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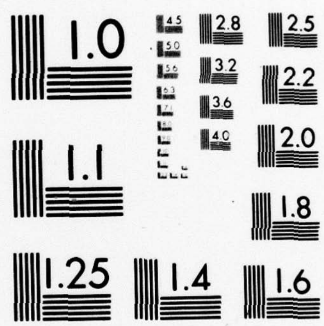

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20. Abstract continued.

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A STATE-SPACE THEORY FOR STATIONARY STOCHASTIC PROCESSES*

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Abstract

Consider a stationary Gaussian stochastic process $\{y(t); t \in R\}$ with a rational spectral density, and let $H(y)$ be the Hilbert space spanned by it. The problem of determining all stationary and purely nondeterministic families of minimal splitting subspaces of $H(y)$ is considered; the splitting subspaces constitute state-spaces for the process y. It is shown that some of these families are Markovian, and they lead to internal stochastic realizations. A complete characterization of all Markovian and non-Markovian families of minimal splitting subspaces is provided. Many of the basic results hold without the assumption of rational spectral density.

1. Introduction

Let $\{y(t); t \in R\}$ be a purely nondeterministic, mean-square continuous, stationary, Gaussian stochastic process with zero mean and defined on a probability space (Ω, F, P). We shall assume that the spectral density Φ of y is rational although, as we shall explain below, many of our results hold without this assumption. Let $H(y)$ be the closed linear hull in $L_2(\Omega, F, P)$ of the stochastic variables $\{y(t); t \in R\}$. Then $H(y)$ is a Hilbert space with inner product $(\xi, \eta) = E\{\xi\eta\}$, where $E\{\cdot\}$ denotes mathematical expectation. The stationarity of y implies that there is a translation group $\{U_t; t \in R\}$ of unitary linear bounded operators $H(y) \rightarrow H(y)$ such that $y(t) = U_t y(0)$ for every $t \in R$. A family $\{S_t; t \in R\}$ of subspaces of $H(y)$ is said to be *stationary* if, for each $t \in R$, $S_t = U_t S_0$.

For each $t \in R$, the Hilbert space $H(y)$ can be written

$$H(y) = H_t^-(y) \vee H_t^+(y), \tag{1}$$

where $H_t^-(y)$ is the *past space*

$$H_t^-(y) = \overline{sp}\{y(\tau); \tau < t\} \tag{2}$$

and $H_t^+(y)$ is the *future space*

$$H_t^+(y) = \overline{sp}\{y(\tau); \tau > t\}\ . \tag{3}$$

Here $\overline{sp}\{\cdot\}$ denotes the closed linear hull, and $X \vee Y$ is the same as $\overline{sp}\{X, Y\}$. Of course, (1) is not an orthogonal decomposition; in fact, the past and future spaces overlap. We shall call

$$H_t^0(y) = H_t^-(y) \cap H_t^+(y) \tag{4}$$

the *present space*. It contains the *germ space*

$$H_{t+}(y) = \overline{sp}\{y(t), \dot{y}(t), \ldots, y^{(r)}(t)\}, \tag{5}$$

i.e. the subspace spanned by $y(t)$ and all its derivatives at t, $y^{(r)}(t)$ being the highest existing derivative defined in mean-square. If Φ has roots on the imaginary axis, $H_t^0(y)$ will also contain some integrals of y over the real line.

For an arbitrary $t \in R$, we wish to determine a subspace of $H(y)$ which, loosely speaking, contains all the information about the past of the process needed in predicting the future or, which is equivalent, all the information about the future required to estimate the past. More precisely stated: Find all (closed) subspaces X which satisfy the condition

$$E\{\eta \mid H_t^-(y) \vee X\} = E\{\eta \mid X\} \quad \text{for all } \eta \in H_t^+(y) \tag{6}$$

or the equivalent condition

$$E\{\eta \mid H_t^+(y) \vee X\} = E\{\eta \mid X\} \quad \text{for all } \eta \in H_t^-(y), \tag{7}$$

where $E\{\eta \mid X\}$ denotes the orthogonal projection of η onto the subspace X, or, in probabilistic terms, the conditional mean of η given (the sigma-field generated by) X. Each of the two conditions (6) and (7) are equivalent to $H_t^-(y)$ and $H_t^+(y)$ being conditionally independent given X [10]. A subspace X with this property is said to be a *splitting subspace* at time t [6, 11]. Obviously $H(y)$ is a splitting subspace, and so are $H_t^-(y)$ and $H_t^+(y)$, but they are too large for our purposes. We shall be

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interested in splitting subspaces X which are *minimal* in the sense that there is no proper subspace of X which is also splitting. We will show that (in the rational case) all such subspaces are finite dimensional.

