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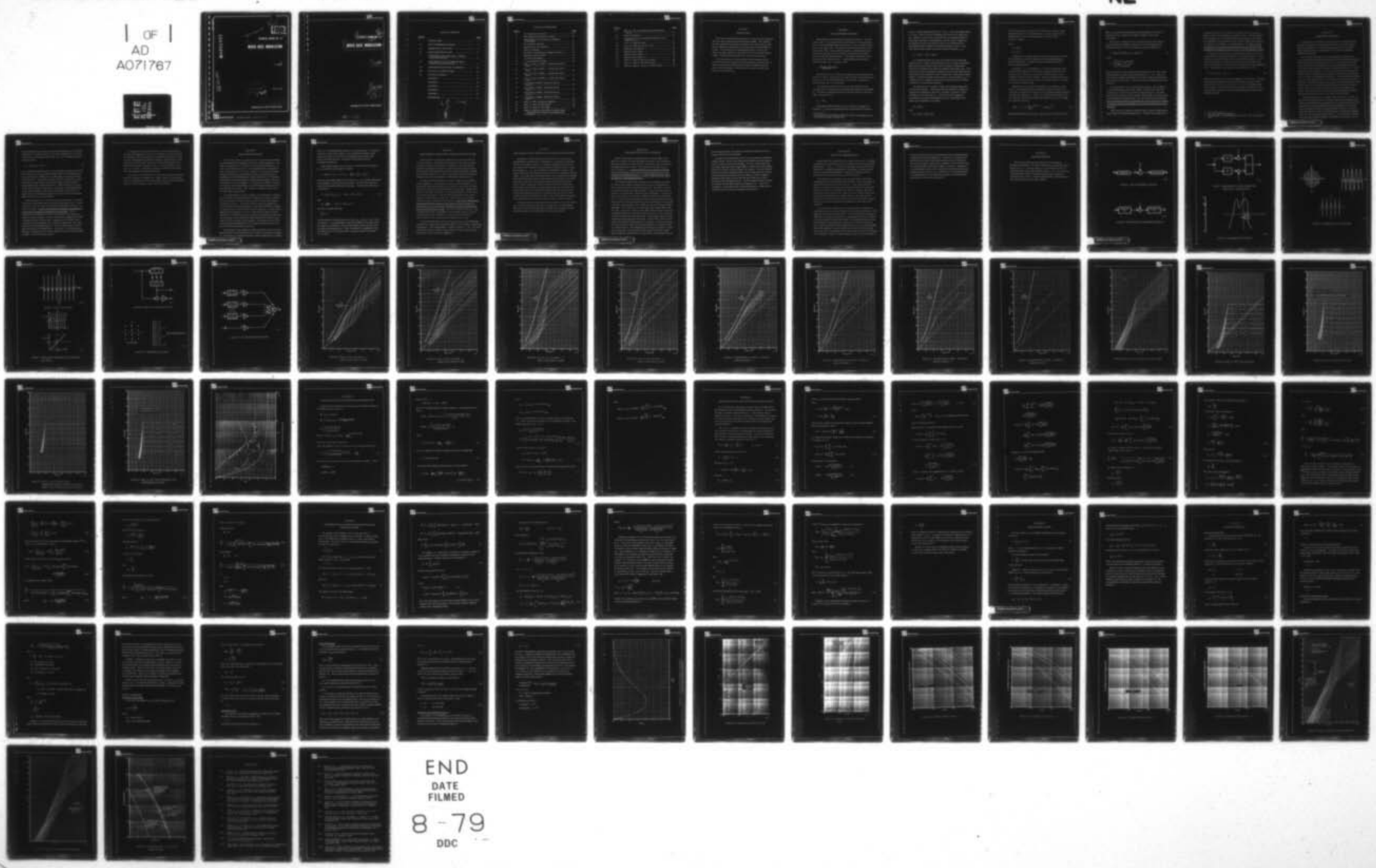

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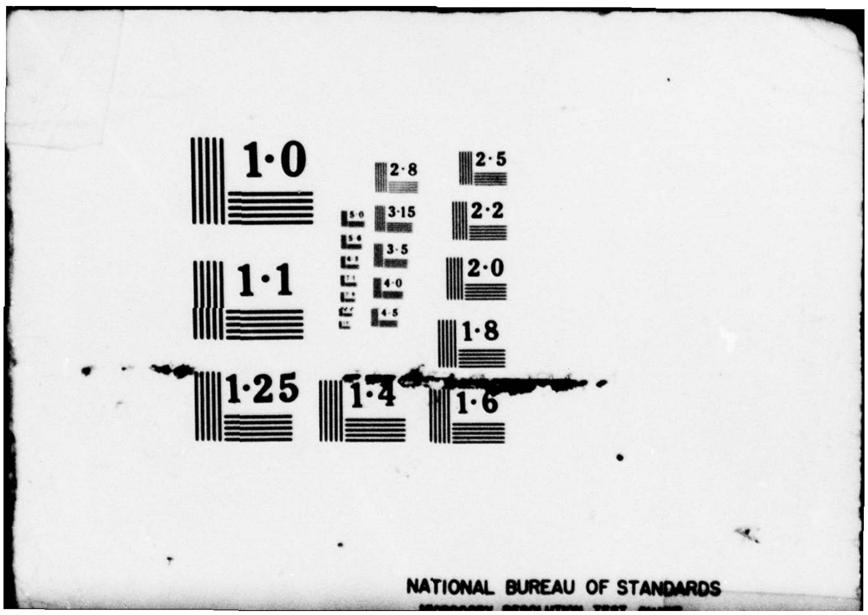
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TECHNICAL REPORT NO. 37

MIXED BASE MODULATION

by:
D. D. McRae

20 MAY 1970

ADVANCED SYSTEMS OPERATIONS

RADIATION INCORPORATED SYSTEMS DIVISION - Melbourne, Florida
SUBSIDIARY OF HARRIS INTERTYPE CORPORATION

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SECTION I

INTRODUCTION

This report presents the performance of a new modulation technique which has been called mixed-base modulation (MBM). This technique provides significant performance advantages over other common modulation techniques for applications where both power and bandwidth are limited. Its performance for bandspreading factors of two to five relative to single-sideband is reasonably close to the ultimate Shannon bounds. Most available techniques are very poor in this region.

The viewpoint which allowed synthesis of this technique will be given before the basic approach and its performance are discussed. A section comparing the theoretical performance of MBM and that of other common techniques is followed by a discussion of some of the practical considerations.

SECTION II

THE TRANSMISSION PROBLEM

The problem to be considered is illustrated in Figure 1. An input signal, m(t), is to be transformed in some manner to a second signal, h(t), to which independent white gaussian noise, n(t), is added to produce the received signal, r(t). This signal is then transformed into an estimate of m(t), $\hat{m}(t)$. We will take the liberty of calling the device that transforms m(t) into h(t) the modulator and the device which transforms r(t) into $\hat{m}(t)$ the demodulator. The fidelity criteria will be the normalized mean-squared error, ϵ, between $\hat{m}(t)$ and m(t) where a known time delay, Td, is allowed. Thus:

$$\epsilon = \frac{E[(m(t) - \hat{m}(t+T_d))^2]}{E[m^2(t)]} \qquad (1)$$

The objective will be to choose the modulation and demodulation processes to produce a small value of ϵ when the spectral characteristics of the source, and the transmitted signal, m(t) and h(t), and the noise spectral height, $N_o/2$, are specified.*

To simplify the problem we will assume the input data to be a stationary process with an ideal flat bandlimitted spectrum of bandwidth, B_v. We then can represent the process with time samples of the input taken at intervals of T_s:

$$T_s = 1/2B_v.$$

If the output of the modulator is also a sequence of samples at intervals, T_o, they can be represented by time functions such that the

* The objective of centering the spectrum of h(t) for propagation will be discussed separately in Section VIII.

output or channel bandwidth will be $B_c = 1/2T_o$. Thus, the bandspreading, $\beta = B_c/B_v$, will be the ratio of sampling rates between the output and input. Further, if the receiver first passes the noisy signal through an ideal sharp cutoff filter of bandwidth B_c, and the result is sampled at the appropriate times, the problem reduces to a transformation of independent random variables as indicated in Figure 2 where i is the time index and the variance of the noise random variables, n_i, is:

$$\sigma_n^2 = E[n_i^2] = B_c N_o = \beta N_o B_v.$$

It is worth noting that our setup of the problem differs from that taken by Ford [1] which models the input source as a finite dimensional Markoff process and allows control theory techniques discussed by Snyder [2] to synthesize the optimum modulator-demodulator pair. Ford was unsuccessful in evaluating the optimum non-linear pair and discusses instead some of the properties of the optimum time-variant linear combinations. It should be pointed out also that the real-time (that is $T_d = 0$) evaluation of error implicit in the model provided by Snyder is usually unrealistic in communication problems.

The simplest case to consider is that of unity bandspreading and a memoryless modulator. This implies that the modulator is simply an operator that modifies the value of m_i to produce h_i and the receiver estimates m_i from each input, r_i. Since the received signals are independent from sample to sample the receiver cannot make use of other received samples. If we define:

$$h_i = 0_i [m_i]$$

and

$$m_i = 0_2 [r_i] = 0_2 [n_i + h_i]$$

then one may wonder if a good choice of non-linear operators might significantly reduce ϵ (as compared with ϵ for linear operators) for a specified value of receiver input signal to noise ratio, $SNR_i = P_s/N_oB_v = 2E_s/N_o$.

where:

$$P_s = E[h_i^2]$$

$$E_s = P_sT_s .$$

In Appendix I an integral equation is derived for the optimum modulator-demodulator pair which depends upon the probability density function of the input. An explicit solution was obtained only for a gaussian density function.

Lechleider [3, 4] and Haddad [5, 6] have considered the transmission of the memoryless time discrete variables and derived some results for the "small noise" case using a truncated Taylor series expansion for the non-linear operator. Both have extended the analysis to consider the memoryful case where each output sample is a function of more than one input sample.

Although the case of unity bandspreading is interesting, it can be shown by information theoretic arguments that where the input, m(t), is gaussian, minimum mean-square error is achieved when the operators are linear and memoryless. Goblick [7] and Brown and Palermo [8] show that the maximum achievable output signal-to-noise ratio, SNR_o, for a gaussian source is:

$$SNR_o = 1/\epsilon = \left(1 + \frac{P_s}{N_oB_c}\right)^{B_c/B_v} = [1 + SNR_i/\beta]^{\beta} . \quad (2)$$

When the bandspread ratio, B_c/B_v, equals one, the maximum value of

SNR_o is achieved when both modulator and demodulator are linear and memoryless and a voltage gain of $SNR_i/(1+SNR_i)$ is used in the demodulator.

If the source is non-gaussian the calculation of the rate distortion bound becomes more difficult. However, by use of bounds given by Shannon and Weaver [9] we may determine bounds for SNR_o to be:

$$(1 + SNR_i/\beta)^{\beta} \leq SNR_o \leq K(1 + SNR_i/\beta)^{\beta} \qquad (3)$$

where:

$$K = \frac{2^{-2\left[\int f(m) \log_2 f(m)dm\right]}}{(2\pi e)\left[\int m^2 f(m)dm\right]}$$

and as before f(m) is the probability density function of m. If the source density function is uniform rather than gaussian, K = 1.42. Thus, the best possible modulator-demodulator for uniform density and bandspreading of one can at most gain 1.5 dB over the linear combination previously discussed.

From equation (3) we can conclude that to supply a value of SNR_o of 40 dB an input signal to noise ratio, SNR_i, of between 38.5 and 40 dB is required for the case of a uniformly distributed source. An output signal-to-noise ratio of 40 dB is achieved with an SNR_i of 40 dB by simply transmitting the modulating signal, m(t), and using r(t) as the estimate. Thus, the ultimate results are achievable (or nearly so) by use of a linear memoryless modulator-demodulator pair where all of the error is due to link error.

This leads one to wonder if relatively simple schemes might provide good results for larger bandspread ratios. Equation (3) indicates that if a bandspread ratio of two is employed a scheme exists for providing a SNR_o of 40 dB with 23 dB SNR_i for a uniform source. This performance is very significant since a standard FM system requires a 22 dB SNR_i for 40 dB SNR_o with a bandspread ratio of 20 and a PSK system requires 21 dB SNR_i for 40 dB SNR_o for a bandspreading of 14.* Thus, a choice of operators must exist that provides performance close to that of FM and PSK with a bandspread ratio of 1/10 or 1/7 of the respective system.

