AD-A071 769 TEXAS UNIV AT AUSTIN DEFENSE RESEARCH LAB F/G 12/1

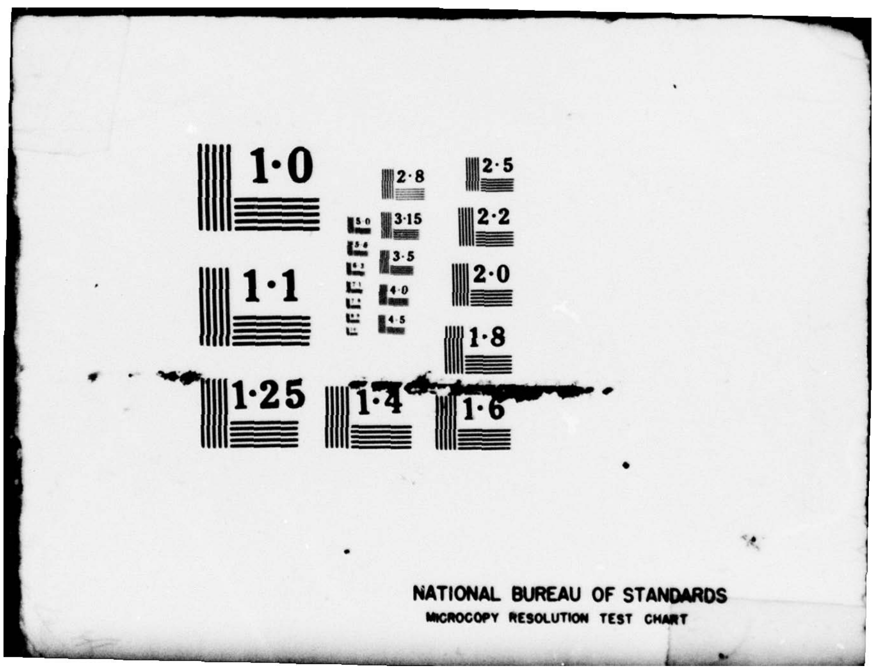
GENERATION OF A TIME SERIES WITH SPECIFIED POWER SPECTRAL DENSI--ETC(U)

MAY 67 J F ENGLAND NOBSR-95181

UNCLASSIFIED DRL-A-283 NL

1 OF 1
AD
A071769

END
DATE
FILMED
8-79
DDC


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5
5·0
NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

EFENSE RESEARC
LABORATORY

LEVEL II

(9) Technical rept.

(14) DRL-A-283

(6) GENERATION OF A TIME SERIES WITH SPECIFIED POWER SPECTRAL DENSITY FUNCTION

by

(10) Joseph F. England

(11) 11 May 1967

Copy No. 1

(12) 34 p.

A Technical Report Prepared under
Naval Ship Systems Command Contract (15) NObsr-95181

DEFENSE RESEARCH LABORATORY
THE UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TEXAS 78712

107500

ABSTRACT

This thesis presents the results of a mathematical study concerning the repeated application of a numerical simulation of a linear filter. The summation of several filter outputs was used to generate samples of a composite output function which had a power spectral density function similar to the sum of the power spectral density functions of the individual filter output functions.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This thesis presents the results of a mathematical study concerning a method for generating samples of a time function which has a specified power spectral density function. The method makes multiple use of a linear processor described by Mr. Joel N. Franklin[6] and programmed for digital solution by Mr. Walter A. Matuska, Jr.[8]

A linear filter with time-invariant elements may be described mathematically as the convolution

$$x(t) = \int_0^t g(t - \tau)w(\tau)d\tau \quad , \tag{1}$$

where

x(t) is the output time function,

w(t) is the input function, and

g(t) is the filter weighting function.

The filter weighting function is the time response of the filter due to the input of the Dirac delta function.

The Fourier transformation of a convolution is the product of the individual Fourier transformations. Therefore, Eq. (1) can be expressed as a function of frequency as

$$X(\omega) = G(\omega)W(\omega) \quad , \tag{2}$$

where $X(\omega)$, $G(\omega)$, and $W(\omega)$ are the Fourier transforms of $x(t)$, $g(t)$, and $w(t)$, respectively. The function $G(\omega)$ is referred to as the transfer function of the filter.

Davenport and Root[4] have shown that the power spectral density functions $S_w(\omega)$ and $S_x(\omega)$ of the filter input and output are related by the equation

$$S_x(\omega) = |G(\omega)|^2 \, S_w(\omega) \quad . \tag{3}$$

Suppose an input function is selected such that its power spectral density function is identically equal to one. Equation (3) suggests that, in this case, the power spectral density function of the output is dependent only on the transfer function of the filter, in that

$$S_x(\omega) = |G(\omega)|^2 \quad . \tag{4}$$

If we assume that $S_x(\omega)$ may be represented by a rational function, then it is possible to find a transfer function which satisfies the condition expressed in Eq. (4). Davenport and Root[4] have shown that the conditions

$$0 \le S_x(\omega) < \infty \quad ,$$

$$S_x(\omega) = S_x(-\omega) \quad ,$$

and

$$S_x(\omega) \to 0 \text{ as } \omega \to \pm\infty$$

insure that the rational function $S_x(\omega)$ can be represented as

$$S_x(\omega) = \left|\frac{P(i\omega)}{Q(i\omega)}\right|^2 \quad , \tag{5}$$

where P and Q are polynomials in $(i\omega)$ with real coefficients, where the degree of P is less than the degree of Q, and where the zeroes of Q do not lie in the positive real half plane. The polynomials are defined as

$$Q(i\omega) = a_0 + a_1(i\omega) + a_2(i\omega)^2 + \cdots + a_n(i\omega)^n \quad , \tag{6}$$

and

$$P(i\omega) = b_0 + b_1(i\omega) + b_2(i\omega)^2 + \cdots + b_m(i\omega)^m \quad , \tag{7}$$

where $m < n$.

The result of combining Eqs. (4) and (5) implies that the transfer function of the filter can be expressed as a ratio of polynomials by

$$G(\omega) = \frac{P(i\omega)}{Q(i\omega)} \quad . \tag{8}$$

The substitution of this equation in Eq. (2) yields

$$X(\omega) = \frac{P(i\omega)}{Q(i\omega)} W(\omega) \quad . \tag{9}$$

If $\emptyset(i\omega)$ is defined to be the ratio $X(\omega)/P(i\omega)$, then Eq. (9) can be written as

$$Q(i\omega)\emptyset(i\omega) = W(\omega) \quad , \tag{10}$$

or

$$a_0\emptyset(i\omega) + a_1(i\omega)\emptyset(i\omega) + a_2(i\omega)^2\emptyset(i\omega) + \cdots + a_n(i\omega)^n\emptyset(i\omega) = W(\omega) \quad . \tag{11}$$

Since the inverse transform of $(i\omega)^k\emptyset(i\omega)$ is $\emptyset^{(k)}(t)$ plus a constant, Eq. (11) can be expressed in the time domain as

$$a_0\emptyset(t) + a_1\emptyset'(t) + a_2\emptyset''(t) + \cdots + a_n\emptyset^{(n)}(t) = w(t) \quad . \tag{12}$$

The function $\emptyset(i\omega)$ was defined as the ratio $X(\omega)/P(i\omega)$ which implies that $X(\omega)$ can be expressed as

$$X(\omega) = P(i\omega)\emptyset(i\omega) \quad , \tag{13}$$

or

$$X(\omega) = b_0\phi(i\omega) + b_1(i\omega)\phi(i\omega) + b_2(i\omega)^2\phi(i\omega) + \cdots + b_m(i\omega)^m\phi(i\omega) \quad . \tag{14}$$

Using the inverse transforms, Eq. (14) becomes

$$x(t) = b_0\phi(t) + b_1\phi'(t) + b_2\phi''(t) + \cdots + b_m\phi^{(m)}(t) \tag{15}$$

in the time domain.

If the differential equation (12) is solved for $\phi(t)$, $\phi'(t)$, $\cdots$, $\phi^{(m)}(t)$ in terms of the input function $w(t)$ and the coefficients of the polynomial $Q(i\omega)$, then the substitution of the solution into Eq. (15) yields the output function $x(t)$ in terms of the coefficients of the two known polynomials and the input function.[6]

It should be remembered that $x(t)$ has the desired power spectral density function $S_x(\omega)$ since the transfer function $G(\omega)$ was chosen such that

$$S_x(\omega) = |G(\omega)|^2 \quad . \tag{16}$$

CHAPTER II

DETERMINATION OF A TRANSFER FUNCTION

The generation of a time series with a specified power spectral density function is a straightforward task as has been pointed out in Chapter I. When a time series with a specified power spectral density function is desired, the transfer function of the system is usually not available. However, in its place there may be a sketch of the power spectral density function. Therefore, a method of determining a transfer function to match the sketch is needed.