The significance of the minimal splitting subspaces is that they will serve as *state spaces*. If X_0 is a minimal splitting subspace at time 0, $X_t = U_t X_0$ is a minimal splitting subspace at time t. Hence $\{X_t; t \in R\}$ is a stationary family. *Any* stationary vector process $\{x(t); t \in R\}$ such that, for every $t \in R$, x(t) is a basis in X_t, is called a *state process*. In the next section we show that $y(t) \in X_t$, and hence there is a constant row vector c such that

$$y(t) = cx(t) \ . \tag{8a}$$

It will be seen in Section 4 that there are families of minimal splitting subspaces which are Markovian. Then any state process x will be a Markov process and there is a representation

$$dx = Axdt + bdu \ , \tag{8b}$$

where A is a constant stability matrix, b is a constant vector and u is a Wiener process defined on the whole real line. In general, there is also a multitude of non-Markovian families of minimal splitting subspaces for which there is no such representation.

A representation (8) is called a *stochastic realization* of y. Usually a white noise term is added in (8a), which leads to considering a stochastic process y with stationary increments instead. The results of this paper can be trivially modified to cover this case [8], the present formulation being for clarity of presentation only. A deterministic version of the stochastic realization problem, concerned with the determination of (A,b,c) in (8) only, has been considered by Anderson [2], Faurre [3] and J. C. Willems [15]. The probabilistic problem of finding all representations (8), i.e. quadruplets (A,b,c,u), was solved (in a more general setting) in Lindquist and Picci [7] and Ruckebusch [13]. A coordinate-free state-space approach, such as the one taken in this paper, was used by Akaike [1], Picci [11] and Rozanov [12], but only state spaces contained in the past or, as for Akaike, in the future were considered. A complete characterization of all Markovian state spaces was obtained by Lindquist and Picci [8] and Ruckebusch [14]. The former paper is written in terms of minimal splitting subspaces, but the non-Markovian splitting subspaces were overlooked. An erratum for [8] is provided in the end of this paper; a revised version is under preparation. The existence of non-Markovian splitting subspaces is established in Lindquist-Picci-Ruckebusch [9]. This result is discussed in Section 5.

In this paper we have limited our attention to *internal* stochastic realizations, i.e. representations (8) whose state spaces are contained in the Hilbert space H(y). If we dispense with this assumption, we obtain *external* realizations. Generalizations of our results to include these representations will be presented elsewhere.

2. Minimal splitting subspaces

Let us begin by considering an arbitrary splitting subspace X contained in the past space $H_t^-(y)$. In view of (6) it must satisfy $E\{\eta \mid H_t^-(y)\} = E\{\eta \mid X\}$ for all $\eta \in H_t^+(y)$. Hence any such splitting subspace must contain

$$H_t^{+/-}(y) = \overline{\mathrm{sp}}\{E\{\eta \mid H_t^-(y)\}; \eta \in H_t^+(y)\} \ , \tag{9}$$

which is itself a splitting subspace. Hence $H_t^{+/-}(y)$ is a minimal splitting subspace. We shall use the shorthand notation

$$H_t^{+/-}(y) = \overline{E}\{H_t^+(y) \mid H_t^-(y)\} \tag{10}$$

instead of (9). As soon as we have established that $H_t^{+/-}(y)$ is finite dimensional, we may remove the bar over the E denoting closure. In the same way, it can be seen that

$$H_t^{-/+}(y) = \overline{E}\{H_t^-(y) \mid H_t^+(y)\} \tag{11}$$

is the minimal splitting subspace contained in $H_t^+(y)$. These are the two state spaces considered in [1, 11, 12].

A generalization of this construction leads to the following two lemmas.

Lemma 1. *Let* S *be a subspace of* H(y). *If* $S \supset H_t^-(y)$, *then* $\overline{E}\{H_t^+(y) \mid S\}$ *is a splitting subspace at time* t. *Similarly, if* $S \supset H_t^+(y)$, $\overline{E}\{H_t^-(y) \mid S\}$ *is a splitting subspace at time* t.

