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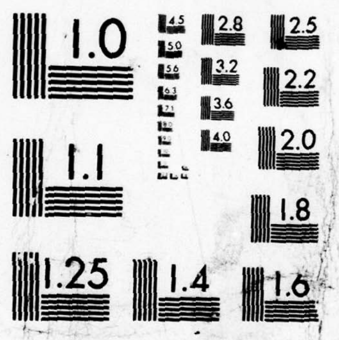

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Approximations of Normal Range Revised with New Tables for d_2 and d_3 - Factors

by

John H. K. Kao

Technical Report No. Poly/MA 79-2

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Patnaik's X-approximation and Cox's X^2-approximation to the normal mean range are revisited and many inaccurate entries discovered. Investigations revealed, among other reasons, that the original Tippett's calculations on $\hat{d_2}$ and $\hat{d_3}$ -factors were of inadequate accuracy. New tables for $\hat{d_2}$ and $\hat{d_3}$ -factors as well as constants accociated with Patnaik's and Cox's approximations are recomputed and tabulated. These new tables will be useful for revising the MIL-STD-414 range plans for inspection by variable and for the analysis of variance tests when ranges or mean ranges are used in place of sample variances or standard deviations, as

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well as for constructing limits for the cumulative sum charts for range variability.

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Approximations of Normal Range Revisited With
New Tables for d_2 and d_3 Factors*

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Introduction

MIL-STD-414, the sampling plans by variables has not been revised since its inception in 1957, while MIL-STD-105 D, the sampling plans by attributes has reached international popularity through 4 revisions. Recently revision activities on MIL-STD-414 have received a great deal of attention. A British proposal BS6002 [16] restored the graphical method for double-specification-limit case and omitted the range method completely. BS6002 was sent to ISO for draft discussion. The American position is to keep the range plans and furthermore to devise a set of variables plans to match the OC curves of plans published in MIL-STD-105 D. This position created some technical difficulties which are under careful scrutiny by both industry and government.

If the range tables (pp. 61-84 of MIL-STD-414) are to be kept, much of the entries(close to 9000 in number) will have to be recomputed. This report indicates the reasons why this is necessary by revisiting the two

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approximations (Patnaik's and Cox's) for the normal range. Chiefly, these approximations gave rise to tables of four constants whose values were based upon d_2 and d_3-factors of inadequate accuracy. This report publishes in its three appendices more accurate values which will be useful not only in revising range table entries in MIL-STD-414 but also useful wherever the normal range is used as an estimate of normal variability.

Sample Range

When the observation number n is not too large, it is appropriate to use the sample range, $R = X_{[n]} - X_{[1]}$ as an estimate of population variability under the normal assumption. In control chart work where the subgroup size n is intentionally kept small (in order to avoid possible inclusion of process shifts within the subgroup), the situation is ideal. However, if the sample size needs to be larger, such as in sampling inspection, a great deal of "information" is lost by using R which ignores all but two extreme observations. Under such circumstances it is wise to break up, if possible, the sample at random into m small subgroups of size n each, i.e., sample size = (m x n) and use, instead, the average range

$$\overline{R} = \sum_{i=1}^{m} R_i/m \qquad (1)$$

where R_i is the sample range of the i^{th} subgroup.

Clearly, with fixed available resources when observing a sample of size (m x n), if n chosen is small, m will be large and the resultant $\overline{R}$ would reflect a loss of inter-subgroup information through massive averaging, although the loss of intra-subgroup information is kept a a low level. On the other hand, if m is reduced, in order to minimize the loss through averaging, a price has to be paid for larger n which results in loss of intra-group information. An optimum n exists, which fact was studied by Grubbs and Weaver [1] and confirmed by Cox [2] and found to be 7 or 8 for the normal case.

Under the normal assumption for X, the distribution of R and its moments can not be expressed in closed form and hence numerical integrations are necessary. This resulted in the famous d_2 and d_3- factors widely used in quality control charts, which are:

$$ER = d_2\sigma' \quad \text{and} \quad \text{Var } R = (d_3\sigma')^2 . \tag{2}$$

Here the quality characteristic X is assumed to be normally distributed with population mean $\bar{x}'$ and population variance σ'^2, i.e., $N(X|\bar{x}', \sigma')$ and $N(Y|0,1)$ for $Y = (X-\bar{x}')/\sigma'$ to use the standard notations of quality control. If one defines the standard range $W = Y[n] - Y[1]$, then $W=R/\sigma'$ and $EW = d_2$ and $\text{Var } W = (d_3)^2$ which are given by the following well-known expressions:

$$d_2 = EW = \int_{-\infty}^{+\infty} [1-F^n(y) - Q^n(y)]\, dy \quad \text{and} \tag{3}$$

$$(d_3)^2 = \text{Var } W = 2\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \left\{1-F^n(y)-Q^n(z)+[F(y)-F(z)]^n\right\} dz-(d_2)^2 \tag{4}$$

where

$$1 - Q(y) = F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

Tippett [3] using Eq. (3) with Gaussian quadrature obtained d_2 for $n = 2(1)1000$ in 5 D. (5D means five decimal places.) Instead of using Eq. (4) directly, he chose an approximate distribution of $\bar{W}$ and computed only a few values for d_3 in 4 D. Tippett's values of d_2 for

$n = 2(1)\,500\,(10)\,1000$ in 5D and d_3 for $n = 2(1)\,20$ in 4D can also be found in Biometrika [4] Tables 27 and 20 respectively. Grubbs and Weaver [1] also tabulated 11 pairs of d_2 and d_3 (their d_n and k_n) for $n = 2(1)\,12$, in 4D and 5D respectively, but some of their values are in error. In need of better accuracy we [5] have recomputed values of d_2 and d_3 by Eqs. (3) and (4) above directly by numerical methods. We herewith present these factors for $n = 2(1)\,50$ in 7D and 8D in Appendix A. Our values check favorably with computations calculated from Teichroew [15] for $n = 2(1)\,20$.

χ-approximation Patnaik [6] approximated $\overline{W} = \overline{R}/\sigma'$ with $c\chi/\sqrt{\nu} = c'\chi$ by equating the first two moments of the latter with the former and obtained 40 pairs of c and ν for $m = 1(1)\,5$ and $n = 3(1)\,10$. The density χ can be easily found from χ^2-distribution as

$$f(\chi) = \frac{\chi^{\nu-1} e^{-\chi^2/2}}{2^{\frac{\nu}{2}-1}\,\Gamma(\frac{\nu}{2})}, \quad \nu > 0,\ \chi > 0, \text{ whose } k^{th} \text{ moment is,} \tag{5}$$

$$E\chi^k = 2^{\frac{k}{2}}\,\Gamma\left(\frac{\nu+k}{2}\right)/\,\Gamma(\nu/2), \text{ which in turn gives,} \tag{6}$$

$$E\chi = \sqrt{2}\,\Gamma\left(\frac{\nu+k}{2}\right)/\,\Gamma(\nu/2) \text{ and} \tag{7}$$

$$\text{Var}\,\chi = \nu - 2\,\Gamma^2\left(\frac{\nu+1}{2}\right)/\,\Gamma^2(\nu/2), \text{ hence} \tag{8}$$

$$E(c\chi/\sqrt{\nu}) = c\sqrt{\frac{2}{\nu}}\,\Gamma\left(\frac{\nu+1}{2}\right)/\,\Gamma(\nu/2) \text{ and} \tag{9}$$

$$\text{Var}\,(c\chi/\sqrt{\nu}) = c^2 - E^2(c\chi/\sqrt{\nu}) \tag{10}$$

On the other hand, from Eq. (2), we have,

$$E(\overline{R}/\sigma') = \frac{1}{m}\sum R/\sigma' = \frac{1}{m}\sum ER/\sigma' = \frac{1}{m}\sum ER/\sigma' = \frac{1}{m}\sum d_2 = d_2 \tag{11}$$

$$V(\bar{R}/\sigma') = \frac{1}{\sigma'^2} \frac{(d_3\sigma')^2}{m} = d_3^2/m . \qquad (12)$$

