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(6) BIDDING IN AUCTIONS WITH MULTIPLICATIVE LOGNORMAL ERRORS: AN EXAMPLE

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BIDDING IN AUCTIONS WITH MULTIPLICATIVE LOGNORMAL ERRORS:

AN EXAMPLE*

by

Richard Engelbrecht-Wiggans

ABSTRACT

Auction models with lognormally-distributed multiplicative errors
are used extensively in models of mineral lease sales. Equilibrium
strategies are typically difficult to calculate; multiplicative strate-
gies are often used as approximations. An example based on a federal
offshore oil lease sale shows that multiplicative strategies may be
quite far from being in equilibrium. However, under a special form of
repetition, such strategies converge very rapidly to an equilibrium.
The effects of any fixed costs, the reservation price, the number of
bidders and the variance of the error are examined briefly for this
example.

Introduction

In a recent survey of auction and bidding models, Engelbrecht-Wiggans [2] notes the extensive use of lognormally-distributed multiplicative errors in models of mineral lease sales. It is generally difficult to calculate equilibrium strategies for such models; indeed, Engelbrecht-Wiggans and Weber [4] prove that there are in general no equilibrium strategies with closed form expressions. It is often assumed that bids are a constant fraction of an unbiased estimate of the true value of the object; there has been little research to determine whether such strategies are approximately in equilibrium.

Rothkopf [6, 7, 8] proves that if individuals have no prior information about the true value, then there are multiplicative equilibrium strategies. Rothkopf [9] also proves that if the prior information is of much greater variance than any subsequent information, then the equilibrium linear strategies are approximately multiplicative. There has apparently been no attempt at determining how close to equilibrium multiplicative strategies are in terms of the expected profit accrued by a bidder.

We model auctions as games with incomplete information. The mathematics of calculating a symmetric equilibrium strategy are outlined. Equilibrium multiplicative strategies and equilibrium multiples of Bayes estimates are calculated. An example based on an actual offshore oil lease sale is used to show that such strategies typically fail significantly to maximize a bidder's expected profit; such strategies are quite far from being in equilibrium. However, if bidders participate in an appropriately naive way in a sequence of auctions, then their strategies rapidly converge to an equilibrium.

The effects of different reservation prices, fixed costs, and numbers of bidders on the equilibrium strategy is examined briefly. The expected profit to bidders and expected revenue to the auctioneer is calculated for several cases with different numbers of bidders and variances of the error. Finally, to verify the plausibility of the example's parameters, and thereby establish the relevance of our observations, the distribution of bids predicted by the model is compared to that actually observed in a recent oil lease auction.

Model

Auctions may be modelled as games with incomplete information. Nature chooses the true value of the object. Although Nature's choice is not revealed to the bidders, they do know the distribution from which the choice was made. In addition, each bidder observes some "private" information; the joint distribution of the true value and private information is known to all bidders. The bidders use the distributions to determine a bidding strategy; a strategy is a function which specifies a real valued bid for each possible outcome of the observed private information.

In particular, let $h(z)$ be the density function of the true value Z. For example, let Z be lognormally-distributed; equivalently assume that the natural logarithm of the true value is normally-distributed with mean u and variance s^2. The private information $\{X_i\}_{i=1}^{i=n}$ of the n bidders are lognormally-distributed multiples of the true value. Assume that the natural logarithm of a bidder's private information is equal to the natural logarithm of the true value plus an error E_i; the errors are independent identically distributed normally-distributed random variables with mean zero

and variance σ^2 . There is a single object which is sold to a highest bidder at a price equal to his bid, except that if all bids are less than the reservation price r , the object remains unsold. The values of the parameters u , s^2 , σ^2 , n , r and the fixed costs c are known to everyone.

General Equilibrium

The expected monetary profit to a bidder observing X = x and bidding B when the remaining bidders use the monotonically increasing strategy $b(\cdot)$ is as follows, where $F(\cdot)$ denotes the cumulative distribution function of an individual's private information conditional on the true value, and $\delta(\cdot)$ is the corresponding density function.

$$E(\$|x) = \frac{\int_z (z - B - c) F^{(n-1)}(b^{-1}(B)|z)\delta(x|z)h(z)dz}{\int_z \delta(x|z)h(z)dz}$$

$$E(\$) = \int_X \int_z (z - B - c) F^{(n-1)}(b^{-1}(B)|z)\delta(x|z)h(z)dz\,dx \ .$$

A necessary condition for there to exist a symmetric Nash [5] equilibrium strategy b(x) is that the derivative of $E(\$|x)$ with respect to B evaluated at B = b(x) is equal to zero. This results in a first order linear differential equation for b(x) ; the initial condition is obtained by noting that b(x) < r if and only if $E(\$|x) < 0$. It must be verified that the solution indeed maximizes expected profit when all opposing bidders bid according to it.