Probably the simplest method of determining a rational algebraic function approximating a specified gain characteristic is a reverse application of the asymptotic technique for plotting gain versus frequency from a given transfer function. The following discussion presents the basic features of this asymptotic procedure. A more complete description can be found in Chestnut and Mayer.[3]

A. Construction of Gain Characteristics from Transfer Functions

It is desired to plot $|G(\omega)|^2$ versus frequency, with the function $G(\omega)$ given. $G(\omega)$ is in general the ratio of two polynomials in $i\omega$, each polynomial containing both real and conjugate complex zeroes. A typical form for $G(\omega)$ is

$$G(\omega) = \frac{K\left(\frac{i\omega}{\omega_a} + 1\right)\left[\left(\frac{i\omega}{\omega_b}\right)^2 + 2\zeta_1 \frac{i\omega}{\omega_b} + 1\right]}{i\omega\left(\frac{i\omega}{\omega_c} + 1\right)\left[\left(\frac{i\omega}{\omega_d}\right)^2 + 2\zeta_2 \frac{i\omega}{\omega_d} + 1\right]} \quad . \tag{17}$$

The multiplication and division of the various numerator and denominator factors are transformed to addition and subtraction by consideration of $\log_{10} |G(\omega)|^2$, rather than $|G(\omega)|^2$ directly. Ordinarily, it is convenient to plot $10 \log |G(\omega)|^2$ which is the gain in decibels. Therefore, Eq. (17) is transformed into

$$\begin{aligned} 20 \log |G(\omega)| = {} & 20 \log K + 20 \log \left|\frac{i\omega}{\omega_a} + 1\right| \\ & + 20 \log \left|\left(\frac{i\omega}{\omega_b}\right)^2 + 2\zeta_1 \frac{i\omega}{\omega_b} + 1\right| \\ & - 20 \log |i\omega| - 20 \log \left|\frac{i\omega}{\omega_c} + 1\right| \\ & - 20 \log \left|\left(\frac{i\omega}{\omega_d}\right)^2 + 2\zeta_2 \frac{i\omega}{\omega_d} + 1\right| \quad . \end{aligned} \tag{18}$$

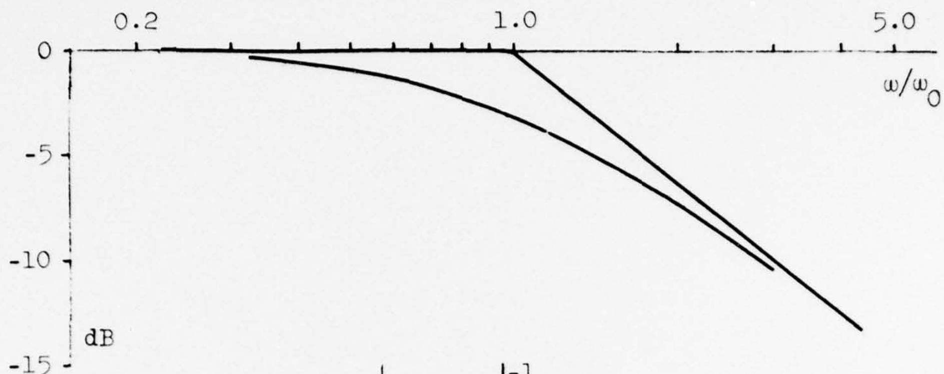
Thus, a plot of the gain vs frequency can be made by simple point-by-point addition of the gain of each individual term in the summation on the right side of Eq. (18).

The constant gain term 20 log K is independent of frequency and can be plotted as a horizontal line on a log-log plot.

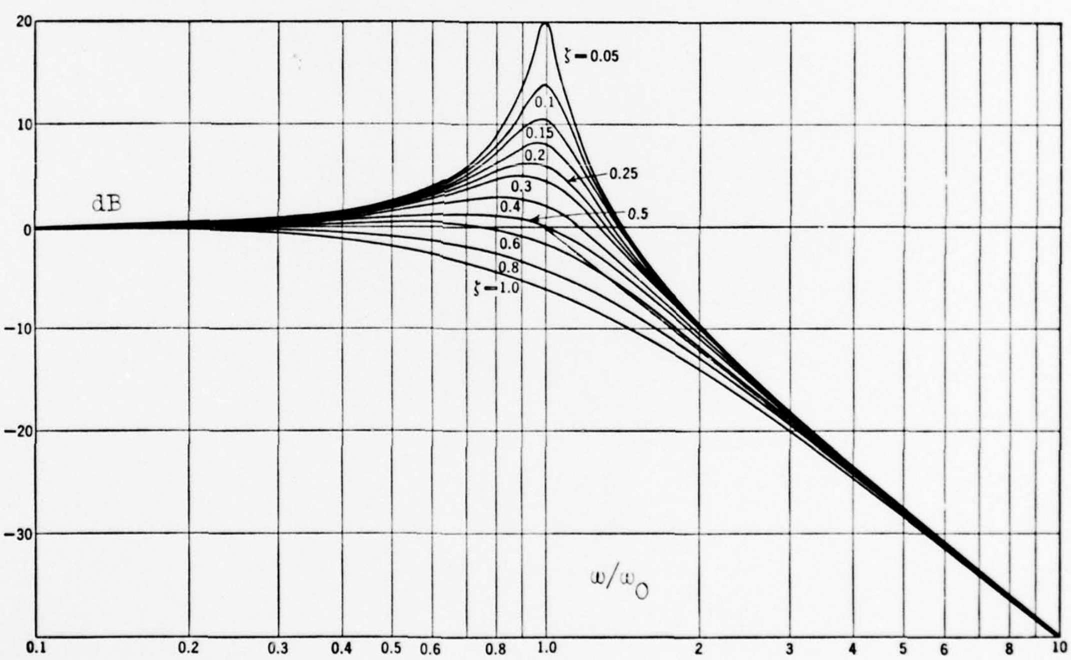
The gain of the linear term $\pm 20 \log \left|\frac{i\omega}{\omega_0} + 1\right|$ can be plotted approximately by using a pair of asymptotes. These asymptotes can be inferred by considering the linear term at frequencies above and below the frequency ω_0. For frequencies $\omega \ll \omega_0$ the term ω/ω_0 may be neglected. Therefore, at these frequencies the gain approaches 0 dB. At frequencies $\omega \gg \omega_0$ the term ω/ω_0 is dominant and the gain becomes asymptotic to a straight line with a slope of ± 20 dB per decade. Positive slopes are used with numerator factors of Eq. (17), while negative slopes are used with denominator factors. The two asymptotes intersect at the frequency ω_0, which is known as the break frequency. Figure 1a is a log-log graph showing these two asymptotes. The actual gain curve is shown below the asymptotes.

A special case of the linear term is the term $\pm 20 \log |i\omega|$. Its gain can be plotted as a single straight line having a slope of ± 20 dB per decade and 0 dB of gain at $\omega = 1$.