Equating Eqs. (9) and (10) with Eqs. (11) and (12) respectively, we obtain

$$c^2 - d_2^2 = d_3^2/m, \quad \text{or } c = \sqrt{d_2^2 + d_3^2/m} \text{ and} \qquad (13)$$

$$\Gamma\left(\frac{\nu+1}{2}\right) / \sqrt{\nu}\ \ \Gamma(\nu/2) = d_2/\sqrt{2(d_2^2 + d_3^2/m)} \ . \qquad (14)$$

Eq. (13) gives an explicit solution for c in terms of d_2 and d_3 and ν can be obtained from Eq. (14) iteratively. But Patnaik [6] and later Resnikoff [7], used the Stirling's large ν formula for gamma function and gave an approximate value of ν from solving the following cubic equation, for $a = d_2m/d_3$,

$$\nu^3 = (a/2)\nu^2 - (a/8)\nu + (a/16) = 0 , \qquad (15)$$

and then solve the value of c afterwards (Instead of solving c directly from Eq. (13)). David [11] used Eq. (13) for c but a similar scheme to Eq. (15) for ν and presented 54 pairs of c and ν for m = 1(1) 5 and 10, n = 2(1) 10 (cf. Table 30 of [4]). Duncan [12] extended David's table to 210 pairs for m = 1(1) 15 and n = 2(1) 15. Duncan renamed c by d_2^* , since $d_2^* \to d_2$ as $m \to \infty$ which is evident from looking at Eq. (13). Unfortunately, both David's and Duncan's values have only 1 to 2 decimals. Thomson [13] interpolated the Γ-function at close intervals and tabulated for n = 2(10) 10 and m = 1, yielding 9 pairs (See values in brackets in Table 1 below) of ν and c which are much better than those obtained by Patnaik, Resnikoff, David and Duncan. The facts remain that in order to obtain accurate c and ν from Eqs. (13) and (14); better d_2 and d_3 than

Tippett's values are necessary (See our values in Appendix A). Patnaik, Resnikoff, David, Duncan and Thomson all used Tippett's d_2 and d_3 values [Tables 27 and 20, Ref. 4] and their results on c and ν check each other only approximately. It is the purpose of this report to publish more accurate values of c and ν from our more accurate values of d_2 and d_3 given by our previous research (Appendix A). Table 1 is a comparison between Patnaik's and our results rounded to one more digit than Patnaik's values indicating many poor Patnaik's values. Table 2: between Resnikoff's and ours. Included in Table 2 are also values of ν (center value) using Eq. (15) the cubic equation used by Resnikoff with Tippett's values of d_2 and d_3 for the coefficients. For some curious reasons, they check only approximately with Resnikoff's original results. Since we have an iterative program for Eq. (14), all we need to do here, is to substitute a subroutine of Eq. (15) in place of that for Eq. (14) and proceed with the computation. Therefore we must conclude that Resnikoff's values are in error. From these two tables, we see that both Patnaik's and Resnikoff's tables need some corrections, especially on values of ν for small n. Thomson's 9 pairs shown in Table 1 are superior to Patnaik's. They contain only occasional errors at the 4th decimal.

χ^2-Approximation Cox [3] approximated $\overline{W} = \overline{R}/\sigma'$ with $c\chi^2/\nu = c'\chi^2$ also by equating the first two moments of the latter with the former random variables and tabulated the values of $2c'(= \theta^2 d$, Cox's notation) and ν for m = 1 (only) and n = 2(1) 10. Since $E\chi^2 = \nu$ and Var $\chi^2 = 2\nu$, values of c and ν are trivial to solve. Equating the two first moments, we have,

$$c(\nu)/\nu = c = d_2 \qquad (16)$$

Equating the two second moments, we have,

Table 1. Values of ν and c for Patnaik's χ-Approximation to $\overline{\overline{W}}$.

	m=1		m=2		m=3		m=4		m=5	
n	ν	c	ν	c	ν	c	ν	c	ν	c
2	1.00000 [1.0000]	1.414214 [1.41421]	1.9195+	1.27930	2.8173	1.23105+	3.7062	1.20620	4.5906	1.19105-
3	1.98463 (1.934) [1.9846]	1.911540 (1.9164) [1.9115-]	3.8337 (3.850)	1.80538 (1.8049)	5.6628 (5.674)	1.76857 (1.7684)	7.4854 (7.499)	1.74988 (1.7498)	9.3050+ (9.312)	1.73857 (1.7385)
4	2.92916 (2.951) [2.9291]	2.238865 (2.2374) [2.23887]	5.6935+ (5.705)	2.15069 (2.1505)	8.4415- (8.450)	2.12049 (2.1204)	11.1846 (11.191)	2.10522 (2.1052)	13.9356 (13.931)	2.09601 (2.0960)
5	3.82651 (3.828) [3.8267]	2.481246 (2.4812) [2.48124]	7.4710 (7.474)	2.40484 (2.4048)	11.1019 (11.107)	2.37883 (2.3788)	14.7288 (14.732)	2.36571 (2.3657)	18.3542 (18.355)	2.35781 (2.3578)
6	4.67716 (4.692) [4.6772]	2.672531 (2.6721) [2.67253]	9.1612 (9.179)	2.60439 (2.6001)	13.6335+ (13.641)	2.58127 (2.5812)	18.1026 (18.109)	2.56964 (2.5696)	22.5704 (22.576)	2.56263 (2.5626)
7	5.48415+ (5.499) [5.4841]	2.829802 (2.8295) [2.82981]	10.7675- (10.779)	2.76779 (2.7677)	16.0405- (16.052)	2.74681 (2.7468)	21.3107 (21.324)	2.73626 (2.7362)	26.5698 (26.596)	2.72991 (2.7299)
8	6.251123 (6.259) [6.2512]	2.962883 (2.9630) [2.96288]	12.2959 (12.297)	2.90562 (2.9056)	18.3315- (18.328)	2.88628 (2.8850)	24.3645+ (24.358)	2.87656 (2.8768)	30.3966 (30.386)	2.87071 (2.8707)
9	6.98207 (6.989) [6.9818]	3.077930 (3.0778) [3.07794]	13.7530 (13.753)	3.02446 (3.0245)	20.5162 (20.511)	3.00642 (3.0064)	27.2769 (27.270)	2.99737 (2.9974)	34.0367 (34.025)	2.99192 (2.9920)
10	7.68007 (7.689) [7.6798]	3.179045+ (3.1789) [3.17905+]	15.1459 (15.151)	3.12869 (3.1287)	22.6041 (22.609)	3.11172 (3.1117)	30.0602 (30.066)	3.10320 (3.1032)	37.5156 (37.523)	3.09808 (3.0981)

Note: Values in parenthesis are Patnaik's [6], in brackets are Thomson's [13].
Values above them are ours.