Proof. Set $X = \overline{E}\{H_t^+(y) \mid S\}$. Let $\eta \in H_t^+(y)$. Then $E\{E\{\eta \mid S\} \mid X\} = E\{\eta \mid S\}$. Therefore, since $S \supset X$,

$$E\{\eta \mid S\} = E\{\eta \mid X\} \ . \tag{12}$$

Assume that $S \supset H_t^-(y)$. Then, $H_t^-(y) \vee X \subset S$. Hence, projecting (12) onto $H_t^-(y) \vee X$ yields (6), i.e. X is a splitting subspace. A symmetric argument yields the second part of the lemma. ∎

Lemma 2. *Let* X *be a minimal splitting subspace at time* t. *Then we have the following two representations.*

$$X = \overline{E}\{H_t^+(y) \mid H_t^-(y) \vee X\} \tag{13}$$

$$X = \overline{E}\{H_t^-(y) \mid H_t^+(y) \vee X\} \tag{14}$$

Proof. By Lemma 1, the right members of (13) and (14) are splitting subspaces at time t. But, in view of (6) and (7), these are contained in the minimal splitting subspace X and therefore (13) and (14) follow. ∎

Now define the *frame space*

$$H_t^{\square}(y) = H_t^{+/-}(y) \vee H_t^{-/+}(y) \ , \tag{15}$$

which is itself a splitting subspace. In fact, $H_t^{\square}(y)$ is the smallest subspace containing all minimal splitting subspaces, as is seen from the following theorem.

Theorem 1. *Let* X *be a minimal splitting subspace at time* t. *Then*

$$H_t^{0}(y) \subset X \subset H_t^{\square}(y) \ . \tag{16}$$

Proof. (i) Let $\eta \in H_t^0(y) = H_t^-(y) \cap H_t^+(y)$. Then $\eta = E\{\eta \mid H_t^-(y) \vee X\}$, which, by the splitting property (6), equals $E\{\eta \mid X\}$. Hence $\eta \in X$.

(ii) Define N_t^- and N_t^+ to be $H_t^-(y) \ominus H_t^{+/-}(y)$ and $H_t^+(y) \ominus H_t^{-/+}(y)$ respectively. It is easy to see that $N_t^- = H_t^-(y) \cap [\bar{H}_t^+(y)]^\perp$ and $N_t^+ = H_t^+(y) \cap [H_t^-(y)]^\perp$. Consequently, we have the orthogonal decomposition

$$H(y) = N_t^- \oplus H_t^\square(y) \oplus N_t^+ \ . \tag{17}$$

For each $\eta^+ \in H_t^+(y)$ define $\eta = E\{\eta^+ \mid H_t^-(y) \vee X\}$. By Lemma 2, X is the closed linear span of all such η. Moreover, $E\{\eta \mid H_t^-(y)\} = E\{\eta^+ \mid H_t^-(y)\} \subset H_t^{+/-}(y)$, which is orthogonal to N_t^-. Since, in addition, $[\eta - E\{\eta \mid H_t^-(y)\}] \perp H_t^-(y) \supset N_t^-$, we have $\eta \perp N_t^-$, i.e. X is orthogonal to N_t^-. In the same way, using representation (14), it is seen that X is orthogonal to N_t^+. Hence $X \subset H_t^\square$. ■

This theorem shows that, as pointed in Section 1, all minimal splitting subspaces at time t contain y(t).

Corollary 1. *Let X be a minimal splitting subspace at time t. Then*

$$E\{X \mid H_t^-(y)\} = H_t^{+/-}(y) \tag{18}$$

and

$$E\{X \mid H_t^+(y)\} = H_t^{-/+}(y) \ . \tag{19}$$

In anticipation of Corollary 2 below and to provide a more suggestive notation, the closure bars have been deleted in (18) and (19). This result will be interpreted in Section 5.

3. Forward and backward Kalman-Bucy filters

Since the stochastic process y is stationary and mean-square continuous, it has a spectral representation

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\hat{y}(\omega), \tag{20}$$

where $d\hat{y}$ is an orthogonal stochastic measure [4], which, in view of the purely nondeterministic assumption, has the property $E|d\hat{y}|^2 = \Phi(i\omega)d\omega$. Here we assume that the spectral density Φ is rational, i.e. $\Phi = p/q$, where p and q are relatively prime polynomials of degrees 2m and 2n respectively; of course $m < n$.