Unfortunately, Engelbrecht-Wiggans and Weber prove that it is generally impossible to obtain a closed form expression equilibrium for

the multiplicative lognormal error model. There appear to be few realistic distributions for which analytic solutions are possible. Even numerical solutions are complicated by the ratios of double integrals which appear as coefficients in the differential equation.
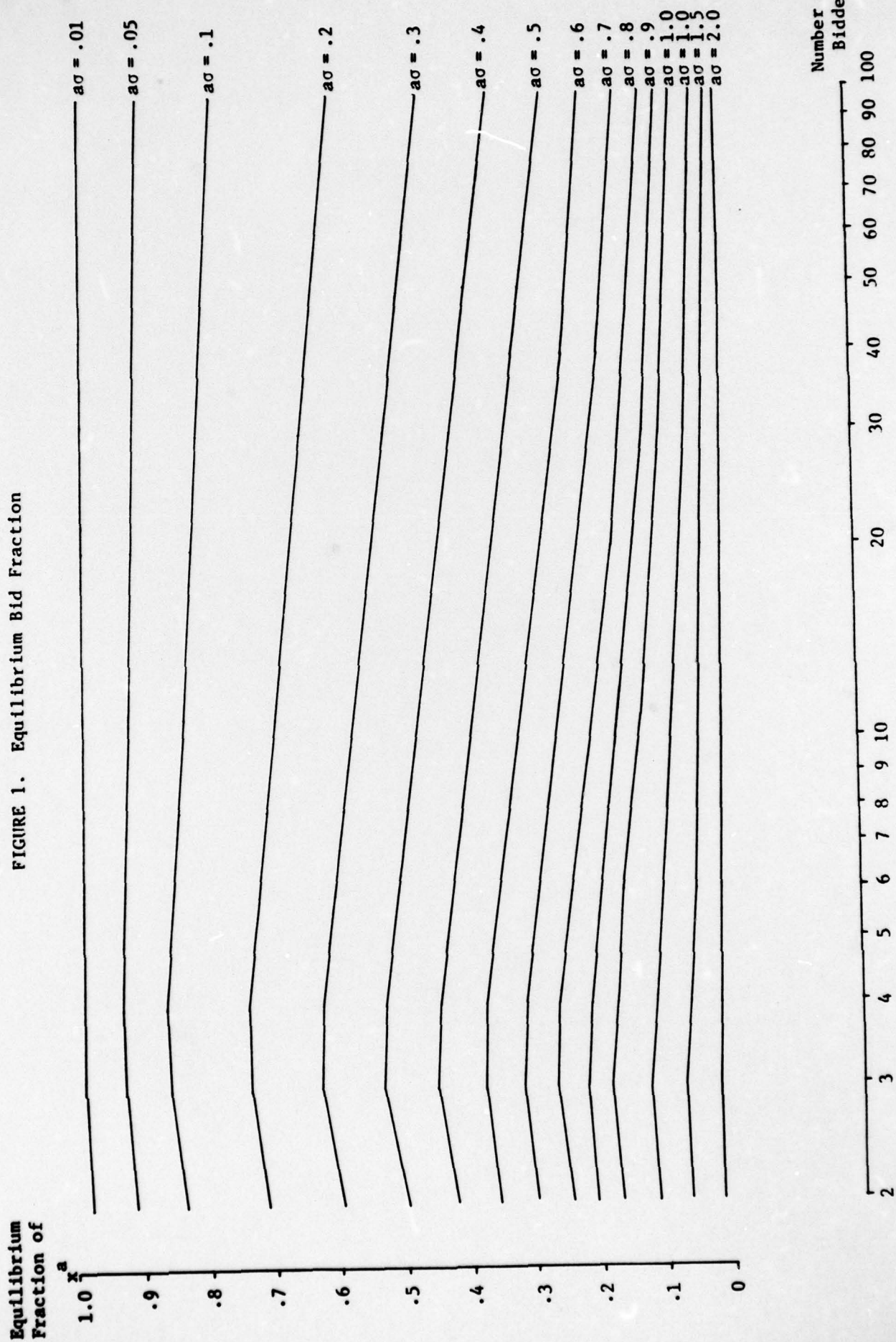
Multiplicative Strategies

A common alternative is to assume that all bidders use multiplicative strategies; i.e., $b(x) = b \cdot x$. Note that under the multiplicative error assumption, the private information is proportional to an unbiased estimate of the true value and thus, multiplicative strategies are also equivalent to bidding a fixed multiple of a particular unbiased estimate. An advantage of restricting bidders to multiplicative strategies is that such strategies are relatively simple to calculate.