Table 2. Values of ν and c for Resnikoff's χ-Approximation to $\overline{W}$.

mn	m	n	c	ν
3	1	3	1.9115+, 1.9117, (1.910)	1.9846, 1.9858, (1.934)
4	1	4	2.2389, 2.2389, (2.234)	2.922, 2.9315+, (2.995)
5	1	5	2.4812, 2.4813. (2.474)	3.8265+ 3.8284, (3.828)
7	1	7	2.8298, 2.8298, (2.830)	5.4842, 5.4856, (5.499)
10	2	5	2.4048, 2.4048, (2.405)	7.4710, 7.4717, (7.474)
15	3	5	2.3788, 2.3788, (2.379)	11,1019, 11.1019, (11,106)
25	5	5	2.3578, 2.3578, (2.358)	18.3542, 18.3536, (18.355)
30	6	5	2.3525+, 2.3535+, (2.353)	21.9787, 21.9780, (21.986)
35	7	5	2.3487, 2.3487, (2.349)	25.6026, 25.6019, (25.611)
40	8	5	2.3459, 2.3459 (2.346)	29.2264, 29.2255-, (29.236)
50	10	5	2.3419, 2.3419, (2.342)	36.4733, 36.4722, (36.486)
60	12	5	2.3393, 2.3393, (2.339)	43.7199, 43.7184, (43.735)
85	17	5	2.3354, 2.3354, (2.335)	61.8354, 61.833, (61.856)
115	23	5	2.3329, 2.3329, (2.333)	83.5733, 83.5704, (83.601)
175	35	5	2.3305+, 2.3305+, (2.331)	127.0482, 127.0439, (127.091)
230	46	5	2.3294, 2.3294, (2.330)	166.8999, 166.8942, (166.958)

Note: Center column from Eq. (15); values in parenthesis are Resnikoff's [7]. Values at left are ours rounded to 4 decimals.

$$(c/\nu)^2 (2\nu) = 2c^2/\nu = 2d_2^2/\nu = d_3^2/m$$

or

$$\nu = 2m\,(d_2/d_3)^2 \qquad (17)$$

And hence,

$$c' = c/\nu = d_3^2/2md_2 \qquad (18)$$

Eqs. (16) and (17) represent a very small number of additional program steps and hence are easily incorporated into our computer program.

In a series of articles on the cumulative sum control charts by Johnson and Leone [8], they needed an approximation for the sample range and included a table of c' and ν for both Patnaik's χ-approximation as well as Cox's χ^2-approximation. Table 3 gives a comparison between values by Johnson and Leone and the results of our program using our more accurate d_2 and d_3 (Appendix A). By comparison Johnson and Leone's values are uneven and some with large differences in the second decimal place and therefore are obvious in need of corrections. The most recent entry for Patnaik's c and ν for m=1(1)15, 20, 30, 50 and n =2(1)15 can be found from Nelson [14]. Unfortunately, Nelson only tabulated them (just like David's and Duncan's) in 1 to 2 decimals. In Appendix B, we give our values for Patnaik's c and ν and in Appendix C, we give our values for Cox's $2c'$ and ν.

Approximate Probability Integrals of W

Since the density function as per Eq. (19) of the sample range from a normal population cannot be expressed in a closed form (except for n = 2, which is shown below), the probability integral, $P\{W \leqq W\}$ can only be evaluated approximately. A table for this probability integral was

Table 3. Values of ν and c' for χ and χ^2- approximation to $\overline{W}$.
(Subscript 1 for Patnaik's χ-approximation and 2 for Cox's χ^2- approximation)

n	χ-approximation (Patnaik's)		χ^2-approximation (Cox's)	
	$c_1' = c_1/\sqrt{\nu}$	$\nu 1$	$c_2' = c_2/\nu_2$	ν_2
2	1.4142	1.000	.3220	3.504
3	1.3569 (1.378)	1.985- (1.93)	.2331 (.233)	7.260 (7.27)
4	1.3081 (1.302)	2.929 (2.95)	.1880 (.188)	10.951 (10.95)
5	1.2684 (1.268)	3.827 (3.83)	.1605+ (.160)	14.491 (14.49)
6	1.2358 (1.237)	4.677 (4.69)	.1419 (.142)	17.863 (17.86)
7	1.2084 (1.207)	5.484 (5.50)	.1284 (.128)	21.069 (21.08)
8	1.1850+ (1.184)	6.251 (6.26)	.1180 (.118)	24.122 (24.11)
9	1.1648 (1.164)	6.982 (6.99)	.1099 (.110)	27.034 (27.01)
10	1.1471 (1.146)	7.680 (7.69)	.1032 (.103)	29.816 (29.82)

Note: Values in parenthesis are Johnson and Leone's [8].
Values above them are ours.

calculated by Pearson and Hartley [9] and reproduced in [4] as Table 23 for n = 2(1) 20 and W = 0(0.5) 7.25 in 4D. The density function of W is,

$$g(w) = n(n-1) \int_{-\infty}^{+\infty} [F(w+y) - F(y)]^{n-2} f(w+y) f(y) dy \qquad (19)$$

where, under the normal case, F(y) is defined in Eq. (4), where n = 2, Eq. (19) may be integrated into:

$$g(w) = \frac{1}{\sqrt{\pi}} e^{-\frac{w^2}{4}}, \qquad \text{for } w > 0 \qquad (20)$$

which is a folded or half-normal density. Furthermore,

$$P\{W \leqq a\} = \int_0^a g(w)\, dw = \int^{\frac{a^2}{2}} y^{\frac{1}{2}-1} e^{-y/2} dy/2^{\frac{1}{2}} \Gamma(\tfrac{1}{2}) \qquad (21)$$

which is a χ^2-integral with one degree of freedom. In fact, the k^{th} moment in this case is,

$$EW^k = \frac{1}{\sqrt{\pi}} \int_0^{\infty} w^k e^{-\frac{w^2}{4}} dw = 2^k \Gamma\left(\frac{k+1}{2}\right) / \sqrt{\pi} \qquad (22)$$

from which one may find, for n = 2,

$$d_2 = EW = 2/\sqrt{\pi} \quad \text{and} \quad (d_3)^2 = \text{Var } W = 2 - 4/\pi . \qquad (23)$$

Eq. (23) is interesting because it shows as expected that for n = 2, $d_2 = 2c_2$ and $d_3 = 2c_3$ where $c_2 = \sqrt{2}\,\Gamma(n/2)/\sqrt{n}\,\Gamma(n/2-1/2)$ and $c_3 = \sqrt{(n-1)/n - c_2^2}$ are factor for sigma control charts widely used in quality control.