The situation is slightly more complicated in the consideration of the quadratic terms $\pm 20 \log \left|\left(\frac{i\omega}{\omega_0}\right)^2 + 2\zeta \frac{i\omega}{\omega_0} + 1\right|$. The gain of these quadratics in $i\omega$ can be plotted approximately by using a pair of asymptotes. One asymptote is a horizontal line constructed at 0 dB, and the other is a straight line with a slope of ± 40 dB per decade. These asymptotes intersect at the frequency ω_0. The closeness of the fit of the actual gain curve to the asymptotes near the break frequency is controlled by the value of ζ. Figure 1b shows a family of actual curves with ζ as the independent parameter. Regardless of



a. $\left| \frac{i\omega}{\omega_0} + 1 \right|^{-1}$



b. $\left| \left(\frac{i\omega}{\omega_0}\right)^2 + 2\zeta \frac{i\omega}{\omega_0} + 1 \right|^{-1}$

ASYMPTOTIC APPROXIMATIONS

FIGURE 1

the value of ζ, the actual curve approaches the asymptotes at frequencies above and below the break frequency. The error between the actual curve and the asymptotes is geometrically symmetric about the break frequency; i.e., the error for $\omega = \alpha\omega_0$ is identical to the error at an ω of ω_0/α.[9]

B. Example Problem

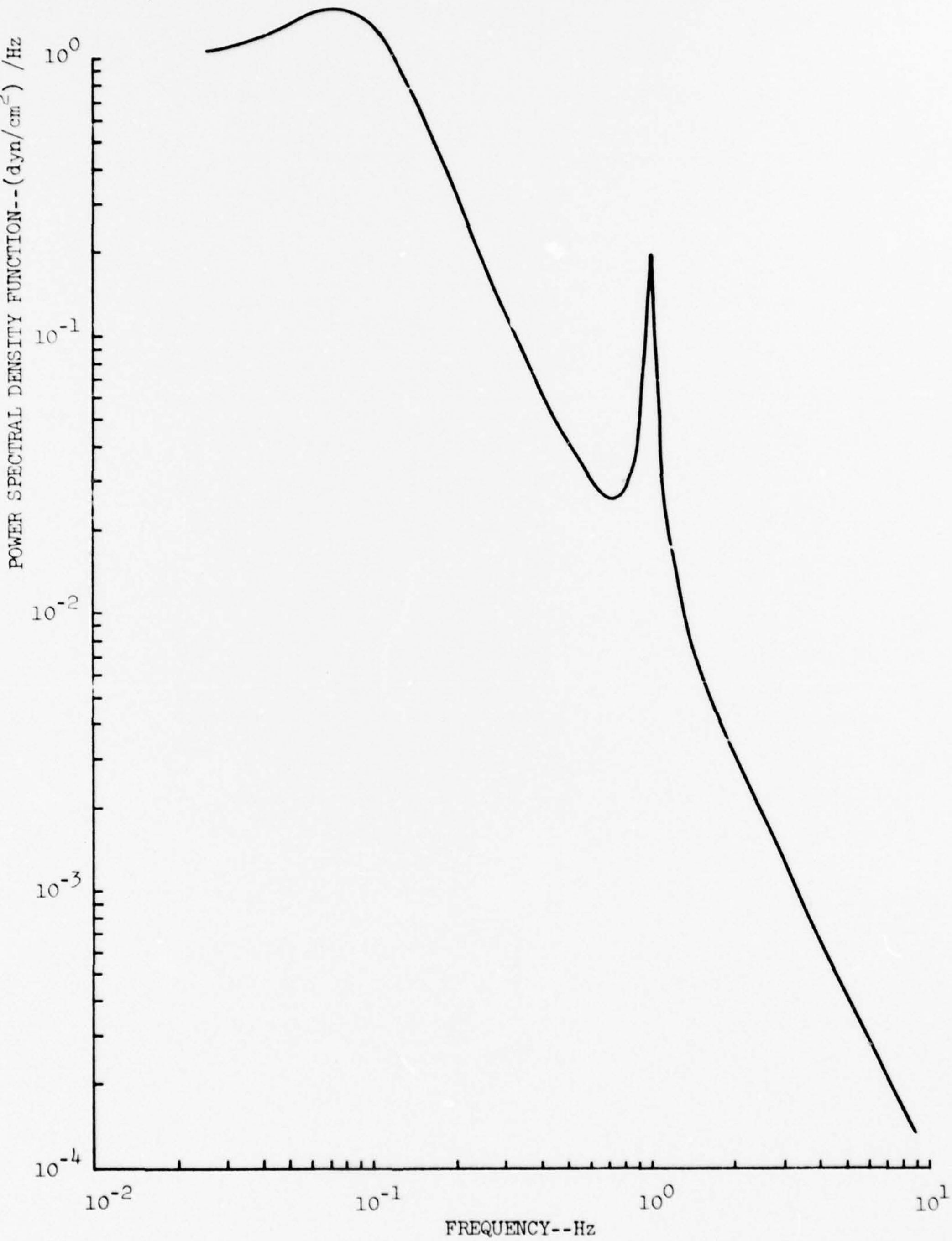
A rational algebraic function is to be found which, when used as a filter transfer function, will cause the output of the filter to have a power spectral density function similar to the curve shown in Fig. 2. The general slope of the curve is to be -20 dB per decade. There are to be two peaks: a slight peak at 0.1 Hz and a sharp peak at 1 Hz.

The rise at 0.1 Hz can be generated by a quadratic factor, $\left(\frac{i\omega}{\omega_1}\right)^2 + 2\zeta_1 \frac{i\omega}{\omega_1} + 1$, in the denominator of the transfer function, where

$\omega_1 = 0.2\pi$ radians, and

$\zeta_1 = 0.6$.

This factor causes a -40 dB per decade slope at frequencies greater than 0.2π. A linear factor $\frac{i\omega}{\omega_1} + 1$ is required in the numerator to decrease the slope to -20 dB per decade.



DESIRED SPECTRUM

FIGURE 2

The peak at 1 Hz can be created by using another quadratic factor, $\left(\frac{i\omega}{\omega_2}\right)^2 + 2\zeta_2 \frac{i\omega}{\omega_2} + 1$, in the denominator,

where

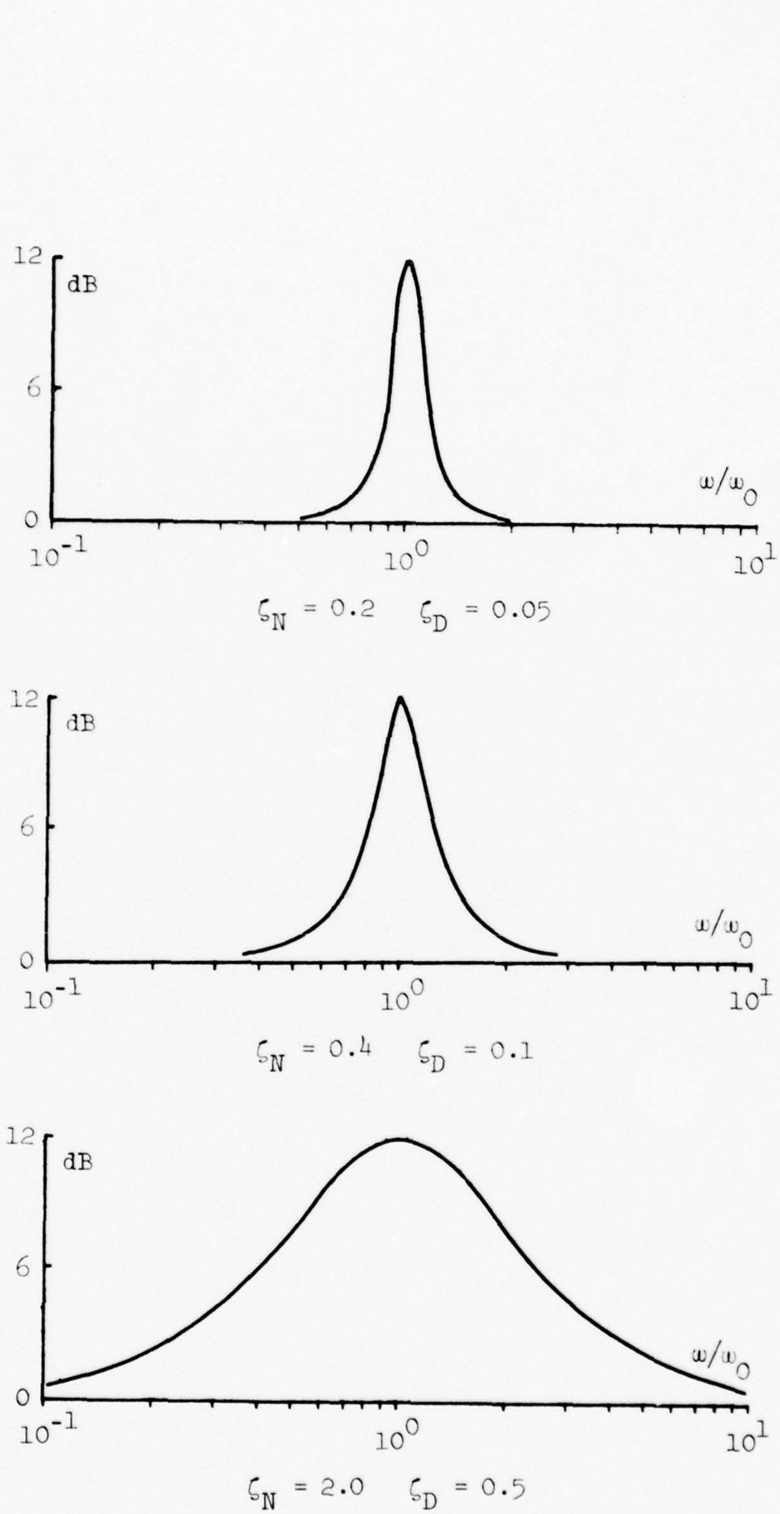
$\omega_2 = 2\pi$ radians, and

$\zeta_2 = 0.05$.