For the symmetric model, the equilibrium fraction b may be determined by differentiating $E(\$)$ with respect to B , evaluating at $B = b \cdot x$, setting the derivative equal to zero, and solving the result for b . Whenever the expected value of Z (prior to observing any private information) is finite, the equilibrium fraction is independent of the distribution $h(z)$ of z . If $F(\cdot)$ and $f(\cdot)$ denote the distribution and density of the ratio X_i/z (which is equal to $\exp(E_i)$), then there is the following expression for b :

$$b = \frac{(n-1)\int_w wF^{(n-2)}(w)f^2(w)dw}{(n-1)\int_w w^2F^{(n-2)}(w)f^2(w)dw + \int_w wF^{(n-1)}(w)f(w)dw} .$$

The equilibrium bid fraction b of x^a (where $a = 1$ for multiplicative bidding) is plotted in Figure 1. The larger the error variance

FIGURE 1. Equilibrium Bid Fraction

Equilibrium
Fraction of
x_a

aσ = .01
aσ = .05
aσ = .1
aσ = .2
aσ = .3
aσ = .4
aσ = .5
aσ = .6
aσ = .7
aσ = .8
aσ = .9
aσ = 1.0
aσ = 1.0
aσ = 1.5
aσ = 2.0

Number of
Bidders

σ^2 , the more conservative one should bid. For very small number of competitors the chance of picking up a bargain results in more conservative bidding than when there are slightly more competitors. For large numbers of bidders, the effect of the "winner's curse" is to force more conservative bidding again. Capen, Clapp and Campbell [1] obtain qualitatively similar results in their decision theoretic analysis.

If the prior distribution of the true value is diffuse, then Rothkopf proves that equilibrium multiplicative strategies are (unrestricted) equilibrium strategies. Winkler and Brooks [12] state an analogous result for additive strategies in models with additive normally-distributed errors. Engelbrecht-Wiggans and Weber establish that such strategies are generally not in equilibrium if the prior distribution is not diffuse. Since, in practical applications, there is always at least a very little prior information about the true value, it is important to establish how close multiplicative strategies are to being in equilibrium.

Consider a particular example based on the Outer Continental Shelf federal offshore oil lease sale #40 [11]; we will focus on the approximately three dozen most promising sites identified by the, admittedly less than perfect, criterium of a non-minimal geological pre-sale estimate. (We attempt to eliminate sites with spurious estimates by eliminating those with fewer than three bids.) Assume $\log_e(Z/\$1000)$ has a mean = 9.0 and variance = 4.0. Let $\log_e(X_i/z)$ have mean = 0.0 and variance = 1.0, let $n = 8$, and $r = c = 0.0$. Figure 2 plots the equilibrium multiplicative strategy and the expected profit maximizing strategy of a single individual when all opposing bidders use the multiplicative strategy. Table 1 indicates that a single bidder can increase his expected profit from about 2.82 to about 4.08 million dollars by
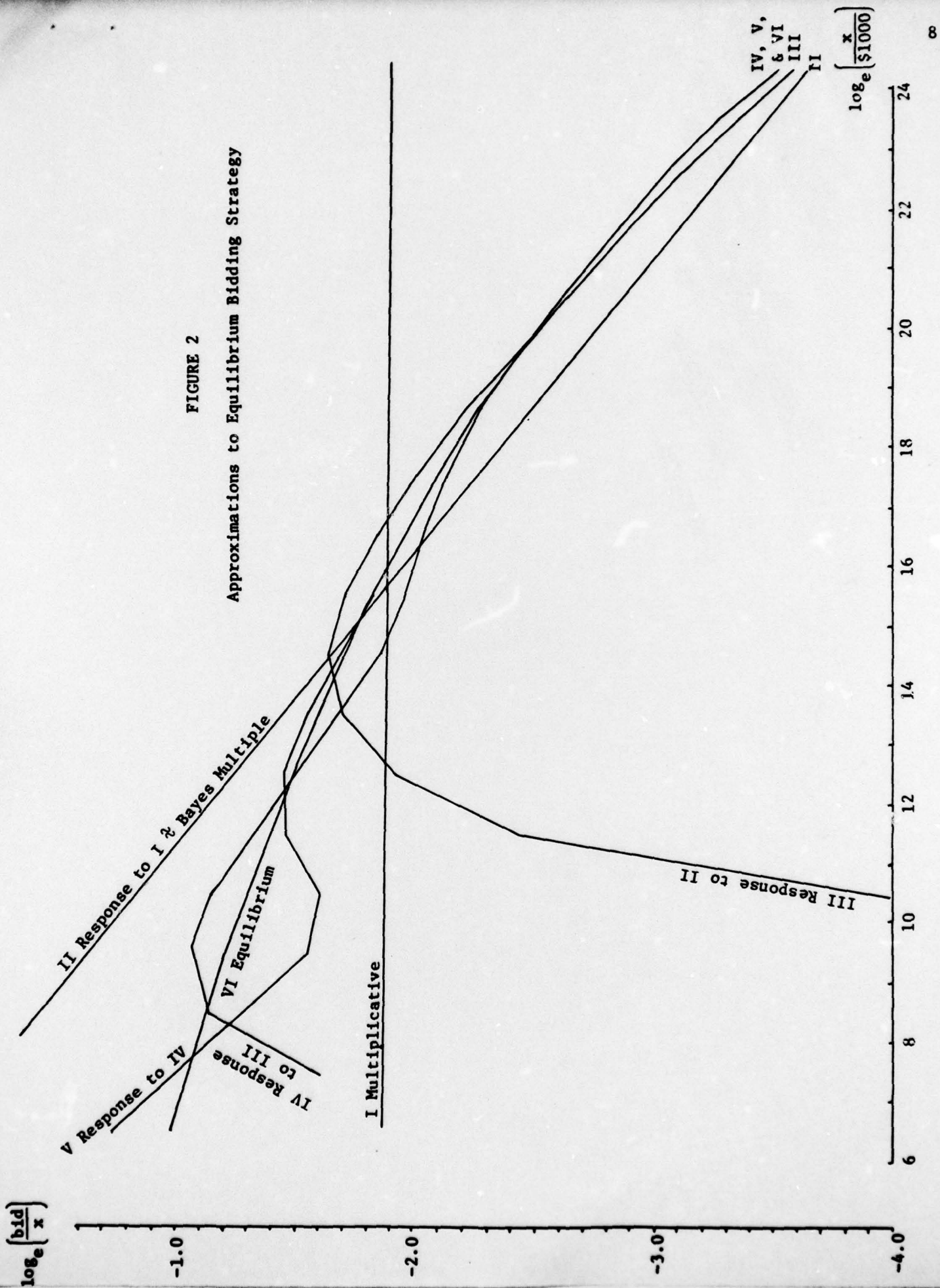
FIGURE 2