The problem of synthesizing a modulator-demodulator pair for a β of two utilizing a memoryless operator is illustrated in Figure 3. Each input pulse is passed through two memoryless transformations, $0_1(m)$** and $0_2(m)$ to produce two output pulses h_1 and h_2 for each input pulse. The received signals r_1 and r_2 are the sum of these values and independent noise. The minimum mean-square error estimate for m, $\hat{m}$, is:

$$\hat{m} = E(m|r_1, r_2) = 0_3(r_1, r_2) \tag{4}$$

Expressions for $\hat{m}$ and ϵ are given in Appendix 1 for this case. Although in principle these equations can be used to provide a solution for the optimum choice of $0_1(m)$, $0_2(m)$ and $0_3(r_1, r_2)$, the computational difficulties appear to be very great even if a computer is employed. Thus the selection of operators for higher bandspreadings than one requires a different approach.

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* For more details see Appendix 5.

** The subscript i has been dropped on m, h and r in the memoryless case.

SECTION III

GEOMETRICAL APPROACH

A notion which was introduced quite early in the information theory literature by Shannon [10] (and suggested earlier in much less explicit fashion by Kotelnikov [11]) and has been discussed more recently by Wozencraft and Jacobs [12] and Sakrison [13] is that of viewing the problem geometrically.

Let us consider the input, m, to be constrained to a finite range (say ± A) and assume that we wish to choose the two memoryless operators, $0_1(m)$ and $0_2(m)$ for a bandspread factor of two. Any choice of $0_1(m)$ and $0_2(m)$ can be viewed as a mapping of a line representing the values of m onto a plane with coordinates h_1 and h_2 as illustrated in Figure 4. Further the pair of received signals, r_1 and r_2 represent a point in the plane that differs from the transmitted pair of signals by the two noise components, n_1 and n_2 which are independent and gaussian. The task of the demodulator, $0_3(r_1, r_2)$ is to estimate which value on the line was sent given a point in the plane. Perhaps the simplest rule (but not necessarily the optimum rule) for the receiver is to select the point on the line closest to the received point. This would correspond to assuming that the most likely noise vector occurred and as such would be the maximum likelihood estimate. Assuming this type of receiver, how should we draw the line?

The first observation we can make is that if the noise is "small" so that we do not confuse one major segment of the line with another and we choose to represent the noise components as one tangent to the line at the estimated value and one perpendicular at the estimated value, the only error will be that due to the component tangent to the line. The error in the estimate of m, however, depends upon how different the estimated value, $\hat{m}$, is from the transmitted signal m_o. If we uniformly mapped m onto a line that was 20A in length (or stretched by ten) then the error in

$\hat{m}$ normalized to the full scale of m will be one tenth the size of the noise component. If, however, the line is only 2A in length the error in $\hat{m}$ for "small" noise will be equal to the noise. Even if we constrain the power in the two pulses:

$$P_s = E[h_1^2 + h_2^2] = E[m^2] \quad (5)$$

we can make the normalized error in the estimate as small as we wish for the "small noise" case by making the length of the line sufficiently long. In this case the gain in SNR_o for a specified SNR_i is equal to the square of stretch factor. The problem with this is that for a fixed P_s we cannot make the line longer without reducing the distance between locations representing largely different values of m. When the noise is such that errors of this nature (anomolies) occur frequently, a thresholding effect is observed in the SNR_o versus SNR_i curve and the closer the line is packed in the two dimensional space the smaller the noise (or the larger SNR_i) necessary to cause thresholding.

This then instructs us as to how we should draw the line. We should make the line sufficiently long to provide the high-signal to noise improvement that we wish, but make it such that the probability of anomoly is as small as possible. <u>It should be noted that the arguments presented place no particular desirability on making the line continuous</u>. This is emphasized since references [12] and [13] both place the constraint of continuity upon the mapping. Figure 5 shows three mappings which seem as though they would be pretty good. Mappings a and b contain no discontinuities whereas mapping c does contain discontinuities. One might expect that the thresholding effect in c would be sharper than in a and b where an anomoly is somewhat fuzzy. In all cases a trade between high SNR gain and threshold can be made by putting in more or less revolutions, sawteeth, or lines as the case may be.

If a bandspread ratio greater than two is used with memoryless operators the mapping problem is one of mapping a line into more than two dimension with the same notions of stretching and spacing. Timor [14] has investigated the problem of mapping a continuous line in five dimensions such that points on the line corresponding to values of m differing by more than a specified value, Δm, are separated by a distance Δr or greater. Under this constraint he maximizes the stretch factor and estimates resulting performance.

If one wishes to consider the use of memoryless operators then the geometrical object to be mapped is no longer a line but a hyper-square. Thus if a bandspread of two is allowed with a memory of two samples the problem is one of mapping a square into four dimensions.

SECTION IV

MIXED BASE MODULATION

By use of the geometric approach one can conjure up many interesting configurations for mapping lines into two or three dimensions. The type of mapping shown in Figure 5c is particularly interesting since it looks as if it should be good, it is easily implemented and its performance is relatively easy to evaluate. Further, it can be easily extended to higher dimensions by noting that the location of parallel lines in n dimensions can be identified by the location of points in n-1 dimensions. If the length of each line segment is 2A then the total length is 2A times the number of such line segments. To minimize the problem of anomaly (or minimize the threshold SNR_i) one must maximize the minimum distance between these points. This, however, is identical to the usual discrete signal design problem.

The implementation of such a mapping can be explained by considering an example of mapping the input into eight lines as shown in Figure 6. The two operators, $0_1(m)$ and $0_2(m)$ required to achieve this mapping are shown in Figure 7. The second operator can be recognized as an 8-ary quantizer and the first operator as one that provides the remainder or quantizing error associated with the 8-ary quantizer. A simple way to synthesize such operators is to use a 3 bit analog-to-digital converter followed by a 3 bit digital-to-analog converter to produce the 8-ary pulse. The remainder may be obtained by subtracting the input from the pulse output as shown in Figure 8 and amplifying. A technique similar to this was recently described by Hill [15] in connection with some tests run for the British Broadcasting Corporation.

The extension of this mapping to higher dimensions can be accommodated simply by using more than one m-ary pulse to subdivide the range of m. For instance, a bandspread-by-three system might achieve a stretch factor by 12 by subdividing the range of m into twelve parts. As indicated in Figure 9, the h_2 pulse could be 3-ary designating which third of the range m occupies, h_1 could be 4-ary designating which fourth of the third of full scale m lies in. The pulse h_0 would then carry the remainder information.

If the range of values of h_0, h_1, and h_2, were the same as that of m we can note that in the example of Figure 9:

$$m = \frac{1}{17}[4 \cdot 3 \cdot h_2 + 4 \cdot h_1 + h_0] = \frac{12}{17} h_2 + \frac{4}{17} h_1 + \frac{1}{17} h_0$$

where the transmitted signal is the n-tuple $[h_2, h_1, h_0]$ and the coefficients may be thought of as the basis for the vector. For this reason the technique has been named mixed base modulation. The selection of the basis is such that if:

$$m = B_n h_n + B_{n-1} h_{n-1} + \cdots B_2 h_2 + B_1 h_1 + B_0 h_0 \qquad (6)$$

then:

$$B_k = K \prod_{k=1}^{n} M_j \qquad \text{for } k \geq 1 \text{ and } B_0 = K$$

where K is selected such that:

$$\sum_{j=0}^{n} B_j = 1$$

and the number of discrete levels of h_n is M_n, h_{n-1} is M_{n-1}, etc., and h_0 is continuous. It is interesting to note if M_1 through M_2 are equal to two the system is equivalent to a binary PCM system with an extra bit which describes the quantizing error. Such a technique was suggested as early as 1948 by Earp [16] and described in a paper by Nomura and Yasuda [17] in 1962.

SECTION V

PERFORMANCE OF MBM USING A SIMPLE RECEIVING SYSTEM

The discussion of mixed base modulation thus far has considered only the modulator. The simplest demodulator is that shown in Figure 10 where each noisy discrete pulse is quantized to select the nearest possible transmitted level and the continuous pulse is simple left alone. The estimates of the transmitted values are then appropriately weighted to provide the estimate of m. This is equivalent to maximum likelihood reception except for an end effect in receiving the continuous pulse.

An analysis of the performance in terms of $(SNR)_o$ versus $(SNR)_i$ is given in Appendix 2. Performance curves are given in Figures 11 - 14 for the case where the peak power in all pulses is made equal and where the number of levels in all discrete pulses is made equal. Bandspread factors of 2 through 5 are given. The probability density function of m(t) is assumed to be uniform. The rate distortion bounds for the uniform source are obtained from equation (3). The performance of a single-sideband system is plotted also for reference. Note that the MBM system trades threshold performance for high signal-to-noise ratio gain in a manner that is very close to that of the optimum modulator-demodulator pair which allows infinite memory. As can be seen from Figure 11, with a bandspreading of two, 40 dB SNR_o can be obtained with an 8-ary and a continuous pulse with 27 dB SNR_i as compared with 40 dB SNR_i for single-sideband. The ultimate bound is between 22 and 23 dB.

The requirement that all pulses have equal energy is not fundamental. The division of energy which maximizes SNR_o for a given SNR_i is determined in Appendix 4. The resulting SNR_o is plotted in Figures 15 - 18. As can be seen in most cases very little is to be gained by unequal energy division. The optimum ratios are given in Appendix 4.

SECTION VI

PERFORMANCE WITH A MINIMUM-MEAN-SQUARE ERROR RECEIVER

The equation for the minimum-mean-square error receiver when bandspreading is present (given in Appendix 1) is somewhat cumbersome. For the case of MBM, however, it is shown in Appendix 3 that the minimum mean square estimate of the transmitted value $\hat{m}$ given the received set of signals, $r_1 - r_n$, can be obtained by weighting the minimum mean square estimate of each transmitted pulse given the corresponding received signal. This argues that the discrete pulses can be operated upon independently and if the discrete pulses in the transmission have the same number of possible levels, the same operation needs to be performed on each pulse. Of more significance is the fact that if the receiver minimizes mean-square error based on a uniform density function for the input, m, then its performance will be better than that of the simple receiver described in the preceding section for all density functions which are zero outside the range of the uniform variable. This, then makes the minimum mean square error receiver of practical as well as academic interest.

Unfortunately the calculated performance of the mmse receiver based upon a uniform density function (given in Appendix 4) indicates that the performance gain is small (generally less than a dB) over that available from the simpler receiver described in the preceding section.

SECTION VII
COMPARISON WITH OTHER TECHNIQUES

Figures 19 through 23 are performance curves for SNR_o versus SNR_i for a variety of common modulation techniques. In all cases the same idealizations were made that have been made in the calculation of the MBM performance curves. The detailed basis for the curves is given in Appendix 5. A significant point to note is that in all of these systems the high SNR_i versus improvement threshold tradeoff is tied directly to the bandspread factor, β, whereas MBM allows the trade to be made for a fixed bandspread factor.

The FM performance given in Figure 19 is calculated on the basis of a peak-to-peak frequency deviation equal to the IF bandwidth and a noise model based upon an unmodulated carrier. The values of β given are the ratios of the IF to video bandwidths. Figure 20 shows the performance of coherent phase-shift keying and quadraphase. In Figure 20 the values of SNR_o for large values of SNR_i are limited by quantization error. In Figure 21 three different coding-decoding techniques are given which provide essentially equivalent bit-error performance versus E_b/N_o (and hence equivalent SNR_o versus SNR_i) but requires different channel bandwidths. The values for β for each depend upon whether PSK or QP are used. The logic in listing all three rather than the most conservative in bandwidth (the 63, 50 block code) is that the 23, 12 is a more commonly used block code and the constraint length 5 convolutional code allows the simplest decoding equipment of the three. The two block codes are BCH codes and a minimum distance decoder is assumed. For the convolutional code of Figures 21 and 22 the decoding strategy is assumed to be that of a Viterbi decoder. The orthogonal and biorthogonal systems of Figure 23 are based upon coherent matched filter detection and hence might apply to digitally coded systems, discrete PPM or banddividing FM such as described by Akima [18] (PFM as described by Rochelle [19]) if a coherent reference can be obtained.