Consider all scalar solutions $\hat{W}$ of the spectral factorization problem

$$\hat{W}(s)\hat{W}(-s) = \Phi(s) \ . \tag{21}$$

For each such $\hat{W}$,

$$d\hat{u} = \hat{W}(i\omega)^{-1} d\hat{y} \tag{22a}$$

is a unitary orthogonal spectral measure and

$$u(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - 1}{i\omega} d\hat{u}(\omega) \tag{22b}$$

is a Wiener process on the real line. Define $H_t^-(du)$ and $H_t^+(du)$ *to be the closed linear spans of past and future increments of* u respectively, i.e. $H_t^-(du) = \overline{sp}\{u(t+\tau) - u(t); \tau \leq 0\}$ and $H_t^+(du) = \overline{sp}\{u(t+\tau) - u(t); \tau \geq 0\}$. Let $\mathcal{U}$ denote the class of all Wiener processes (22) and $\mathcal{U}^+(\mathcal{U}^-)$ the subset of those $u \in \mathcal{U}$ which correspond to a $\hat{W}$ for which the function $\omega \to \hat{W}(i\omega)$ is of Hardy class $H_2^+(H_2^-)$. Let W be the inverse Fourier transform of $\omega \to \hat{W}(i\omega)$. Then

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{W}(i\omega)d\hat{u} = \int_{-\infty}^{\infty} W(t-\tau)du(\tau) \ . \tag{23}$$

Now, $W(t) \leq 0$ if and only if $u \in \mathcal{U}^+$. Therefore, $\mathcal{U}^+$ contains precisely those u for which $H_t^-(du) \supset H_t^-(y)$. Let $\hat{W}_*$ be the unique rational spectral factor with all poles *and* zeros in the left complex half-plane, and let u_* be the corresponding Wiener process. Then $u_* \in \mathcal{U}^+$. Moreover, since $\hat{W}_*^{-1}$ has all its poles *in the left closed half-plane, with some effort it can be seen from* (22) that $H_t^-(du_*) \subset H_t^-(y)$. Hence $H_t^-(du_*) = H_t^-(y)$, and u_* is called the *innovation process*. In the same way we can see that $\mathcal{U}^-$ is the class of those u for which $H_t^+(du) \subset H_t^+(y)$, with equality for the *backward innovation process* $\bar{u}_*$, i.e. the Wiener process corresponding to the spectral factor $\hat{W}_*(-s)$. Finally, we note that $H(u) = H(y)$ for all $u \in \mathcal{U}$.

Proposition 1. *Let $u \in \mathcal{U}^+$ and let $\hat{W}$ be the corresponding spectral factor. Then $E\{H_t^+(y) \mid H_t^-(du)\}$ is finite dimensional if and only if $\hat{W}$ is rational, i.e. $\hat{W} = \pi/\chi$, where π and χ are real relatively prime polynomials. In this case*

$$E\{H_t^+(y) \mid H_t^-(du)\} = \int_{-\infty}^{\infty} e^{i\omega t}\overline{sp}\left\{\frac{\rho(i\omega)}{\pi(i\omega)} \mid \deg(\rho) < k\right\} d\hat{y}, \tag{24}$$

where $k = \deg(\chi)$, and where we have taken the closed linear span in $L_2(R)$ over all real polynomials ρ of degree less than k.

Proof. First note that, in view of (23),

$$E\{y(t+\tau) \mid H_t^-(du)\} = \int_{-\infty}^{t} W(t+\tau-\sigma)du(\sigma) \ . \tag{25}$$

(if): Suppose that $\hat{W} = \pi/\chi$ is rational, and let (A,b,c) be a minimal realization of $\hat{W}$. Then $\hat{W}(s) = c(sI-A)^{-1}b$ and, in view of the H_2^+ condition, $W(t) = ce^{At}b1(t)$, where 1(t) is one for $t \geq 0$ and zero for $t < 0$. This inserted into (25) yields

$$E\{y(t+\tau) \mid H_t^-(du)\} = \int_{-\infty}^{\infty} e^{i\omega t}\frac{\rho_\tau(i\omega)}{\pi(i\omega)} d\hat{y}(\omega)$$

where $\rho_\tau = \chi(s)ce^{A\tau}(sI-A)^{-1}b$ is a polynomial, for χ is the characteristic polynomial of A. Since (c,A) is observable, span $\{\rho_\tau; \tau \geq 0\}$ consists of all real polynomials of degree less than k, and consequently (24) holds. The dimension of (24) is k.

(only if): We use a technique found in [5]. Suppose that $\dim E\{H_t^+(y) \mid H_t^-(du)\} = k < \infty$. Let $\mathcal{W}$ be the closed span of $\{W_\tau; \tau \geq 0\}$ in $L_2(0,\infty)$, where $W_\tau(t) = W(t+\tau)$. Then, in view of (25), $\dim \mathcal{W} = k$. Let $\{e_1, e_2, \ldots, e_k\}$ be a basis in $\mathcal{W}$, and let e be the column vector of these functions. Since $\mathcal{W}$ is closed under the shift $W(t) \to W(t+\tau)$, there is an invertible matrix function T such that $e(t+\tau) = T(\tau)e(t)$; T is continuous, for the shift is continuous in L_2. Since $T(t+\tau) = T(t)T(\tau)$, $T(t) = e^{At}$ for some matrix A. Then there is a row vector c such that, for $t \geq 0$, $W(t) = ce(t) = ce^{At}e(0)$, and hence $\hat{W}$ is rational. ■

Using a symmetric argument we can prove a backward version of this proposition.