Approximations to Equilibrium Bidding Strategy

TABLE 1

Relative Expected Profit as a Function of Bidding Strategy Versus
Strategy Used by Remaining Bidders
(Each unit is approximately $1,000,000)

Bidding Strategy of Remaining Players	Bidding Strategy			
	I Multiplicative	II Multiple of Bayes Estimate*	III Best Response to II	VI Equilibrium
I	2.82	4.06	3.83	3.79
II	2.49	2.32	2.81	2.71
VI	1.95	2.11	2.28	2.31

*Within the accuracy of the computations, this strategy is identical with best response to strategy I. (This equivalence is coincidental; different parameters of the model result in distinctly different strategies for the best response to multiplicative bids and the equilibrium multiple of Bayes estimates.)

unilaterally deviating from the multiplicative strategy. Although these figures are based on $r = c = 0$, we will observe that the bidding strategies are substantially the same for more realistic choices of these parameters and thus one can expect similar disequilibrium in the expected profits for different fixed costs and reservation prices. In addition, since strategies are relatively insensitive to the number of bidders, small changes in the number of bidders should have little effect. Multiplicative strategies are not close to being in equilibrium.

Bayes Strategies

An alternative to multiplicative bidding is to restrict bids to be a fixed multiple of the Bayes, equivalently, posterior, estimate of the true value; in particular, let $b(x) = b \cdot x^a$, where $a = s^2/(s^2 + \sigma^2)$. Teisberg [10] proves that if strategies are restricted to functions of x^a, then there is a strategy of the form $b \cdot x^a$ which is in equilibrium among functions of x^a. Equilibrium Bayes multiples are plotted in Figure 1, where the comments are the same as for multiplicative strategies, except that Bayes strategies do depend on the prior distribution through the parameter a.

Engelbrecht-Wiggans and Weber prove that multiples of Bayes estimates are generally not in equilibrium if $h(z)$ is not diffuse. In our example, an individual can increase his expected profit from about 2.32 million to about 2.81 million dollars by unilaterally deviating from the "multiple of Bayes estimate" strategy. Although such strategies are closer to equilibrium than multiplicative strategies, half a million dollars is still a substantial amount.

One might ask what would happen is all bidders initially used

multiplicative strategies (or multiples of Bayes estimates) and in sub-
sequent auctions naively reacted to their mistakes in previous auctions.
In particular, assume that each bidder determines what his best strategy
would have been against the strategies used by the opposing bidders in
the most recent previous auction. Assume also that each bidder uses
this "reactive" strategy in the next auction; such an assumption becomes
more plausible if one considers the vast amount of literature which sug-
gests determining one's strategy through a decision theoretic analysis
of past auctions. Figure 2 plots several iterations of this procedure.
The strategies converge to within a few percent (in terms of expected
profit) to equilibrium in two or three iterations. Indeed, it is possible
that a learning process similar to the above actually occurred in the
early offshore oil lease sales.

Parameters

In actual oil lease sales, there is a substantially non-zero reser-
vation price. In addition, there may be substantial fixed costs. (It
may however be difficult to separate out the truely fixed costs from
any slightly uncertain "fixed" costs; the latter would not appear ex-
plicitly but be implicit in the uncertain true value of the object.)
Observe that the equilibrium bid where the reservation price is r and
the fixed costs c is precisely c less than the bid when the reser-
vation price is $r+c$ and there are no fixed costs. Thus, in terms of
the mathematics, the fixed costs together with the reservation price
constitute only one free parameter.

