M(M) = \left(\frac{1}{2\pi}\right)^2 \int_{-2\pi}^{0} e^{-Y\cos\gamma}(-1)^M \cos^M\gamma \left\{ (2\pi+\gamma) + \sum_{k=1}^{\infty} b_k^2 \left[ \left(\frac{2\pi+\gamma}{2}\right)\cos(k\gamma) + \frac{1}{2k}\sin(k\gamma)\right] \right.$$

$$\left. + 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2}\left[k\sin(k\gamma) - q\sin(q\gamma)\right]\right\} d\gamma$$

$$+ \left(\frac{1}{2\pi}\right)^2 \int_{0}^{2\pi} e^{-Y\cos\gamma}(-1)^M \cos^M\gamma \left\{ (2\pi-\gamma) + \sum_{k=1}^{\infty} b_k^2 \left[ \left(\frac{2\pi-\gamma}{2}\right)\cos(k\gamma) - \frac{1}{2k}\sin(k\gamma)\right] \right.$$

$$\left. - 2\sum_{k=1}^{\infty}\sum_{q=0}^{k-1} b_k b_q \frac{(-1)^{k+q}}{k^2-q^2}\left[k\sin(k\gamma) - q\sin(q\gamma)\right]\right\} d\gamma. \qquad \text{(B-25)}$$

Comparing (B-25) with (B-3) we see that the only differences are a multiplicative factor of $(-1)^M$ and a change of sign in the argument of the exponential. This latter difference will lead to a change in sign of the arguments of the modified Bessel functions, allowing us to write the results of (B-25) by analogy to (B-15) as

$$W_M(M) = \frac{(-1)^{2M+\upsilon(M)}}{2^{M+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \frac{\varepsilon_{\mu+\upsilon(M)}}{\varepsilon_k} b_k^2 \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot\left[I_{2\mu+\upsilon(M)-k}(-Y) + I_{2\mu+\upsilon(M)+k}(-Y)\right] \quad . \qquad \text{(B-26)}$$

Since $I_n(-x) = (-1)^n I_n(x)$, for n an integer,

$$W_M(M) = \frac{(-1)^{2[M+\upsilon(M)]}}{2^{M+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} (-1)^{2\mu} \frac{\varepsilon_{\mu+\upsilon(M)}}{\varepsilon_k} b_k^2 \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot \left[(-1)^{-k} I_{2\mu+\upsilon(M)-k}(Y) + (-1)^k I_{2\mu+\upsilon(M)+k}(Y)\right] . \qquad \text{(B-27)}$$

Furthermore, since an even power of -1 is equal to 1, and $(-1)^{-k} = (-1)^k$, (B-27) becomes

$$W_M(M) = \frac{1}{2^{M+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \frac{\varepsilon_{\mu+\upsilon(M)}}{\varepsilon_k} (-1)^k b_k^2 \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot \left[ I_{2\mu+\upsilon(M)-k}(Y) + I_{2\mu+\upsilon(M)+k}(Y) \right] . \tag{B-28}$$

Therefore, from (B-21) and (B-28) when a "mark" is transmitted

$$V_{m,n} = \sum_{\ell=0}^{m} \sum_{r=0}^{n} \binom{m}{\ell}\binom{n}{r}(-1)^r Z^{\ell+r} \frac{1}{2^{\ell+r+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{\ell+r-\upsilon(\ell+r)}{2}} \frac{\varepsilon_{\mu+\upsilon(\ell+r)}}{\varepsilon_k}$$

$$\cdot (-1)^k b_k^2 \binom{\ell+r}{[\ell+r-\upsilon(\ell+r)-2\mu]/2}$$

$$\cdot \left[ I_{2\mu+\upsilon(\ell+r)-k}(Y) + I_{2\mu+\upsilon(\ell+r)+k}(Y) \right] . \tag{B-29}$$

We again use (B-17) to introduce infinite limits on the summations and interchange the order of summations, summing first over k and $\mu$ then over $\ell$ and r. Then we apply Bailey's theorem to diagonalize the sums over $\ell$ and r. Letting $s=\ell+r$ and $t=r$, (B-29) becomes

$$V_{m,n} = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{s} \binom{m}{s-t}\binom{n}{t}(-1)^t \left(\frac{Z}{2}\right)^s (-1)^k \frac{\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k} b_k^2$$

$$\cdot \binom{s}{[s-\upsilon(s)-2\mu]/2} \left[ I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y) \right] . \tag{B-30}$$

The summation over t in (B-30) becomes a Gauss hypergeometric function [12,p.17] as was shown in (B-19). Therefore,

$$v_{m,n} = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{s=0}^{m} \sum_{\mu=0}^{s} \binom{s}{[s-\upsilon(s)-2\mu]/2} \left(\frac{Z}{2}\right)^{s} (-1)^{k}\, b_k^2\, \frac{\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}$$

$$\cdot \left[I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y)\right]$$

$$\cdot \binom{m}{s}\ {}_2F_1(-n,-s;m-s+1;-1) \qquad \text{(B-31)}$$

where we have interchanged the order of summations over s and $\mu$ and have again used (B-17) to introduce finite summation limits.

APPENDIX C

EVALUATION OF EQUATIONS (23a) AND (23b)

## C.1 EVALUATION OF (23a)

Using (20a) of the main text in (23a) of the main text,

$$P(e;\rho|\text{space}) = \int_{-\infty}^{0} \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2+h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!n!} V_{m,n} \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)\left(h_1^2+h_2^2\right)\right]^m$$

$$\cdot \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)\left(h_1^2+h_2^2\right)\right]^n \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] G_n^m\left[-\frac{4y}{\sigma_d^2(1-\rho^2)}\right] dy \tag{C-1}$$

where $V_{m,n}$ is not a function of $y$ (see Appendix B) and the function $G_n^m(\cdot)$ is defined by

$$G_n^m(x) = \sum_{j=0}^{n} \binom{n+m-j}{m} \frac{x^j}{j!} . \tag{C-2}$$

Interchanging the order of summation and integration in (C-1) we obtain

$$P(e;\rho|\text{space}) = \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2+h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{V_{m,n}}{m!n!} \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)\left(h_1^2+h_2^2\right)\right]^m$$

$$\cdot \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)\left(h_1^2+h_2^2\right)\right]^n U_{m,n} \tag{C-3}$$

where

$$U_{m,n} = \int_{-\infty}^{0} \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] G_n^m\left[-\frac{4y}{\sigma_d^2(1-\rho^2)}\right] dy. \tag{C-4}$$

Using (C-2) in (C-4) and interchanging the order of summation and integration,

$$U_{m,n} = \sum_{j=0}^{n} \binom{n+m-j}{m} \frac{(-1)^j}{j!} \left[\frac{4}{\sigma_d^2(1-\rho^2)}\right]^j \int_{-\infty}^{0} y^j \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] dy. \qquad \text{(C-5)}$$

Making the substitution x=-y in (C-5) and using [9, eq. 3.381.4] we can evaluate the integral in (C-5) to obtain

$$U_{m,n} = \left(\frac{1-\rho}{2}\right) \sigma_d^2 \sum_{j=0}^{n} \binom{n+m-j}{m} \left(\frac{2}{1+\rho}\right)^j . \qquad \text{(C-6)}$$

Equation (C-6) can be summed using [12,p.17] and [10,eq. 15.4.1] to yield

$$U_{m,n} = \left(\frac{1-\rho}{2}\right) \sigma_d^2 \binom{n+m}{m} \, {}_2F_1\left(-n,1;-n-m; \frac{2}{1+\rho}\right) . \qquad \text{(C-7)}$$

Substituting (C-7) into (C-3) and rearranging terms gives (26a) of the main text.

## C.2 EVALUATION OF (23b)

Using (20b) of the main text in (23b) of the main text,

$$P(e;\rho|\text{mark}) = \int_0^{\infty} \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2 + h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!n!} V_{m,n} \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)\left(h_1^2 + h_2^2\right)\right]^m$$

$$\cdot \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)\left(h_1^2 + h_2^2\right)\right]^n \exp\left[\frac{-2y}{\sigma_d^2(1+\rho)}\right] G_m^n\left[\frac{4y}{\sigma_d^2(1-\rho^2)}\right] dy. \qquad \text{(C-8)}$$

Interchanging the order of summation and integration in (C-8) we obtain

$$P(e;\rho|\text{mark}) = \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2 + h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{V_{m,n}}{n!m!} \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)\left(h_1^2 + h_2^2\right)\right]^m$$

$$\cdot \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)\left(h_1^2 + h_2^2\right)\right]^n T_{m,n} \qquad \text{(C-9)}$$

where

$$T_{m,n} = \int_0^{\infty} \exp\left[\frac{-2y}{\sigma_d^2(1+\rho)}\right] G_m^n \left[\frac{4y}{\sigma_d^2(1-\rho^2)}\right] dy \; . \tag{C-10}$$

Using (C-2) in (C-10) and interchanging the order of summation and integration,

$$T_{m,n} = \sum_{j=0}^{m} \binom{m+n-j}{n} \frac{1}{j!} \left[\frac{4}{\sigma_d^2(1-\rho^2)}\right]^j \int_0^{\infty} y^j \exp\left[\frac{-2y}{\sigma_d^2(1+\rho)}\right] dy. \tag{C-11}$$

Using [9, eq. 3.381.4] to evaluate the integral in (C-11) we obtain

$$T_{m,n} = \left(\frac{1+\rho}{2}\right) \sigma_d^2 \sum_{j=0}^{m} \binom{n+m-j}{n} \left(\frac{2}{1-\rho}\right)^j \; . \tag{C-12}$$

Equation (C-12) can be summed in the same way that (C-6) was summed to yield

$$T_{m,n} = \left(\frac{1+\rho}{2}\right) \sigma_d^2 \binom{m+n}{m} {}_2F_1\left(-m,1;-m-n;\frac{2}{1-\rho}\right) \; . \tag{C-13}$$

Substituting (C-13) into (C-9) and rearranging terms gives (26b) of the main text.

# APPENDIX D

## ALTERNATE DERIVATION OF ERROR PROBABILITY EQUATIONS

This appendix presents an alternate derivation of the conditional error probability expressions. For brevity, only the case of a "space" being transmitted is considered.

The probability density function of the decision variable conditioned on the phase difference between the two inputs to the multiplier is given by (14) of the main text.

The phase difference $\phi$ is a random variable with some density function $f_\phi(\beta)$. Thus the unconditional p.d.f. of the decision variable is determined from

$$p(y) = \int_{-\infty}^{\infty} f_y(y;\rho|\phi=\beta) f_\phi(\beta) d\beta. \tag{D-1}$$

Our task, then, is to find $f_\phi(\beta)$ so that (14) and (D-1) may be used to find the pdf of y, which may then be integrated to determine the error rate performance of the DPSK system.

### PROBABILITY DENSITY FUNCTION OF PHASE DIFFERENCE $\phi$

The probability density function of $\phi$ can be obtained from the p.d.f.'s of $\phi_i$, i=1,2:

$$f_\phi(\beta) = \int_{-\infty}^{\infty} f_{\phi_1}(\beta+\alpha) f_{\phi_2}(\alpha) d\alpha. \tag{D-2}$$

In reality, the functions $f_{\phi_1}$ and $f_{\phi_2}$ depend upon the transmitted signal phases, $\theta_1$ and $\theta_2$, respectively, and hence should be written as conditional densities $f_{\phi_1}(\alpha|\theta_1)$ and $f_{\phi_2}(\alpha|\theta_2)$. Then $f_\phi(\beta)$ becomes $f_\phi(\beta|\theta_1,\theta_2)$ and, as expected, the result must be averaged over the a priori symbol probabilities.

The phases $\phi_1$ and $\phi_2$ are identically distributed random variables, with density function [2], [3]

$$f_{\phi_i}(\alpha|\theta_i) = \frac{1}{2\pi}\sum_{k=0}^{\infty} b_k \cos\left[k(\alpha-\theta_i)\right], \quad i=1,2, \qquad |\alpha-\theta_i|\le\pi \tag{D-3}$$

where

$$b_k = \epsilon_k \frac{R_u^k}{k!}\Gamma\left(\frac{k}{2}+1\right){}_1F_1\left(\frac{k}{2};\ k+1;\ -R_u^2\right)$$

$R_u^2$ = uplink signal to noise ratio (at input to bandpass limiter)

$$\epsilon_k = \begin{cases} 1, & k=0 \\ 2, & k>0 \end{cases}$$

and ${}_1F_1(a;b;z)$ is the confluent hypergeometric function.

The p.d.f. of the decision variable conditioned on $\theta_1-\theta_2$ is then

$$p(y) = \int_{-\infty}^{\infty} f_y(y;\rho|\phi=\beta)f_\phi(\beta|\theta_1-\theta_2)d\beta$$

$$= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} f_y(y;\rho|\phi=\beta)f_{\phi_1}(\beta+\alpha|\theta_1)f_{\phi_2}(\alpha|\theta_2)d\alpha d\beta . \tag{D-4}$$

The p.d.f. of $\phi_i$, i=1,2, is given by (D-3). We first determine the limits of the integration

$$f_\phi(\beta|\theta_1-\theta_2) = \int_{-\infty}^{\infty} f_{\phi_1}(\beta+\alpha|\theta_1)f_{\phi_2}(\alpha|\theta_2)d\alpha. \tag{D-5}$$

In view of the restricted range over which the $f_{\phi_i}$'s are non-zero, (D-5) must be treated as four separate cases (see Figure D-1). If $-\beta+\theta_1+\pi<\theta_2-\pi$, or $\theta_1-\theta_2+2\pi<\beta$, then there is no overlap of the two $f_{\phi_i}$'s and $f_\phi \equiv 0$. In the second case, we have $\theta_2-\pi<-\beta+\theta_1+\pi<\theta_2+\pi$, which implies that

$$f_\phi(\beta|\theta_1-\theta_2) = \int_{\theta_2-\pi}^{\theta_1+\pi-\beta} f_{\phi_1}(\beta+\alpha|\theta_1)f_{\phi_2}(\alpha|\theta_2)d\alpha, \quad \theta_1-\theta_2\le\beta\le\theta_1-\theta_2+2\pi. \tag{D-6}$$

In the third case, $-\beta+\theta_1-\pi\le\theta_2+\pi\le-\beta+\theta_1+\pi$, which implies that

$$f_\phi(\beta|\theta_1-\theta_2) = \int_{\theta_1-\pi-\beta}^{\theta_2+\pi} f_{\phi_1}(\beta+\alpha|\theta_1)f_{\phi_2}(\alpha|\theta_2)d\alpha, \quad \theta_1-\theta_2-2\pi\le\beta\le\theta_1-\theta_2 . \tag{D-7}$$

In the fourth case $-\beta+\theta_1-\pi>\theta_2+\pi$, or $\beta<\theta_1-\theta_2-2\pi$, there is no overlap and $f_\phi(\beta|\theta_1-\theta_2) \equiv 0$.