Proposition 2. *Let* $u \in \mathcal{U}^-$ *and let* $\hat{W}$ *be the corresponding spectral factor. Then* $E\{H_t^-(y) \mid H_t^+(du)\}$ *is finite dimensional if and only if* $\hat{W}$ *is rational, in which case* $E\{H_t^-(y) \mid H_t^+(du)\}$ *is given by the right member of (24), where* π *is the numerator polynomial of* $\hat{W}$ *and* k *the degree of the denominator polynomial.*

Since $H_t^-(y) = H_t^-(du_*)$, Proposition 1 yields a representation of $H_t^{+/-}(y)$, namely

$$H_t^{+/-}(y) = \int_{-\infty}^{\infty} e^{i\omega t}\, \overline{sp}\left\{ \frac{\rho(i\omega)}{\pi_*(i\omega)} \mid \deg(\rho) < n \right\} d\hat{y}\ , \qquad (26)$$

where π_* is the numerator polynomial of $\hat{W}_*$. Now, using the procedure outlined in the end of Section 4, we can see that there is an equivalence class of representations

$$\begin{aligned} dx_* &= A_* x_* dt + b_* du_* \\ y_* &= c_* x_* \end{aligned} \qquad (27)$$

of type (8), such that, for every $t \in R$, $x_*(t)$ is a basis of $H_t^{+/-}(y)$. This is the (steady-state) *Kalman-Bucy filter*. Likewise, the representation

$$H_t^{-/+}(y) = \int_{-\infty}^{\infty} e^{i\omega t}\, \overline{sp}\left\{ \frac{\rho(i\omega)}{\pi_*(-i\omega)} \mid \deg(\rho) < n \right\} d\hat{y} \qquad (28)$$

follows from Proposition 2, using the fact that $H_t^+(y) = H_t^+(d\bar{u}_*)$, and we obtain an equivalence class of *backward Kalman-Bucy filters*

$$\begin{aligned} d\bar{x}_* &= \bar{A}_* \bar{x}_* dt + \bar{b}_* d\bar{u}_* \\ \bar{y}_* &= \bar{c}_* x_*\ , \end{aligned} \qquad (29)$$

which evolve backward in time starting at $t = \infty$, for $\mathrm{Re}\{\lambda(\bar{A}_*)\} > 0$ (see Section 4). Now the second part of Theorem 1 can be rephrased to read: Let X be a minimal splitting subspace at time t. Then $X \subset \overline{sp}\{x_*(t), \bar{x}_*(t)\}$, i.e. any (minimal) state vector can be expressed in terms of the forward and backward Kalman-Bucy estimates; c.f. Lindquist-Picci [7] and Ruckebusch [13]. In view of (26) and (28), the following proposition is immediate.

Proposition 3. *Let* $\Phi = p/q$, *where* p *and* q *are relatively prime polynomials of degrees* $2m$ *and* $2n$ *respectively. Then the frame space is given by*

$$H_t^{\square}(y) = \int_{-\infty}^{\infty} e^{i\omega t}\, \overline{sp}\left\{ \frac{\rho(i\omega)}{p(i\omega)} \mid \deg(\rho) < n+m \right\} d\hat{y}\ . \qquad (30)$$

Corollary 2. *Given the same assumptions as in Proposition 3, let* X *be a minimal splitting subspace at time* t. *Then* $n \le \dim X \le n+m$.

Proof. Since $X \subset H_t^{\square}$ (Theorem 1), $\dim X \le n+m$. In view of (18), $\dim X \ge \dim H_t^{+/-}(y) = n$. ∎

4. Markovian families of minimal splitting subspaces

A family $\{X_t; t \in R\}$ of subspaces is said to be *Markovian* if, for every $t \in R$, $X_t^- := \bigvee_{\tau \le t} X_\tau$ and $X_t^+ := \bigvee_{\tau \ge t} X_\tau$ are conditionally independent given X_t. This is equivalent to each of the two conditions

$$\text{(i)}\quad E\{\eta \mid X_t^-\} = E\{\eta \mid X_t\} \qquad \text{for all } \eta \in X_t^+ \qquad (31)$$

$$\text{(ii)}\quad E\{\eta \mid X_t^+\} = E\{\eta \mid X_t\} \qquad \text{for all } \eta \in X_t^-\ , \qquad (32)$$

either of which can be used as an alternative definition.