In Figure 3, the equilibrium strategy is plotted for various different values of the fixed costs when the reservation price is $1000e^5 (approximately the typical actual reservation price for offshore oil leases). As the fixed cost increases, bidders observing low estimates of the true value must bid more conservatively to assure themselves of a non-negative profit. The effect of non-zero fixed costs and reservation prices on the equilibrium bidding strategy is mainly for the cases of low estimates. For a typical object, several bidders are likely to receive a moderate estimate, and thus the distribution of the winning bid is relatively insensitive to the precise values of r and c; if all estimates are small, then the object is likely to be of low true value and any effects of the reservation price or fixed costs on the sale price of such objects has relatively little effect on the average sale price of objects.

The effect of the number of bidders is examined in Figure 4; equilibrium strategies are relatively insensitive, especially for the larger values of the information variable to the exact number of bidders. However, with more bidders using basically the same strategy, the expected selling price should increase and therefore the expected profit to bidders should decrease. In the example, the expected profit per bidder is approximately 4.67, 2.31, and 1.76 million dollars if there are 6, 8, and 10 bidders, respectively, using the corresponding equilibrium strategies (where $r = c = 0$).

Notice that equilibrium strategies are significantly affected by changes in the variance of the error. As the variance of E_i increases from .5 to 1.0 to 1.5, the expected profit increases from approximately 1.96 to 2.31 to 3.11 million dollars; for this range of variances in the error, better information to each of the bidders results in lower
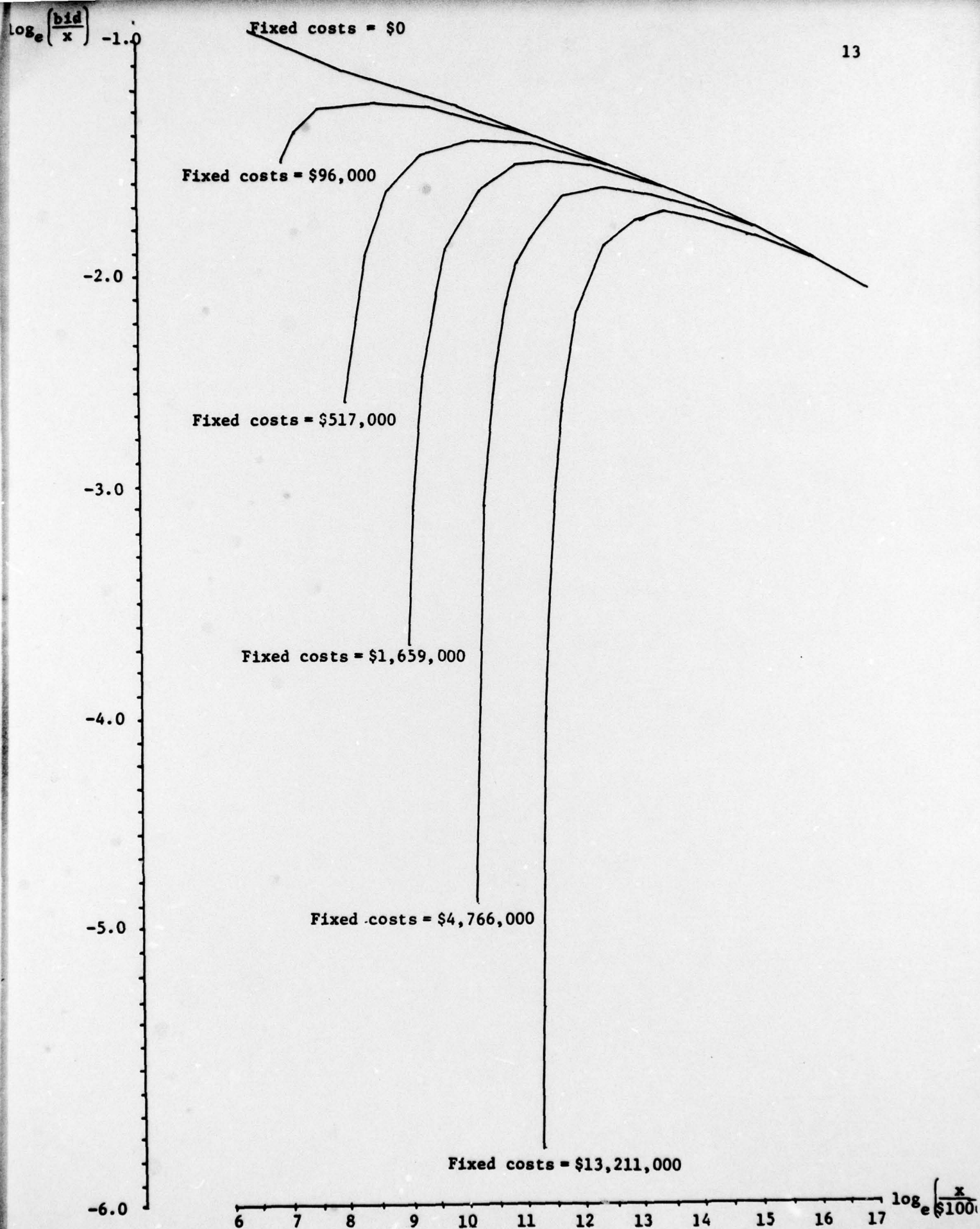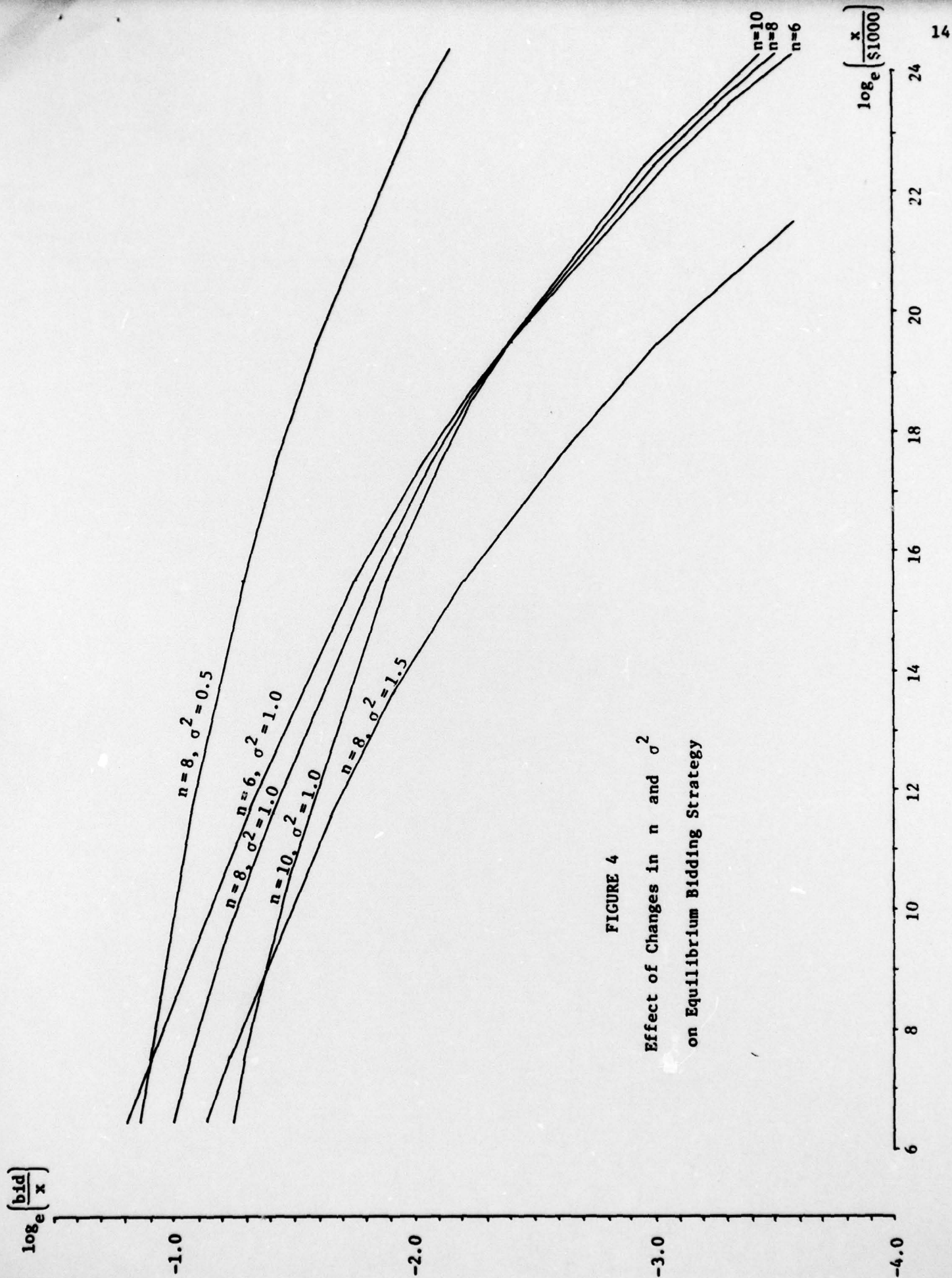
FIGURE 4

Effect of Changes in n and σ^2 on Equilibrium Bidding Strategy

expected profits to each. Note, however, that this relationship depends on the range of parameters. (If there is no private information (i.e., the variance σ^2 is infinitely large), than the equilibrium strategy is for each individual to bid $b = E(z)$, the prior expected true value. This results in zero expected profit.)