Accordingly, the range of $\alpha$ and $\beta$ over which the integration (D-4) should be performed is as shown by the shaded area in Figure D-2:

$$p(y) = \int_{\theta_2-\pi}^{\theta_2+\pi}\int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} f_{\phi_2}(\alpha|\theta_2)f_y(y;\rho|\phi=\beta)f_{\phi_1}(\beta+\alpha|\theta_1)d\beta d\alpha. \tag{D-8}$$

To calculate the error probability, consider the case where a space is transmitted, i.e., there is no change of transmitted phase from one symbol to the next, $\theta_1-\theta_2=0$. Thus an error is made if the decision variable $y<0$, and we have

$$P(e;\rho|\text{space}) = \int_{-\infty}^{0} p(y)dy$$

$$= \int_{-\infty}^{0}\int_{\theta_2-\pi}^{\theta_2+\pi}\int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} f_{\phi_1}(\alpha|\theta_1)f(y;\rho|\phi=\beta)f_{\phi_1}(\beta+\alpha|\theta_1)d\beta d\alpha dy . \tag{D-9}$$

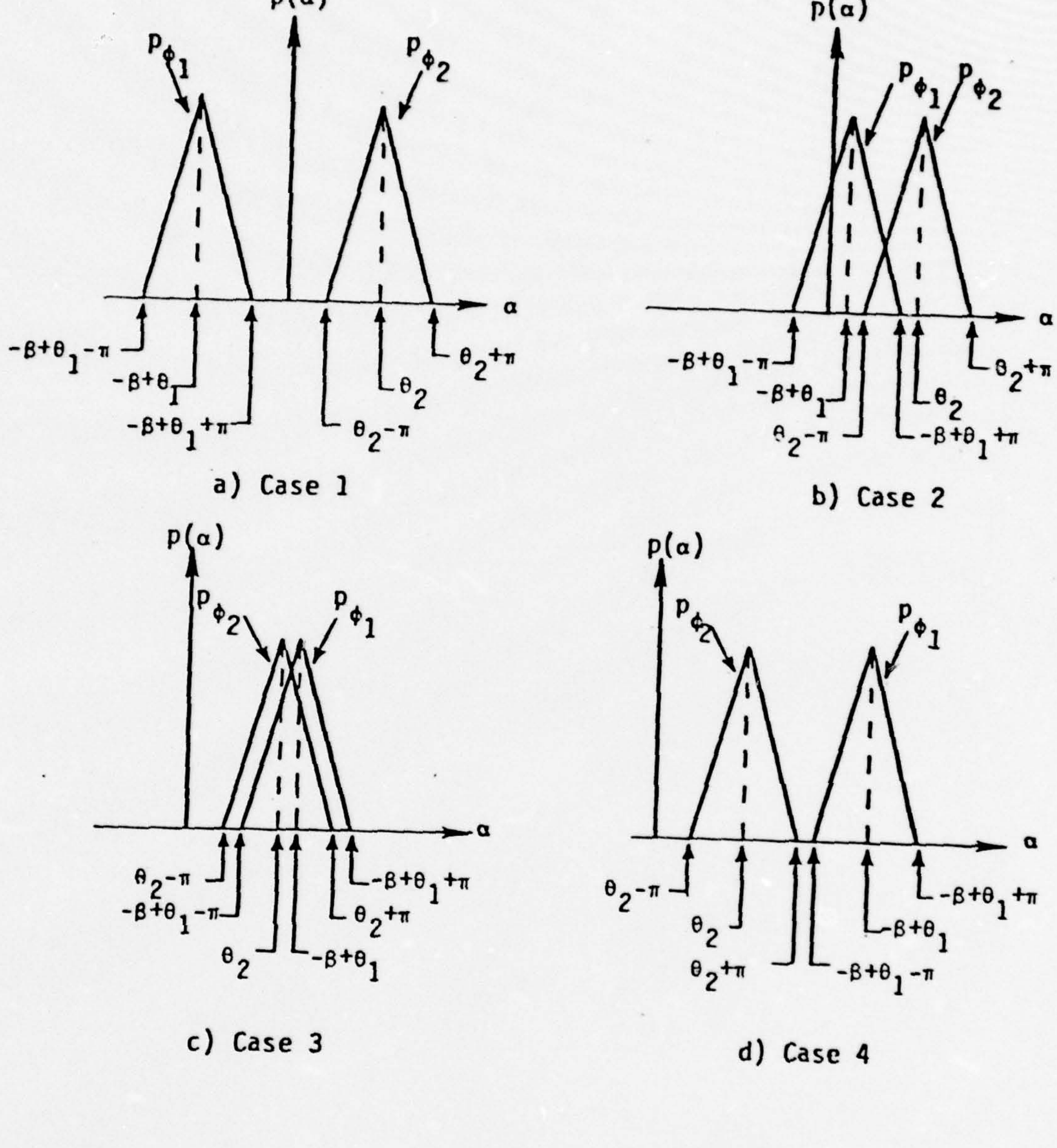


FIGURE D-1 DETERMINATION OF INTEGRATION LIMITS FOR PROBABILITY DENSITY OF $\phi_1-\phi_2$ AS A FUNCTION OF $\beta$

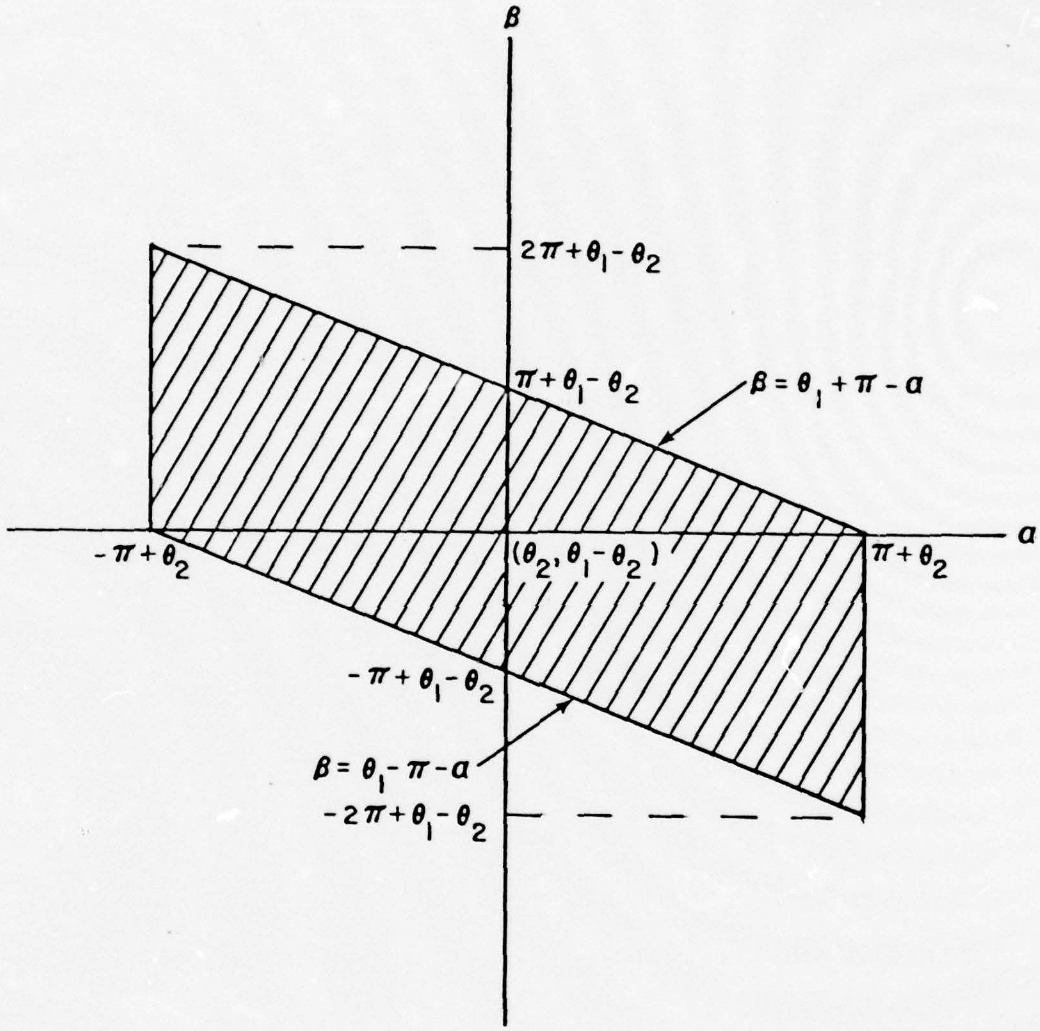


FIGURE D-2 INTEGRATION RANGE OF α AND β

Putting (14) of the main text into (D-9) we have, after some algebraic manipulation,

$$P(e;\rho|\text{space}) = \frac{1}{\sigma_d^2} \exp\left[-\frac{h_1^2+h_2^2}{1-\rho^2}\right] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{n!} \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)(h_1^2+h_2^2)\right]^m$$

$$\cdot \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)(h_1^2+h_2^2)\right]^n \int_{-\infty}^{0} \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] G_n^m \left[\frac{-4y}{\sigma_d^2(1-\rho^2)}\right] dy$$

$$\cdot \int_{\theta_2-\pi}^{\theta_2+\pi} \int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} e^{Y\cos\beta}\left[1+Z\cos\beta\right]^m\left[1-Z\cos\beta\right]^n$$

$$\cdot f_{\phi_1}(\alpha|\theta_1) f_{\phi_1}(\beta+\alpha|\theta_1)\, d\beta\, d\alpha \tag{D-10}$$

where

$$Y = \frac{2\rho h_1 h_2}{1-\rho^2}$$

and

$$Z = \frac{2h_1 h_2}{h_1^2+h_2^2} .$$

Let us denote the double integral in (D-10) by $I_2^S$. Then

$$I_2^S = \int_{\theta_2-\pi}^{\theta_2+\pi} f_{\phi_1}(\alpha|\theta_1) I_3^S(\alpha)\, d\alpha \tag{D-11}$$

where

$$I_3^S(\alpha) = \int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} \exp\left[Y\cos\beta\right]\left[1+Z\cos\beta\right]^m\left[1-Z\cos\beta\right]^n f_{\phi_1}(\beta+\alpha|\theta_1)\, d\beta. \tag{D-12}$$

We can use the binomial theorem to write (D-12) as

$$I_3^S = \sum_{\ell=0}^{m} \sum_{r=0}^{n} \binom{m}{\ell}\binom{n}{r} (-1)^r Z^{\ell+r} V(\ell+r) \tag{D-13}$$

where

$$V(M) = \int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} e^{Y\cos\beta} \cos^M\beta f_{\phi_1}(\beta+\alpha|\theta_1)d\beta \tag{D-14}$$

and $M = \ell + r$.

Substituting (D-3) into (D-14) we have

$$V(M) = \frac{1}{2\pi}\sum_{k=0}^{\infty} b_k \int_{\theta_1-\alpha-\pi}^{\theta_1-\alpha+\pi} e^{Y\cos\beta} \cos^M\beta\cos[k(\beta+\alpha-\theta_1)]d\beta. \tag{D-15}$$

Using [9, 1.320.5 and 1.320.7] we can expand the power of the cosine in terms of cosines of multiple arguments

$$\cos^M\zeta = \frac{1}{2^M}\sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \varepsilon_{\mu+\upsilon(M)} \binom{M}{[M-\upsilon(M)-2\mu]/2} \cos\{[2\mu+\upsilon(M)]\zeta\} \tag{D-16}$$

where

$$\varepsilon_\mu = \begin{cases} 1, & \mu=0 \\ 2, & \mu>0 \end{cases}$$

and

$$\upsilon(q) = \begin{cases} 0, & q \text{ even} \\ 1, & q \text{ odd} \end{cases} .$$

Using the identity

$$\cos z \cos \zeta = \frac{1}{2}\left[\cos(z+\zeta) + \cos(z-\zeta)\right] \tag{D-17}$$

in connection with (D-16) allows us to reduce the integral in (D-15) to the form [13]

$$v = \int_{\phi}^{2\pi+\phi} e^{Y\cos\alpha}\cos N\alpha \, d\alpha = 2\pi I_N(Y) \tag{D-18}$$

where $\phi$ is an arbitrary angle.