In Figure 24 the performance of the various systems are plotted for a SNR_o of 40 dB. In the case of the discrete systems only the minimum value of β which can provide 40 dB SNR_o is shown. Other values of β are either larger and require higher SNR_i or cannot provide 40 dB SNR_o. Only the 63, 50 block code from Figure 18 is shown. As can be seen the bandspread region occuppied by MBM is not available from the common discrete transmission systems and the continuous systems capable of operating in that region are not very efficient in their use of these relatively small bandspread factors. As the bandspread ratio increases the straight-forward MBM system would appear to be unlikely to be competitive in performance with coded quadraphase but might still be a reasonable choice based upon equipment consideration.

SECTION VIII

PRACTICAL CONSIDERATIONS

Several practical considerations are worthy of discussion. The MBM system analysis was based upon use of cardinal functions for pulses with bandwidth $B_c = 1/(2T_o)$. To retain this bandwidth, but to transform this spectrum to a frequency appropriate for propagation it is necessary for either a single sideband technique or quadrature modulation (such as that used for the color subcarrier in color TV) be used.

If one takes into account the use of real pulse shapes rather than cardinal functions, the bandwidth required in the receiver preceeding the matched filter may be much wider than $1/T_o$. This implies that the spectral occupancy of the system by almost any definition is larger than the values of B_c given. The same fact, however, applies to the other systems discussed in Section VII. A thumb rule for a PSK system which uses square modulating pulses is that a filter of 4 times the bit rate should be used before the matched filter to avoid mismatch and crosstalk. The spectral occupany of FM by most definitions is wider than the IF bandwidth.

It is reasonable to argue that the crosstalk problem in MBM due to bandlimitting will be more severe than that of a binary transmission, since MBM may use pulses with more levels. For that reason pulse shaping of MBM pulses will almost certainly be a requirement and must be considered in the complexity tradeoff with binary (or quadraphase) systems where pulse shaping is rarely employed. However, it is felt that reasonable pulse shaping techniques can be employed which allow use of IF bandwidths and provide spectral occupancies which are closer to the values quoted than is the case for quadraphase and bi-phase systems where pulse shaping is not employed. Means for synchronization of the receiver demodulator and for providing timing for the matched filter and for demultiplexing in MBM have not been accounted for. Insertion of a reference pulse periodically could serve both of these functions plus the additional function of providing a reference for calibration, which would be likely to be required when large values of SNR_o are desired. In general, it is felt that MBM systems can be constructed such that the total implementation losses are held to 1 to 2 dB which is a reasonable estimate for implementation losses in the other systems over the theoretical performance curves given in Section VII.

SECTION IX

ACKNOWLEDGEMENT

The work discussed here was sponsored in its entirety by Radiation Incorporated. The author would like to recognize the contributions of Dr. C. J. Palermo and Mr. G. M. Pelchat who were of fundamental help in the original MBM concepts. Dr. C. A. Baird was of considerable assistance in interpreting the state variable type references given. Mrs. C. Manders performed the calculation associated with the optimum power split of MBM.

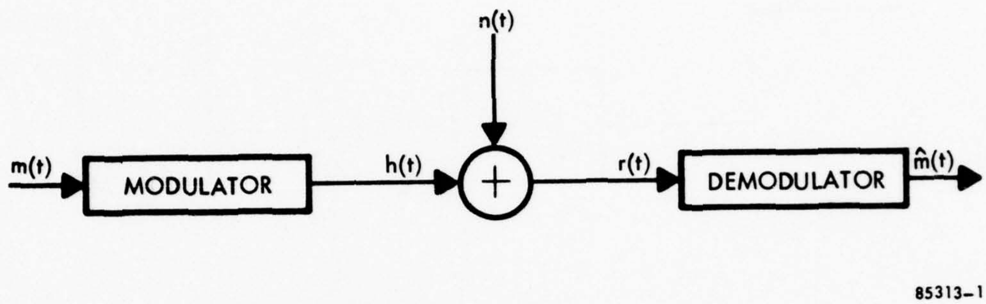

FIGURE 1 THE TRANSMISSION PROBLEM

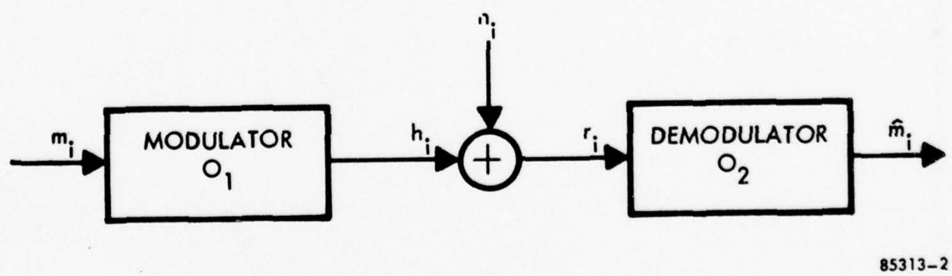

FIGURE 2 TIME SAMPLED TRANSMISSION PROBLEM

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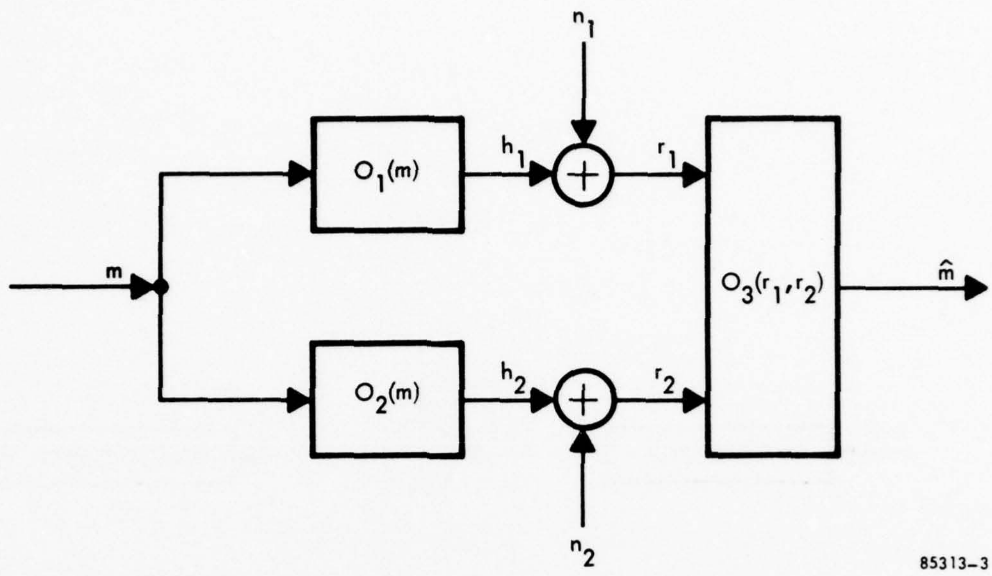