The following are probability integrals using the two approximations revisited by this report:

(a) Using Patnaik's χ-approximation, $W = c_1 \chi / \sqrt{\nu_1} = c_1' \chi$, the distribution of W may be found from $f(\chi)$ given by Eq. (5) as, (we omit for clarity the subscripts for c and ν)

$$f(w) = \frac{w^{\nu-1} e^{-w^2/2c'^2}}{c'^{\nu} 2^{\nu/2-1} \Gamma(\nu/2)}, \quad w > 0, \ \nu > 0. \tag{24}$$

From this, $P\{W \leqq a\} = \int_0^a f(w)dw$ can be obtained by letting $y = w^2/2c'^2$ as a gamma integral,

$$P\{W \leqq a\} = \int_0^{\frac{a^2}{a} c'^2} y^{\nu/2-1} e^{-y} dy/\Gamma(\nu/2) = G\left(\frac{a^2}{2c_1'^2} \Big| \frac{\nu}{2}\right) \tag{25}$$

Incidentally, by changing the variable of integration in Eq. (21) from y to $z = y/2$, the probability integral of w for $n = 2$ can be seen as $G\left(\frac{a^2}{4} \Big| \frac{1}{2}\right)$ which says that,

$$\nu_1 = 1, \ c_1 = c_1' = \sqrt{2} \ \text{ for } n = 2 \, . \tag{26}$$

Of course Eq. (25) checks exactly the entries shown in Table 1.

(b) Using Cox's χ^2-approximation, $W = c_2 \chi^2/\nu_2 = c_2' \chi^2$, the distribution of W may be found from a χ^2-distribution as,

$$f(w) = \frac{w^{\nu/2-1} e^{-w/2c'}}{(2c')^{\nu/2} \Gamma(\nu/2)}, \quad w > 0, \ \nu > 0. \tag{27}$$

By letting $y = w/2c'$, the probability integral for W is,

$$P\{W \leqq b\} = \int_0^{b/2c'} y^{\nu/2-1} e^{-y}/\Gamma(\nu/2) = G\left(\frac{b}{2c'} \Big| \frac{\nu}{2}\right) \tag{28}$$

With a gamma integral subroutine, both approximate probability integrals given by Eqs. (25) and (28) may be easily evaluated by inputting our more accurate values for c_1, ν_1, c_2 and ν_2 shown in Appendices B and C.

Pearson [10] made a comparison of the above two approximate probability integrals with those obtained by Table 23 of [4] which be called "True P.I.". Table 4 of this report reproduces his results together with the output of our program. The agreements are good and very few corrections are needed on Pearson's calculations. Had we used David's [11] or Johnson and Leone's [8] c and ν for inputs, the results will be way off.

Table 4. Probability Integral Approximation

n=4

W= R/σ'	True P.I.	χ-approx.	χ^2-approx.
.35	(.0053)	.00575+, (.0058)	.00112, (.0012)
.45	(.0111)	.01184, (.0119)	.00358, (.0036)
.75	(.0483)	.04972, (.0499)	.03063, (.0306)
1.00	(.1057)	.10724, (.1074)	.08716, (.0877)
1.30	(.2054)	.20631, (.2065)	.19727, (.1973)
2.00	(.5096)	.50786, (.5079)	.53026, (.5303)
2.80	(.8045)	.80318, (.8031)	.81532, (.8153)
3.25	(.9016)	.90144, (.9013)	.90207, (.9021)
3.65	(.9516)	.95207, (.9520)	.94702, (.9470)
4.40	(.9899)	.99051, (.9904)	.98487, (.9849)
4.70	(.9951)	.99549, (.9955)	.99111, (.9911)

n=6

W= R/σ'	True P.I.	χ-approx.	χ^2-approx.
.75	(.0050)	.00602, (.0060)	.00182, (.0018)
.90	(.0117)	.01336, (.0134)	.00584, (.0058)
1.25	(.0495)	.05252, (.0526)	.03813, (.0383)
1.50	(.1031)	.10600, (.1061)	.09250+, (.0925)
1.80	(.2000)	.20116, (.2012)	.19691, (.1969)
2.45	(.4899)	.48572, (.4858)	.50457, (.5046)
3.25	(.8053)	.80303, (.8030)	.81186, (.8119)
3.65	(.8981)	.89817, (.8982)	.89775+, (.8978)
4.05	(.9519)	.95332, (.9533)	.94809, (.9481)
4.75	(.9898)	.99104, (.9910)	.98616, (.9862)
5.05	(.9952)	.99605- (.9960)	.99249, (.9925)

n=10

W= R/σ'	True P.I.	χ-approx.	χ^2-approx.
1.35	(.0054)	.00750-, (.0076)	.00339, (.0034)
1.50	(.0117)	.01485, (.0149)	.00847, (.0085)
1.85	(.0479)	.05300, (.0530)	.04255-, (.0425)
2.10	(.1015)	.10604, (.1061)	.09708, (.0971)
2.40	(.2025)	.20338, (.2034)	.20222, (.2022)
3.00	(.4878)	.47994, (.4800)	.49544, (.4954)
3.75	(.8602)	.80268, (.8026)	.80896, (.8089)
4.15	(.9038)	.90476, (.9047)	.90305+, (.9030)
4.45	(.9474)	.94997, (.9500)	.94528, (.9453)
5.15	(.9898)	.99193, (.9919)	.98799, (.9880)
5.40	(.9948)	.99623, (.9962)	.99338, (.9934)

n=15

W= R/σ	True P.I.	χ-approx.	χ^2-approx.
1.80	(.0049)	.00773, (.0077)	.00411, (.0041)
1.95	(.0108)	.01506, (.0150)	.00963, (.0096)
2.30	(.0468)	.05355-, (.0535)	.04512, (0451)
2.55	(.1026)	.10825-, (.1083)	.10150+, (.1015)
2.85	(.2103)	.21051, (.2105)	.21087, (.2109)
3.40	(.4885)	.47751, (.4775)	.49089, (.4909)
4.10	(.8036)	.79906, (.7990)	.80426, (.8043)
4.45	(.8964)	.89753, (.8975)	.89614, (.8962)
4.80	(.9505)	.95424, (.9543)	.94977, (.9498)
5.45	(.9900)	.99279, (.9928)	.98950+, (.9895)
5.70	(.9950)	.99686, (.9968)	.99462, (.9946)

Note: Values in the parenthesis are Pearson's [10]. Values in front of them are ours.