This factor will cause an additional -40 dB per decade drop in gain at frequencies greater than 2π. Therefore, a quadratic factor $\left(\frac{i\omega}{\omega_2}\right)^2 + 2\zeta_3 \frac{i\omega}{\omega_2} + 1$ is required in the numerator to restore the general slope to -20 dB per decade.

The use of a ratio of quadratic factors with the same break frequency causes no gain except in the vicinity of the break frequency. The gain in dB at the break frequency appears to be approximately 20 log (ζ_N/ζ_D), where ζ_N and ζ_D come from the numerator and denominator, respectively, and are of reasonable magnitude. When the gain for a peak or null is given, the ratio ζ_N/ζ_D is fixed. The initial choice of one of the ζ not only implies a value for the other but also sets the "width" of the peak or null. An examination of Fig. 3 will indicate that for a fixed ratio, ζ_N/ζ_D of 4, several "widths" are created which depend upon the choice of one of the ζ.

The preceding description of factors indicates that the required rational algebraic function or transfer function should be

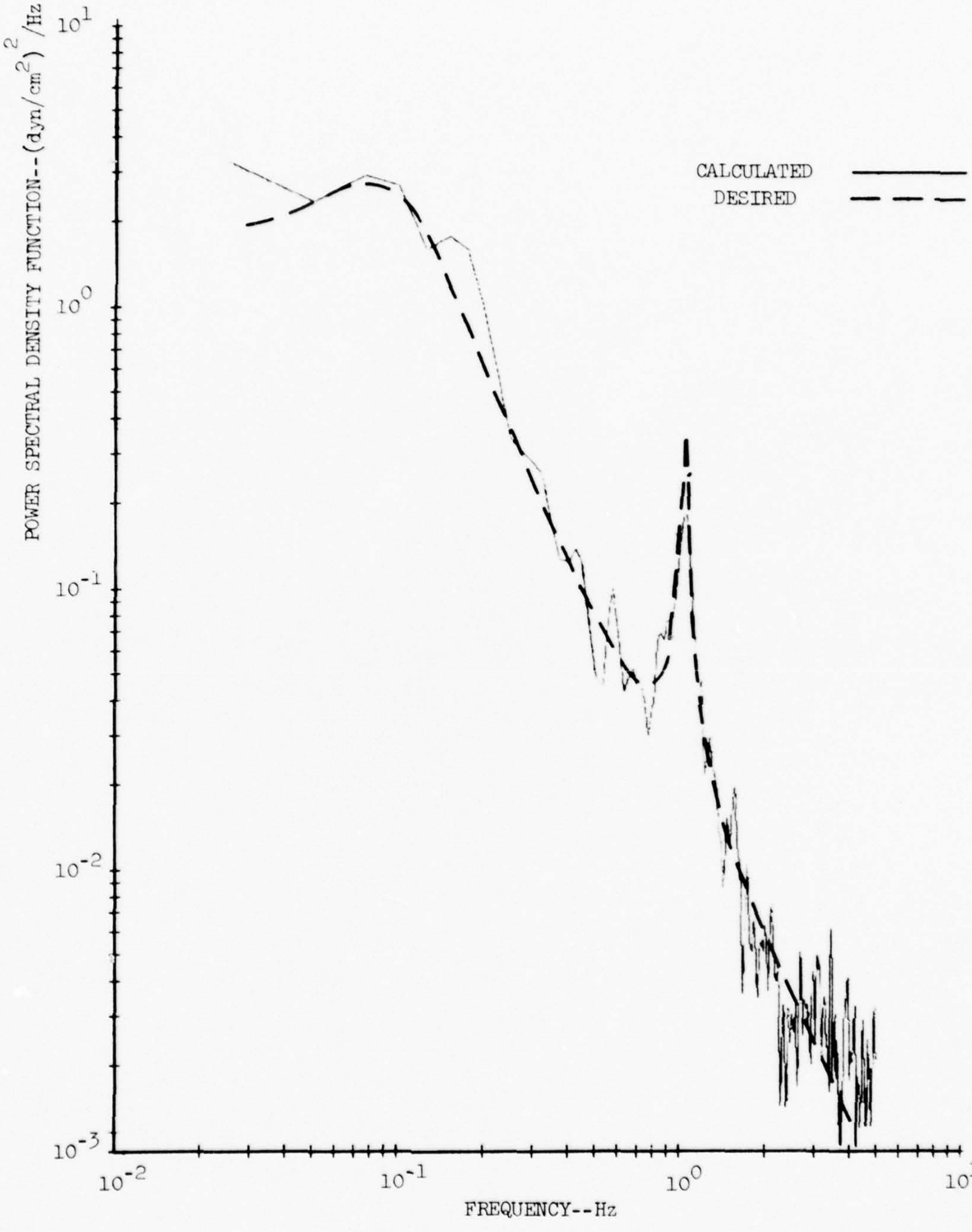


GAIN FOR RATIO OF QUADRATIC FACTORS

FIGURE 3

$$G(\omega) = \frac{\left(\frac{i\omega}{\omega_1} + 1\right)\left[\left(\frac{i\omega}{\omega_2}\right)^2 + 0.4\,\frac{i\omega}{\omega_2} + 1\right]}{\left[\left(\frac{i\omega}{\omega_1}\right)^2 + 1.2\,\frac{i\omega}{\omega_1} + 1\right]\left[\left(\frac{i\omega}{\omega_2}\right)^2 + 0.1\,\frac{i\omega}{\omega_2} + 1\right]} \quad . \tag{19}$$

Figure 4 is a log-log plot of the power spectral density function obtained when the transfer function, Eq. (19), was used in Franklin's method. A comparison of the dashed line which represents the desired power spectral density function with the solid curve indicates that this is indeed the necessary transfer function.



SINGLE FILTER OUTPUT

FIGURE 4

CHAPTER III

EXPERIMENTATION

The method of creating samples of a time function with a given power spectral density function as described in Chapter I requires the transfer function. A method of determining the transfer function from a sketch of the desired power spectral density function was described in Chapter II. Therefore, it would seem that the task of creating these samples of the time function is a straightforward procedure. However, Franklin's method of solving the differential equation, Eq. (12), is quite inefficient with respect to the ratio of the number of output samples to the number of input samples. The generation of each output sample requires n samples of the input where n is the degree of the polynomial in the denominator of the transfer function. This requirement may be a severe limitation when the power spectral density function is not simple.

A. Parallel Filters--Same Input

In an effort to generate samples of a time function with a complicated power spectral density function and still have a large number of output samples, two filters with simple transfer functions were used in parallel. These filters shared samples of a common input function $w_1(t)$ whose power spectral density function was flat. The samples of the output time functions $x_1(t)$ and $x_2(t)$ were summed in a point-by-point fashion to yield samples of the time function $y_1(t)$, where

