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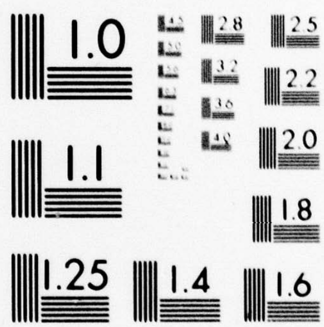


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# ON THE NON-EXISTENCE OF MULTIPLICATIVE EQUILIBRIUM BIDDING STRATEGIES

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# ON THE NON-EXISTENCE OF MULTIPLICATIVE EQUILIBRIUM BIDDING STRATEGIES*

by

Richard Engelbrecht-Wiggans and Robert J. Weber

## ABSTRACT

Extensive use has been made of multiplicative bidding strategies in the literature relating to auctions of mineral leases. It is shown that, generally, multiplicative strategies are not in equilibrium. It is also established that an equilibrium bidding strategy is not a function merely of a sufficient statistic for the true value, or of any other of a class of related statistics, if an individual bidder observes more than one piece of information. The general insufficiency of a single simple statistic is briefly discussed and a special class of models is identified in which a single simple statistic is indeed strategically sufficient.

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## Introduction

In a recent survey (Engelbrecht-Wiggans [1]), it is noted that multiplicative lognormal error models and multiplicative bidding strategies are used extensively in the literature relating to auctions of mineral leases. In many of these models, each individual bidder is assumed to have available only, at most, a single piece of proprietary information about the true value of the object. Under relatively restrictive conditions, Rothkopf [3] proves that a symmetric Nash equilibrium point arises when each bidder bids a fixed multiple of his informational random variable; Winkler and Brooks [5] prove an analogous result for the existence of equilibrium additive strategies in models with additive errors.

In more general situations, such simple strategies need not be in equilibrium. Using a numerical example based on an Outer Continental Shelf oil lease sale, Engelbrecht-Wiggans [2] indicates that multiplicative strategies may be quite inferior when all opposing bidders use their equilibrium strategies. We will verify that in general there is no simple statistic on which equilibrium strategies may be based, if each individual observes more than one piece of information; in particular, this establishes that strategies which are multiples of such a statistic are not in equilibrium.

An interesting corollary is obtained by considering our result in conjunction with a result stated by Teisberg [4; pp. 47-50]. He considers summarizing all of an individual's information in a single statistic; with an appropriate choice of parameters, the particular statistic he uses is the posterior mean of the true value. If the bidders are restricted to bidding strategies which are functions only of such a statistic, then an equilibrium bidding strategy for each bidder is to bid a multiple of his statistic. Upon comparison with our result, it must be concluded that several pieces of information should not in general be condensed into a single statistic of the form considered by Teisberg.

This raises the question of whether there is any single statistic which captures all the information necessary to calculate an equilibrium bid. On one level, the answer is trivially affirmative: the equilibrium bidding strategy itself is such a statistic of the available information. Alternatively, one could restrict attention to a more limited class of "simple" statistics. For example, one might only consider different averages of the information; the statistics used by Teisberg are of this form. In this case, equilibrium bidding strategies do not generally depend only on a "simple" statistic. However, we will characterize several special classes of models in which each equilibrium strategy depends only on a single simple statistic, an average of the several pieces of information.

## The Auction Model

Let $Z$ be the true (but unknown) value of the single object being sold; $Z$ is chosen by Nature from the known distribution $F_Z(z)$ . It is traditional to assume that $Z$ has a "diffuse" distribution; that is, there is no prior information available about the value of $Z$ . Each of $n$ individuals observes the outcome of one or more informational random variables; individual $i$ is told the vector $\underline{X}_i = (X_{i,1}, \ldots, X_{i,k_i})$ . The distribution $F_{\underline{X}_1, \ldots, \underline{X}_n | Z}(\underline{x}_1, \ldots, \underline{x}_n | z)$ of the information, conditional on the true value, is known to all bidders.

Each individual must choose a bidding strategy $b_i : R^{k_i} \to R$ specifies what real-valued bid will be submitted when any particular vector of informational variables is observed. The rules of the auction specify that the object is sold to the highest bidder at a price equal to the amount bid, whenever this amount exceeds the seller's reservation price; the object remains unsold if no bid exceeds the reservation price.