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Appendix A. Mean d_2 and Standard Deviation d_3 of the Standard Range W from a Normal Population

n	EW = d_2	Std W = d_3	n	EW = d_2	Std W = d_3
-	-	-	26	3.9643157	.70498834
2	1.1283792	.85250247	27	3.9965386	.70169659
3	1.6925688	.88836800	28	4.0274138	.69855282
4	2.0587507	.87980820	29	4.0570443	.69554570
5	2.3259289	.86408194	30	4.0855217	.69266510
6	2.5344127	.84803967	31	4.1129282	.68990192
7	2.7043568	.83320534	32	4.1393377	.68724797
8	2.8472006	.81983149	33	4.1648167	.68469583
9	2.9700263	.80783427	34	4.1894255+	.68223881
10	3.0775055-	.79705067	35	4.2132189	.67987079
11	3.1728727	.78731462	36	4.2362466	.67758623
12	3.2584553	.77847834	37	4.2585541	.67538004
13	3.3359804	.77041620	38	4.2801829	.67324760
14	3.4067631	.76302310	39	4.3011713	.67118462
15	3.4718269	.75621143	40	4.3215544	.66918720
16	3.5319828	.74990809	41	4.3413644	.66725172
17	3.5878840	.74405178	42	4.3606312	.66537485-
18	3.6400638	.73859085+	43	4.3793825+	.66355350+
19	3.6889630	.73348150-	44	4.3976439	.66178482
20	3.7349501	.72868635-	45	4.4154391	.66006617
21	3.7783358	.72417334	46	4.4327903	.65839507
22	3.8193846	.71991481	47	4.4497181	.65676923
23	3.8583234	.71588674	48	4.4662418	.65518651
24	3.8953481	.71206818	49	4.4823792	.65364492
25	3.9306292	.70844077	50	4.4981473	.65214259

APPENDIX B PATNAIK'S CHI APPROXIMATION

N	DF(M=1)	C(M=1)	DF(M=2)	C(M=2)	DF(M=3)	C(M=3)
2	1.0000000D 00	1.4142136D 00	1.9195218D 00	1.2793044D 00	2.8172896D 00	1.2310537D 00
3	1.9846345D 00	1.9115404D 00	3.8337216D 00	1.8053775D 00	5.6627792D 00	1.7685743D 00
4	2.9291550D 00	2.2388651D 00	5.6935391D 00	2.1506942D 00	8.4414647D 00	2.1204894D 00
5	3.8265132D 00	2.4812462D 00	7.4710477D 00	2.4048418D 00	1.1101853D 01	2.3788284D 00
6	4.6771607D 00	2.6725305D 00	9.1612114D 00	2.6043873D 00	1.3633502D 01	2.5812732D 00
7	5.4841533D 00	2.8298016D 00	1.0767465D 01	2.7677900D 00	1.6040465D 01	2.7468083D 00
8	6.2512253D 00	2.9628829D 00	1.2295944D 01	2.9056175D 00	1.8331451D 01	2.8862766D 00
9	6.9820658D 00	3.0779299D 00	1.3753301D 01	3.0244593D 00	2.0516182D 01	3.0064245D 00
10	7.6800658D 00	3.1790454D 00	1.5145891D 01	3.1286874D 00	2.2604051D 01	3.1117203D 00
11	8.3482531D 00	3.2690955D 00	1.6479508D 01	3.2213434D 00	2.4603661D 01	3.2052679D 00
12	8.9893018D 00	3.3501581D 00	1.7759327D 01	3.3046248D 00	2.6522720D 01	3.2893070D 00
13	9.6055619D 00	3.4237854D 00	1.8989931D 01	3.3801680D 00	2.8368068D 01	3.3655033D 00
14	1.0199099D 01	3.4911659D 00	2.0175372D 01	3.4492227D 00	3.0145753D 01	3.4351278D 00
15	1.0771736D 01	3.5532292D 00	2.1319235D 01	3.5127639D 00	3.1861139D 01	3.4991714D 00
16	1.1325083D 01	3.6107153D 00	2.2424695D 01	3.5715660D 00	3.3518972D 01	3.5584205D 00
17	1.1860566D 01	3.6642222D 00	2.3494576D 01	3.6262540D 00	3.5123481D 01	3.6135093D 00
18	1.2379459D 01	3.7142403D 00	2.4531397D 01	3.6773391D 00	3.6678434D 01	3.6649561D 00
19	1.2882898D 01	3.7611758D 00	2.5537411D 01	3.7252444D 00	3.8187207D 01	3.7131900D 00
20	1.3371905D 01	3.8053694D 00	2.6514648D 01	3.7703241D 00	3.9652839D 01	3.7585698D 00
21	1.3847399D 01	3.8471089D 00	2.7464931D 01	3.8128773D 00	4.1078062D 01	3.8013983D 00
22	1.4310213D 01	3.8866407D 00	2.8389915D 01	3.8531594D 00	4.2465355D 01	3.8419341D 00
23	1.4761096D 01	3.9241752D 00	2.9291094D 01	3.8913884D 00	4.3816955D 01	3.8803980D 00
24	1.5200737D 01	3.9598957D 00	3.0169837D 01	3.9277542D 00	4.5134915D 01	3.9169818D 00
25	1.5629765D 01	3.9939618D 00	3.1027394D 01	3.9624218D 00	4.6421109D 01	3.9518526D 00
26	1.6048749D 01	4.0265128D 00	3.1864901D 01	3.9955351D 00	4.7677239D 01	3.9851557D 00

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N	DF(M=1)	C(M=1)	DF(M=2)	C(M=2)	DF(M=3)	C(M=3)
26	1.60487490 01	4.02651280 00	3.18649010 01	3.99553510 00	4.76772390 01	3.98515570 00
27	1.64582120 01	4.05767120 00	3.26833990 01	4.02722070 00	4.89048640 01	4.01701920 00
28	1.68586440 01	4.08754630 00	3.34838650 01	4.05759120 00	5.01054490 01	4.04755690 00
29	1.72504900 01	4.11623490 00	3.42671820 01	4.08674660 00	5.12803200 01	4.07686980 00
30	1.76341620 01	4.14382340 00	3.50341780 01	4.11477580 00	5.24307130 01	4.10504760 00
31	1.80100280 01	4.17038810 00	3.57855830 01	4.14175740 00	5.35577280 01	4.13216980 00
32	1.83784690 01	4.19600110 00	3.65221570 01	4.16776560 00	5.46625010 01	4.15831120 00
33	1.87397910 01	4.22072270 00	3.72445080 01	4.19286250 00	5.57459460 01	4.18353450 00
34	1.90943210 01	4.24461250 00	3.79532940 01	4.21710930 00	5.68090470 01	4.20790160 00
35	1.94423260 01	4.26772020 00	3.86490440 01	4.24055700 00	5.78525990 01	4.23146390 00
36	1.97840870 01	4.29009370 00	3.93323190 01	4.26325490 00	5.88774410 01	4.25427100 00
37	2.01198540 01	4.31177640 00	4.00036180 01	4.28524760 00	5.98843240 01	4.27636810 00
38	2.04498620 01	4.33280830 00	4.06634130 01	4.30657590 00	6.08739520 01	4.29779620 00
39	2.07743210 01	4.35322350 00	4.13121180 01	4.32727520 00	6.18469500 01	4.31859110 00
40	2.10934650 01	4.37305870 00	4.19502030 01	4.34738270 00	6.28040200 01	4.33879030 00
41	2.14074640 01	4.39234170 00	4.25780090 01	4.36692720 00	6.37456750 01	4.35842270 00
42	2.17165210 01	4.41110260 00	4.31959370 01	4.38593940 00	6.46725150 01	4.37751950 00
43	2.20207970 01	4.42936690 00	4.38043130 01	4.40444540 00	6.55850290 01	4.39610690 00
44	2.23204640 01	4.44715910 00	4.44034770 01	4.42247040 00	6.64837270 01	4.41421010 00
45	2.26156830 01	4.46450290 00	4.49937510 01	4.44003850 00	6.73690920 01	4.43185370 00
46	2.29065910 01	4.48141820 00	4.55754120 01	4.45717030 00	6.82415390 01	4.44905830 00
47	2.31933370 01	4.49792530 00	4.61487520 01	4.47388630 00	6.91015080 01	4.46584460 00
48	2.34760510 01	4.51404220 00	4.67140370 01	4.49020510 00	6.99493940 01	4.48223130 00
49	2.37548670 01	4.52978730 00	4.72715290 01	4.50614550 00	7.07855920 01	4.49823730 00
50	2.40298950 01	4.54517510 00	4.78214500 01	4.52172220 00	7.16104360 01	4.51387750 00