$$y_1(t) = x_1(t) + x_2(t) \quad . \tag{20}$$

The transfer functions of the two filters are

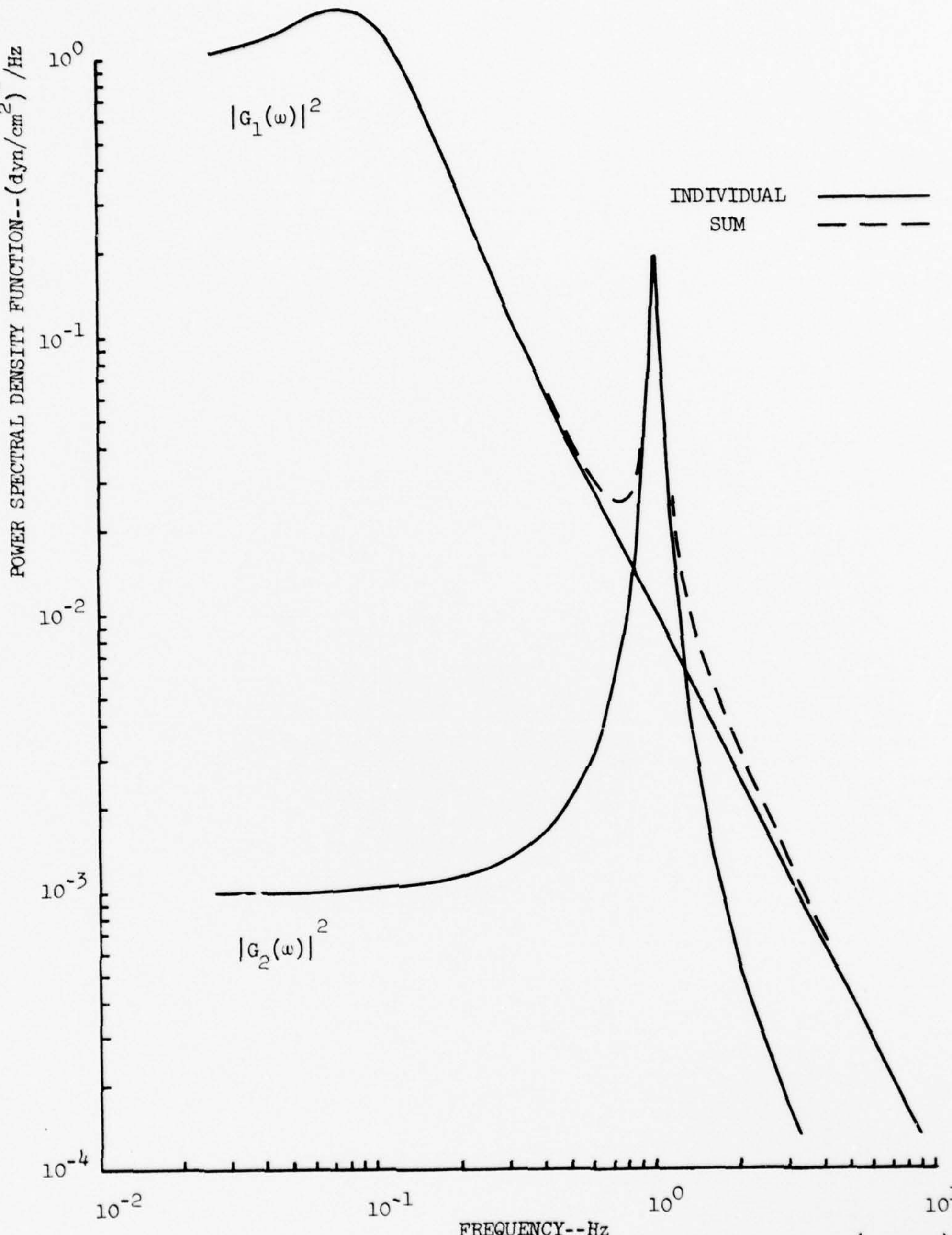
$$G_1(\omega) = \frac{\frac{i\omega}{\omega_1} + 1}{\left(\frac{i\omega}{\omega_1}\right)^2 + 2\zeta_1 \frac{i\omega}{\omega_1} + 1} \tag{21}$$

where the values of ω_1 and ζ_1 are 0.2π and 0.6, respectively, and

$$G_2(\omega) = \frac{K\left(\frac{i\omega}{\omega_2} + 1\right)}{\left(\frac{i\omega}{\omega_2}\right)^2 + 2\zeta_2 \frac{i\omega}{\omega_2} + 1} \quad , \tag{22}$$

where the values of ω_2 and ζ_2 are 2π and 0.05, respectively. The constant gain factor K has the value $0.01\sqrt{10}$.

The square of the magnitude of each of the transfer functions is shown in Fig. 5. The dashed lines near the intersection are the approximate sums of the two functions. These two transfer functions were selected in an effort to create samples of a time function with a power spectral density function similar to the curve shown in Fig. 2.



$$G_1(\omega) = \frac{\frac{i\omega}{\omega_1} + 1}{\left(\frac{i\omega}{\omega_1}\right)^2 + 1.2\,\frac{i\omega}{\omega_1} + 1}$$

$$\omega_1 = 0.2\pi$$

$$\omega_2 = 2.0\pi$$

$$G_2(\omega) = \frac{0.01\,/10\left(\frac{i\omega}{\omega_2} + 1\right)}{\left(\frac{i\omega}{\omega_2}\right)^2 + 0.1\,\frac{i\omega}{\omega_2} + 1}$$

FIGURE 5

The power spectral density function of the sampled output time function from the filters is shown in Figs. 6 and 7. The dashed lines are the plots of the functions $|G(\omega)|^2$.

All power spectral density functions shown in this thesis were produced by a spectral analysis program similar to the program SPECT, as found in a report by Ellis and Boston.[5] Under certain conditions the power spectral density function of a time function x(t) can be estimated by

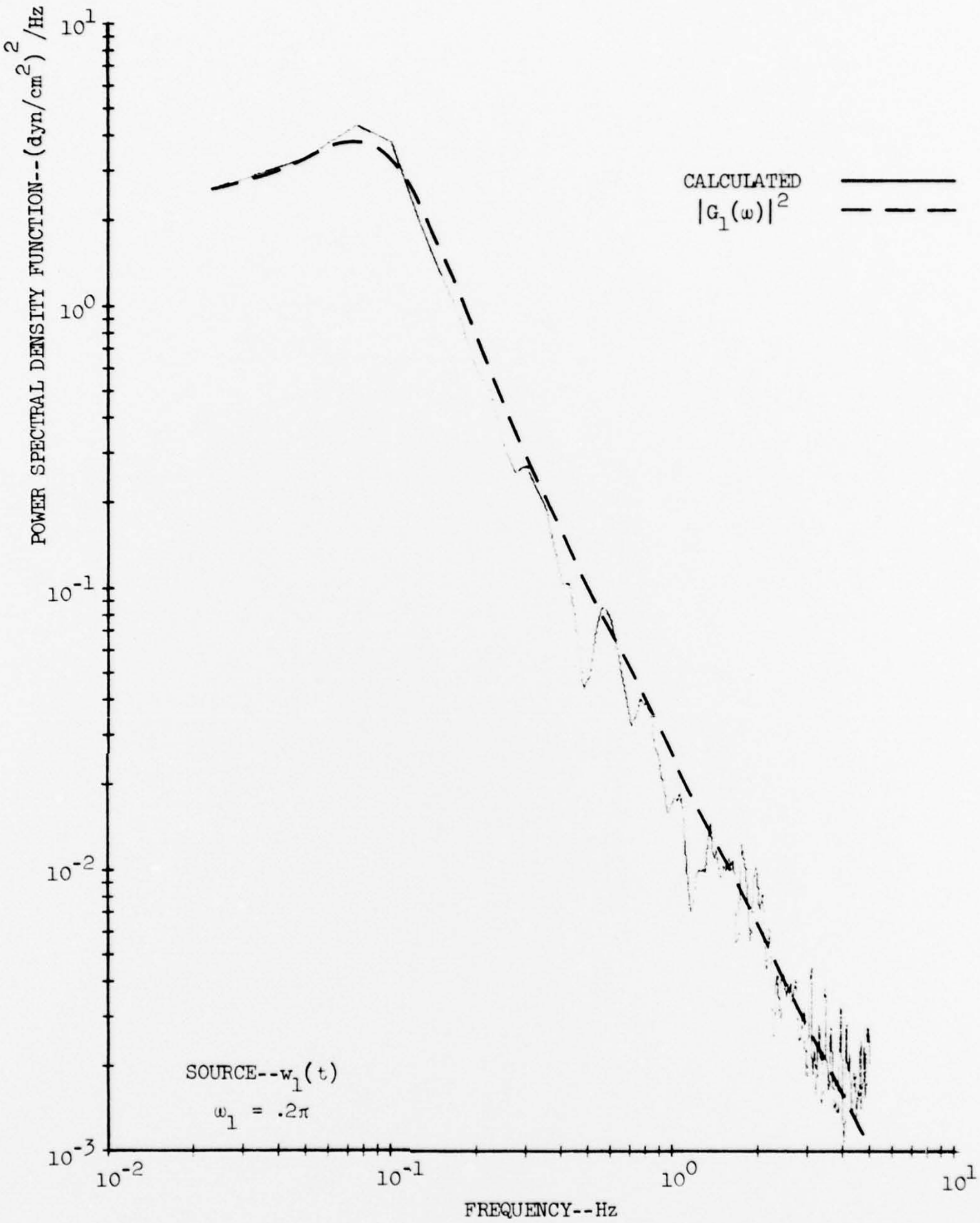
$$PSDF_x = \int_{-\infty}^{\infty} R_{xx}\, e^{-i\omega t}\, dt \quad , \tag{23}$$

where R_{xx} is the autocorrelation function of x(t). The autocorrelation function is defined by

$$R_{xx} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t - \tau)dt \quad , \tag{24}$$