FIGURE 3 BANDSPREAD-BY-TWO MEMORYLESS MODULATOR - DEMODULATOR

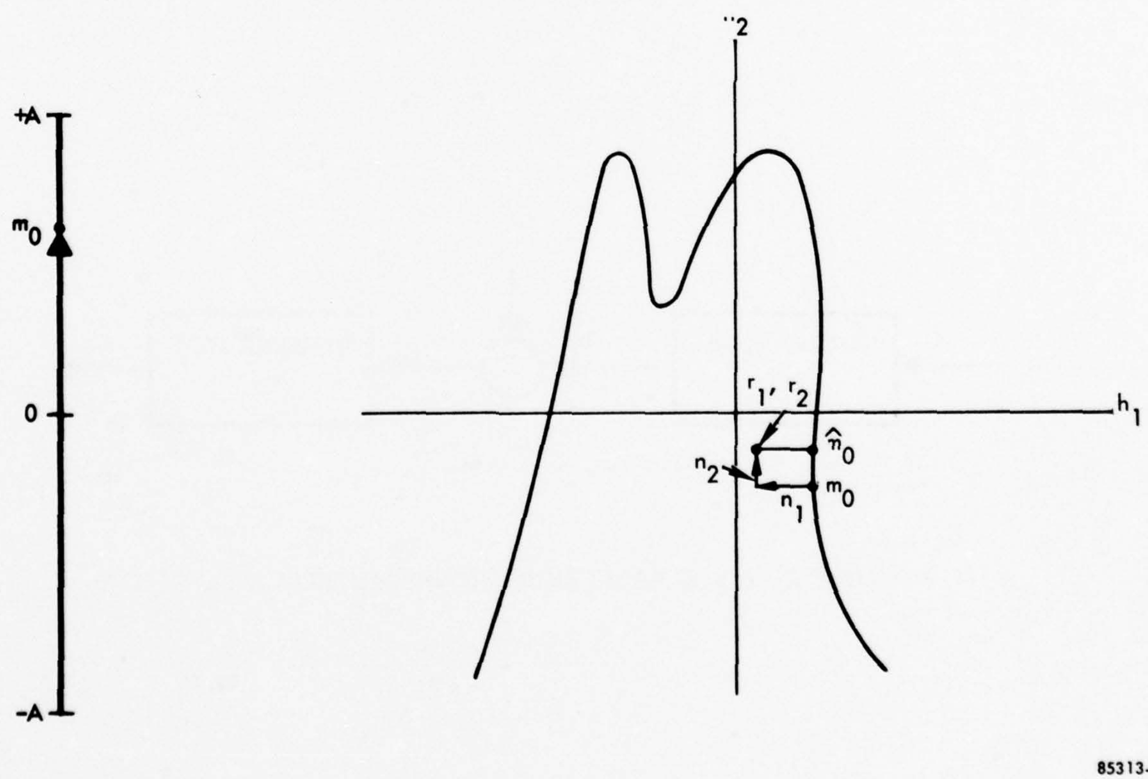

FIGURE 4 THE GEOMETRIC APPROACH

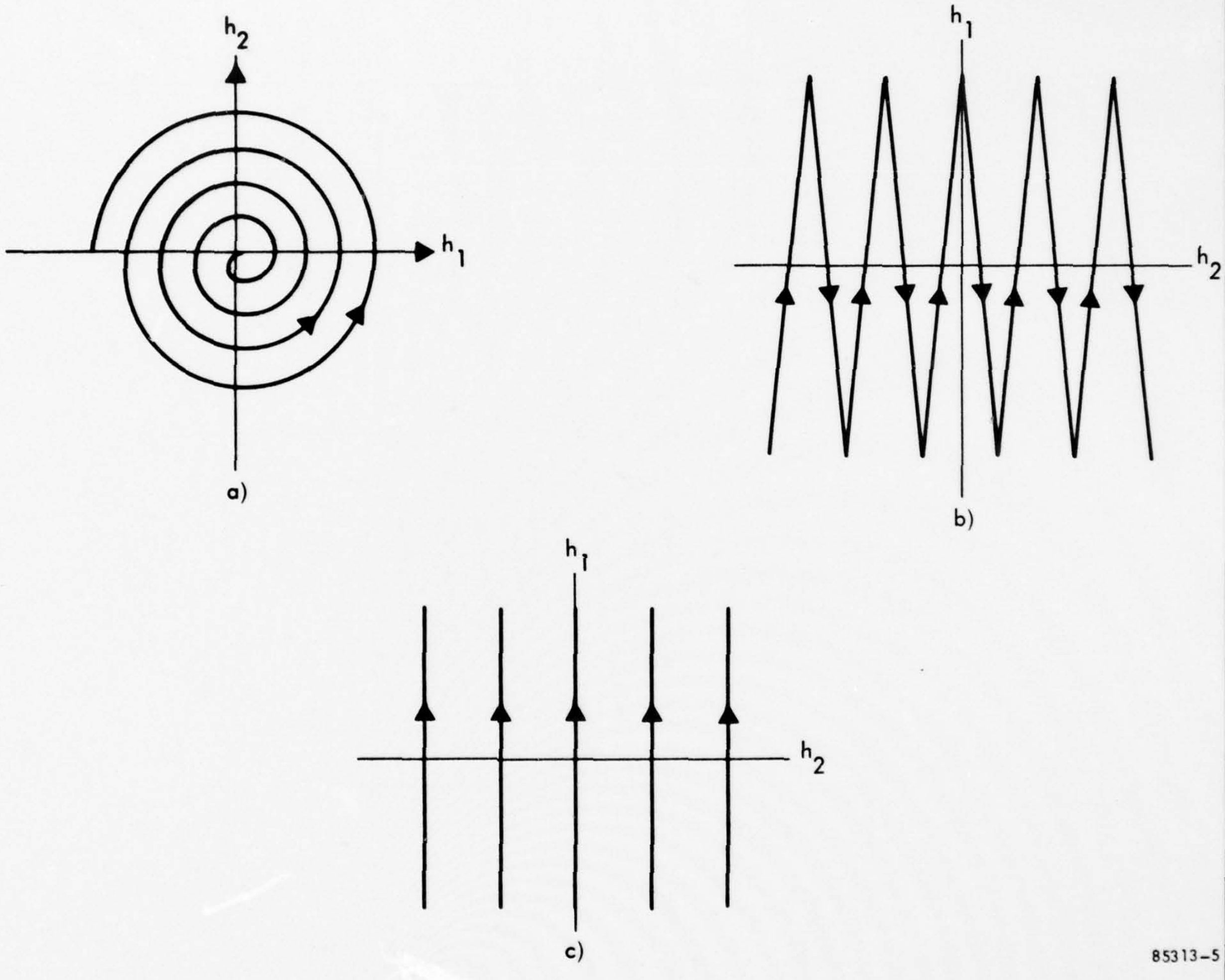

FIGURE 5 BANDSPREAD BY TWO MAPPINGS

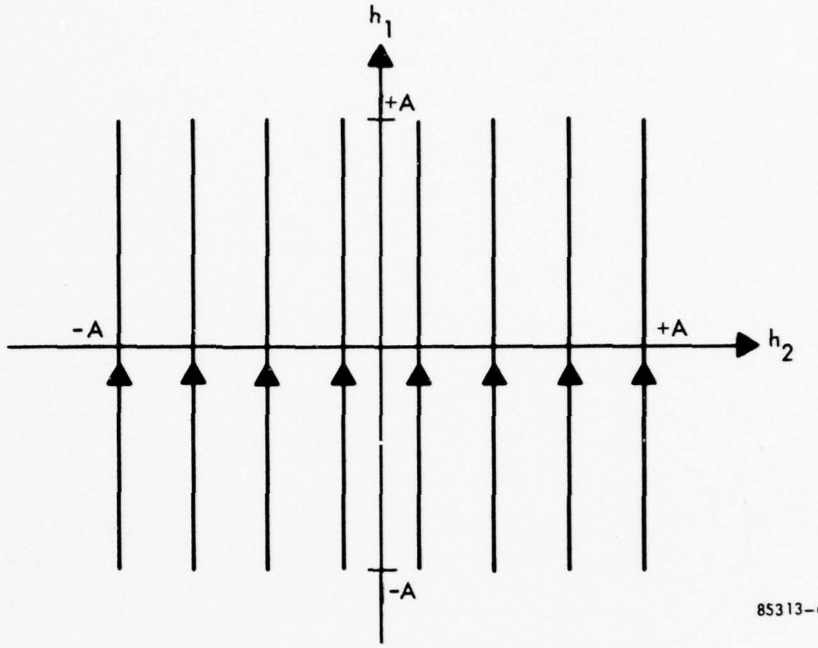

FIGURE 6 EIGHT-ARY MAPPING

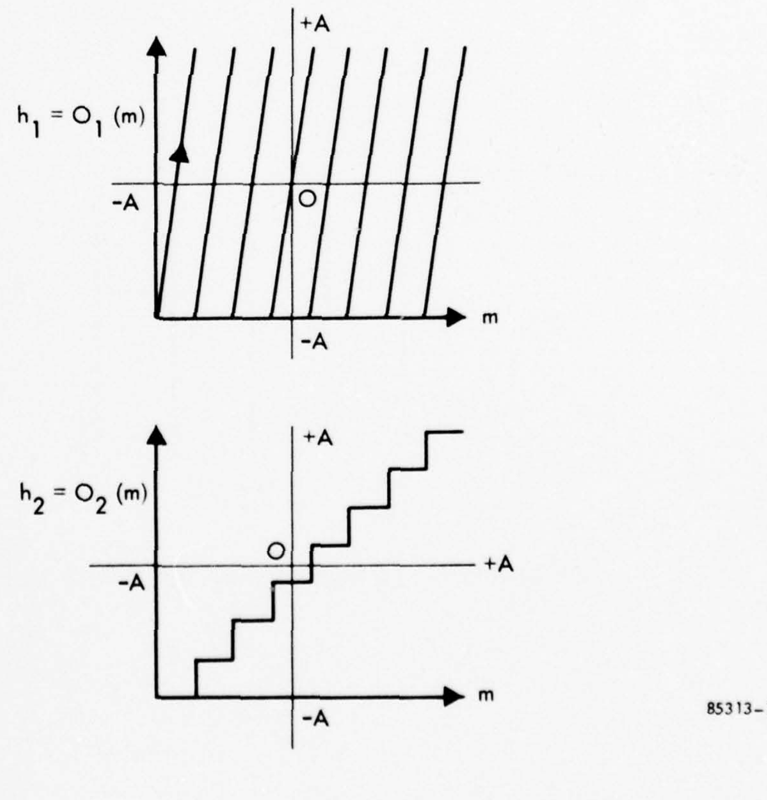

FIGURE 7 OPERATORS CORRESPONDING TO MAPPING OF FIGURE 6