It is convenient to restrict our consideration to models in which each $X_{i,j}$ is a lognormally-distributed multiple of the true value $Z$ . (We specifically do not assume that the $X_{i,j}$'s are conditionally independent.) It may be noted, by considering the logarithms of $Z$ and of the $X_{i,j}$'s , that many observations about the multiplicative model will translate directly into analogous results for bidding models with additive normally-distributed errors.

The simplest possible symmetric case arises when each individual observes one piece of "strictly private" information; $\underline{X}_1, \ldots, \underline{X}_n$ are independent (conditional on $Z$ ), identically-distributed random variables. For this case, Rothkopf proves (for the multiplicative-information model, when the reservation price is zero) that there is a constant $c$ such that the bidding strategies $b_i(x_i) = cx_i$ constitute a symmetric Nash equilibrium. Winkler and Brooks consider the analogous case in which the information is a random error added to the true value and prove that there is a constant $c$ such that the strategies $b_i(x_i) = c + x_i$ are in equilibrium.

## Public and Private Information

Perhaps the next simplest case is that in which each individual observes two pieces of information, one of which is strictly private and the other of which is "public": each $\underline{X}_i = (X_{i,1}, X_{i,2})$ where $X_{1,2} = \ldots = X_{n,2} = W$, and $(X_{1,1}, \ldots, X_{n,1}, W)$ are conditionally independent. In traditional auction models, this case corresponds to the situation in which each individual receives only one piece of strictly private information, but where the prior distribution of the true value is not diffuse. (The public information is used to update the diffuse prior distribution, yielding a proper (non-diffuse) distribution of the true value prior to observing the strictly private information.)

In our subsequent analysis we assume that the private informational variables $(X_1, \ldots, X_n)$ are identically distributed, that they and the public informational variable $W$ are independent lognormally-distributed multiples of the true value $Z$, and that the prior distribution of $Z$ is a diffuse lognormal distribution. We make the further assumption that the seller's reservation price is zero. In this case, one might hope to find a symmetric Nash equilibrium in which each bidder's strategy is an increasing function of his private information. It is proven in the appendix that if such a strategy exists, it cannot be expressed in closed form; specifically, if there were a closed-form expression for such equilibrium strategies, it could be used to obtain a closed-form expression for the cumulative distribution of a normally-distributed random variable.

There is at least one important implication of the result. Teisberg considered the private-public information model described above, and restricted each bidder to strategies which are functions of a statistic of the form $X_i^a W^{(1-a)}$ ; for appropriate choices of the parameter $a$, this statistic reduces to the private information $X_i$, the public information $W$, or a constant multiple of the expected true value of the object posterior to the observation of the random variables $X_i$ and $W$. Under this restriction on the bidding strategies, Teisberg proved that, for some $c$, the strategies $b_i(x_i, w) = cx_i^a w^{(1-a)}$

constitute a Nash equilibrium point; note that each strategy $b_i$ is an increasing function of the variable $x_i$, and has a closed-form expression.

When the two preceding results are combined, it follows that equilibrium strategies in the unrestricted auction are not functions merely of statistics of the form $X_i^a W^{(1-a)}$. Thus, despite any intuition to the contrary (e.g., Teisberg [4; p. 49]), an individual should not base his bid solely on a single statistic sufficient for estimating the true value; such a single statistic does not capture everything of relevance in the several pieces of information.

There is an intuitive explanation for why a single statistic, sufficient for estimating the true value, does not capture everything revealed by the several pieces of information. In an auction, a bidder is not only concerned with the true value of the object, but is also concerned with his chances of winning. Whether a bidder wins depends on the strategies used by the opposing individuals and on the information they observe. Thus, even if each other bidder's bid were to be a function of a single statistic of his information, the remaining bidder should not only be concerned with the true value, but also with these statistics of the opposing individuals. In general, a single statistic sufficient for the true value is not sufficient for the other bidders' statistics; thus bidding strategies should not be based merely on a sufficient statistic for the true value.