Substitution of (D-16), (D-17), and (D-18) into (D-15) yields, after some algebraic manipulation,

$$V(M) = \frac{1}{2^{M+1}} \sum_{k=0}^{\infty} \sum_{\mu=0}^{\frac{M-\upsilon(M)}{2}} \cos[k(\alpha-\theta_1)]\epsilon_{\mu+\upsilon(M)} b_k \binom{M}{[M-\upsilon(M)-2\mu]/2}$$

$$\cdot \left[I_{2\mu+\upsilon(M)-k}(Y) + I_{2\mu+\upsilon(M)+k}(Y)\right] \qquad \text{(D-19)}$$

where $b_k$, $\epsilon_k$, and $\upsilon(k)$ are as defined previously. Using (D-13), (D-19) and (D-3) in (D-11) yields

$$I_2^S = \frac{1}{4\pi} \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{m} \sum_{r=0}^{n} \sum_{\mu=0}^{\frac{\ell+r-\upsilon(\ell+r)}{2}} \frac{\epsilon_{\mu+\upsilon(\ell+r)}}{2^{\ell+r}} \binom{m}{\ell}\binom{n}{r} (-1)^r Z^{\ell+r} \binom{\ell+r}{\frac{\ell+r-\upsilon(\ell+r)-2\mu}{2}}$$

$$\cdot \left[I_{2\mu+\upsilon(\ell+r)-k}(Y) + I_{2\mu+\upsilon(\ell+r)+k}(Y)\right] b_p b_k \int_{\theta_2-\pi}^{\theta_2+\pi} \cos\left[p(\alpha-\theta_1)\right]\cos\left[k(\alpha-\theta_1)\right] d\alpha. \qquad \text{(D-20)}$$

The integral in (D-20) is equal to $2\pi$ for $p=k=0$, $\pi$ for $p=k\neq 0$, and disappears for $p\neq k$.

Thus we have

$$I_2^S = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{m}\sum_{r=0}^{n}\sum_{\mu=0}^{\frac{\ell+r-\upsilon(\ell+r)}{2}} \frac{1}{\varepsilon_k}\binom{m}{\ell}\binom{n}{r} Z^{\ell+r}(-1)^r \frac{b_k^2\varepsilon_{\mu+\upsilon(\ell+r)}}{2^{\ell+r}}$$

$$\cdot\binom{\ell+r}{[\ell+r-\upsilon(\ell+r)-2\mu]/2}\left[I_{2\mu+\upsilon(\ell+r)-k}(Y) + I_{2\mu+\upsilon(\ell+r)+k}(Y)\right]. \quad \text{(D-21)}$$

Making use of the property of the binomial coefficients,

$$\binom{a}{b} = 0 \text{ if } b<0 \text{ or } b>a,$$

to extend the limits of the finite sums to infinity and then applying Bailey's theorem [11, pp. 58-59] to the sums over $\ell$ and r, we can write (D-21) as

$$I_2^S = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}}\binom{s}{[s-\upsilon(s)-2\mu]/2}\left(\frac{Z}{2}\right)^s \frac{b_k^2\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}(-1)^s$$

$$\cdot\left[I_{2\mu+\upsilon(s)-k}(Y)+I_{2\mu+\upsilon(s)+k}(Y)\right]\sum_{t=0}^{s}\binom{m}{t}\binom{n}{s-t}(-1)^t. \quad \text{(D-22)}$$

It can be shown [12, p. 17] that the summation over t in (D-22) can be expressed as a Gauss hypergeometric function with a negative numerator parameter. Making use of this, we have

$$I_2^S = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{n}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}}\binom{s}{[s-\upsilon(s)-2\mu]/2}\left(\frac{Z}{2}\right)^s \frac{b_k^2\varepsilon_{\mu+\upsilon(s)}}{\varepsilon_k}(-1)^{\upsilon(s)}$$

$$\cdot\left[I_{2\mu+\upsilon(s)-k}(Y)+I_{2\mu+\upsilon(s)+k}(Y)\right]$$

$$\cdot\binom{n}{s}{}_2F_1(-m,-s;\, n-s+1;\, -1) \quad \text{(D-23)}$$

where we have again used the properties of the binomial coefficients to introduce finite summation limits and have used $(-1)^s = (-1)^{\nu(s)}$.

Letting $I_1^S$ denote the first integral in (D-10), from the definition of the polynomial $G_n^m(x)$ we have

$$I_1^S = \sum_{j=0}^{n} \binom{n+m-j}{m} \frac{1}{j!}(-1)^j \left[\frac{4}{\sigma_d^2(1-\rho^2)}\right]^j \int_{-\infty}^{0} \exp\left[\frac{2y}{\sigma_d^2(1-\rho)}\right] y^j dy. \tag{D-24}$$

The integral in (D-24) may be evaluated by making the substitution $x = -y$ and using [9, eq. 3.381.4] to obtain

$$I_1^S = \left(\frac{1-\rho}{2}\right)\sigma_d^2 \sum_{j=0}^{n} \binom{n+m-j}{m}\left(\frac{2}{1+\rho}\right)^j . \tag{D-25}$$

Using [12], we can sum (D-25) to yield

$$I_1^S = \left(\frac{1-\rho}{2}\right)\sigma_d^2 \binom{n+m}{m} {}_2F_1\left(-n,1;\ -n-m;\ \frac{2}{1+\rho}\right). \tag{D-26}$$

Substituting (D-23) and (D-26) into (D-10), and rearranging terms for clarity, yields:

$$P(e;\rho|\text{space}) = \pi\left(\frac{1-\rho}{4}\right)\exp\left[-\frac{2R_d^2}{1-\rho^2}\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{n}\sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}}$$

$$\frac{\epsilon_k\epsilon_{\mu+\upsilon(s)}(-1)^{\upsilon(s)}}{m!n!\left[\Gamma\left(\frac{k+1}{2}\right)\right]^2}\binom{n+m}{m}$$

$$\cdot\binom{n}{s}\binom{s}{[s-\upsilon(s)-2\mu]/2}\left(\frac{R_u}{2}\right)^{2k}\left(\frac{\lambda}{1+\lambda^2}\right)^s\left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)R_d^2\ (1+\lambda^2)\right]^m$$

$$\cdot\left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)R_d^2(1+\lambda^2)\right]^n\left[I_{2\mu+\upsilon(s)-k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)+I_{2\mu+\upsilon(s)+k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\right]$$

$$\cdot\ {}_2F_1\left(-n,1;-n-m;\frac{2}{1+\rho}\right){}_2F_1(-m,-s;\text{[illegible]}-s+1;-1)$$

$$\cdot\left[{}_1F_1\left(\frac{k}{2};k+1;-R_u^2\right)\right]^2 \qquad \text{(D-27)}$$

where

$$\epsilon_q = \begin{cases}1, & q=0\\ 2, & q>0\end{cases}$$

and

$$\upsilon(q) = \begin{cases}0, & q \text{ even}\\ 1, & q \text{ odd}\end{cases}$$

and where we have defined the signal-to-noise ratio parameters

$$R_d^2 = h_1^2 = \text{direct channel SNR into the multiplier}$$

and

$$\lambda^2 = \frac{h_2^2}{h_1^2} = \text{power imbalance between multiplier inputs}$$

and we have used the relation $(2z)! = \Gamma(2z+1) = \frac{2^{2z}}{\sqrt{\pi}}\Gamma\left(z+\frac{1}{2}\right)\Gamma(z+1)$ to simplify the form arising from the coefficients $(b_k)^2$ which occur in (D-21). Equation (D-27) is identical to (26a) of the main text.

## APPENDIX E

## ALTERNATE FORMS FOR ERROR PROBABILITY EQUATION

The error probability equations (26a) and (26b) of the main text may be written in several alternate forms. Although these alternate forms do not appear to be any more computationally efficient, they are presented here for the benefit of those readers who may wish to pursue further the mathematical properties of the error behavior of a DPSK system. For brevity we present only the forms for a "space" being transmitted; the forms for "mark" can easily be written by analogy.

First, we consider the inner-most summation of (26a):

$$\Delta = \sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}} \epsilon_{\mu+\upsilon(s)} \binom{s}{[s-\upsilon(s)-2\mu]/2} \left[ I_{2\mu+\upsilon(s)-k}(Y) + I_{2\mu+\upsilon(s)+k}(Y) \right] \quad \text{(E-1)}$$

where $Y = 2\rho\lambda R_d^2/(1-\rho^2)$. The expression in (E-1) may be summed as shown below.

From [10, eq. 9.6.29] we find that

$$I_\nu^{(p)}(z) = \frac{1}{2^p} \left\{ I_{\nu-p}(z) + \binom{p}{1} I_{\nu-p+2}(z) + \binom{p}{2} I_{\nu-p+4}(z) + \dots + I_{\nu+p}(z) \right\}, \quad p=0,\ 1,\ 2,\dots \quad \text{(E-2)}$$

where

$$I_\nu^{(p)}(z) = \frac{d^p}{dz^p} I_\nu(z) .$$

For $p=0$, $I_\nu^{(0)}(z) = I_\nu(z)$. Also, from [10, eq. 9.6.6]

$$I_{-n}(z) = I_n(z) \quad \text{(E-3)}$$

where n is any integer. Since

$$\binom{p}{j} = \binom{p}{p-j} \tag{E-4}$$

we can use (E-3) to re-arrange (E-2). If p is even, we begin with the term

$$\binom{p}{p/2} I_\nu(z) \tag{E-5a}$$

and proceed with terms of the form, using (E-3) and (E-4),

$$\binom{p}{p/2-j}\left[I_{\nu-2j}(z) + I_{\nu+2j}(z)\right] \tag{E-5b}$$

with all the terms of (E-2) being accounted for by the time we reach $j=p/2$. If p is odd, then the first term is

$$\binom{p}{[p-1]/2}\left[I_{\nu-1}(z) + I_{\nu+1}(z)\right] \tag{E-6a}$$

and we proceed with terms of the form, again using (E-3) and (E-4),

$$\binom{p}{[p-1/2]-j}\left[I_{2j+1-\nu}(z) + I_{2j+1+\nu}(z)\right] \qquad \text{(E-5b)}$$

for $j=1,2,\ldots,(p-1)/2$.

We now notice that (E-5b) and (E-6b) can be combined by use of the function $\upsilon(\cdot)$ which we introduced in the main text. However, (E-6a) yields twice the quantity of (E-5a) if we attempt to combine them through use of the $\upsilon(\cdot)$ function. To circumvent this problem, we can double all the other terms by introducing the Neumann factor $\varepsilon_{j+\upsilon(p)}$ which is 1 when p is even and j=0 and is 2 otherwise. The entire result is then halved. Thus

$$I_\nu^{(p)}(z) = \frac{1}{2^{p+1}} \sum_{j=0}^{\frac{p-\upsilon(p)}{2}} \varepsilon_{j+\upsilon(p)} \binom{p}{[p-\upsilon(p)-2j]/2}\left[I_{2j+\upsilon(p)-\nu}(z) + I_{2j+\upsilon(p)+\nu}(z)\right]. \tag{E-7}$$

Using (E-7) in (E-1), we find that

$$\Delta = 2^{s+1} I_k^{(s)}(\gamma) . \tag{E-8}$$

Putting (E-8) into (26a) of the main text, we have the alternate expression

$$P(e;\rho,\lambda|\text{space}) = \pi\left(\frac{1-\rho}{2}\right) \exp\left[-\frac{2R_d^2}{1-\rho^2}\right] \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{n} \frac{\varepsilon_k(-1)^{\upsilon(s)}}{m!n!\left[\Gamma\left(\frac{k+1}{2}\right)\right]^2}\binom{n+m}{m}\binom{n}{s}$$

$$\cdot\left(\frac{R_u}{2}\right)^{2k}\left(\frac{2\lambda}{1+\lambda^2}\right)^s\left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right)R_d^2(1+\lambda^2)\right]^m\left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right)R_d^2(1+\lambda^2)\right]^n$$

$$\cdot\; I_k^{(s)}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\, {}_2F_1\left(-n,1;-n-m;\frac{2}{1+\rho}\right)$$

$$\cdot\; {}_2F_1(-m,-s;n-s+1;-1)\left[{}_1F_1\left(\frac{k}{2};k+1;-R_u^2\right)\right]^2 \tag{E-9}$$

for the error probability given that a "space" was transmitted.

Another formulation for the error probability may be written by using the relationship [14]

$${}_1F_1(n+\tfrac{1}{2};2n+2;-R_u^2) = \frac{\exp(-R_u^2/2)\Gamma(n+1)2^{2n}}{\left(R_u^2\right)^n}\left[I_n\left(\frac{R_u^2}{2}\right)+I_{n+1}\left(\frac{R_u^2}{2}\right)\right]. \tag{E-10}$$

Setting $k/2 = n+\frac{1}{2}$, the confluent hypergeometric function in (26a) of the main text may be expressed as

$${}_1F_1(\tfrac{k}{2};k+1;-R_u^2) = \frac{\exp(-R_u^2/2)\,\Gamma\left(\frac{k+1}{2}\right)2^{k-1}}{\left(R_u^2\right)^{(k-1)/2}}\left[I_{(k-1)/2}\left(\frac{R_u^2}{2}\right)+I_{(k+1)/2}\left(\frac{R_u^2}{2}\right)\right]. \tag{E-11}$$

Putting (E-11) into (26a) yields a second alternate formulation for the error probability

$$P(e;\rho,\lambda|\text{space}) = \pi R_u^2 \left(\frac{1-\rho}{16}\right) \exp\left[-\frac{2R_d^2}{1-\rho^2}\right] \exp\left[-R_u^2\right]$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{n} \sum_{\mu=0}^{\frac{s-\upsilon(s)}{2}} \frac{\varepsilon_k\, \varepsilon_{\mu+\upsilon(s)} (-1)^{\upsilon(s)}}{m!n!} \binom{n+m}{m} \binom{n}{s} \left(\frac{\lambda}{1+\lambda^2}\right)^s$$

$$\cdot \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right) R_d^2 \,(1+\lambda^2)\right]^m \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right) R_d^2 \,(1+\lambda^2)\right]^n$$

$$\cdot \left[I_{2\mu+\upsilon(s)-k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right) + I_{2\mu+\upsilon(s)+k}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)\right]$$

$$\cdot \left[I_{(k-1)/2}\left(\frac{R_u^2}{2}\right) + I_{(k+1)/2}\left(\frac{R_u^2}{2}\right)\right]^2$$

$$\cdot {}_2F_1\left(-n,1;-n-m;\frac{2}{1+\rho}\right) {}_2F_1(-m,-s;n-s+1;-1). \qquad \text{(E-12)}$$

A third alternative form may be derived by using both (E-8) and (E-11) in (26a) to yield

$$P(e;\rho,\lambda|\text{space}) = \pi R_u^2 \left(\frac{1-\rho}{16}\right) \exp\left[-\frac{2R_d^2}{1-\rho^2}\right] \exp\left[-R_u^2\right]$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{n} \frac{\varepsilon_k}{m!n!} \binom{n+m}{m} \binom{n}{s} \left(\frac{-2\lambda}{1+\lambda^2}\right)^s$$

$$\cdot \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right) R_d^2 \,(1+\lambda^2)\right]^m \left[\frac{1}{4}\left(\frac{1+\rho}{1-\rho}\right) R_d^2 \,(1+\lambda^2)\right]^n I_k^{(s)}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right)$$

$$\cdot \left[I_{(k-1)/2}\left(\frac{R_u^2}{2}\right) + I_{(k+1)/2}\left(\frac{R_u^2}{2}\right)\right]^2$$

$$\cdot {}_2F_1\left(-n,1;-n-m;\frac{2}{1+\rho}\right) {}_2F_1(-m,-s;n-s+1;-1) . \qquad \text{(E-13)}$$

where we also have used $(-1)^s = (-1)^{\upsilon(s)}$.