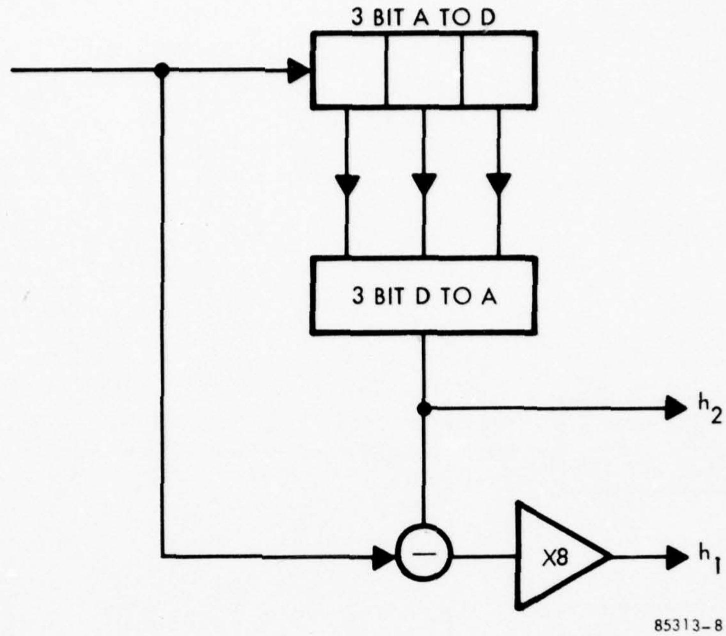

FIGURE 8 MODULATOR IMPLEMENTATION

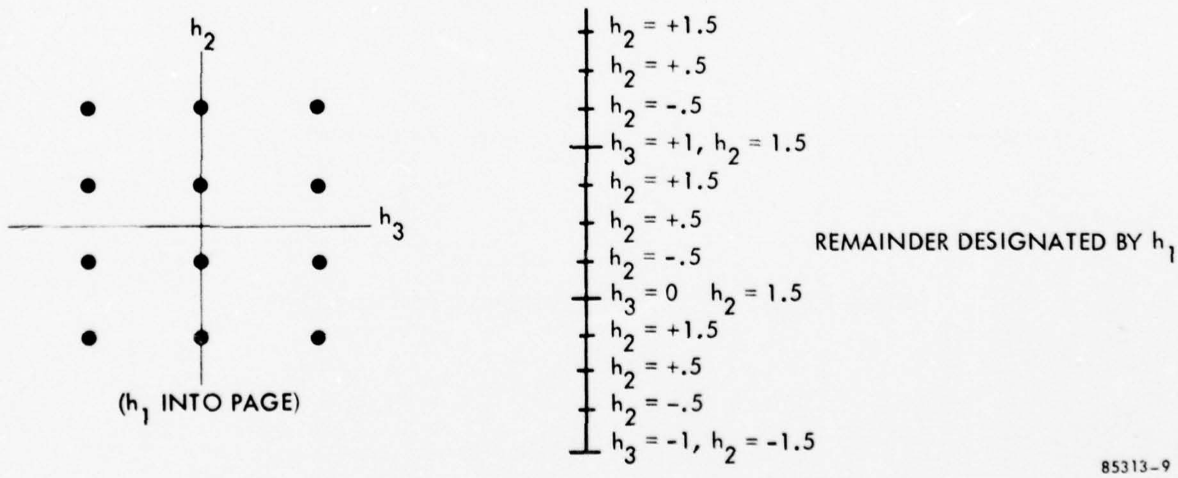

FIGURE 9 3 DIMENSIONAL MAPPING

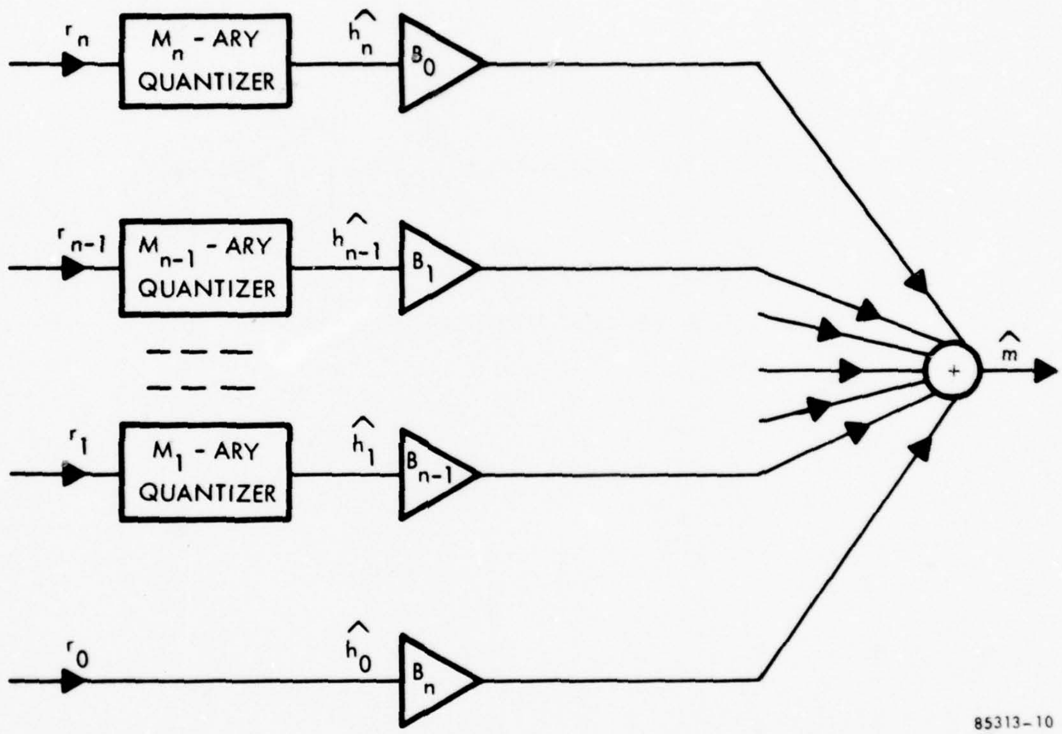

FIGURE 10 RECEIVING SYSTEM FOR MBM

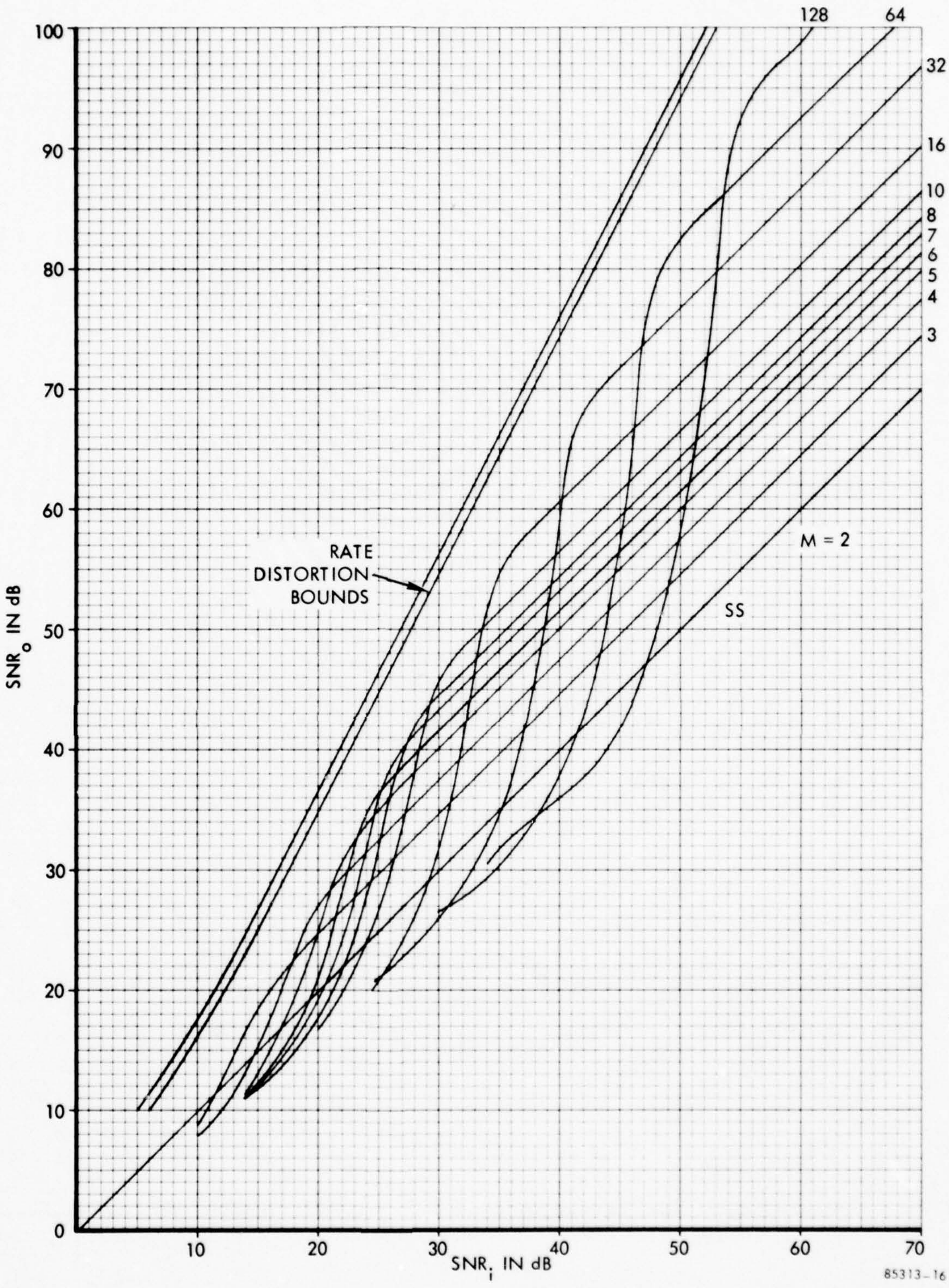

FIGURE 11 SNR_o VS. SNR_i FOR MBM $\beta = 2$
EQUAL PEAK POWER PULSES

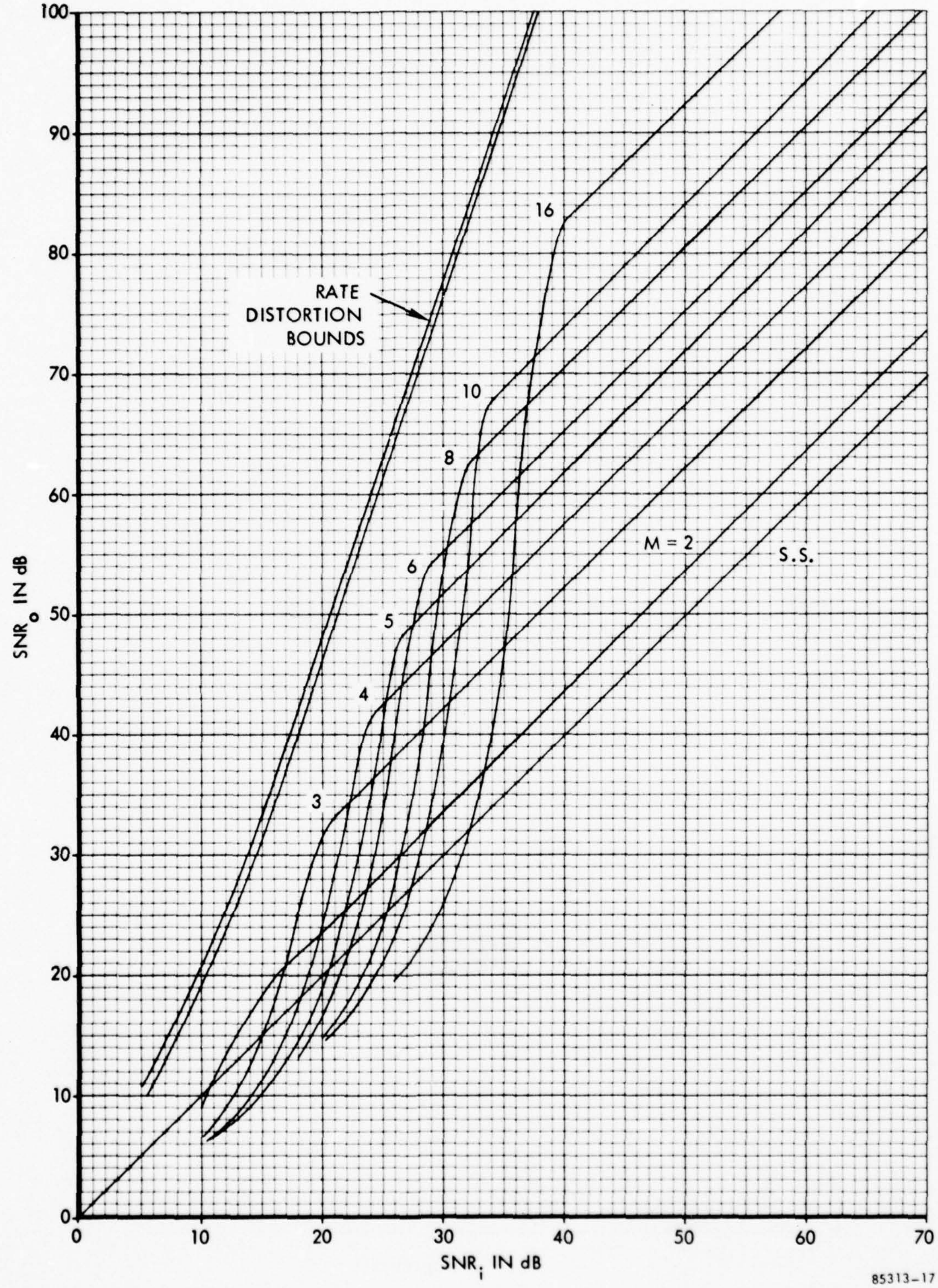

FIGURE 12 S/N$_o$ VS. S/N$_i$ FOR MBM β = 3