A relatively special class of model can be identified, in which a single "natural" statistic is indeed sufficient for both the true value and for the opposing individuals' values for their statistics. Define a model to be information-symmetric both with respect to individuals and with respect to information components if each individual receives the same number k of pieces of information and if $F_{\underline{X}|Z}(\underline{x}|z) = F_{\underline{X}|Z}(\underline{x}'|z)$ where $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$ and $\underline{x}'$ denotes any vector obtained by interchanging two components of the vector $\underline{x}$ . For such symmetric models, when all informational variables are lognormal random multiples of the true value, the statistic $\Pi_{j=1}^{j=k} X_{i,j}$ is sufficient (in the eyes of bidder $i$ ) both for the true value and for all the information of the opposing individuals. (After one takes the natural logarithms of all random variables, the verification entails straightforward, but occasionally messy, manipulations of multivariate normal distributions.)

In another vein, one can eliminate the assumption of information-symmetry, and assume instead that the bidders observe only strictly private information (that is, the random vectors $\underline{X}_1, \ldots, \underline{X}_n$ are conditionally independent). Then it is clear that a sufficient statistic for the true value contains all the information a player must consider for strategic purposes.

## Conclusion

This paper verifies that if each individual receives more than one piece of information (or, equivalently, if the prior distribution of the true value is not diffuse) then there is not in general a symmetric Nash equilibrium point in which each bidder's strategy is a function of a single simple statistic of his information.

Although the discussion and the particular results are related to a multiplicative-information model, the results can easily be extended to models with additive normally-distributed errors. More important, however, is the observation that the insufficiency of a single simple statistic appears to depend on the fact that it cannot summarize both the true value, and other bidders' information, simultaneously. Since this is also true for more general models, the results and observations in the above discussion should provide insight into the problem of obtaining functional forms of equilibrium bidding strategies in a general context.

## APPENDIX

It will be shown, by contradiction, that there is no closed-form symmetric equilibrium point, each component of which is a strictly-increasing differentiable function of the corresponding private information variable. For simplicity, we will consider the case where $n$ , the number of bidders, is two; $W|(Z=z)$ and both $X_i|(Z=z)$ are assumed to be normal with mean $z$ and variance one; and $Z|(X_i = x_i, W = w)$ is normal with mean $(x + w)/2$ and variance one-half. (Note, the above random variables correspond to the natural logarithms of those used in the body of the paper; the above conditional distribution of $Z$ is consistent with $W|(Z=z)$ being normal with mean $z$ and variance one (i.e., additive errors) and the marginal prior of $Z$ being normal with infinite variance).

Consider any differentiable function $b(x, w)$ which is strictly increasing in $x$ . If one individual observes $W = w$ and $X_1 = x$ and bids $B$ , while the other individual bids $b(X_2, w)$ , then the expected monetary profit to the first is

$$E(x,w,B) = \int_{z=-\infty}^{z=\infty} \int_{t:b(t,w)\le B} (e^z - B) f_{(X_2,Z|X_1,W)}(t,z|X_1=x, W=w)\, dt\, dz$$

Thus, a necessary condition for a symmetric equilibrium is that the derivative of $E(x, w, B)$ with respect to $B$ , evaluated at $B = b(x, w)$, is zero; that is,

$$\left[ \int_{z=-\infty}^{z=\infty} e^{z} f_{(X_2,Z|X_1,W)}(t,z|X_1=x, W=w)\, dz \right.$$

$$\left. - b(x,w) \int_{z=-\infty}^{z=\infty} f_{(X_2,Z|X_1,W)}(t,z|X_1=x, W=w)\, dz \right] \Big/ \left[ \frac{d}{dt} b(x,w) \right]_{t=x}$$

$$= \int_{t=-\infty}^{t=x} \int_{z=-\infty}^{z=\infty} f_{(X_2,Z|X_1,W)}(t,z|X_1=x,W=w)\, dz\, dt \ .$$

If the involved random variables have the previously-assumed normal densities, and if the function $b$ can be expressed in closed form, then the left-hand side of the above condition may be reduced to a closed-form expression, while the right-hand side reduces to

$$\int_{t=-\infty}^{t=x} \frac{1}{\sqrt{3\Pi}} \exp\left\{ -\frac{1}{2} \left[ \frac{t - (w+x)/2}{\sqrt{3/2}} \right]^{2} \right\} dt \ .$$

However, this is simply the probability that a normal random variable with mean $(w + x)/2$ and variance $3/2$ is less than $x$ ; since there is no closed-form expression for the cumulative distribution of a normal random variable, the desired result follows.

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