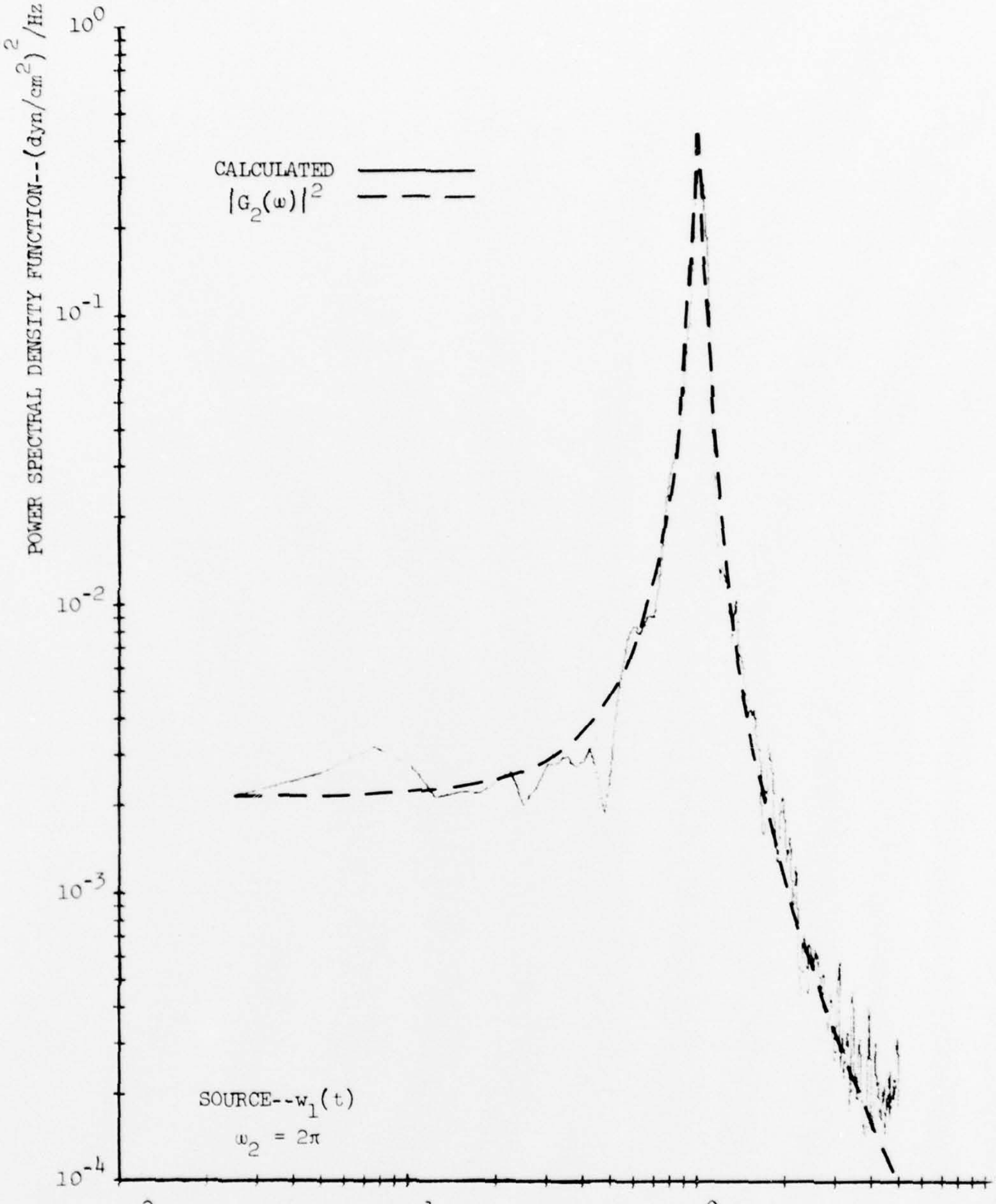
if the limit exists. The limit in Eq. (24) was dropped in the program SPECT because the time functions to be analyzed were all of finite duration.

The power spectral density function of the summed samples of the two filter outputs, $x_1(t)$ and $x_2(t)$, was somewhat different than had been expected. Figure 8 shows the power spectral density function of the output samples of the parallel filters. The dashed line in this figure is the approximate sum of the two curves shown in Fig. 5.



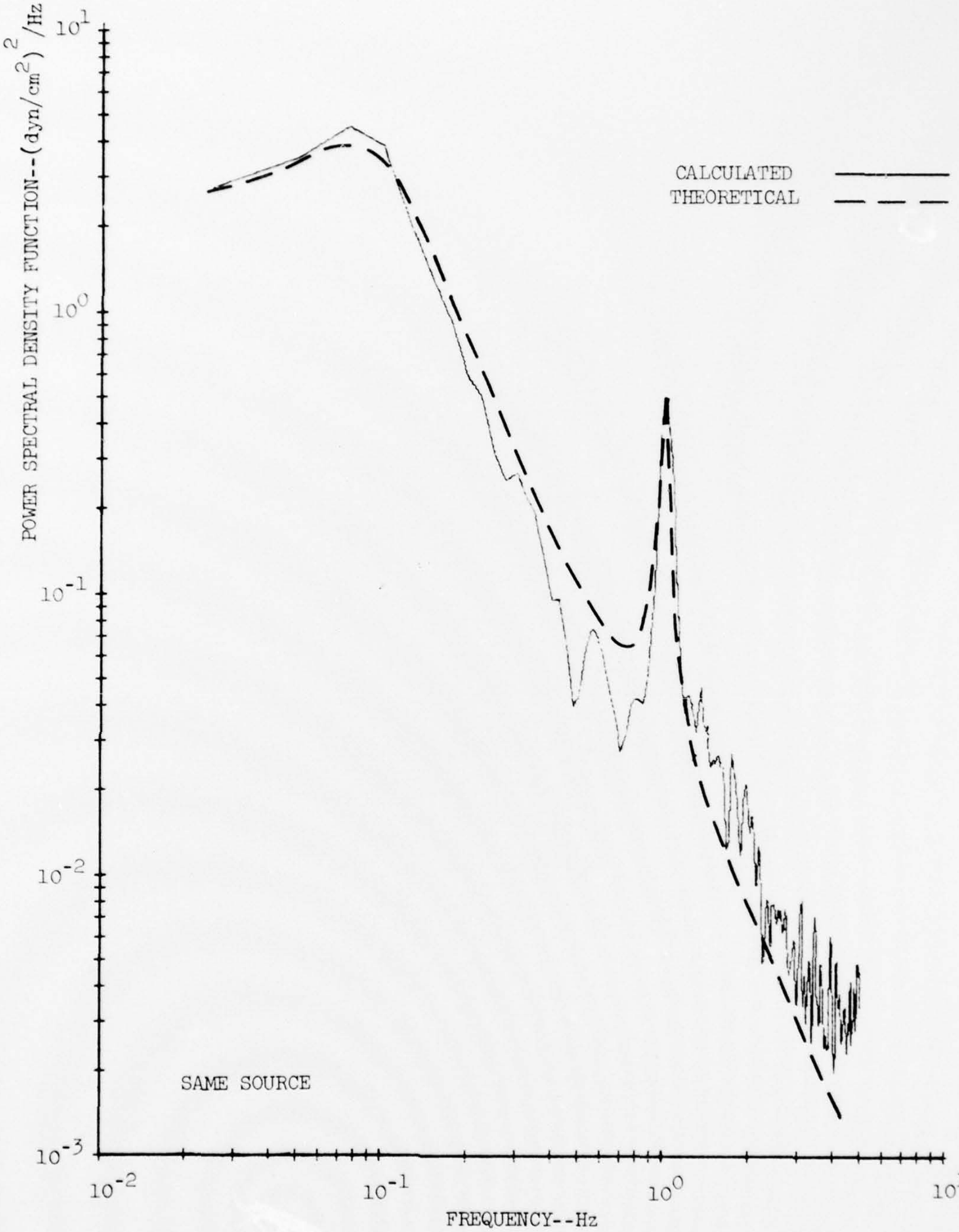
$$G_1(\omega) = \frac{\frac{i\omega}{\omega_1} + 1}{\left(\frac{i\omega}{\omega_1}\right)^2 + 1.2\,\frac{i\omega}{\omega_1} + 1}$$