EQUAL PEAK POWER PULSES

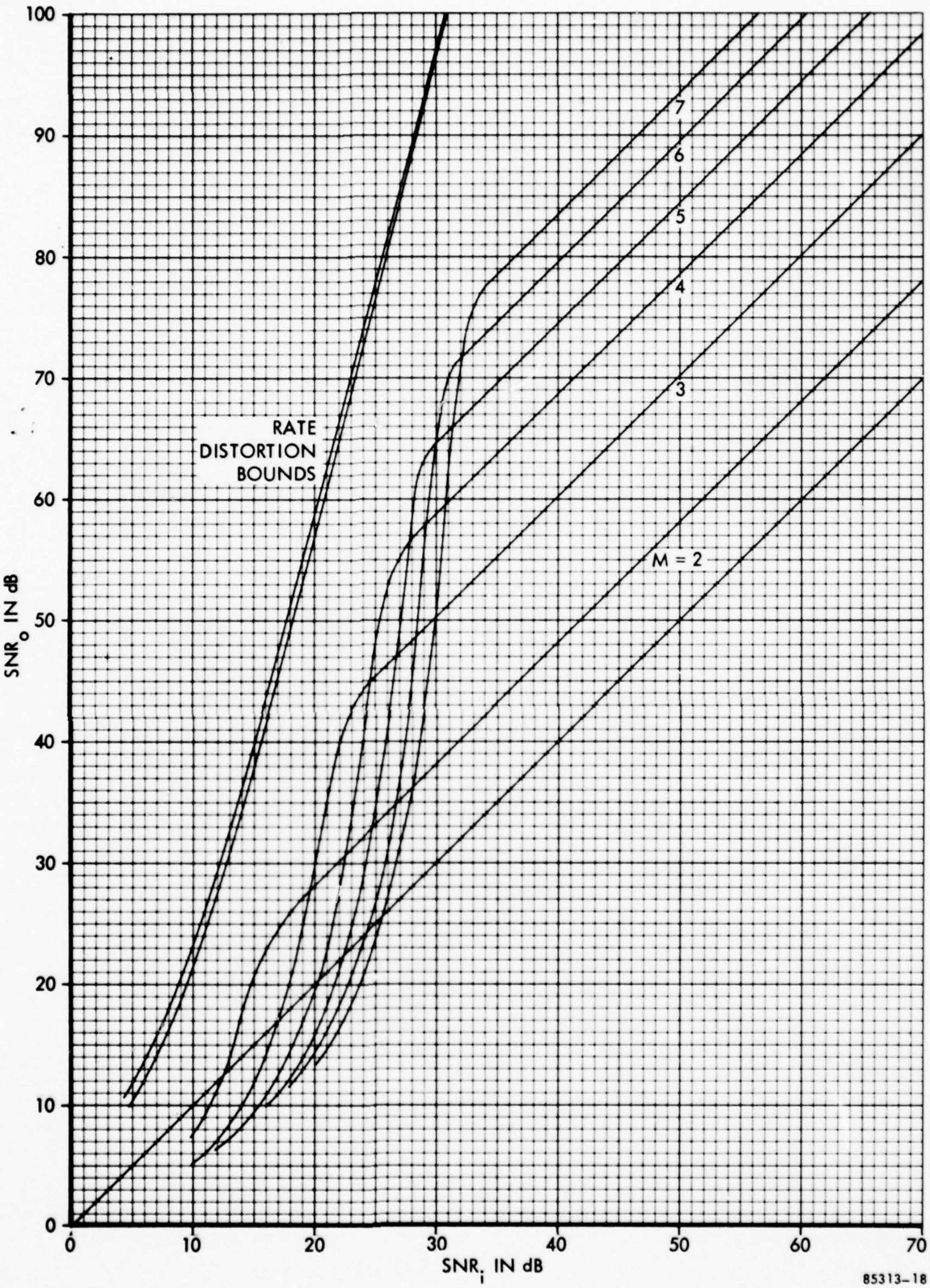

FIGURE 13 S/N$_o$ VS. S/N$_i$ FOR MBM β = 4
EQUAL PEAK POWER PULSES

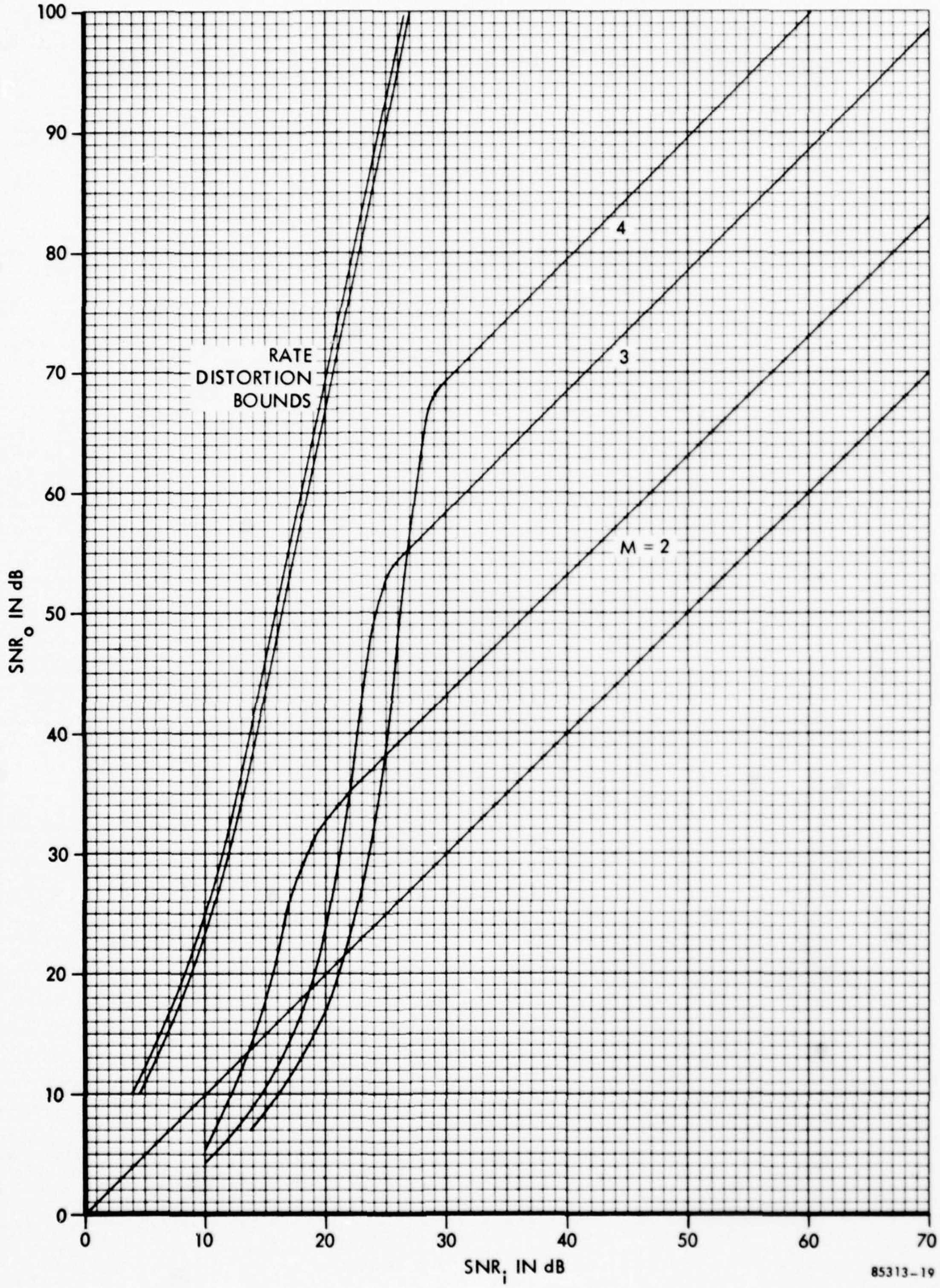

FIGURE 14 S/N_o VS. S/N_i FOR MBM $\beta = 5$
EQUAL PEAK POWER PULSES

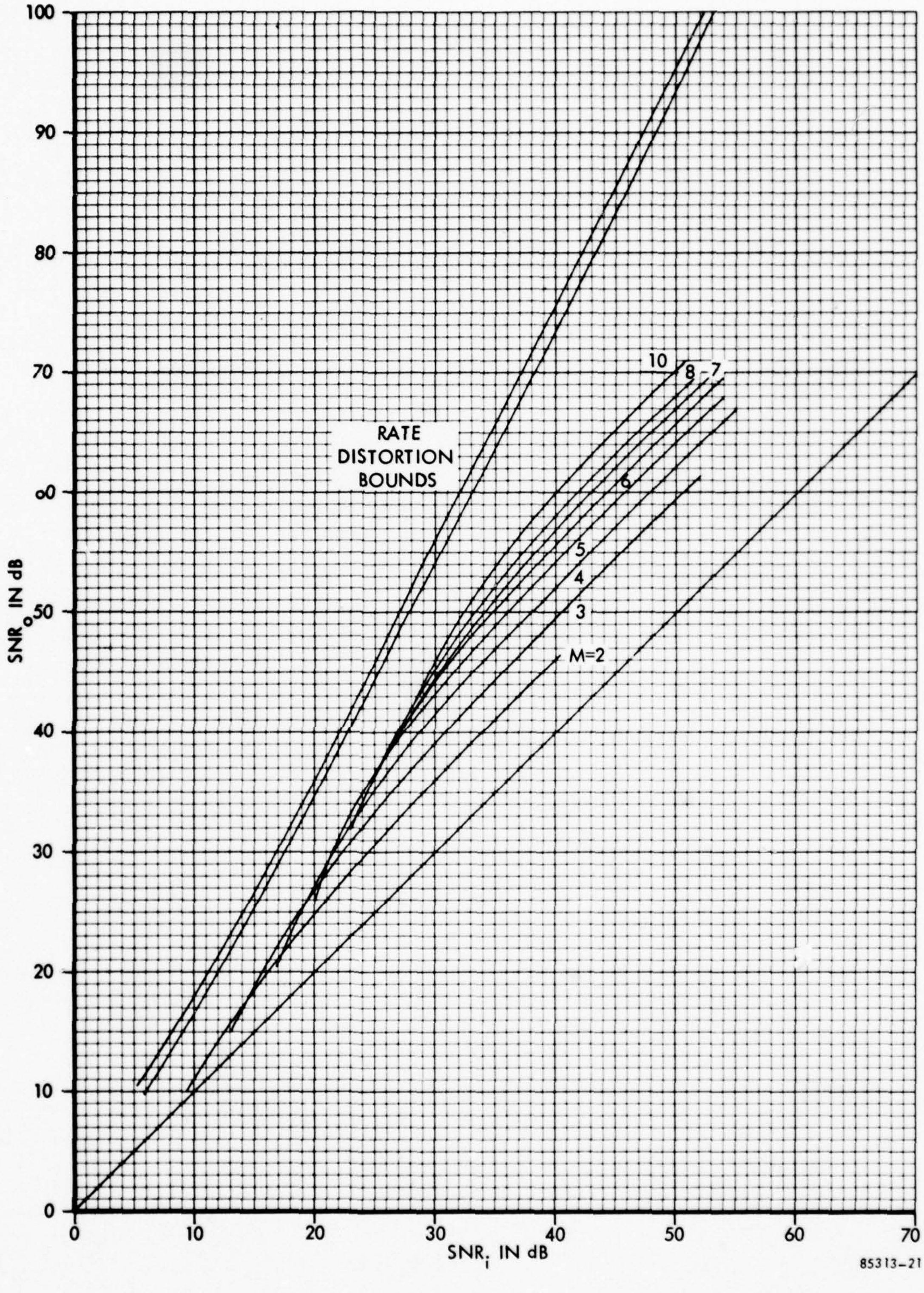

FIGURE 15 PERFORMANCE OF MBM - OPTIMIZED ENERGY RATIOS β = 2

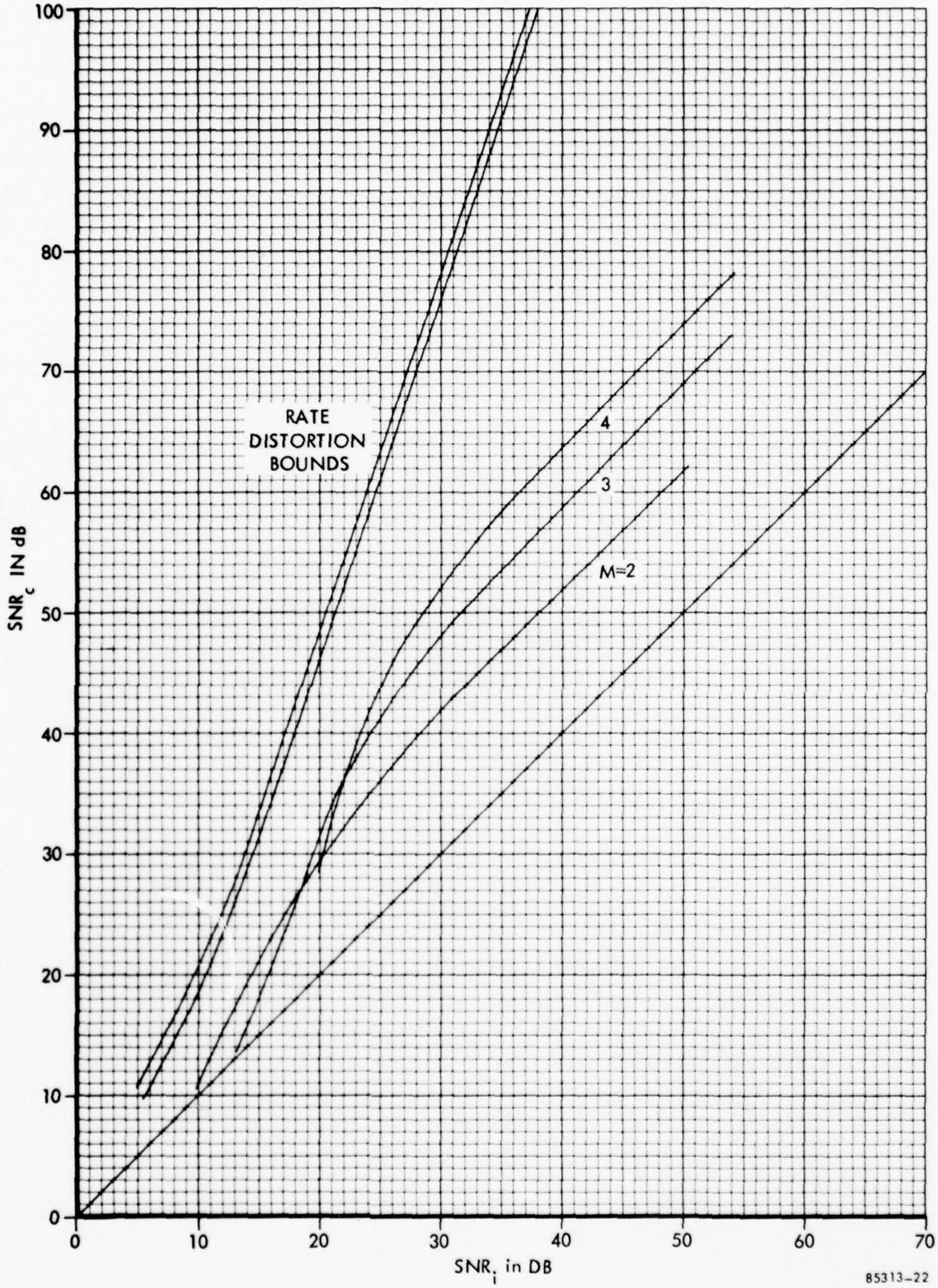