Comparison to Data

The distribution of bids predicted by the model under the particular choice of parameters is plotted in Figure 5, as is the actual distribution of bids in Sale #40 on leases which received at least three bids and had non-minimal Geological Survey pre-sale estimates. The correspondence is sufficiently close so that it is difficult to fault the choice of parameters on this basis. This suggests that the various numbers calculated in this example are very possibly indicative of the corresponding true values. In particular, multiplicative strategies are likely to be quite far from being in equilibrium.

Finally, observe that a substantial portion of the variance in the number of bids submitted on an object may result from one or more potential bidders observing information which makes them expect to loose money even if they could obtain the object at the reservation price. This variance is of course in addition to any variance in the number of potential bidders. (A similar effect has been modeled by Engelbrecht-Wiggans, Dougherty and Lohrenz [3].)

In particular, consider the case in which an individual submits a bid if and only if he expected a non-negative profit by bidding above the reservation price. We consider two examples, in the first there are 7, 8, and 9 potential bidders on each of 25%, 50%, and 25% of the leases, and in the second case there are 7, 8, 9, and 10 bidders on 16%, 52%, 28%, and 4% of the
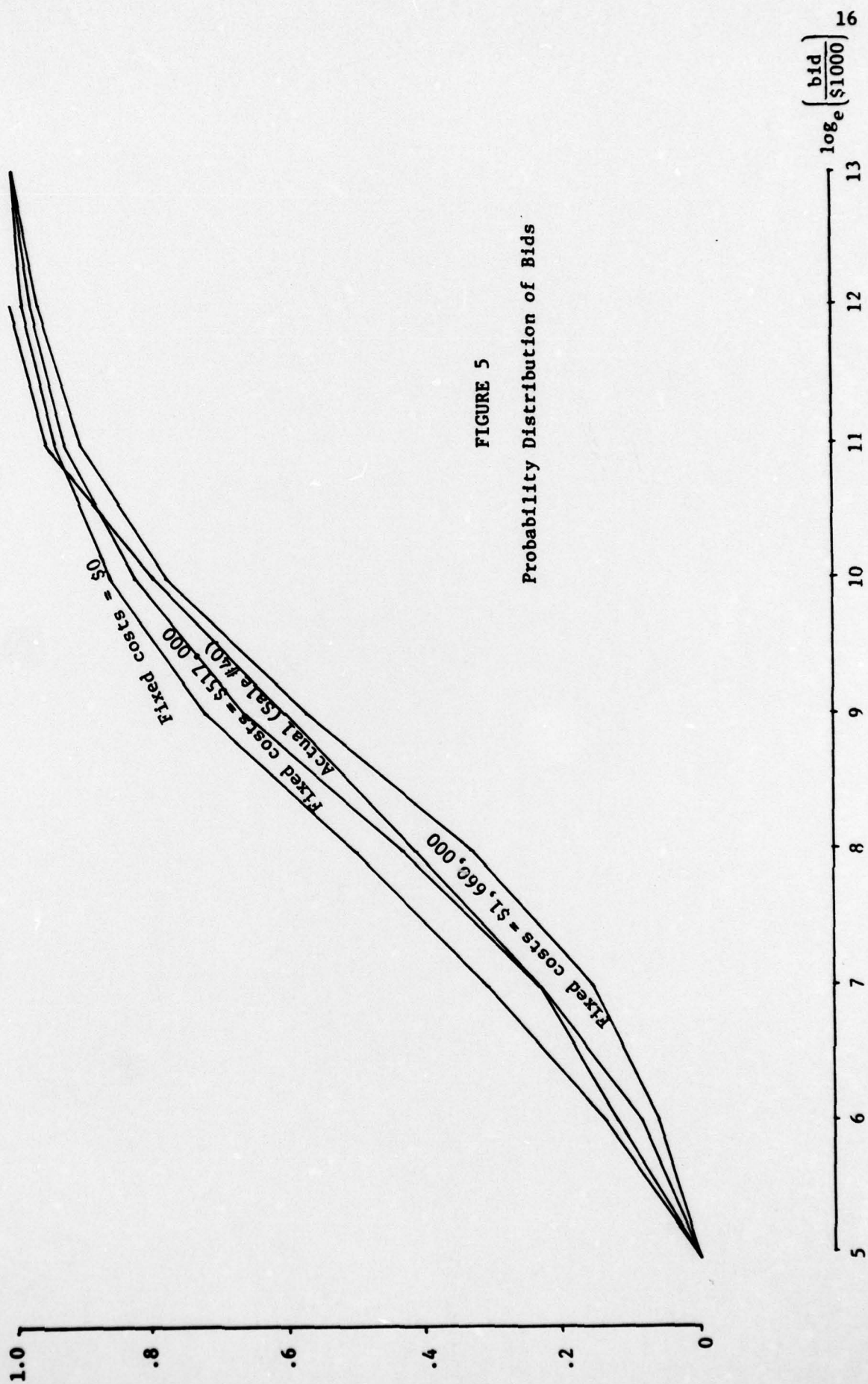
FIGURE 5

Probability Distribution of Bids

leases, respectively. In both cases, the reservation price is $1000e^5$ and the fixed costs are about $100,000; the expected fraction of leases receiving different numbers of bids is plotted along with the actual distribution of Sale #40 in Figure 6.