FIGURE 6



$$G_2(\omega) = \frac{0.01\sqrt{10}\left(\frac{i\omega}{\omega_2} + 1\right)}{\left(\frac{i\omega}{\omega_2}\right)^2 + 0.1\frac{i\omega}{\omega_2} + 1}$$

FIGURE 7



SUMMED FILTER OUTPUTS

FIGURE 8

The difference can be explained as follows. The autocorrelation of the sum of the functions is

$$R_{y_1y_1} = \frac{1}{2T}\int_{-T}^{T} y_1(t)y_1(t - \tau)dt \quad . \tag{25}$$

But the function $y_1(t)$ is the sum of the functions $x_1(t)$ and $x_2(t)$. Therefore, Eq. (25) can be expressed as

$$R_{y_1y_1} = \frac{1}{2T}\int_{-T}^{T}\left[x_1(t) + x_2(t)\right]\left[x_1(t - \tau) + x_2(t - \tau)\right] dt \quad , \tag{26}$$

or as

$$\begin{aligned} R_{y_1y_1} = \frac{1}{2T}\int_{-T}^{T}\Big[x_1(t)x_1(t - \tau) &+ x_1(t)x_2(t - \tau) \\ &+ x_2(t)x_1(t - \tau) + x_2(t)x_2(t - \tau)\Big] dt \quad . \end{aligned} \tag{27}$$

Since the integral of the sum is the same as the sum of the integrals, Eq. (27) can be written as

$$\begin{aligned} R_{y_1y_1} = \frac{1}{2T}\int_{-T}^{T} x_1(t)x_1(t - \tau)dt &+ \frac{1}{2T}\int_{-T}^{T} x_2(t)x_2(t - \tau)dt \\ + \frac{1}{2T}\int_{-T}^{T} x_1(t)x_2(t - \tau)dt &+ \frac{1}{2T}\int_{-T}^{T} x_2(t)x_1(t - \tau)dt \quad . \end{aligned} \tag{28}$$

This can be expressed entirely in terms of correlation functions as

$$R_{y_1y_1} = R_{x_1x_1} + R_{x_2x_2} + R_{x_1x_2} + R_{x_2x_1} \quad . \tag{29}$$

Therefore, the power spectral density function of $y_1(t)$ produced by the spectral analysis program SPECT is the sum of the power spectral density function of each of the filter outputs and the cross-spectral density function due to the cross-correlation terms $R_{x_1x_2}$ and $R_{x_2x_1}$.

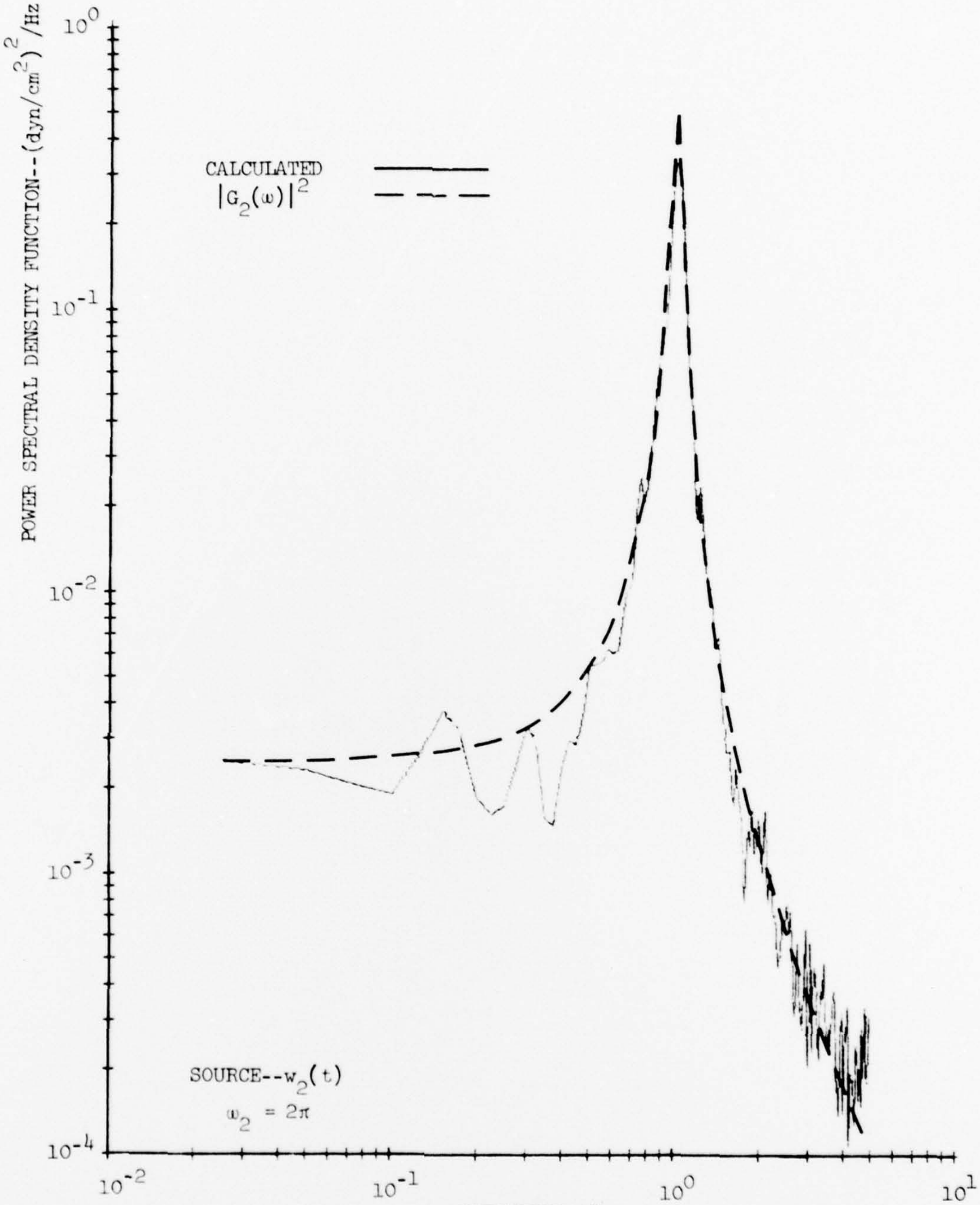
Since the filters shared the same input samples, there is a possibility that the filter output samples are somewhat correlated. If this is true, then the cross-correlation terms in Eq. (29) will cause the cross-spectral density functions to contribute significantly to the power spectral density function of the sum of the two filter outputs.

B. Parallel Filters--Separate Inputs

In an attempt to minimize these cross-spectral density functions, separate uncorrelated inputs were used with the same pair of filters. It was hoped that since the inputs were uncorrelated, the outputs would also be uncorrelated. If this were the case, the terms $R_{x_1x_2}$ and $R_{x_2x_1}$ in Eq. (29) would be small, thereby reducing the effect of cross-spectral density functions.