An interesting fourth transformation arises from the use of Bailey's theorem [11] to replace the infinite sums over n and m in (E-13) with one infinite sum and one finite sum. Letting v=n and w=n+m we obtain from (E-13), after a little algebra,

$$P(e;\rho,\lambda|\text{space}) = \pi R_u^2 \left(\frac{1-\rho}{16}\right) \exp\left[-R_u^2\right] \exp\left[-\frac{2R_d^2}{1-\rho^2}\right]$$

$$\cdot \sum_{k=0}^{\infty} \sum_{w=0}^{\infty} \sum_{v=0}^{w} \sum_{s=0}^{v} \frac{\varepsilon_k}{w!} \left[\binom{w}{v}\right]^2 \binom{v}{s} \left(\frac{-2\lambda}{1+\lambda^2}\right)^s \left[\frac{1}{4}\left(\frac{1-\rho}{1+\rho}\right) R_d^2 \, (1+\lambda^2)\right]^w$$

$$\cdot \left(\frac{1+\rho}{1-\rho}\right)^{2v} I_k^{(s)}\left(\frac{2\rho\lambda R_d^2}{1-\rho^2}\right) \left[I_{(k-1)/2}\left(\frac{R_u^2}{2}\right) + I_{(k+1)/2}\left(\frac{R_u^2}{2}\right)\right]^2$$

$$\cdot \; {}_2F_1\left(-v,1;-w;\frac{2}{1+\rho}\right) \; {}_2F_1(v-w,-s;v-s+1;-1). \qquad \text{(E-14)}$$

The form in (E-14) begins to resemble summation formulas for generalized hypergeometric functions, for example [15] or [16]. This is an area of mathematics which is not yet fully explored; hence the implications of forms such as (E-14) cannot be fully stated at this time.

Other forms can be written by applying various transformation formulas to the Gauss hypergeometric functions in (26a), (26b), (E-9), (E-12), (E-13), and (E-14). Examples of applicable transformation can be found, for example, in chapter 15 of [10]. It seems of little value, though, to write out in full the multitude of forms thus derivable.

# APPENDIX F

## A FURTHER CONSIDERATION ON POWER IMBALANCE

The power imbalance between adjacent pulses arises due to such effects as delay line attenuation, phase error, and intersymbol interference. The power imbalance may also arise from sources other than the receiver itself such as fading. The effect of delay line attenuation or fading on power imbalance is straightforward and need not be mentioned. The phase error can arise, for example, from an improper delay length in the delay circuit of a phase detector. The intersymbol interference is, however, a complicated problem resulting from the combined effects of the delay circuit and the transfer function (filter characteristics) of the system under consideration.

In this appendix we want to enhance our understanding of the possible relationships between power imbalance, intersymbol interference, and phase error associated with non-ideal delay lines in the differential phase detector.

Hubbard [17] analyzed the effect of intersymbol interference on the probability of error (for the case of no noise correlation) under the assumption that the intersymbol interference comes only from adjacent pulses, and showed that the probability of error is a function of power imbalance (between direct and delayed channel) caused by intersymbol interference [17, eq. 14].

Another way of viewing the power imbalance is to relate it to phase error. The intersymbol interference is assumed to manifest itself in the form of a perturbation in the phase of the signal at the sampling instant. More precisely, the intersymbol interference introduces a phase shift $\Delta\phi$ so that the phase change in a time slot is $\phi \pm \Delta\phi$ instead of $\phi$ ($\phi=0$ for a "space" and $\phi=\pi$ for a "mark"). The value of $\Delta\phi$ depends on the details of the signal waveform and the filter characteristics of the system. The determination of the $\Delta\phi$'s (for a particular system) which describe the intersymbol interference phenomena in a complicated manner is a rather difficult problem, and even if it is theoretically possible (see [18] for example), the actual measurement of this quantity will be a formidable task. For this reason we will establish an analytical background to replace $\Delta\phi$ by an easily measurable quantity, power (or equivalently SNR) imbalance $\lambda^2$ which is used as a basic parameter throughout the report. By relating $\Delta\phi$ to $\lambda^2$, all the possible degradation factors mentioned above (attenuation, fading, phase error, and intersymbol interference) which otherwise appear to be different attributes are merged into one quantity $\lambda^2$. The statistics of $\lambda^2$ may then be experimentally determined for a particular system.

We now show the relation of $\Delta\phi$ to $\lambda^2$. To do this we note from the definitions following (14) in the main text that the parameters $h_3^2$ and $h_4^2$ are all that we need in describing the error performance in terms of power imbalance. For simplicity, consider the case of no noise correlation ($\rho=0$). We may write then

$$H_3^2 \triangleq 2h_3^2 = h_1^2 + h_2^2 + 2h_1h_2 \cos\phi \tag{F-1a}$$

and

$$H_4^2 \triangleq 2h_4^2 = h_1^2 + h_2^2 - 2h_1h_2 \cos\phi. \tag{F-1b}$$

Let us define the average SNR (or equivalently average power)* $R^2$ by**

$$R^2 = \frac{h_1^2 + h_2^2}{2} \tag{F-2}$$

and a differential phase (or phase error) $\Delta\phi$ such that

$$\cos\Delta\phi = \frac{2h_1h_2}{h_1^2+h_2^2} \quad . \tag{F-3}$$

Then (F-1) can be written

$$H_3^2 = h_1^2 + h_2^2 + (h_1^2+h_2^2) \frac{2h_1h_2}{h_1^2+h_2^2} \cos\phi$$

$$= 2R^2 + 2R^2 \cos\Delta\phi \cos\phi \tag{F-4a}$$

and

$$H_4^2 = 2R^2 - 2R^2 \cos\Delta\phi \text{ oos}\phi . \tag{F-4b}$$

Also, defining the power imbalance $\lambda^2$ by

$$\lambda^2 \triangleq \frac{h_2^2}{h_1^2} ,$$

we can write (F-3) as

$$\cos\Delta\phi = \frac{2\lambda}{1+\lambda^2}$$

or

$$\lambda^2 = \frac{1-\sin\Delta\phi}{1+\sin\Delta\phi} \quad . \tag{F-5}$$

---

*In binary communication systems, error performance of a correlation receiver (optimum) depends only on the average signal energy $E=(E_0+E_1)/2$ where $E_0$ and $E_1$ are energy associated with bits "0" and "1" respectively provided the level of noise spectral density is the same in two inputs and the two input signals are uncorrelated [19, p. 163].

** This definition of R differs from that used in Appendix H of this report.

We note that (F-1) describes $H_3$ and $H_4$ in terms of $h_1$ and $h_2$ (two different input SNR's), while (F-4) describes the very same parameters by R and $\Delta\phi$ (constant average power and differential phase error). We may call the former a "power imbalance model" (since $h_1 \neq h_2$ in general), and the latter a "phase imbalance model." The important fact is that the two models are equivalent under the constraints of (F-2) and (F-5). A geometrical interpretation will be more instructive in grasping physical understanding of the relationships between the parameters involved.

Let us consider, for simplicity, the case that a "mark" is transmitted; $\phi=\theta_1-\theta_2=\pi$. Then since $\cos\phi=-1$ and $\sin\phi=0$, we may write (F-1) and (F-4) as

$$H_3 = h_1-h_2 \tag{F-6a}$$

$$H_4 = h_1+h_2 \tag{F-6b}$$

and

$$H_3 = \sqrt{R^2+R^2-2R\cdot R\cos\Delta\phi} \tag{F-7a}$$

$$H_4 = \sqrt{R^2+R^2+2R\cdot R\cos\Delta\phi}\ . \tag{F-7b}$$

We now show that the geometry in Figure F-1 satisfies all the relationships given by (F-6), (F-7), and also (F-2) and (F-5). Noting that $\overline{CP}=\overline{EP}=h_1$, we have (F-6a), while (F-6b), (F-7a) and (F-7b)* are obvious from the figure. Equation (F-2) follows from the fact that $\overline{CD}^2=\overline{CP}^2+\overline{DP}^2=h_1^2+h_2^2$; and $\overline{CD}^2+\overline{AD}^2=2\overline{CD}^2=\overline{AC}^2=(2R)^2$. The validity of (F-5) can be shown as follows:

$$\lambda = \frac{h_2}{h_1} = \frac{\overline{DP}}{\overline{CP}} = \text{tangent of angle DCP} = \tan\,(45^\circ-\Delta\phi/2).$$

* (F-7a) and (F-7b) are from the law of cosines.

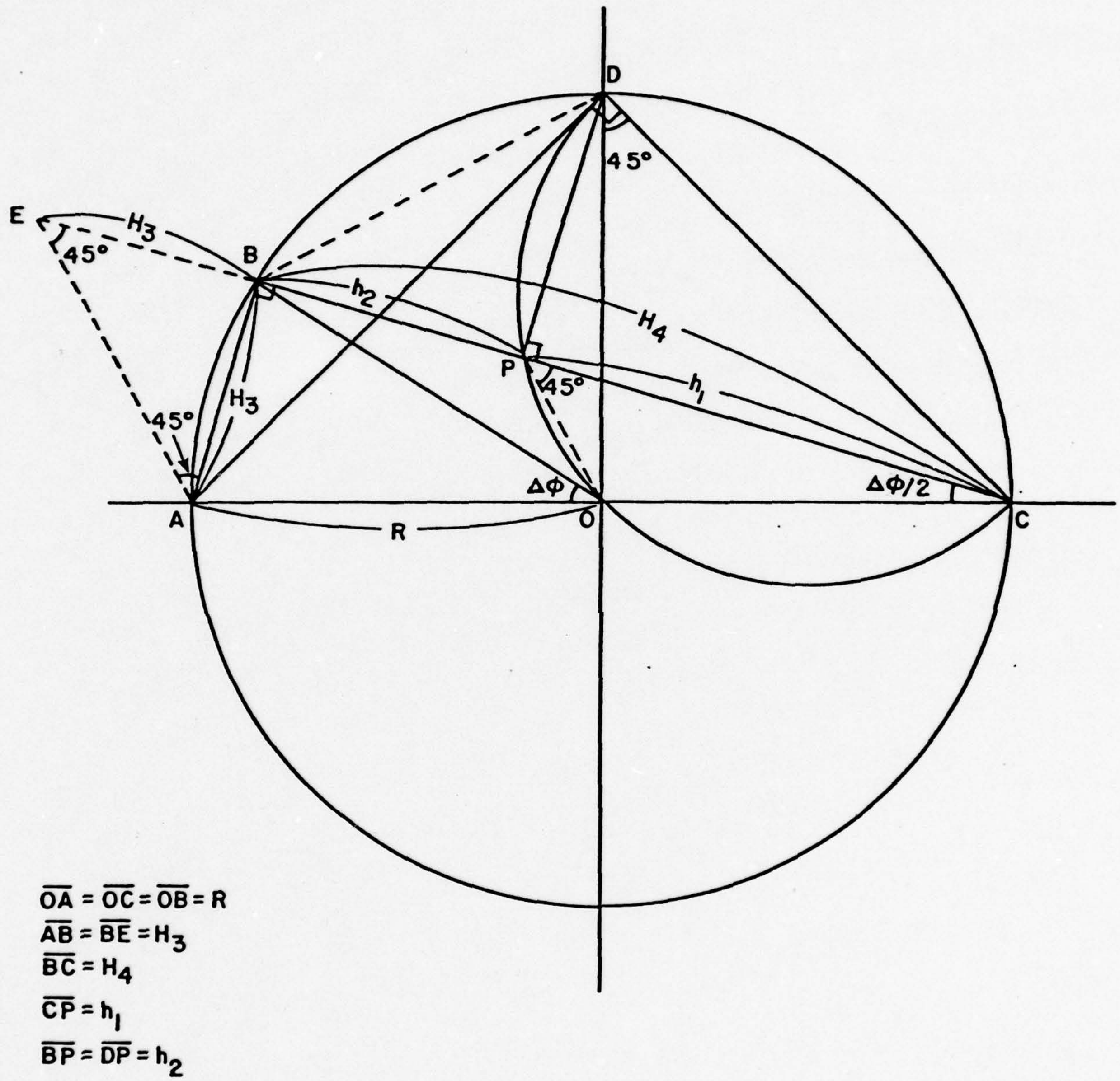


FIGURE F-1 GEOMETRICAL REPRESENTATION OF PARAMETERS $H_3$ AND $H_4$ IN TERMS OF $h_1$ AND $h_2$ (POWER IMBALANCE MODEL), AND R AND $\Delta\phi$ (PHASE IMBALANCE MODEL)

Thus

$$\lambda^2 = \tan^2(45^\circ - \Delta\phi/2) = \frac{\sin^2(45^\circ - \Delta\phi/2)}{\cos^2(45^\circ - \Delta\phi/2)} = \frac{1-\cos(90^\circ - \Delta\phi)}{1+\cos(90^\circ - \Delta\phi)}$$

$$= \frac{1-\sin\Delta\phi}{1+\sin\Delta\phi} \quad . \tag{F-8}$$

Equation (F-8) or (F-5) is significant in that it bridges, under the constraints of constant average power (F-2), the gap between the power imbalance model and the phase imbalance model. The relationship between $\lambda$ and $\Delta\phi$ as a function of the degree of imbalance can also be shown from the figure. When point "P" is close to point "O", we can see that $h_1 \approx h_2 \approx R$ and $\Delta\phi \approx 0$ which represents an ideal situation. As the point "P" moves toward point "D" along the arc OPD, the value of $\lambda = h_2/h_1$ decreases (power imbalance increases) and accordingly $\Delta\phi$ increases. As "P" approaches "D", $\lambda$ becomes zero (large power imbalance) and $\Delta\phi$ becomes $90^\circ$. Thus we can see that change of $\lambda$ from 1 to 0 corresponds to change of $\Delta\phi$ from $0^\circ$ to $90^\circ$.