Lemma 3. *Let* $\{S_t; t \in R\}$ *be a nondecreasing family of subspaces of* $H(y)$, *i.e.* $S_t \subset S_{t+\epsilon}$ *for all* $t \in R$ *and* $\epsilon > 0$. *Then* $\{\bar{E}\{H_t^+(y) \mid S_t\}; t \in R\}$ *is Markovian.*

Proof. Set $X_t = \bar{E}\{H_t^+(y) \mid S_t\}$. Let $\eta^+ \in H_\tau^+(y)$, where $\tau > t$. Define $\eta = E\{\eta^+ \mid S_\tau\}$. Since X_t^+ is the closed span of all such η, it just remains to show that $E\{\eta \mid X_t^-\} = E\{\eta \mid X_t\}$ to prove (31). But, as in (12), $E\{\eta^+ \mid S_t\} = E\{\eta^+ \mid X_t\}$. Since $S_\tau \supset S_t \supset X_t$, this implies $E\{\eta \mid S_t\} = E\{\eta \mid X_t\}$, which projected onto X_t^- yields the desired result, for $X_t^- \subset S_t$. ∎

Consider a family $\{X_t; t \in R\}$ of minimal splitting subspaces. Then, by Lemma 2 (and Corollary 2), $X_t = E\{H_t^+(y) \mid S_t\}$ for all $t \in R$, where $S_t = H_t^-(y) \vee X_t$. Now, could we find a Wiener process u such that $S_t = H_t^-(du)$, Proposition 1 would yield a representation (24) of $\{X_t; t \in R\}$. The following three lemmas show under what conditions this will happen.

Lemma 4. *Let* $\{X_t; t \in R\}$ *be a Markovian family of minimal splitting subspaces, and, for every* $t \in R$, *define* $S_t = H_t^-(y) \vee X_t$. *Then the family* $\{S_t; t \in R\}$ *is nondecreasing.*

Proof. By Theorem 1, $y(t) \in X_t$, and hence $X_t^- \supset H_t^-(y) \vee X_t = S_t$. Define $Z = X_t^- \ominus S_t$. Let $\tau > t$. Then, using the splitting property (6), $E\{y(\tau) \mid X_t^-\} = E\{y(\tau) \mid X_t\} + E\{y(\tau) \mid Z\}$. Hence, since $y(\tau) \in X_\tau$, the Markov property (31) implies that $E\{y(\tau) \mid Z\} = 0$. Therefore $E\{H_t^+(y) \mid Z\} = 0$ which together with the trivial fact $E\{H_t^-(y) \mid Z\} = 0$ yields $Z = 0$. Consequently, $S_t = X_t^-$, which is nondecreasing. ∎

Somewhat differently stated, Lemma 4 can be found in [9], where an alternative proof is given.

A family $\{X_t; t \in R\}$ of subspaces is said to be *purely nondeterministic* if $\bigcap X_t = 0$. By assumption, $\{H_t^-(y); t \in R\}$ is purely nondeterministic. The following result is immediate.

Lemma 5. *Let* $X_t = E\{H_t^+(y) \mid S_t\}$, *where* $S_t = H_t^-(y) \vee X_t$. *Then* $\{X_t; t \in R\}$ *is stationary and purely nondeterministic if and only if* $\{S_t; t \in R\}$ *has these properties.*

Lemma 6. *Let* $\{S_t; t \in R\}$ *be a stationary, purely nondeterministic, and nondecreasing family of subspaces of* $H(y)$ *such that* $S_t \supset H_t^-(y)$ *for all* $t \in R$. *Then there exists a Wiener process* u *defined on all of* R *such that, for every* $t \in R$, $H_t^-(du) = S_t$. *Moreover,* $u \in \mathcal{U}^+$.

The proof of this lemma is a bit lengthy and technical, and it will be given in the revised version of [8]. Once the existence of a u such that $H_t^-(du) = S_t$ has been established, it is easy to see that it must belong to $\mathcal{U}^+$. In fact, since $H_t^-(y) \subset H_t^-(du)$, there is an L_2-function W such that

$$y(t) = \int_{-\infty}^{t} W(t-\sigma)du(\sigma) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{W}(i\omega)d\hat{u}(\omega).$$

Hence $\hat{W}$ is a spectral factor of class H_2^+, and therefore $u \in \mathcal{U}^+$.