FIGURE 16 PERFORMANCE OF MBM - OPTIMIZED ENERGY RATIOS β = 3

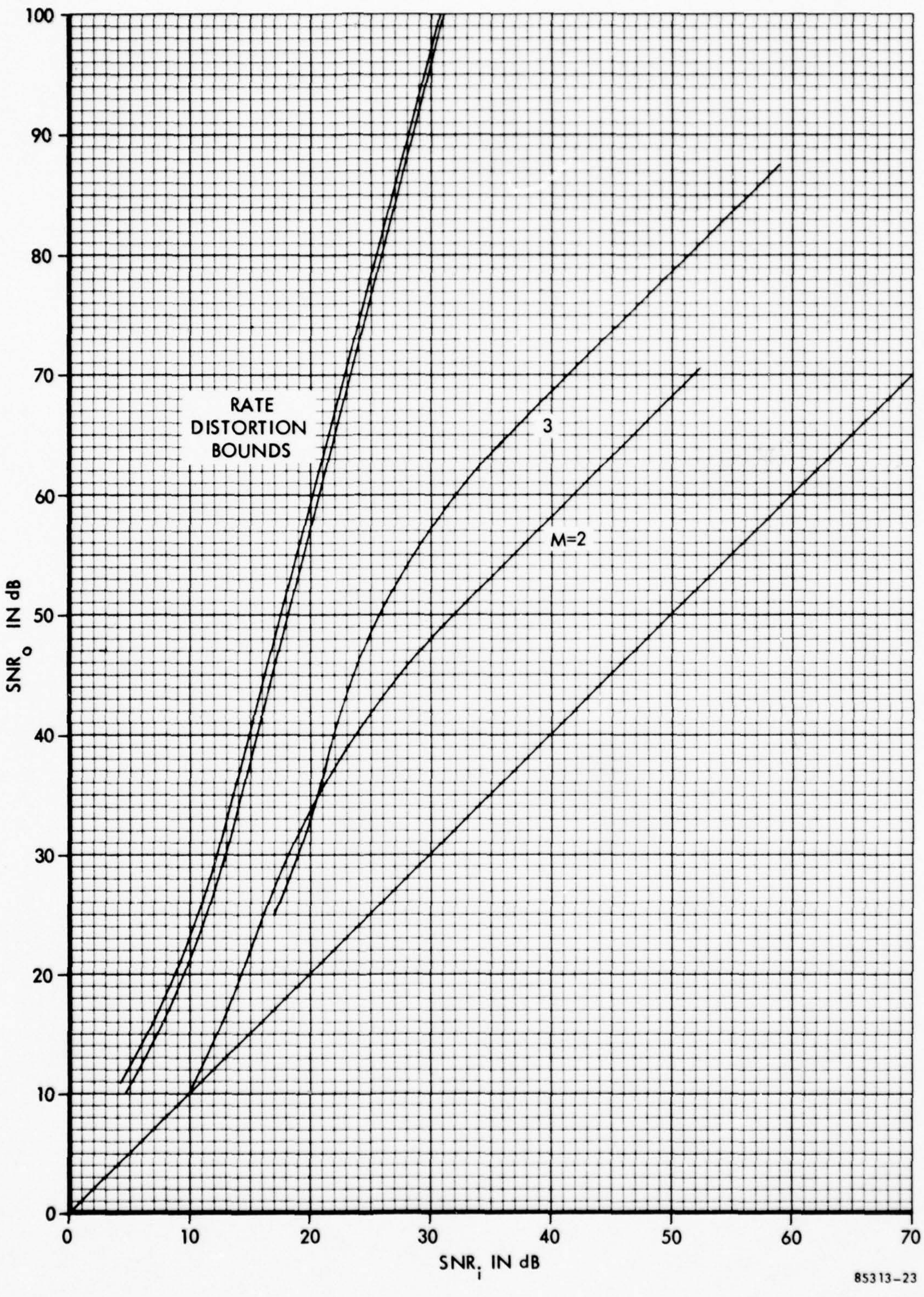

FIGURE 17 PERFORMANCE OF MBM - OPTIMIZED ENERGY RATIOS β = 4

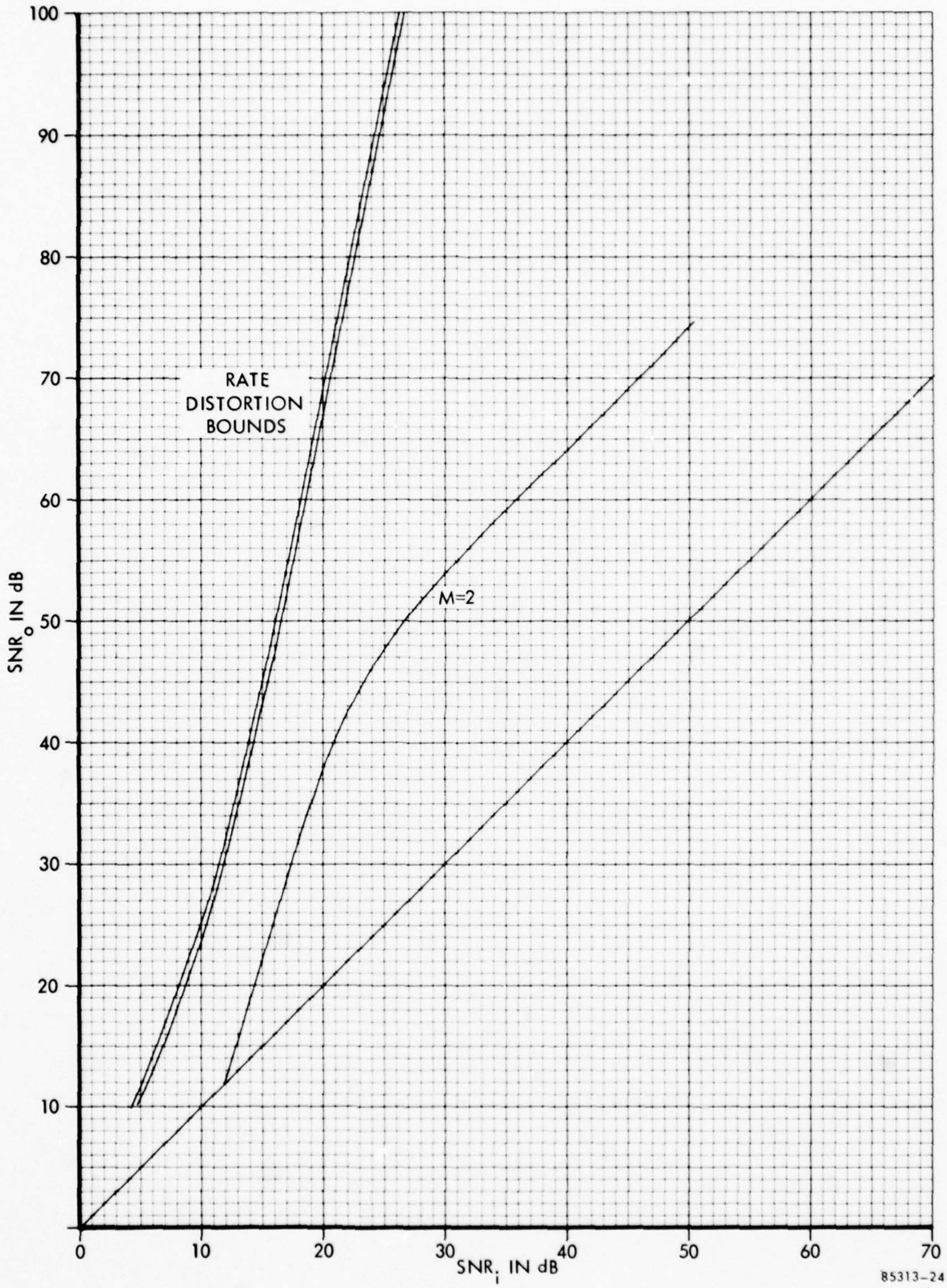

FIGURE 18 PERFORMANCE OF MBM - OPTIMIZED ENERGY RATIOS β = 5

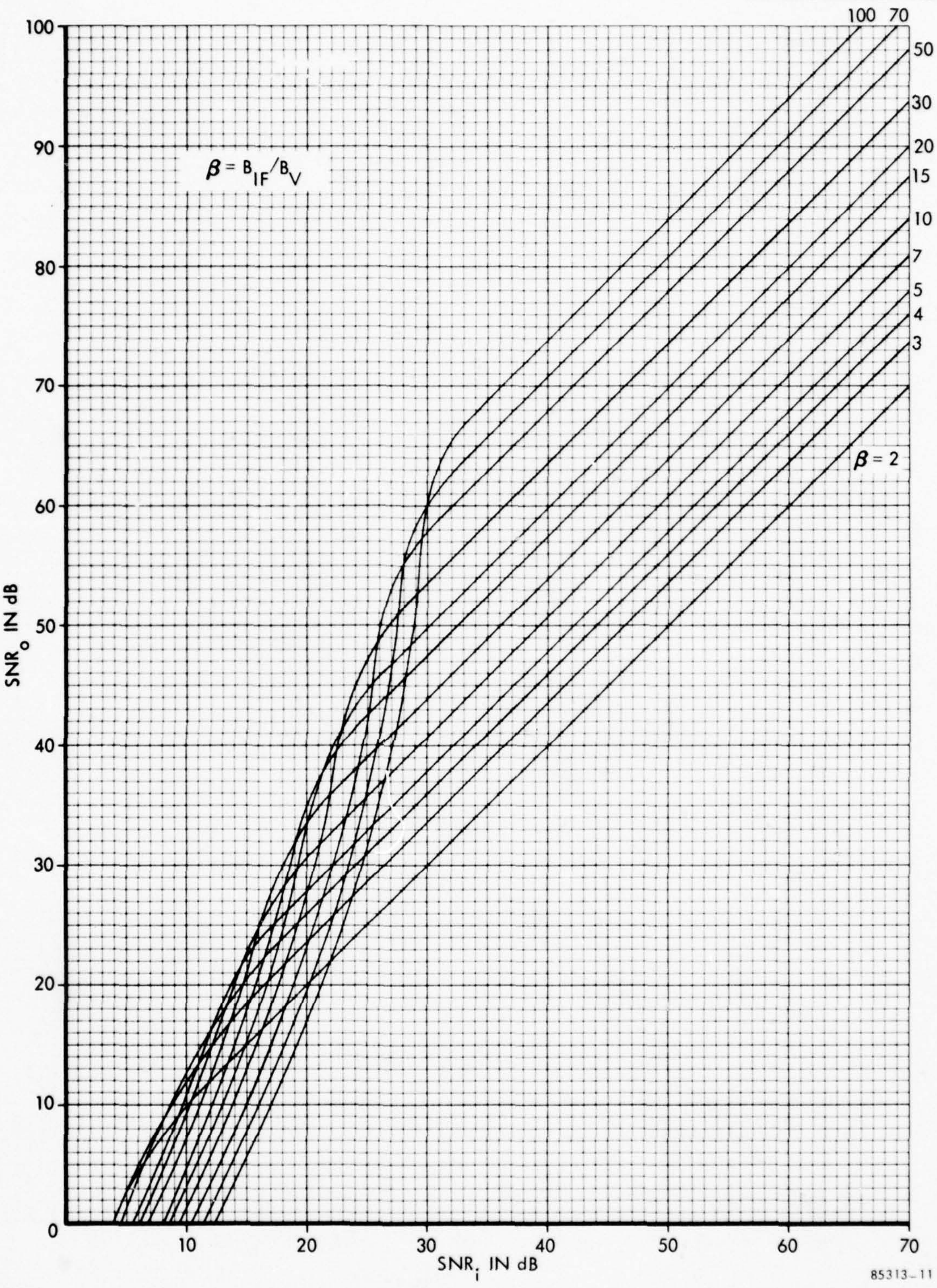

FIGURE 19 SNR$_o$ VS. SNR$_i$ FREQUENCY MODULATION

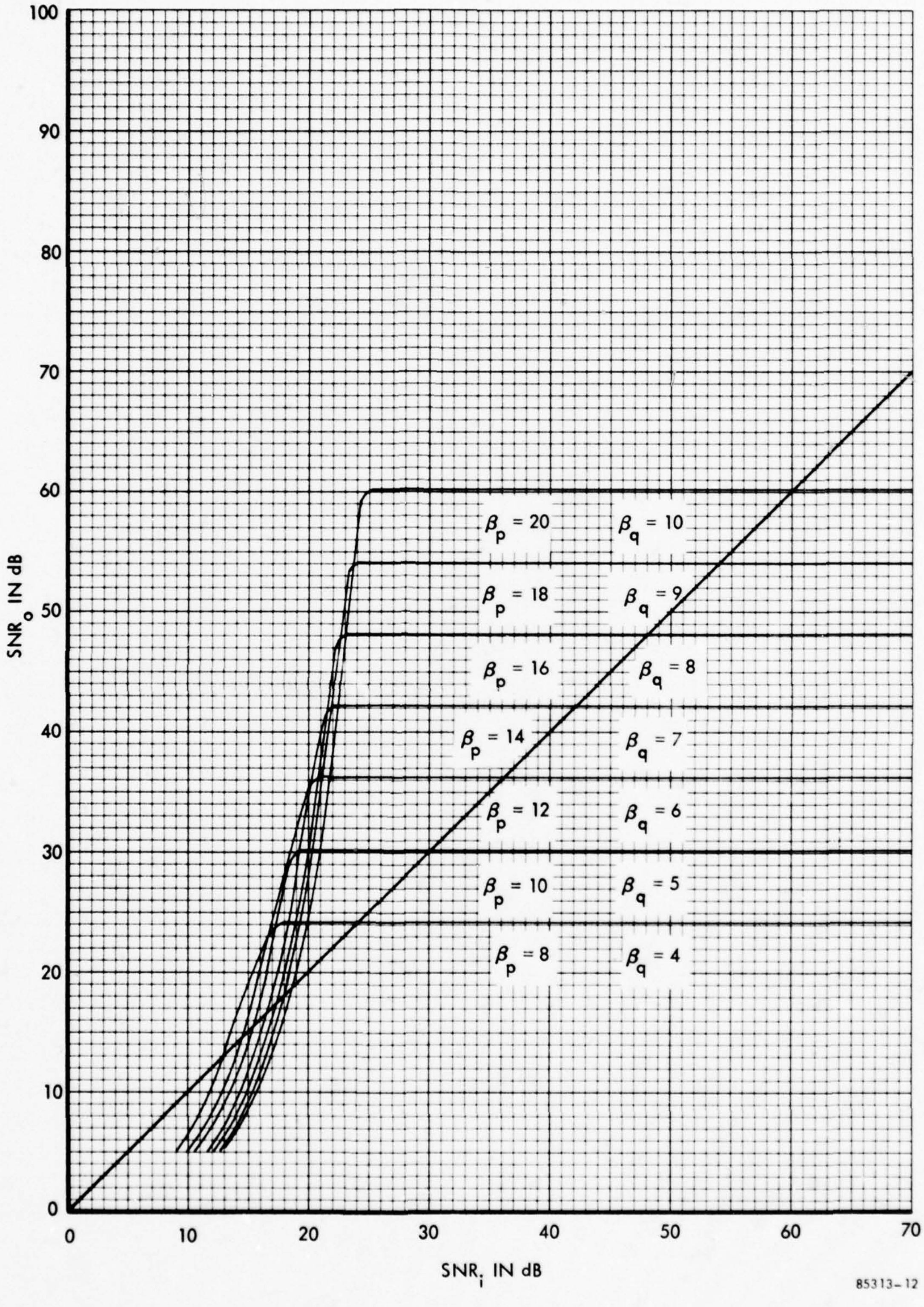