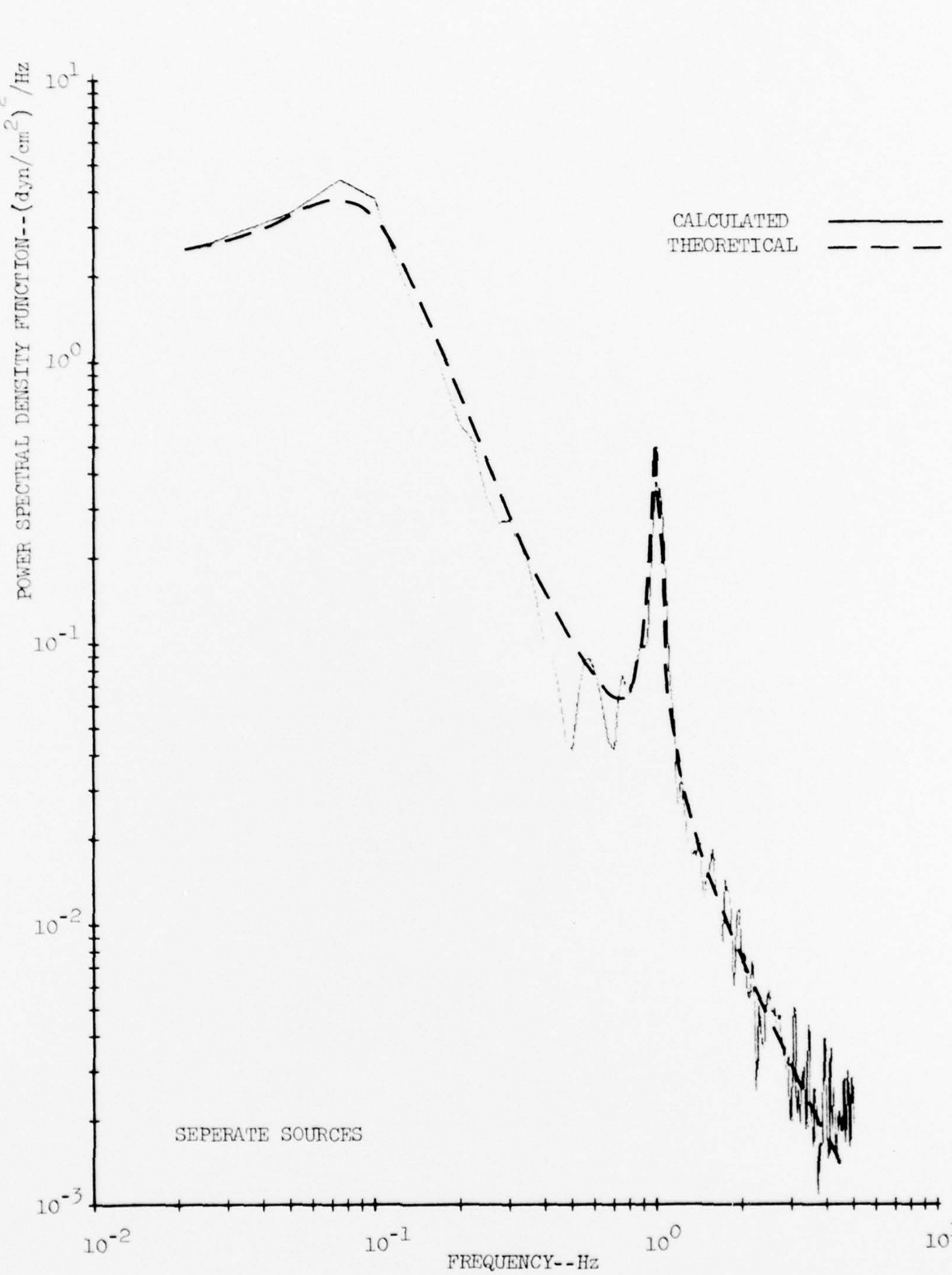
The power spectral density function of the second set of input samples was flat. These samples were used as input to the filter with the transfer function expressed in Eq. (22). Figure 9 depicts the power spectral density function of the samples of the filter output $x_3(t)$. The dashed line is the plot of the function $|G_2(\omega)|^2$.

An examination of Fig. 10 will indicate that there was only a slight difference between the power spectral density function of the summed samples of the two filter outputs, $x_1(t)$ and $x_3(t)$, and the approximate sum of the two curves shown in Fig. 5. The dashed line is the desired shape and the solid line is the power spectral density function of the summed samples of the two filter outputs.



$$G_2(\omega) = \frac{0.01\sqrt{10}\left(\frac{i\omega}{\omega_2} + 1\right)}{\left(\frac{i\omega}{\omega_2}\right)^2 + 0.1\frac{i\omega}{\omega_2} + 1}$$

FIGURE 9



SUMMED FILTER OUTPUTS

FIGURE 10

CHAPTER IV

CONCLUSIONS

A transfer function for a filter can be determined which will yield an output function with any specified power spectral density function. Franklin's method of generating the samples of the filter output function is inefficient if the transfer function has very many factors in the denominator. To circumvent this limitation, the outputs of several filters can be summed. These filters should have simple transfer functions and separate uncorrelated input functions.

The summation of several filter outputs can be used to generate a composite output function which has a power spectral density function similar to the sum of the power spectral density functions of the individual filter output functions. A library of time function samples with various power spectral density functions, generated from uncorrelated input samples, could be of value. Time functions from the library could be selected according to their power spectral density functions and summed in order to produce a time function with most any desired power spectral density function. These library time functions could also be used to alter recorded samples of natural occurrences to suit the desires of the user.

APPENDIX

A reference in the thesis was made concerning the inefficiency of Franklin's method for solving the differential Eq. (12). His solution requires n samples of the input to generate each sample of the output where n is the degree of the polynomial in the denominator of the transfer function.

It was necessary for Franklin to define a sequence of n-dimensional vectors, one vector to be used for each output sample. These vectors were defined as

$$w^{(0)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix}, \quad w^{(1)} = \begin{bmatrix} w_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ w_{2n} \end{bmatrix}, \quad w^{(2)} = \begin{bmatrix} w_{2n+1} \\ \cdot \\ \cdot \\ \cdot \\ w_{3n} \end{bmatrix}, \ldots,$$

where w_1, w_2, w_3, ..., are the input samples.

It was noted that the method might have been more efficient had the vectors been defined as

$$w^{(0)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} , \quad w^{(1)} = \begin{bmatrix} w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_{n+1} \end{bmatrix} , \quad w^{(2)} = \begin{bmatrix} w_3 \\ \cdot \\ \cdot \\ \cdot \\ w_{n+2} \end{bmatrix} , \quad \ldots \quad .$$

The difference between the number of input samples and output samples for the last definition of vectors is n - 1; the difference resulting from Franklin's method is $N(1 - \frac{1}{n})$, where N is the number of input samples.

All claims made by Franklin concerning the various characteristics of the linear processor which he described apply equally well with either choice of vector definition. This is true because the arguments really apply to the individual vector components rather than to the vectors themselves.

BIBLIOGRAPHY

1. Bendat, J. S., Principles and Applications of Random Noise Theory (John Wiley and Sons, New York, 1958).

2. Blackman, R. B., and J. W. Tukey, The Measurement of Power Spectra (Dover Publications, Inc., New York, 1958).

3. Chestnut, Harold, and Robert W. Mayer, Servomechanisms and Regulating System Design (John Wiley and Sons, New York, 1951), 2nd Ed., Vol. I, Chapter 11, pp. 327-340.

4. Davenport, W. B., and W. L. Root, An Introduction to the Theory of Random Signals and Noise (McGraw-Hill, New York, 1958), pp. 182-183, 232-234, 375-376.

5. Ellis, G. E., and R. L. Boston, "Program SPECT: Power Spectral Analysis of a Random Process Using the CDC 1604 Computer," Defense Research Laboratory, The University of Texas, Austin, Texas, Acoustical Report A-252 (December, 1965).

6. Franklin, J. N., "Numerical Simulation of Stationary and Non-stationary Gaussian Random Processes," SIAM Review 7, No. 1, pp. 68-80 (January, 1965).

7. Lee, Y. W., Statistical Theory of Communication (John Wiley and Sons, New York, 1960).

8. Matuska, Walter A., and George S. Innis, "Generation of a Stationary Gaussian Random Process with a Specified Power Spectral Density Function," Defense Research Laboratory, The University of Texas, Austin, Texas, Acoustical Report A-258 (July, 1966).

9. Truxal, John G., Automatic Feedback Control System Synthesis (McGraw-Hill, New York, 1955), Chapter 6, pp. 344-360.

11 May 1967

DISTRIBUTION LIST FOR DRL-A-283
UNDER CONTRACT NObsr-95181

Copy No.	
1 - 4	Commander, Naval Ship Systems Command (Ships 1622C) Department of the Navy Washington, D. C. 20360 Attn: Mr. C. D. Smith
5	Commanding Officer U. S. Navy Mine Defense Laboratory Panama City, Florida 32402 Attn: Mr. Carl M. Bennett
6	Commanding Officer U. S. Naval Oceanographic Office Washington, D. C. 20390 Attn: Mr. R. C. Guthrie (Code 7240)
7	Commanding Officer U. S. Navy Electronics Laboratory San Diego, California 92152 Attn: Code 3180
8	Office of Naval Research Resident Representative The University of Texas 2507 Main Building Austin, Texas 78712
9	Acoustics Division, DRL/UT
10	Signal Physics Branch, DRL/UT
11	Underwater Missiles Division, DRL/UT
12	E. Maurice Davis, DRL/UT
13	K. J. Diercks, DRL/UT
14	G. E. Ellis, DRL/UT
15	J. F. England, DRL/UT
16	D. W. Evertson, DRL/UT
17	K. W. Harvel, DRL/UT
18	Walter Matuska, DRL/UT
19	S. P. Pitt, DRL/UT
20	Library, DRL/UT