Theorem 2. *Let $\Phi = p/q$, where p and q are relatively prime polynomials of degrees $2m$ and $2n$ respectively and $m < n$. Then $\{X_t; t \in R\}$ is a stationary, purely nondeterministic, Markovian family of minimal splitting subspaces if and only if*

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t}\overline{sp}\left\{\frac{\rho(i\omega)}{\pi(i\omega)} \mid \deg(\rho) < n\right\} d\hat{y} \quad \text{for all } t \in R \tag{33}$$

for some real polynomial π satisfying

$$\pi(s)\pi(-s) = p(s) \ . \tag{34}$$

Proof. (if): Let χ_* be the polynomial solution of

$$\chi(s)\chi(-s) = q(s) \tag{35}$$

having all its zeros in the left open half-plane. Let $\hat{W} = \pi/\chi_*$, and define u to be the Wiener process with stochastic spectral measure $d\hat{u} = \hat{W}(i\omega)^{-1}d\hat{y}$. Then $u \in \mathcal{U}^+$, and therefore $H_t^-(du) \supset H_t^-(y)$. Set $S_t = H_t^-(du)$. Clearly, $\{S_t; t \in R\}$ is a stationary, purely nondeterministic, nondecreasing family of subspaces such that $S_t \supset H_t^-(y)$. Moreover, by Proposition 1, $X_t = E\{H_t^+(y) \mid S_t\}$, for $\deg(\chi_*) = n$. Hence, by Lemmas 1, 3, and 5, $\{X_t; t \in R\}$ is a stationary, purely nondeterministic, and Markovian family of splitting subspaces. Since $\dim X_t = n$, these splitting subspaces are minimal (Corollary 2).
(only if): Let $\{X_t; t \in R\}$ be a stationary, purely nondetministic, Markovian family of minimal splitting subspaces. By Corollary 2, $\dim X_t < \infty$, and, by Lemma 2, $X_t = E\{H_t^+(y) \mid S_t\}$ where $S_t = H_t^-(y) \vee X_t$. The family $\{S_t; t \in R\}$ is nondecreasing (Lemma 4), stationary and purely nondeterministic (Lemma 5) and $S_t \supset H_t^-(y)$. Hence, by Lemma 6, there is a $u \in \mathcal{U}^+$ such that $S_t = H_t^-(du)$ for all $t \in R$. Then, since $\dim X_t < \infty$, Proposition 1 yields

$$X_t = \int_{-\infty}^{\infty} e^{i\omega t}\overline{sp}\left\{\frac{\rho(i\omega)}{\pi(i\omega)\varphi(i\omega)} \mid \deg(\rho) < n + \deg(\varphi)\right\} d\hat{y} \tag{36}$$

In fact, a rational spectral factor must have the form

$$\hat{W}(s) = \frac{\pi(s)\varphi(s)}{\chi(s)\varphi(-s)} \ ,$$

where π and χ satisfy (34) and (35) respectively and φ is some other polynomial. But the right member of (36) contains (33), which we have just shown to be a splitting subspace. Hence $\varphi = 1$. ∎

Corollary 3. *Let $\{X_t; t \in R\}$ be the family of minimal splitting subspaces defined by (33). Let χ be a polynomial solution of (35), and let u be the process in $\mathcal{U}$ corresponding to $\hat{W} = \pi/\chi$. Then, (i) $X_t^- \subset H_t^-(du)$ if and only if $\chi = \chi_*$ (the solution of (35) with all its zeros having negative real parts), in which case $X_t^- = H_t^-(du)$; and (ii) $X_t^+ \subset H_t^+(du)$ if and only if $\chi(s) = \chi_*(-s)$, in which case $X_t^+ = H_t^+(du)$.*

Proof. The two parts are symmetric, so we only need to prove (i). First observe that $u \in \mathcal{U}^+$ if and only if $\chi = \chi_*$.
(if): It follows from the proof of Theorem 2 that $H_t^-(du) = H_t^-(y) \vee X_t$. *But it is seen from the proof of Lemma 4 that* $H_t^-(y) \vee X_t = X_t^-$. Hence $X_t^- = H_t^-(du)$.
(only if): The inclusion $X_t^- \subset H_t^-(du)$ implies that $H_t^-(y) \subset H_t^-(du)$ (Theorem 1). This condition is equivalent to $u \in \mathcal{U}^+$ (Section 3), which can only be the case if $\chi = \chi_*$. ∎