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N	DF(M=4)	C(M=4)	DF(M=5)	C(M=5)	DF(M=6)	C(M=6)
2	3.7061654D 00	1.2062047D 00	4.5906034D 00	1.1910465D 00	5.4725311D 00	1.1808329D 00
3	7.4853527D 00	1.7498824D 00	9.3050652D 00	1.7385709D 00	1.1123274D 01	1.7309888D 00
4	1.1184553D 01	2.1052245D 00	1.3925593D 01	2.0960122D 00	1.6665579D 01	2.0898480D 00
5	1.4728813D 01	2.3657144D 00	1.8354174D 01	2.3578110D 00	2.1978719D 01	2.3525273D 00
6	1.8102587D 01	2.5696382D 00	2.2570352D 01	2.5626318D 00	2.7037447D 01	2.5579503D 00
7	2.1310703D 01	2.7362572D 00	2.6579809D 01	2.7299069D 00	3.1848343D 01	2.7256652D 00
8	2.4364518D 01	2.8765573D 00	3.0396589D 01	2.8707100D 00	3.6428158D 01	2.8668052D 00
9	2.7276869D 01	2.9973664D 00	3.4036663D 01	2.9919184D 00	4.0796008D 01	2.9882808D 00
10	3.0060210D 01	3.1032020D 00	3.7515557D 01	3.0980797D 00	4.4970495D 01	3.0946602D 00
11	3.2725969D 01	3.1971999D 00	4.0847529D 01	3.1923493D 00	4.8968712D 01	3.1891115D 00
12	3.5284397D 01	3.2816213D 00	4.4045379D 01	3.2770012D 00	5.2806011D 01	3.2739175D 00
13	3.7744595D 01	3.3581469D 00	4.7120472D 01	3.3537253D 00	5.6496021D 01	3.3507744D 00
14	4.0114618D 01	3.4280586D 00	5.0082869D 01	3.4238101D 00	6.0050812D 01	3.4209748D 00
15	4.2401605D 01	3.4923554D 00	5.2941490D 01	3.4882593D 00	6.3481084D 01	3.4855260D 00
16	4.4611881D 01	3.5518295D 00	5.5704238D 01	3.5478691D 00	6.6796317D 01	3.5452263D 00
17	4.6751078D 01	3.6071200D 00	5.8378148D 01	3.6032810D 00	7.0004953D 01	3.6007194D 00
18	4.8824218D 01	3.6587489D 00	6.0969497D 01	3.6550195D 00	7.3114522D 01	3.6525312D 00
19	5.0835799D 01	3.7071481D 00	6.3483904D 01	3.7035182D 00	7.6131766D 01	3.7010963D 00
20	5.2789869D 01	3.7526788D 00	6.5926431D 01	3.7491398D 00	7.9062758D 01	3.7467786D 00
21	5.4690071D 01	3.7956457D 00	6.8301628D 01	3.7921900D 00	8.1912959D 01	3.7898845D 00
22	5.6539709D 01	3.8363091D 00	7.0613625D 01	3.8329302D 00	8.4687322D 01	3.8306759D 00
23	5.8341763D 01	3.8748910D 00	7.2866148D 01	3.8715831D 00	8.7390319D 01	3.8693763D 00
24	6.0098971D 01	3.9115845D 00	7.5062615D 01	3.9083425D 00	9.0026053D 01	3.9061797D 00
25	6.1813830D 01	3.9465573D 00	7.7206150D 01	3.9433768D 00	9.2598269D 01	3.9412550D 00
26	6.3488608D 01	3.9799559D 00	7.9299587D 01	3.9768327D 00	9.5110370D 01	3.9747492D 00

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N	DF(M=4)	C(M=4)	DF(M=5)	C(M=5)	DF(M=6)	C(M=6)
26	6.34886080 01	3.97995590 00	7.92995870 01	3.97683270 00	9.51103700 01	3.97474920 00
27	6.51253840 01	4.01190870 00	8.13455240 01	4.00883930 00	9.75654720 01	4.00679180 00
28	6.67261100 01	4.04253040 00	8.33464000 01	4.03951160 00	9.99665030 01	4.03749770 00
29	6.82925550 01	4.07192240 00	8.53044270 01	4.06895110 00	1.02316120 02	4.06696900 00
30	6.98263660 01	4.10017480 00	8.72216640 01	4.09724840 00	1.04616780 02	4.09529630 00
31	7.13290090 01	4.12736760 00	8.90999430 01	4.12448360 00	1.06870700 02	4.12255980 00
32	7.28019990 01	4.15357590 00	9.09411560 01	4.15073210 00	1.09080140 02	4.14883520 00
33	7.42465530 01	4.17886270 00	9.27468270 01	4.17605710 00	1.11246930 02	4.17418570 00
34	7.56639850 01	4.20329020 00	9.45185950 01	4.20052090 00	1.13373040 02	4.19867370 00
35	7.70553520 01	4.22691000 00	9.62577840 01	4.22417530 00	1.15460050 02	4.22235110 00
36	7.84217760 01	4.24977200 00	9.79657940 01	4.24707020 00	1.17509650 02	4.24526810 00
37	7.97642550 01	4.27192140 00	9.96438750 01	4.26925120 00	1.19523340 02	4.26747010 00
38	8.10837300 01	4.29339970 00	1.01293200 02	4.29075960 00	1.21502520 02	4.28899860 00
39	8.23810330 01	4.31424250 00	1.02914810 02	4.31163130 00	1.23448440 02	4.30988950 00
40	8.36570990 01	4.33448770 00	1.04509880 02	4.33190410 00	1.25362510 02	4.33018080 00
41	8.49126130 01	4.35416430 00	1.06079260 02	4.35160720 00	1.27245760 02	4.34990170 00
42	8.61483750 01	4.37330350 00	1.07623950 02	4.37077190 00	1.29099370 02	4.36908340 00
43	8.73650370 01	4.39193170 00	1.09144760 02	4.38942460 00	1.30924340 02	4.38775250 00
44	8.85632790 01	4.41007420 00	1.10642550 02	4.40759080 00	1.32721680 02	4.40593440 00
45	8.97437430 01	4.42775570 00	1.12118120 02	4.42529500 00	1.34492350 02	4.42365380 00
46	9.09069860 01	4.44499680 00	1.13572160 02	4.44255810 00	1.36237200 02	4.44093150 00
47	9.20535910 01	4.46181830 00	1.15005400 02	4.45940080 00	1.37957080 02	4.45778840 00
48	9.31840860 01	4.47823900 00	1.16418510 02	4.47584200 00	1.39652800 02	4.47424320 00
49	9.42989990 01	4.49427790 00	1.17812140 02	4.49190070 00	1.41325150 02	4.49031510 00
50	9.53987740 01	4.50995000 00	1.19186850 02	4.50759190 00	1.42974800 02	4.50601910 00