# APPENDIX G

## COMPUTER PROGRAM LISTING
## FOR
## ERROR RATE PERFORMANCE
## OF
## DPSK OVER HARD-LIMITING
## SATELLITE LINK
## WITH
## POWER IMBALANCE AND CORRELATED NOISE
## AT THE PHASE DETECTOR

```
FORTRAN IV G LEVEL  21                    MAIN              DATE = 79038          16/55/57                    PAGE 0001

                C...COMPUTE P(E) FOR HARD LIMITER DPSK BY NUMERICAL INTEGRATION
0001                  DIMENSION RUSODB(16),RDSODB(3),FAC(53),B(53),V1(53),V2(53),
                     1DENS(51),F(51),WORK1(100),WORK2(100),OUT(16,3,3),IOUTT(16,3,2)
0002                  REAL*8 REXP(51),Z1(51),Z2(51)
0003                  DATA PI/3.141592654/
0004                  CALL FCTRL(FAC)
0005                  NOPNT=51
0006                  NPTS2=2*(NOPNT-1)
0007                  N2=NOPNT-2
0008                  DELB=2.*PI/FLOAT(NOPNT-1)
0009                  D3=DELB/3.
                C...SET NUM=NUMBER OF TERMS
0010                  NUM=50
0011                  ERR=1E-4
0012            9000  READ(5,301,END=9999)RHO,RAMDDB,NRD,NRU
0013                  RAMD=10.**(RAMDDB/10.)
0014                  SRAMD=SQRT(RAMD)
0015                  OPLAM=1.+RAMD
0016                  READ(5,302,END=9998)(RDSODB(IN),IN=1,NRD)
0017                  READ(5,302,END=9998)(RUSODB(IN),IN=1,NRU)
0018                  DO 10 II=1,NRU
0019                  RUSQ=10.**(RUSODB(II)/10.)
0020                  CALL DENST(NOPNT,NPTS2,DELB,RUSQ,DENS,WORK1,WORK2,KODE)
0021                  IF(KODE.EQ.0)GOTO998
0022                  WRITE(6,205) KODE
0023                  GOTO 10
0024             998  INDEX=1
0025             101  OPRHO=1.+RHO
0026                  OMRHO=1.-RHO
0027                  OMR2=1.-RHO*RHO
0028                  RHO2=2./OPRHO
0029                  CALL ARRY(RHO2,B)
0030                  C2=OPRHO*OPRHO/OMRHO/OMRHO
0031                  CALL ARRY(C2,V2)
0032                  DO 11 JJ=1,NRD
0033                  RDSQ=10.**(RDSODB(JJ)/10.)
0034                  A=RDSQ*OPLAM/OMR2
0035                  C1=.25*OMRHO*RDSQ*OPLAM/OPRHO
0036                  Y=2.*RHO*SRAMD*RDSQ/OMR2
0037                  Z=2.*SRAMD/OPLAM
0038                  CALL ARRY(C1,V1)
0039                  CALL COSFN(NOPNT,DELB,Y,Z,REXP,Z1,Z2)
                C...SUMMATIONS
0040                  SUM=0.
0041                  DO 20 K1=1,NUM
0042                  K=K1-1
0043                  SUML=0.
0044                  DO 25 L1=1,K1
0045                  L=L1-1
0046                  SUMJ=0.
```

```
FORTRAN IV G LEVEL  21                MAIN              DATE = 79038          16/55/57              PAGE 0002

                DO 30 J1=1,L1
                J=J1-1
         30     SUMJ=SUMJ+FAC(K-J+1)/FAC(K-L+1)/FAC(L-J+1)*B(J+1)
         C...ARRAY OF FUNCTION TO BE INTEGRATED
                DO 40 N=1,NOPNT
                Z1P=1.
                IF(.NOT.(Z1(N).EQ.0DC.AND.K-L.EQ.0))Z1P=Z1(N)**(K-L)
                Z2P=1.
                IF(.NOT.(Z2(N).EQ.0DC.AND.L.EQ.0))Z2P=Z2(N)**L
         40     F(N)=REXP(N)*Z1P*Z2P*DENS(N)
         C...SIMPSON'S RULE NUMERICAL INTEGRATION
                SINT=D3*(F(1)+4.*F(2)+F(3))
                IF(NOPNT.LE.4)GOTO3
                S1=0.
                S2=F(2)
                DO 1 I=3,N2,2
                S1=S1+F(I)
                S2=S2+F(I+1)
         1      CONTINUE
                SINT=D3*(F(1)+2.*S1+4.*S2+F(N))
         3      CONTINUE
                TERML=V2(L+1)/FAC(K-L+1)/FAC(L+1)*SUMJ*2.*SINT
                SUML=SUML+TERML
         25     CONTINUE
                TERMK=V1(K+1)*SUML
                SUM=SUM+TERMK
                RATIO=TERMK/SUM
         C       IF(MOD(K1,5).NE.0)GOTO999
         C       TEMP=.5*(1.-RHO)*EXP(-A)*SUM
         C       WRITE(6,203)K,RATIO,TEMP
         999    IF(ABS(TERMK).LE.ERR*ABS(SUM))GOTO22
         20     CONTINUE
                WRITE(6,204)NUM
         22     ANS=.5*(1.-RHO)*EXP(-A)*SUM
                OUT(II,JJ,INDEX)=ANS
                IOUTT(II,JJ,INDEX)=K
         11     CONTINUE
                IF(INDEX.EQ.2)GOTO12
                IF(RHO.EQ.0.)GOTO10
                INDEX=2
                RHO=-RHO
                GOTO 101
         12     RHO=-RHO
         10     CONTINUE
                IF(RHO.NE.0.)GOTO102
         C...SPECIAL CASE RHO=0.
                DO 110 KOPY=1,NRU
                DO 110 KOPIA=1,NRO
                IOUTT(KOPY,KOPIA,2)=IOUTT(KOPY,KOPIA,1)
                OUT(KOPY,KOPIA,2)=OUT(KOPY,KOPIA,1)
```

```
FORTRAN IV G LEVEL  21                MAIN                DATE = 79038        16/55/57              PAGE 0003

 0091         110   OUT(KOPY,KOPIA,3)=OUT(KOPY,KOPIA,1)
 0092               GOTO 103
 0093         102   DO 120 IAU=1,NRU
 0094               DO 120 IAD=1,NRD
 0095         120   OUT(IAU,IAD,3)=0.5*(OUT(IAU,IAD,1)+OUT(IAU,IAD,2))
 0096         103   DO 50 IOD=1,NRD
 0097               WRITE(6,206) RHO,RAMDDB,RDSQDB(IOD),(RUSQDB(IO),
                   $OUT(IO,IOD,1),IOUTT(IO,IOD,1),OUT(IO,IOD,2),IOUTT(IO,IOD,2),
                   $OUT(IO,IOD,3),IO=1,NRU)
 0098          50   CONTINUE
 0099               GOTO 9000
              C...PREMATURE EOF EXIT--WRITE MESSGE, STOP
 0100        9998   WRITE(6,207)
              C...FORMAT STATEMENTS
              C203    FORMAT(1X,'K=',I3,3X,'RATIO, TEMP P(E) = ',1P2E15.4)
 0101         204   FORMAT(' ERROR TEST NOT SATISFIED IN ',I3,' TERMS.')
 0102         205   FORMAT(' ERROR CODE =',I4)
 0103         206   FORMAT(///' RHO =',F5.2,5X,'LAMPDA**2 = ',F5.2,' DB',5X,
                   $'RD**2 = ',F5.1,' DB'/' SNR(UPLINK) (DB)',T25,'P(E/SPACE)',T40,
                   $'TERMS',T50,'P(E/MARK)',T65,'TERMS',T77,'P(E)'/16(6X,0PF5.1,T25,
                   $1PE11.4,T41,I3,T50,1PE11.4,T66,I3,T75,1PE11.4/))
 0104         207   FORMAT(' INSUFFICIENT DATA ENTERED.')
 0105         301   FORMAT(2F5.1,2I3)
 0106         302   FORMAT(16F5.1)
 0107        9999   STOP
 0108               END
```

```
FORTRAN IV G LEVEL  21                FCTRL               DATE = 79038        16/55/57              PAGE 0001

 0001               SUBROUTINE FCTRL(X)
 0002               DIMENSION X(53)
 0003               X(1)=1.
 0004               X(2)=1.
 0005               DO 10 J=3,53
 0006          10   X(J)=X(J-1)*(J-1)
 0007               RETURN
 0008               END
```

```
FORTRAN IV G LEVEL  21                ARRY                DATE = 79038        16/55/57              PAGE 0001

 0001               SUBROUTINE ARRY(CI,VI)
 0002               DIMENSION VI(53)
 0003               VI(1)=1.
 0004               DO 10 I=2,53
 0005          10   VI(I)=VI(I-1)*CI
 0006               RETURN
 0007               END
```

```
FORTRAN IV G LEVEL  21                COSFN               DATE = 79038        16/55/57              PAGE 0001

 0001               SUBROUTINE COSFN(NOPNT,DELB,Y,Z,REXP,Z1,Z2)
 0002               REAL*8 REXP(NOPNT),Z1(NOPNT),Z2(NOPNT),BETA,DCBETA,IMI
 0003               DO 10 I=1,NOPNT
 0004               IMI=I-1
 0005               BETA=DELB*IMI
 0006               DCBETA=DCOS(BETA)
 0007               REXP(I)=DEXP(Y*DCBETA)
 0008               Z1(I)=1.D0+Z*DCBETA
 0009               Z2(I)=1.D0-Z*DCBETA
 0010          10   CONTINUE
 0011               RETURN
 0012               END
```

SEE APPENDIX I
FOR LISTING
OF FUNCTION
SUBPROGRAM ONEF1.

```
FORTRAN IV G LEVEL  21                 DENST             DATE = 79038          16/55/57              PAGE 0001

              SUBROUTINE DENST(NOPNT,NPTS2,DELBET,RUSQ,DENS,SNN,B,KODE)
              DIMENSION SNN(NPTS2),B(NPTS2),DENS(NOPNT)
              DATA PI/3.14159265/,TWOPI/6.283185307/,TPSQ/39.4784176 1/
              KODE=0
              RU=SQRT(RUSQ)
              RUSQN=-RUSQ
              PB=2.*RU
              GB=0.886226926
        C GENERATE TABLE OF B(M)
              B(1)=PB*GB*ONEF1(.5,2.,RUSQN)
              NPTS3=NPTS2-1
              DO 850 IB=3,NPTS3,2
              F=IB
              PB=PB*RUSQ/(F*(F-1.))
              GB=GB*F*.5
              B(IB)=PB*GB*ONEF1(F*.5,F+1.,RUSQN)
        850   CONTINUE
              PB=2.
              GB=1.
              DO 860 IB=2,NPTS2,2
              F=IB
              PB=PB*RUSQ/(F*(F-1.))
              GB=GB*F*.5
              B(IB)=PB*GB*ONEF1(F*.5,F+1.,RUSQN)
        860   CONTINUE
              DO 700 IBETA=1,NOPNT
              IBM1=IBETA-1
              BETA=DELBET*IBM1
              P=TWOPI-BETA
        C GENERATE TABLE OF SIN(M*BETA)
              DO 701 ISU=1,NPTS2
              F=ISU
        701   SNN(ISU)=SIN(F*BETA)
              SUM=0.
              COEF=TWOPI-BETA
              DO 600 M=1,100
              FM=M
              BETAM=FM*BETA
              TERM=B(M)*B(M)*.5*(COEF*COS(BETAM)-SNN(M)/FM)
              SUM=SUM+TERM
              IF(ABS(TERM).LE.1E-5*ABS(SUM))GOTO630
        600   CONTINUE
              WRITE(6,2)
        2     FORMAT(' FIRST SUM DID NOT CONVERGE.')
              KODE=IBETA
              RETURN
        630   P=P+SUM
              PM1M=1.
              SUMM=0.
              DO 500 M=1,100

FORTRAN IV G LEVEL  21                 DENST             DATE = 79038          16/55/57              PAGE 0002

              FM=M
              PM1M=-PM1M
              SQM=FM*FM
              SM2M=FM*SNN(M)
        C HANDLE N=0 SPECIALLY
              N=0
              SUMN=SNN(M)/FM
        C SET UP FOR N.GE.1
              IF(M.EQ.1)GOTO530
              PM1N=1.
              MM1=M-1
              DO 400 N=1,MM1
              FN=N
              SQN=FN*FN
              PM1N=-PM1N
              SN2N=FN*SNN(N)
              TERMN=B(N)*PM1N*(SM2M-FN*SNN(N))/(SQM-SQN)
              SUMN=SUMN+TERMN
        400   CONTINUE
        530   TERM=B(M)*PM1M*SUMN
              SUMM=SUMM+TERM
              IF(ABS(TERM).LE.ABS(SUMM)*1E-5)GOTO540
        500   CONTINUE
              WRITE(6,3)
        3     FORMAT(' DOUBLE SUM DID NOT CONVERGE.')
              KODE=-IBETA
              RETURN
        540   P=P-2.*SUMM
              P=P/TPSQ
              IF(P.LT.1E-7)P=0.
              DENS(IBETA)=P
        700   CONTINUE
              RETURN
              END
```

## APPENDIX H

## ERROR BEHAVIOR OF A BINARY DPSK SYSTEM DUE TO INTERSYMBOL INTERFERENCE AND CORRELATED NOISE*

It is well known that the classical result of the error probability for differentially coherent detection of binary PSK applies only to the ideal situation where the received symbol signal energy from pulse to pulse is assumed equal and the noises at the sampling instants uncorrelated. The purpose of this appendix is to show the error behavior of a differential phase shift-keying (DPSK) system under practical assumptions where the signal powers at sampling instants between the adjacent pulses are unequal and the noises correlated. Error performance is presented graphically for different levels of power imbalance. A significant result observed is that the probability of error is independent of noise correlation for all degrees of intersymbol interference (power imbalance). This result (based on our computations) is clearly a departure from the previous beliefs.