It follows from Theorem 2 that there are at most 2^m stationary, purely nondeterministic, Markovian families of minimal splitting subspaces. We shall now see that each of these corresponds to an equivalence class of stochastic realizations (8). To this end define the stationary stochastic processes

$$x_k(t) = \int_{-\infty}^{\infty} e^{i\omega t}\,\frac{(i\omega)^{k-1}}{\pi(i\omega)}\,d\hat{y}; \quad k = 1, 2, \ldots, n. \tag{37}$$

Then, for each $t \in R$, the random vector $x(t)$ is a basis in X_t. Let the components of the row vector c be defined by

$$\pi(s) = \sum_{k=1}^{n} c_k s^{k-1} \ . \tag{38}$$

Hence, in view of (20),

$$y(t) = cx(t) \ . \tag{39a}$$

To get a representation for x, let

$$\chi(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n \tag{40}$$

be a solution of (35) and let u be the Wiener process in $\mathcal{U}$ corresponding to $\hat{W} = \pi/\chi$. Then

$$x_k(t) = \int_{-\infty}^{\infty} e^{i\omega t}\,\frac{(i\omega)^{k-1}}{\chi(i\omega)}\,d\hat{u} \tag{41}$$

and therefore it is easy to see that x satisfies

$$dx = Axdt + bdu \ , \tag{39b}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

The system (39) is a Markovian representation of y. But for (39b) to evolve forward in time, we must have $X_t^- \subset H_t^-(du)$. This condition, which is equivalent to $X_t^- \perp H_t^+(du)$, characterizes the forward property, and to satisfy it we must choose $\chi = \chi_*$ (Corollary 3). Then $\mathrm{Re}\{\lambda(A)\} < 0$, and we have a stochastic realization of type (8). Likewise, by imposing the condition $X_t^+ \subset H_t^+(du)$, i.e. $\chi(s) = \chi_*(-s)$ (Corollary 3), we obtain a *backward stochastic realization* [7] with $\mathrm{Re}\{\lambda(A)\} > 0$. By making coordinate changes of the type $x(t) \to Tx(t)$ in X_t, where T is a nonsingular constant matrix, we obtain equivalent representations with (A,b,c) exchanged for (TAT^{-1}, Tb, cT^{-1}), but there is no such relationship between realizations (8) corresponding to *different* families of minimal splitting subspaces.

5. Non-Markovian families of minimal splitting subspaces

The following theorem, presented in Lindquist-Picci-Ruckebusch [9], gives a complete characterization of all minimal splitting subspaces.

Theorem 3. *Let* $X = \bar{E}\{H_t^+(y) \mid S\}$. *Then* X *is a minimal splitting subspace at time* t *if and only if*

$$H_t^-(y) \subset S \subset H_t^-(y) \vee H_t^{\square}(y) , \qquad (41)$$

in which case $S = H_t^-(y) \vee X$.

For the moment, let us set $t = 0$ in Theorem 3. We have seen that there is only a finite number of X which belong to Markovian families, namely those given by (33). Let us call the set of these $\mathcal{M}_0$. However, in general, there is an infinite number of subspaces S satisfying (41). Since there is a one to one correspondence between S and X, there are in general minimal splitting subspaces $X \notin \mathcal{M}_0$. For such an X, define $X_t = U_t X$ for all $t \in R$. Then $\{X_t; t \in R\}$ is a non-Markovian family of minimal splitting subspaces.

Although, in general, an arbitrary state process $\{x(t); t \in R\}$ will not be Markov, there is always a representation (39a), and, in view of Corollary 2, the relations

$$E\{x(t) \mid H_t^-(y)\} = x_*(t) \qquad (42)$$

and

$$E\{x(t) \mid H_t^+(y)\} = \bar{x}_*(t) \qquad (43)$$

where $x_*(t)$ and $\bar{x}_*(t)$ are the forward and backward Kalman-Bucy estimates, will always hold.

Finally, let us remark that many of the results of this paper do not require a rational spectral density; this assumption enters only in Section 3, Theorem 2 and Corollary 3. In fact, Section 2, Lemmas 3 and 4 and Theorem 3 do not even require the stationary and purely nondeterministic assumptions.

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Erratum for Reference 8 (CDC version):

p. 44: The exponential in (3.3) should be replaced by a general inner function. However, by first showing that the frame space is finite dimensional, as we do in this paper, only rational W need to be considered.

p. 44: Add the assumption $\bigcap_t S_t = 0$ in Lemma 3.1 and elsewhere where this result is used.

p. 48: It is claimed in the proof of Lemma 7.2 that the family $\{S_t; t \in R\}$ is increasing. Actually this condition is equivalent to $\{X_t; t \in R\}$ being completely Markovian [9]. Hence this property must be assumed in Lemma 7.2 and Theorem 5.1.