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=1)	2C'(M=1)	DF(M=2)	2C'(M=2)	DF(M=3)	2C'(M=3)
2	3.5038770D 00	6.4407467D-01	7.0077539D 00	3.2203734D-01	1.0511631D 01	2.1469156D-01
3	7.2600037D 00	4.6627216D-01	1.4520007D 01	2.3313608D-01	2.1780011D 01	1.5542405D-01
4	1.0951195D 01	3.7598650D-01	2.1902390D 01	1.8799325D-01	3.2853584D 01	1.2532883D-01
5	1.4491489D 01	3.2100620D-01	2.8982978D 01	1.6050310D-01	4.3474467D 01	1.0700207D-01
6	1.7862915D 01	2.8376250D-01	3.5725830D 01	1.4188125D-01	5.3588745D 01	9.4587500D-02
7	2.1069483D 01	2.5670841D-01	4.2138967D 01	1.2835421D-01	6.3208450D 01	8.5569471D-02
8	2.4122201D 01	2.3606474D-01	4.8244403D 01	1.1803237D-01	7.2366604D 01	7.8688247D-02
9	2.7033734D 01	2.1972742D-01	5.4067468D 01	1.0986371D-01	8.1101203D 01	7.3242472D-02
10	2.9816441D 01	2.0643010D-01	5.9632883D 01	1.0321505D-01	8.9449324D 01	6.8810034D-02
11	3.2481693D 01	1.9536375D-01	6.4963386D 01	9.7681875D-02	9.7445079D 01	6.5121250D-02
12	3.5039707D 01	1.8598645D-01	7.0079414D 01	9.2993224D-02	1.0511912D 02	6.1995483D-02
13	3.7499559D 01	1.7792105D-01	7.4999119D 01	8.8960523D-02	1.1249868D 02	5.9307015D-02
14	3.9869269D 01	1.7089661D-01	7.9738578D 01	8.5448303D-02	1.1960787D 02	5.6965535D-02
15	4.2156025D 01	1.6471320D-01	8.4312049D 01	8.2356601D-02	1.2646807D 02	5.4904401D-02
16	4.4366082D 01	1.5921996D-01	8.8732164D 01	7.9609978D-02	1.3309825D 02	5.3073318D-02
17	4.6505087D 01	1.5430071D-01	9.3010174D 01	7.7150355D-02	1.3951526D 02	5.1433570D-02
18	4.8578057D 01	1.4986453D-01	9.7156114D 01	7.4932264D-02	1.4573417D 02	4.9954843D-02
19	5.0589486D 01	1.4583912D-01	1.0117897D 02	7.2919559D-02	1.5176846D 02	4.8613039D-02
20	5.2543420D 01	1.4216624D-01	1.0508684D 02	7.1083118D-02	1.5763026D 02	4.7388745D-02
21	5.4443498D 01	1.3879841D-01	1.0888700D 02	6.9399206D-02	1.6333050D 02	4.6266137D-02
22	5.6293024D 01	1.3569655D-01	1.1258605D 02	6.7848275D-02	1.6887907D 02	4.5232183D-02
23	5.8094976D 01	1.3282812D-01	1.1618995D 02	6.6414058D-02	1.7428493D 02	4.4276038D-02
24	5.9852090D 01	1.3016580D-01	1.1970418D 02	6.5082900D-02	1.7955627D 02	4.3388600D-02
25	6.1566862D 01	1.2768651D-01	1.2313372D 02	6.3843253D-02	1.8470059D 02	4.2562169D-02
26	6.3241561D 01	1.2537058D-01	1.2648312D 02	6.2685288D-02	1.8972468D 02	4.1790192D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=1)	2C'(M=1)	DF(M=2)	2C'(M=2)	DF(M=3)	2C'(M=3)
26	6.3241561D 01	1.2537058D-01	1.2648312D 02	6.2685288D-02	1.8972468D 02	4.1790192D-02
27	6.4878263D 01	1.2320115D-01	1.2975653D 02	6.1600573D-02	1.9463479D 02	4.1067049D-02
28	6.6478919D 01	1.2116362D-01	1.3295784D 02	6.0581812D-02	1.9943676D 02	4.0387874D-02
29	6.8045300D 01	1.1924539D-01	1.3609060D 02	5.9622693D-02	2.0413590D 02	3.9748462D-02
30	6.9579051D 01	1.1743540D-01	1.3915810D 02	5.8717698D-02	2.0873715D 02	3.9145132D-02
31	7.1081637D 01	1.1572405D-01	1.4216327D 02	5.7862024D-02	2.1324491D 02	3.8574683D-02
32	7.2554574D 01	1.1410273D-01	1.4510915D 02	5.7051366D-02	2.1766372D 02	3.8034244D-02
33	7.3999078D 01	1.1256399D-01	1.4799816D 02	5.6281997D-02	2.2199724D 02	3.7521331D-02
34	7.5416462D 01	1.1110109D-01	1.5083292D 02	5.5550543D-02	2.2624939D 02	3.7033696D-02
35	7.6807785D 01	1.0970812D-01	1.5361557D 02	5.4854058D-02	2.3042336D 02	3.6569372D-02
36	7.8174167D 01	1.0837969D-01	1.5634833D 02	5.4189847D-02	2.3452250D 02	3.6126565D-02
37	7.9516605D 01	1.0711105D-01	1.5903321D 02	5.3555524D-02	2.3854982D 02	3.5703683D-02
38	8.0836042D 01	1.0589788D-01	1.6167208D 02	5.2948941D-02	2.4250813D 02	3.5299294D-02
39	8.2133308D 01	1.0473632D-01	1.6426662D 02	5.2368161D-02	2.4639992D 02	3.4912108D-02
40	8.3409340D 01	1.0362279D-01	1.6681868D 02	5.1811394D-02	2.5022802D 02	3.4540929D-02
41	8.4664821D 01	1.0255414D-01	1.6932964D 02	5.1277070D-02	2.5399446D 02	3.4184713D-02
42	8.5900551D 01	1.0152743D-01	1.7180110D 02	5.0763714D-02	2.5770165D 02	3.3842476D-02
43	8.7117183D 01	1.0054003D-01	1.7423437D 02	5.0270016D-02	2.6135155D 02	3.3513344D-02
44	8.8315396D 01	9.9589501D-02	1.7663079D 02	4.9794750D-02	2.6494619D 02	3.3196500D-02
45	8.9495832D 01	9.8673615D-02	1.7899166D 02	4.9336807D-02	2.6848750D 02	3.2891205D-02
46	9.0659048D 01	9.7790345D-02	1.8131810D 02	4.8895173D-02	2.7197714D 02	3.2596782D-02
47	9.1805627D 01	9.6937795D-02	1.8361125D 02	4.8468897D-02	2.7541688D 02	3.2312598D-02
48	9.2936098D 01	9.6114233D-02	1.8587220D 02	4.8057116D-02	2.7880829D 02	3.2038078D-02
49	9.4050987D 01	9.5318063D-02	1.8810197D 02	4.7659032D-02	2.8215296D 02	3.1772688D-02