### ANALYSIS MODEL

The binary DPSK system under consideration is depicted in Figure H-1. The primary objectives of this appendix are the considerations of the error performance calculations under the assumptions where $(SNR)_1 \neq (SNR)_2$ and the noises $n_1(t)$ and $n_2(t)$ are statistically dependent (see Figure H-1). The situation where $(SNR)_1$ is different from $(SNR)_2$ is brought about by unequal signal powers at the phase detector of the DPSK demodulator, due to

* This appendix is based largely on [5] with a change in the definition of the SNR parameter. Minor typographical errors in [5] have also been corrected.

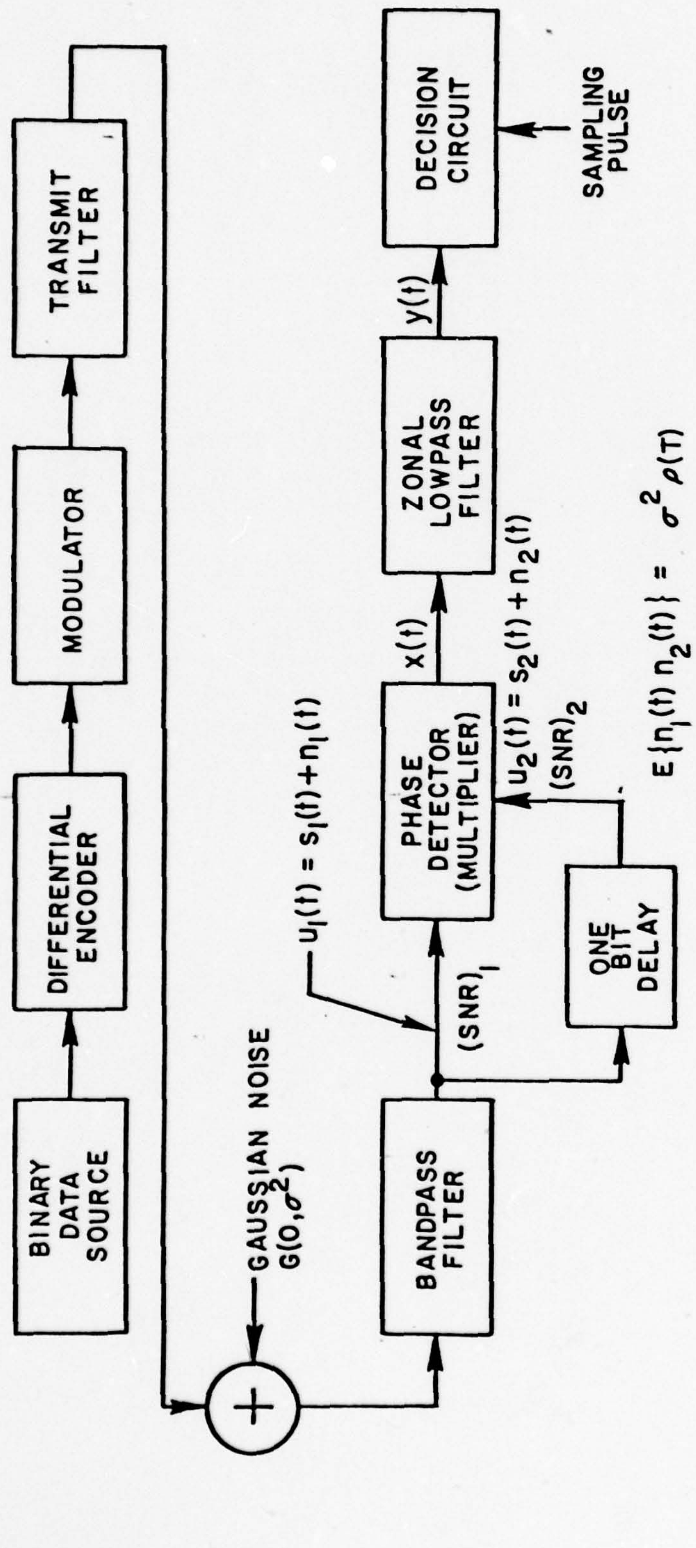


FIGURE H-1 BLOCK DIAGRAM OF BINARY DPSK SYSTEM UNDER CONSIDERATION such phenomena as intersymbol interference, phase errors in the delay circuit and/or delay circuit attenuation. Thus, the error performance analysis based on the unequal signal powers over two consecutive symbol signal pulses would constitute an upper bound of error probability in the presence of intersymbol interference. The noise correlation is a practical assumption since the noise is necessarily bandlimited in practice.

In Figure H-1, the two inputs at the phase detector (multiplier), $u_1(t)$ and $u_2(t)$, represent the noisy received waveforms for two consecutive source symbols:

$$u_1(t) = S_1(t) + n_1(t) \tag{H-1}$$

$$u_2(t) = S_2(t) + n_2(t) \tag{H-2}$$

where $S_i(t)$, i=1,2, are the bandpass information carrying signals; and $n_i(t)$, i=1,2, bandpass noises. Assume that the "present" and the "preceding" source symbols are identified with carrier phases $\theta_1$ and $\theta_2$, respectively. Then we may write

$$S_1(t) = \sqrt{2P_1}\cos(\omega t - \theta_1) \tag{H-3}$$

$$\begin{aligned} S_2(t) &= \sqrt{2P_2}\cos[\omega(t-T) - \theta_2] \\ &= \sqrt{2P_2}\cos(\omega t - \theta_2) \end{aligned} \tag{H-4}$$

where the carrier frequency is assumed to be chosen such that $\omega T = 2\pi k$, k integer, and $P_1$ and $P_2$ are the carrier powers at the phase detector. We shall be primarily concerned with the case where $P_1 \neq P_2$.

The bandpass Gaussian noises are expressed in the forms:

$$n_1(t)=X_1(t)\cos\omega t + Y_1(t)\sin\omega t \tag{H-5}$$

$$\begin{aligned} n_2(t) &= X_1(t-T)\cos[\omega(t-T)] + Y_1(t-T)\sin[\omega(t-T)] \\ &\equiv X_2(t)\cos\omega t + Y_2(t)\sin\omega t \end{aligned} \tag{H-6}$$

and the noise correlation is defined by

$$E\{n_1(t)n_2(t)\} = E\{n_1(t)n_1(t-T)\} = \sigma^2\rho(T) \qquad \text{(H-7)}$$

where $\rho(T)$ is the normalized correlation with T equal to symbol duration, and the noise power is given by

$$\sigma^2 \triangleq E\{n_1^2(t)\} = E\{X_i^2(t)\} = E\{Y_i^2(t)\};\ i=1,2\,. \qquad \text{(H-8)}$$

The decision variable Y(t) at time t is the output of the zonally low pass filtered version of

$$\begin{aligned} X(t) &= u_1(t) \times u_2(t) \\ &= [\sqrt{2P_1}\cos(\omega t - \theta_1) + X_1(t)\cos\omega t + Y_1(t)\sin\omega t] \\ &\quad \cdot [\sqrt{2P_2}\cos(\omega t - \theta_2) + X_2(t)\cos\omega t + Y_2(t)\sin\omega t]\,. \end{aligned} \qquad \text{(H-9)}$$

Since $\theta_1 - \theta_2$ is either 0 or $\pm\pi$ in a binary DPSK system, the binary decision is based on the comparison of the decision variable Y(t) with "zero" threshold, for, when noises are assumed to be absent at the input, the decision variable is given by

$$Y(t) = \sqrt{P_1P_2}\cos(\theta_1 - \theta_2)\,. \qquad \text{(H-10)}$$

## DECISION VARIABLE STATISTICS

The problem of obtaining the probability density function (pdf) of the lowpass filter output for the system model that fits our situation, depicted in Figure H-1, was solved by Miller and Lee [4] in great generality. Our need here is a special case of the problem treated in [4] . From [4, eq.(25)] we have the pdf for the decision variables y(t) as follows:

$$f(y;\rho|\theta_1-\theta_2) = \frac{1}{\sigma^2}\exp\left[-(H_3^2+H_4^2)\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{1}{2}(1-\rho)H_3^2\right]^m\frac{1}{n!}\left[\frac{1}{2}(1+\rho)H_4^2\right]^n$$

$$\cdot\begin{cases} \exp\left[\dfrac{-2y}{\sigma^2(1+\rho)}\right]G_m^n\left[\dfrac{4y}{\sigma^2(1-\rho^2)}\right], & y\geq 0 \\ \exp\left[\dfrac{2y}{\sigma^2(1-\rho)}\right]G_n^m\left[\dfrac{-4y}{\sigma^2(1-\rho^2)}\right], & y<0 \end{cases} \qquad \text{(H-11)}$$

where

$$H_3^2 \triangleq \frac{1}{2(1+\rho)}\left[h_1^2 + h_2^2 + 2h_1h_2\cos(\theta_1 - \theta_2)\right] \quad \text{(H-12a)}$$

$$H_4^2 \triangleq \frac{1}{2(1-\rho)}\left[h_1^2 + h_2^2 - 2h_1h_2\cos(\theta_1 - \theta_2)\right] \quad \text{(H-12b)}$$

$$G_m^n(z) \triangleq \sum_{k=0}^{m}\binom{m+n-k}{n}\frac{z^k}{k!} \quad \text{(H-12c)}$$

and where

$$h_i^2 \triangleq \frac{P_i}{\sigma^2} \equiv (SNR)_i;\ i=1,2 \quad \text{(H-12d)}$$

and $P_i$, $\sigma^2$, $\rho\equiv\rho(T)$, are the signal power, noise power and noise correlation, respectively, as defined earlier.

If we assume that the reference phase $\theta_2$ is 0, then $\theta_1$ is either 0 or $\pi$, depending upon whether the symbol following the reference symbol is 0 (space) or 1 (mark), so that $\theta_1 - \theta_2 = 0$ or $\theta_1 - \theta_2 = \pi$. The conditional pdf's are thus obtained as follows:

$$f(y;\rho|\text{space}) = f(y;\rho|\theta_1-\theta_2=0) \quad \text{(H-13a)}$$

$$f(y;\rho|\text{mark}) = f(y;\rho|\theta_1-\theta_2=\pi). \quad \text{(H-13b)}$$

## CONDITIONAL ERROR PROBABILITIES

From (H-13a) and (H-13b) one can obtain the conditional probabilities of error from the expressions

$$P(e;\rho|\text{space}) = \text{Prob}\{y<0|\theta_1-\theta_2=0\} = \int_{-\infty}^{0} f(y;\rho|\theta_1-\theta_2=0)dy \quad \text{(H-14a)}$$

and

$$P(e;\rho|\text{mark}) = \text{Prob}\{y>0|\theta_1-\theta_2=\pi\} = \int_{0}^{\infty} f(y;\rho|\theta_1-\theta_2=\pi)dy. \quad \text{(H-14b)}$$

Carrying out the integrations indicated by (H-14a) and (H-14b) using the density functions given in (H-11), one obtains the following results:

$$P(e;\rho|\text{space}) = \frac{1-\rho}{2} \exp\left[-\frac{1}{1-\rho^2}(h_1^2 + h_2^2 - 2\rho h_1 h_2)\right]$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \frac{2^{-(m+n)}}{m!n!} \binom{n+m-k}{m}$$

$$\cdot \left[\frac{1}{2}\left(\frac{1-\rho}{1+\rho}\right)(h_1 + h_2)^2\right]^m \left[\frac{1}{2}\left(\frac{1+\rho}{1-\rho}\right)(h_1 - h_2)^2\right]^n \left(\frac{2}{1+\rho}\right)^k \qquad \text{(H-15a)}$$

and

$$P(e;\rho|\text{mark}) = \frac{1+\rho}{2} \exp\left[-\frac{1}{1-\rho^2}(h_1^2 + h_2^2 + 2\rho h_1 h_2)\right]$$

$$\cdot \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{2^{-(m+n)}}{m!n!} \binom{m+n-k}{n}$$

$$\cdot \left[\frac{1}{2}\left(\frac{1-\rho}{1+\rho}\right)(h_1 - h_2)^2\right]^m \left[\frac{1}{2}\left(\frac{1+\rho}{1-\rho}\right)(h_1 + h_2)^2\right]^m \left(\frac{2}{1-\rho}\right)^k . \qquad \text{(H-15b)}$$

It is interesting to observe a "symmetry" property of

$$P(e;\rho|\text{space}) = P(e;-\rho|\text{mark}) . \qquad \text{(H-16)}$$

## ERROR PROBABILITY EXPRESSIONS

To compute the error rates using the equations (H-15a) and (H-15b), we need to define an important variable. In a DPSK system, a single symbol decision is made using two symbols. Since we are considering a situation where each symbol signal energy is not equal (due to intersymbol interference, for example), it is appropriate to define the signal-to-noise power ratio per pulse (or symbol) as follows:

$$R^2 = h_1^2 = \text{direct channel SNR.} \qquad \text{(H-17)}$$

When $h_1^2 = h_2^2$ (a classical case), there is no difference between this definition and the conventional SNR definition per symbol.