One should also note that in the offshore oil lease sale, seven major firms each bid on at least three fourths of the sites with non-minimal pre-sale estimates, two additional firms each bid on about half, and numerous other firms relatively rarely submitted a bid; matters are slightly complicated by the fact that two or more of the bigger firms occasionally submitted joint bids. The data however appears to be consistent with the possibility that there are several firms which bid with very high probability on any site with a sufficiently large expected value, and a very few bids on the more valuable sites are from other firms. Therefore the variance of the number of bids observed may be significantly larger than the number of potential bidders (who bid whenever the object appears "worthwhile" bidding on).

Conclusion

A numerical example is used to analyze bidding in auctions with multiplicative lognormally-distributed errors. The choice of parameters in the example is based on an actual offshore oil lease sale. Multiplicative strategies fail substantially to be in equilibrium. However, under the appropriately naive decision theoretic reaction in repeated auctions, the strategies converge very rapidly to being in equilibrium. Bidding strategies appear to be relatively insensitive to the exact number of bidders.

Strategies are more sensitive to the quality of information observed

FIGURE 6

Relative Number of Objects Receiving K Bids

K = Number
of Bids
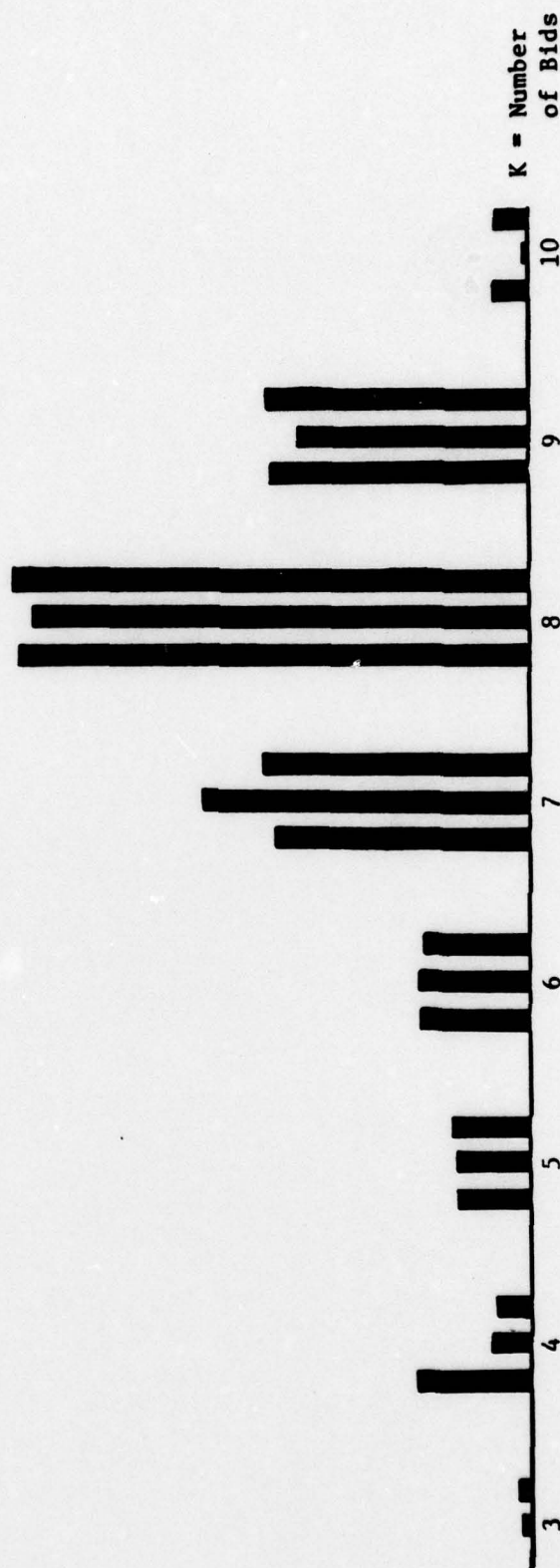
The bars in each group indicate:
Bar 1: Sale #40 (Sites with > 2 Bids and Non-Minimal Geological Survey Presale Estimates).
Bar 2: Mean Predicted for a 1-2-1 mix of 7, 8, 9 potential bidders.
Bar 3: Mean Predicted for a 4-13-7-1 mix of 7, 8, 9, 10 potential bidders.

by bidders; within the range of parameters considered, there is less profit to be made if all bidders are well informed than when they are slightly less well informed. Finally, it is noted that the variance in the number of bids submitted may be considerably larger than the variance in the number of potential bidders on an object.

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