50	9.5150739D 01	9.4547810D-02	1.9030148D 02	4.7273905D-02	2.8545222D 02	3.1515937D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=4)	2C'(M=4)	DF(M=5)	2C'(M=5)	DF(M=6)	2C'(M=6)
2	1.4015508D 01	1.6101867D-01	1.7519385D 01	1.2881493D-01	2.1023262D 01	1.0734578D-01
3	2.9040015D 01	1.1656804D-01	3.6300019D 01	9.3254431D-02	4.3560022D 01	7.7712026D-02
4	4.3804779D 01	9.3996625D-02	5.4755974D 01	7.5197300D-02	6.5707169D 01	6.2664417D-02
5	5.7965956D 01	8.0251550D-02	7.2457445D 01	6.4201240D-02	8.6948934D 01	5.3501033D-02
6	7.1451660D 01	7.0940625D-02	8.9314575D 01	5.6752500D-02	1.0717749D 02	4.7293750D-02
7	8.4277933D 01	6.4177103D-02	1.0534742D 02	5.1341682D-02	1.2641690D 02	4.2784735D-02
8	9.6488805D 01	5.9016185D-02	1.2061101D 02	4.7212948D-02	1.4473321D 02	3.9344123D-02
9	1.0813494D 02	5.4931854D-02	1.3516867D 02	4.3945483D-02	1.6220241D 02	3.6621236D-02
10	1.1926577D 02	5.1607525D-02	1.4908221D 02	4.1286020D-02	1.7889865D 02	3.4405017D-02
11	1.2992677D 02	4.8840938D-02	1.6240847D 02	3.9072750D-02	1.9489016D 02	3.2560625D-02
12	1.4015883D 02	4.6496612D-02	1.7519853D 02	3.7197290D-02	2.1023824D 02	3.0997741D-02
13	1.4999824D 02	4.4480261D-02	1.8749780D 02	3.5584209D-02	2.2499736D 02	2.9653508D-02
14	1.5947716D 02	4.2724151D-02	1.9934645D 02	3.4179321D-02	2.3921573D 02	2.8482768D-02
15	1.6862410D 02	4.1178301D-02	2.1078012D 02	3.2942640D-02	2.5293615D 02	2.7452200D-02
16	1.7746433D 02	3.9804989D-02	2.2183041D 02	3.1843991D-02	2.6619649D 02	2.6536659D-02
17	1.8602035D 02	3.8575178D-02	2.3252544D 02	3.0860142D-02	2.7903052D 02	2.5716785D-02
18	1.9431223D 02	3.7466132D-02	2.4289029D 02	2.9972906D-02	2.9146834D 02	2.4977421D-02
19	2.0235794D 02	3.6459780D-02	2.5294743D 02	2.9167824D-02	3.0353691D 02	2.4306520D-02
20	2.1017368D 02	3.5541559D-02	2.6271710D 02	2.8433247D-02	3.1526052D 02	2.3694373D-02
21	2.1777399D 02	3.4699603D-02	2.7221749D 02	2.7759682D-02	3.2666099D 02	2.3133069D-02
22	2.2517210D C2	3.3924138D-02	2.8146512D 02	2.7139310D-02	3.3775814D 02	2.2616092D-02
23	2.3237991D 02	3.3207029D-02	2.9047488D 02	2.6565623D-02	3.4856986D 02	2.2138019D-02
24	2.3940836D 02	3.2541450D-02	2.9926045D 02	2.6033160D-02	3.5911254D 02	2.1694300D-02
25	2.4626745D 02	3.1921627D-02	3.0783431D 02	2.5537301D-02	3.6940117D 02	2.1281084D-02
26	2.5296624D 02	3.1342644D-02	3.1620780D 02	2.5074115D-02	3.7944936D 02	2.0895096D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=4)	2C'(M=4)	DF(M=5)	2C'(M=5)	DF(M=6)	2C'(M=6)
26	2.5296624D 02	3.1342644D-02	3.1620780D 02	2.5074115D-02	3.7944936D 02	2.0895096D-02
27	2.5951305D 02	3.0800287D-02	3.2439131D 02	2.4640229D-02	3.8926958D 02	2.0533524D-02
28	2.6591568D 02	3.0290906D-02	3.3239460D 02	2.4232725D-02	3.9887352D 02	2.0193937D-02
29	2.7218120D 02	2.9811347D-02	3.4022650D 02	2.3849077D-02	4.0827180D 02	1.9874231D-02
30	2.7831620D 02	2.9358849D-02	3.4789525D 02	2.3487079D-02	4.1747431D 02	1.9572566D-02
31	2.8432655D 02	2.8931012D-02	3.5540819D 02	2.3144810D-02	4.2648982D 02	1.9287341D-02
32	2.9021830D 02	2.8525683D-02	3.6277287D 02	2.2820546D-02	4.3532744D 02	1.9017122D-02
33	2.9599631D C2	2.8140998D-02	3.6999539D 02	2.2512799D-02	4.4399447D 02	1.8760666D-02
34	3.0166585D 02	2.7775272D-02	3.7708231D 02	2.2220217D-02	4.5249877D 02	1.8516848D-02
35	3.0723114D 02	2.7427029D-02	3.8403893D 02	2.1941623D-02	4.6084671D 02	1.8284686D-02
36	3.1269667D 02	2.7094923D-02	3.9087083D 02	2.1675939D-02	4.6904500D 02	1.8063282D-02
37	3.1806642D 02	2.6777762D-02	3.9758303D 02	2.1422210D-02	4.7709963D 02	1.7851841D-02
38	3.2334417D 02	2.6474471D-02	4.0418021D 02	2.1179576D-02	4.8501625D 02	1.7649647D-02
39	3.2853323D 02	2.6184081D-02	4.1066654D 02	2.0947265D-02	4.9279985D 02	1.7456054D-02
40	3.3363736D 02	2.5905697D-02	4.1704670D 02	2.0724558D-02	5.0045604D 02	1.7270465D-02
41	3.3865928D 02	2.5638535D-02	4.2332410D 02	2.0510828D-02	5.0798892D 02	1.7092357D-02
42	3.4360221D 02	2.5381857D-02	4.2950276D 02	2.0305485D-02	5.1540331D 02	1.6921238D-02
43	3.4846873D 02	2.5135008D-02	4.3558591D 02	2.0108006D-02	5.2270310D 02	1.6756672D-02
44	3.5326158D 02	2.4897375D-02	4.4157698D 02	1.9917900D-02	5.2989238D 02	1.6598250D-02
45	3.5798333D 02	2.4668404D-02	4.4747916D 02	1.9734723D-02	5.3697499D 02	1.6445602D-02
46	3.6263619D 02	2.4447586D-02	4.5329524D 02	1.9558069D-02	5.4395429D 02	1.6298391D-02
47	3.6722251D 02	2.4234449D-02	4.5902814D 02	1.9387559D-02	5.5083376D 02	1.6156299D-02
48	3.7174439D 02	2.4028558D-02	4.6468049D 02	1.9222847D-02	5.5761659D 02	1.6019039D-02
49	3.7620395D 02	2.3829516D-02	4.7025494D 02	1.9063613D-02	5.6430592D 02	1.5886344D-02
50	3.8060296D 02	2.3636953D-02	4.7575370D 02	1.8909562D-02	5.7090443D 02	1.5757968D-02