Define, further, an SNR difference measure by

$$\lambda^2 = \frac{h_2^2}{h_1^2} \equiv \frac{(SNR)_2}{(SNR)_1} . \qquad \text{(H-18)}$$

In terms of $R^2$ and $\lambda^2$, the conditional error probabilities of (H-15a) and (H-15b) are given by

$$P(e;\rho;\lambda|\text{space}) = \frac{1-\rho}{2} \exp\left[-\frac{R^2}{1-\rho^2}\left(1+\lambda^2-2\rho\lambda\right)\right]$$

$$\cdot \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{2^{-(m+n)}}{m!n!}\binom{n+m-k}{m}\left(\frac{1-\rho}{1+\rho}\right)^{m-n}$$

$$\cdot \left[\frac{1}{2}(1+\lambda)^2\right]^m \left[\frac{1}{2}(1-\lambda)^2\right]^n \left(\frac{2}{1+\rho}\right)^k (R^2)^{m+n} \qquad \text{(H-19a)}$$

and

$$P(e;\rho;\lambda|\text{mark}) = \frac{1+\rho}{2} \exp\left[-\frac{R^2}{1-\rho^2}\left(1+\lambda^2+2\rho\lambda\right)\right]$$

$$\cdot \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{2^{-(m+n)}}{m!n!}\binom{m+n-k}{n}\left(\frac{1-\rho}{1+\rho}\right)^{m-n}$$

$$\left[\frac{1}{2}(1+\lambda)^2\right]^m \left[\frac{1}{2}(1-\lambda)^2\right]^n \left(\frac{2}{1-\rho}\right)^k (R^2)^{m+n} . \qquad \text{(H-19b)}$$

The total unconditional probability of error is the weighted sum of (H-19a) and (H-19b) given by

$$P(e;\rho;\lambda) = P_S P(e;\rho;\lambda|\text{space}) + P_M P(e;\rho;\lambda|\text{mark}) \qquad \text{(H-20)}$$

where $P_S$ and $P_M$ are the <u>a priori</u> probabilities of space (0) and mark (1).

## ERROR BEHAVIOR UNDER SPECIAL CONDITIONS

If $\lambda^2=1$ is substituted into (H-19a) and (H-19b), we obtain

$$P(e;\rho|\text{space}) = \frac{1-\rho}{2}\exp(-R^2) \tag{H-21a}$$

$$P(e;\rho|\text{mark}) = \frac{1+\rho}{2}\exp(-R^2) \tag{H-21b}$$

and the unconditional error probability of (H-20) becomes

$$P(e;\rho) = \frac{1}{2}\left[1 + (P_M - P_S)\rho\right]e^{-R^2}. \tag{H-22}$$

This is the identical result given by Lee and Miller [7]. The significance of the result given by (H-22) is that the probability of error depends on noise correlation $\rho$ only if a priori probabilities are unequal. When $P_S = P_M = \frac{1}{2}$, (H-22) reduces to the classical error rate expression of $P(e) = \frac{1}{2}e^{-R^2}$.

Now, let us assume that mark and space are equi-probable: $P_M = P_S = \frac{1}{2}$. When one observes the conditional probabilities of error as given in (H-19a) and (H-19b), it appears certain that the total unconditional probability of error given in (H-20) still depends upon noise correlation $\rho$. In fact, it has been remarked in the previous publications [6], [7] that the error probability of a binary DPSK system depends on the correlation if there is intersymbol interference and that, in the absence of intersymbol interference, the error probability is independent of the noise correlation provided that $P_M = P_S = \frac{1}{2}$.

We have computed the probability of error expression (H-20) with $P_M = P_S = \frac{1}{2}$ for various values of $\lambda^2$ (different degrees of intersymbol interference) and $\rho$. We have observed the surprising results that the computed probabilities are independent of noise correlation $\rho$ for all values of $\lambda^2$ considered! The conditional probabilities, however, were dependent upon noise correlations. When these conditional error probabilities were added with equal weighting ($P_M = P_S = \frac{1}{2}$), the result converged to the values of error probability for the case $\rho=0$. To show the error mechanism, we have plotted in Figures H-2 and H-3 some specific cases of conditional error probabilities. Figure H-4 shows the error probabilities of a DPSK system for $\lambda^2$ = 0 dB (no intersymbol interference), -1 dB, -2 dB, and -3 dB. Note that $\lambda^2$ = 0 dB corresponds to the ideal classical case. Our results indicate that the probability of error for a binary DPSK system depends on noise correlation $\rho$ only when the message symbol probabilities are unequal. Whenever the prior probabilities are equal the error rate is independent of noise correlation whether or not there is intersymbol interference. As stated earlier, this is a departure from the previous beliefs. It must be stressed, however, our findings are based on the computational results. The mathematical complexities of the error rate expressions did not lend themselves to an analytical verification of the computational results observed.

## ALTERNATE FORMS FOR THE ERROR RATE EXPRESSIONS

The error rate expression in (H-19a) and (H-19b) may be written in several alternate forms. First, we can use the relation

$$\binom{n+m-k}{m} \triangleq 0, \quad k>n \tag{H-23}$$

to replace the finite upper limit of the summation over k in (H-19a) and (H-19b) by infinity. If we then let $\ell = n-k$ in (H-19a) we obtain

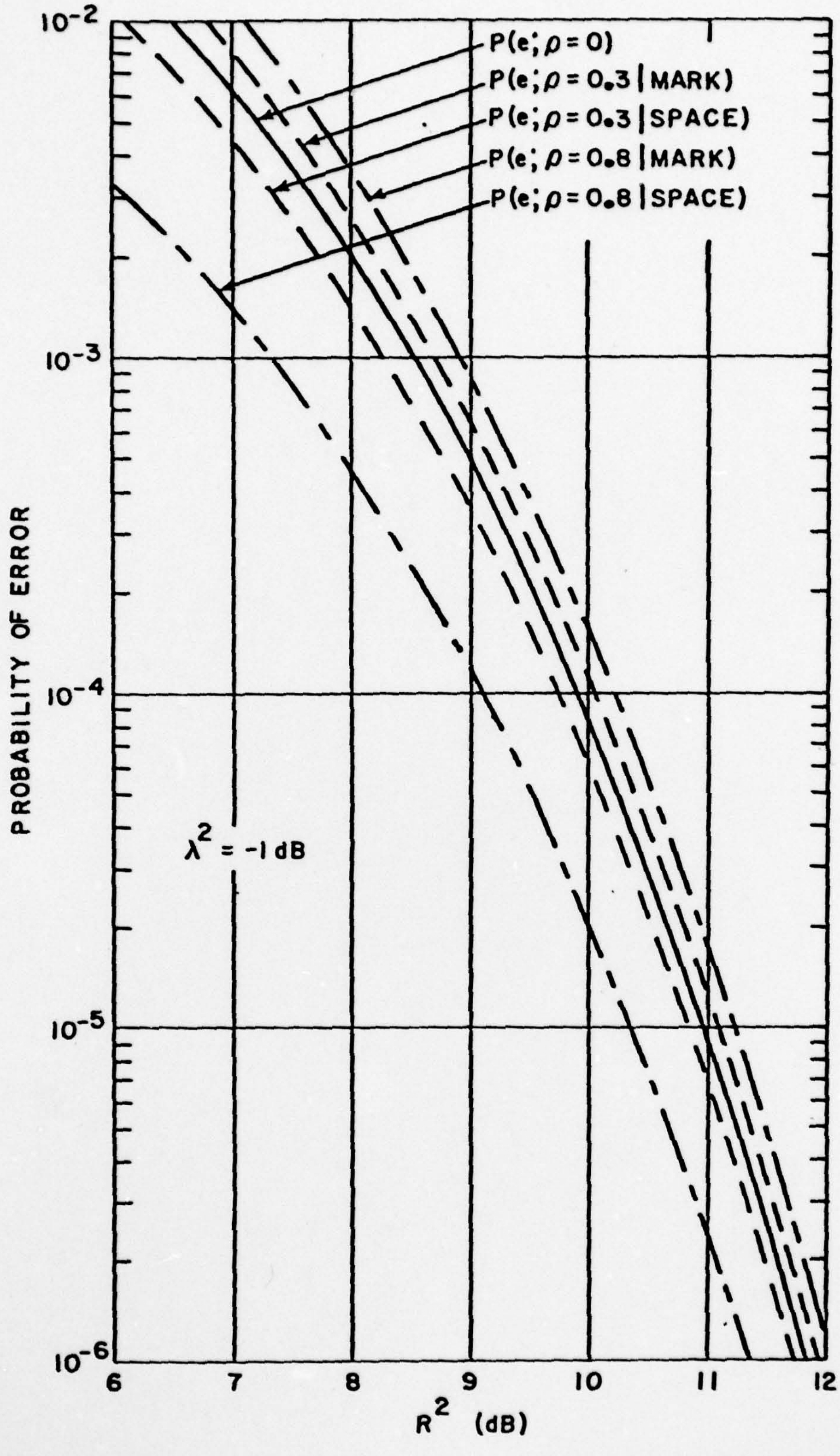


FIGURE H-2 INFLUENCE OF NOISE CORRELATION ON CONDITIONAL ERROR PROBABILITIES FOR POWER IMBALANCE $\lambda^2$= -1 dB

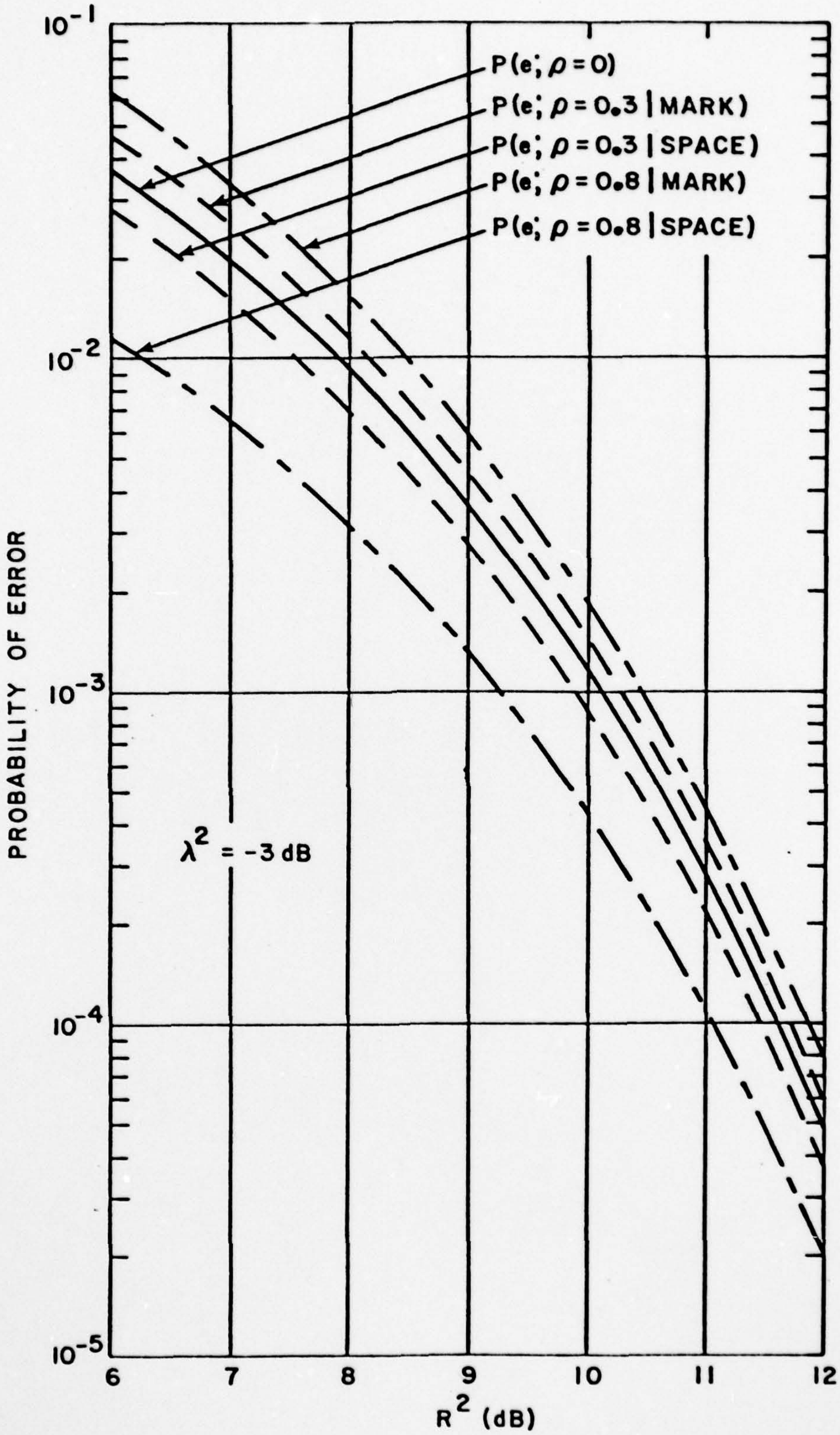


FIGURE H-3 INFLUENCE OF NOISE CORRELATION ON CONDITIONAL ERROR PROBABILITIES FOR POWER IMBALANCE $\lambda^2 = -3$ dB

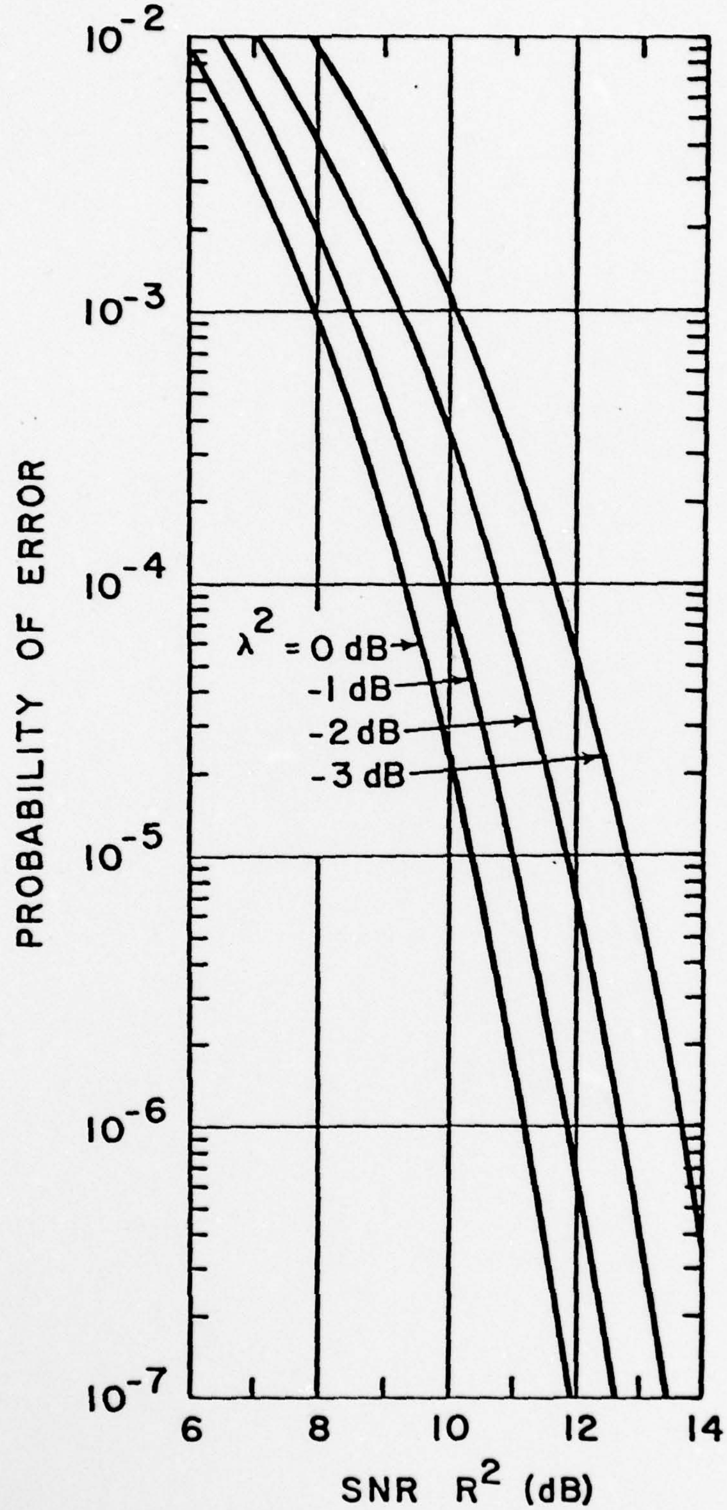


FIGURE H-4 INFLUENCE OF POWER IMBALANCE (DUE TO INTERSYMBOL INTERFERENCE) ON PROBABILITY OF ERROR

$$P(e;\rho,\lambda|\text{space}) = \frac{1-\rho}{2} \exp\left[-\frac{R^2}{1-\rho^2}(1+\lambda^2 - 2\rho\lambda)\right]$$

$$\cdot \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{m!(k+\ell)!} \binom{m+\ell}{m} \left[\frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda^2)\right]^m$$

$$\cdot \left[\frac{R^2}{4}\left(\frac{1+\rho}{1-\rho}\right)(1-\lambda)^2\right]^{\ell+k} \left(\frac{2}{1+\rho}\right)^k . \qquad \text{(H-24)}$$

A similar form can also be obtained from (H-19b) for the case of a mark being transmitted; for brevity we will not go into the details of that case in the following discussions.

Interchanging the order of summations over k and $\ell$ in (H-24), the summation over k is recognized as a confluent hypergeometric function:

$$\sum_k = \frac{1}{\ell!}\, {}_1F_1\left[1;\ \ell+1;\ \frac{R^2}{2}\ \frac{(1-\lambda)^2}{1-\rho}\right] . \qquad \text{(H-25a)}$$

Applying Kummer's transformation [10, eq. 13.1.27] to (H-25a), we obtain

$$\sum_k = \frac{1}{\ell!} \exp\left[\frac{R^2}{2}\ \frac{(1-\lambda)^2}{1-\rho}\right] {}_1F_1\left[\ell;\ell+1;-\frac{R^2}{2}\ \frac{(1-\lambda)^2}{1-\rho}\right] . \qquad \text{(H-25b)}$$

We next interchange the order of the summations over $\ell$ and m and recognize the summation over m as a confluent hypergeometric function:

$$\sum_m = {}_1F_1\left[\ell+1;1;\ \frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda)^2\right] . \qquad \text{(H-26a)}$$

Applying Kummer's transformation [10, 13.1.27] to (H-26a) we find that (H-26a) can also be written as a Laguerre polynomial [10, 13.6.9]:

$$\sum_m = \exp\left[\frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda)^2\right] L_\ell\left[-\frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda)^2\right] \qquad \text{(H-26b)}$$

Using (H-25b) and (H-26b) in (H-24) we find the alternate form

$$P(e;\rho,\lambda|\text{space}) = \frac{1-\rho}{2}\exp\left[-\frac{R^2}{4}(1+\lambda)^2\right]\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left[\frac{R^2}{4}\left(\frac{1+\rho}{1-\rho}\right)(1-\lambda)^2\right]^{\ell} L_{\ell}\left[-\frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda)^2\right]$$

$$\cdot\ {}_1F_1\left[\ell;\ell+1;\ -\frac{R^2}{2}\ \frac{(1-\lambda)^2}{1-\rho}\right]. \qquad \text{(H-27a)}$$

Similarly for mark we can derive the form

$$P(e;\rho,\lambda|\text{mark}) = \frac{1+\rho}{2}\exp\left[-\frac{R^2}{4}(1+\lambda)^2\right]\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left[\frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1-\lambda)^2\right]^{\ell} L_{\ell}\left[-\frac{R^2}{4}\left(\frac{1+\rho}{1-\rho}\right)(1+\lambda)^2\right]$$

$$\cdot\ {}_1F_1\left[\ell;\ell+1;\ -\frac{R^2}{2}\ \frac{(1-\lambda)^2}{1+\rho}\right]. \qquad \text{(H-27b)}$$

The forms (H-27a) and (H-27b) are more computationally efficient than (H-19a) and (H-19b) and are the forms implemented in the computer program listing contained in Appendix I.

Returning to (H-24), we can write another alternate form for the error rate expressions. Using the relation [2, eq. A.1.44c]

$$n! = (1)_n \qquad \text{(H-28)}$$

where $(z)_n$ is Pochhammer's symbol [10, eq. 6.1.22] the triple summation in (H-24) can be recognized as a confluent hypergeometric function of three variables [16, eq. 2.13] and we find that

$$P(e;\rho,\lambda|\text{space}) = \frac{1-\rho}{2}\exp\left[-\frac{R^2}{1-\rho^2}(1+\lambda^2-2\rho\lambda)\right]\ {}_3\Phi_F^{(3)}(1,1,1;1,1,1;X_s,Y_s,Z_s) \qquad \text{(H-29a)}$$

where

$$X_s = \frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1+\lambda)^2,$$

$$Y_s = \frac{R^2(1-\lambda)^2}{2(1-\rho)} ,$$

and

$$Z_s = \frac{R^2}{4}\left(\frac{1+\rho}{1-\rho}\right)(1-\lambda)^2 .$$

Similarly we can derive the other case

$$P(e;\rho,\lambda|\text{mark}) = \frac{1+\rho}{2} \exp\left[-\frac{R^2}{1-\rho^2}(1+\lambda^2+2\rho\lambda)\right] {}_3\Phi_F^{(3)}(1,1,1;1,1,1;X_m,Y_m,Z_m) \tag{H-29b}$$

where

$$X_m = \frac{R^2}{4}\left(\frac{1+\rho}{1-\rho}\right)(1+\lambda)^2,$$

$$Y_m = \frac{R^2(1-\lambda)^2}{2(1+\rho)} ,$$

and

$$Z_m = \frac{R^2}{4}\left(\frac{1-\rho}{1+\rho}\right)(1-\lambda)^2.$$

The forms (H-29a) and (H-29b) are more compact than other notations, but their full analytical importance is unknown at this time since the mathematical properties of these functions have not been fully explored.

## CONCLUSIONS

In this appendix we have presented the error behaviors of a binary DPSK system under influences of intersymbol interference and noise correlations. The graphically presented error probability curves are applicable to system performance evaluations when SNR imbalance at the phase detector is known. Our results show that in a DPSK system, the error probability does not depend on noise correlation when the symbol probabilities are equi-probable regardless whether there is intersymbol interference or not. The noise correlation affects the error probability only when the message symbol probabilities are unequal.

# APPENDIX I

## COMPUTER PROGRAM LISTING
## FOR
## ERROR RATE PERFORMANCE
## OF
## DPSK OVER A TERRESTRIAL LINK
## WITH
## POWER IMBALANCE AND CORRELATED NOISE
## AT THE PHASE DETECTOR

```
FORTRAN IV G LEVEL  21             MAIN             DATE = 79040        08/53/58            PAGE 0001

                IMPLICIT REAL*8(A-H,L,O-Z)
                DIMENSION RSQLIS(16)
        1000    READ(5,1,END=9000) RHO,LSQDB,NRSQ
        1       FORMAT(2F5.1,I3)
                READ(5,2001)(RSQLIS(INP),INP=1,NRSQ)
        2001    FORMAT(16F5.1)
                WRITE(6,2)RHO,LSQDB
        2       FORMAT(///' RHO=',F5.1,5X,'LAMBDA**2=',F5.1,' DB'/
               *' R**2 (DB)',T23,'P(E)',T40,'P(E(MARK)',T60,'P(E(SPACE)')
                LSQ=1D1**(LSQDB/1D1)
                LAMBDA=DSQRT(LSQ)
                OPRHO=1D0+RHO
                OMRHO=1D0-RHO
                OPOOM=OPRHO/OMRHO
                OMOOP=OMRHO/OPRHO
                OPL=1.D0+LAMBDA
                OPL2=OPL*OPL
                OML=1.D0-LAMBDA
                OML2=OML*OML
                DO 900 IRS=1,NRSQ
                RSQDB=RSQLIS(IRS)
                RSQ=1D1**(RSQDB/1D1)
                XARG=-0.25D0*RSQ*OPL2
                TP=.25D0*OMOOP*OML2*RSQ
                TN=.25D0*OPOOM*OML2*RSQ
                LP=-.25D0*RSQ*OPOOM*OPL2
                LN=-.25D0*RSQ*OMOOP*OPL2
                FP=-.5D0*RSQ*OML2/OPRHO
                FN=-.5D0*RSQ*OML2/OMRHO
                XX=DEXP(XARG)
                CALL DOSUM(TP,LP,FP,SP)
                PEP=.5D0*XX*OPRHO*SP
                CALL DOSUM(TN,LN,FN,SN)
                PEN=.5D0*XX*OMRHO*SN
                PE=.5D0*(PEP+PEN)
                WRITE(6,3)RSQDB,PE,PEP,PEN
        3       FORMAT(3X,F4.1,T20,1PD10.3,T40,1PD10.3,T60,1PD10.3)
        900     CONTINUE
                GOTO 1000
        9000    STOP
                END
```

```
FORTRAN IV G LEVEL  21             DOSUM            DATE = 79040        08/53/58            PAGE 0001

                SUBROUTINE DOSUM(T,ARGL,ARGF,SUM)
                IMPLICIT REAL*8(A-H,L,O-Z)
        C M=0 TERM
                SUM=1D0
                COEF=1D0
                TERM=1D0
                MSTOP=5D1*DABS(T)
                IF(MSTOP.LT.102)MSTOP=102
                DO 100 M=1,MSTOP
                FM=M
                FMP1=FM+1D0
                COEF=COEF*(T/FM)
                OTERM=TERM
                LGR=LAGERR(M,ARGL)
                OFO=ONEF1(FM,FMP1,ARGF)
                TERM=COEF*LGR*OFO
                SUM=SUM+TERM
                IF(DABS(TERM).GT.DABS(OTERM))GOTO100
                IF(DABS(TERM).LT.DABS(SUM)*1D-6)GOTO200
        100     CONTINUE
                WRITE(6,1)MSTOP
        1       FORMAT(' THE SUM DID NOT CONVERGE, MSTOP=',I10)
                STOP
        200     RETURN
                END
```

```
FORTRAN IV G LEVEL  21                LAGERR            DATE = 79040        08/53/58           PAGE 0001

              REAL FUNCTION LAGERR*8(N,X)
              IMPLICIT REAL*8(A-H,O-Z)
              IF(N)1,1,2
      1       LAGERR=1D0
              RETURN
      2       S=1D0
              IF(X)3,4,3
      3       T=1D0
              DO 100 M=1,N
              A=M
              T=T*(N-M+1D0)/(A*A)
              T=T*X
              T=-T
              S=S+T
    100       CONTINUE
      4       LAGERR=S
              RETURN
              END
```

```
FORTRAN IV G LEVEL  21                ONEF1             DATE = 79040        08/53/58           PAGE 0001

              REAL FUNCTION ONEF1*8(A,B,Z)
              IMPLICIT REAL*8(A-H, O-Z)
              T=1D0
              S=1D0
              AM1=A-1D0
              BM1=B-1D0
              DO 100 I=1,100
              C=I
              T=T*Z/C
              T=T*((AM1+C)/(BM1+C))
              S=S+T
              IF(DABS(T/S).LT.1D-8) GOTO200
    100       CONTINUE
              WRITE(6,1)A,B,Z
      1       FORMAT(' 1F1(',D15.8,',',D15.8,',',D15.8,') NOT EVALUATED TO 10**-
             #8 ACCURACY')
    200       ONEF1=S
              RETURN
              END
```

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ERROR PERFORMANCE OF DIFFERENTIALLY COHERENT DETECTION OF BINAR--ETC(U)
AUG 79 J S LEE, R H FRENCH, Y K HONG N00014-77-C-0056

UNCLASSIFIED JTR-79-06 NL

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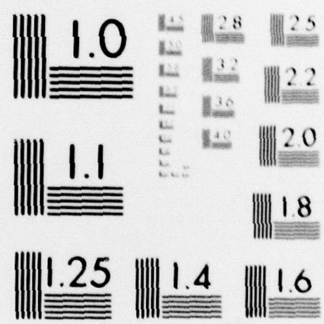


MICROCOPY RESOLUTION TEST CHART

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