FIGURE 20 SNR$_o$ VS. SNR$_i$ FOR PSK AND QP

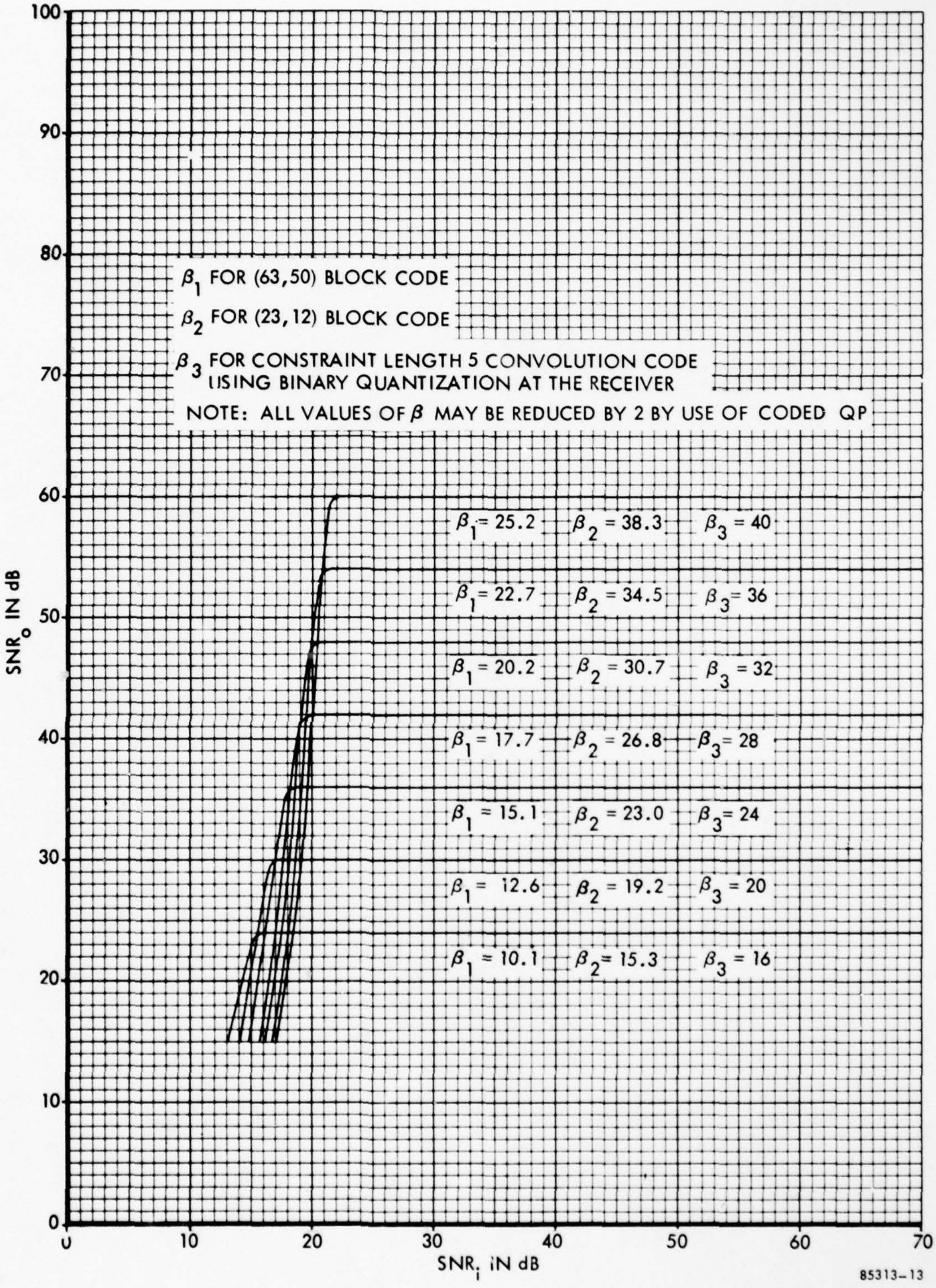

FIGURE 21 SNR_o VS. SNR_i FOR CODED PSK

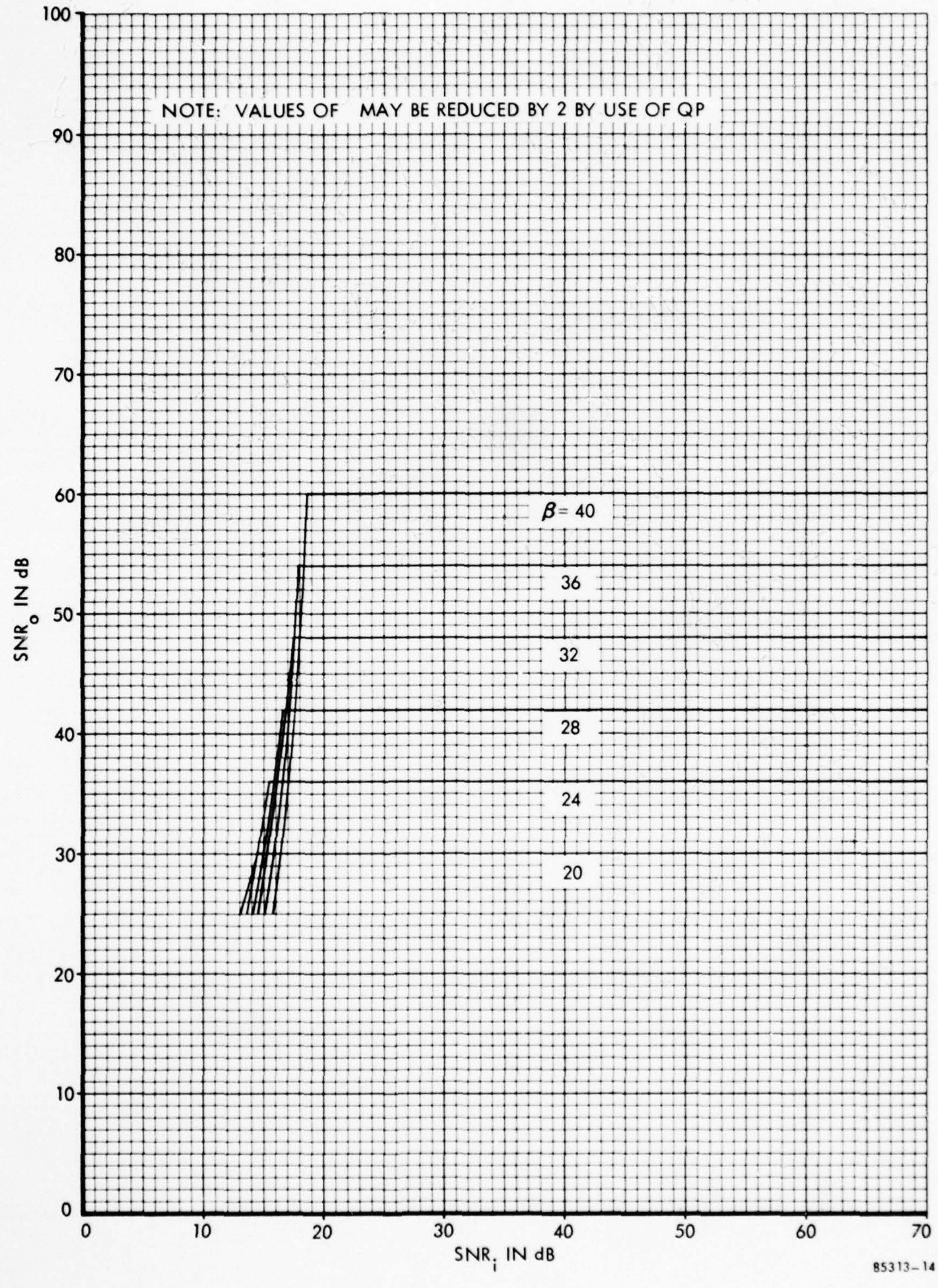

FIGURE 22 SNR_o VS. SNR_i FOR CODED PSK
CONSTRAINT LENGTH 6 CONVOLUTION CODE
USING 8 LEVEL QUANTIZATION AT RECEIVER

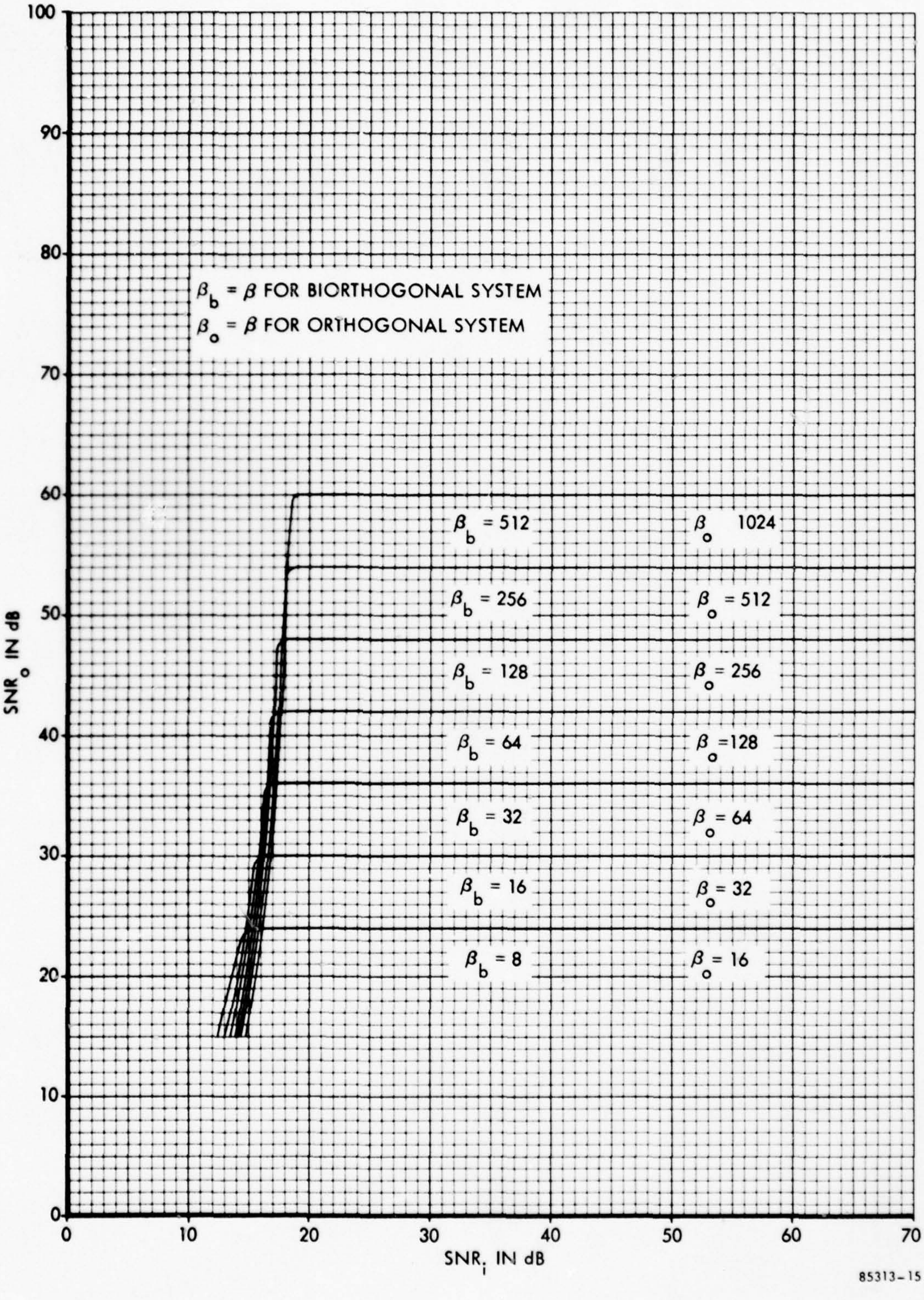

FIGURE 23 SNR_o VS. SNR_i FOR ORTHOGONAL AND BIORTHOGONAL SYSTEMS

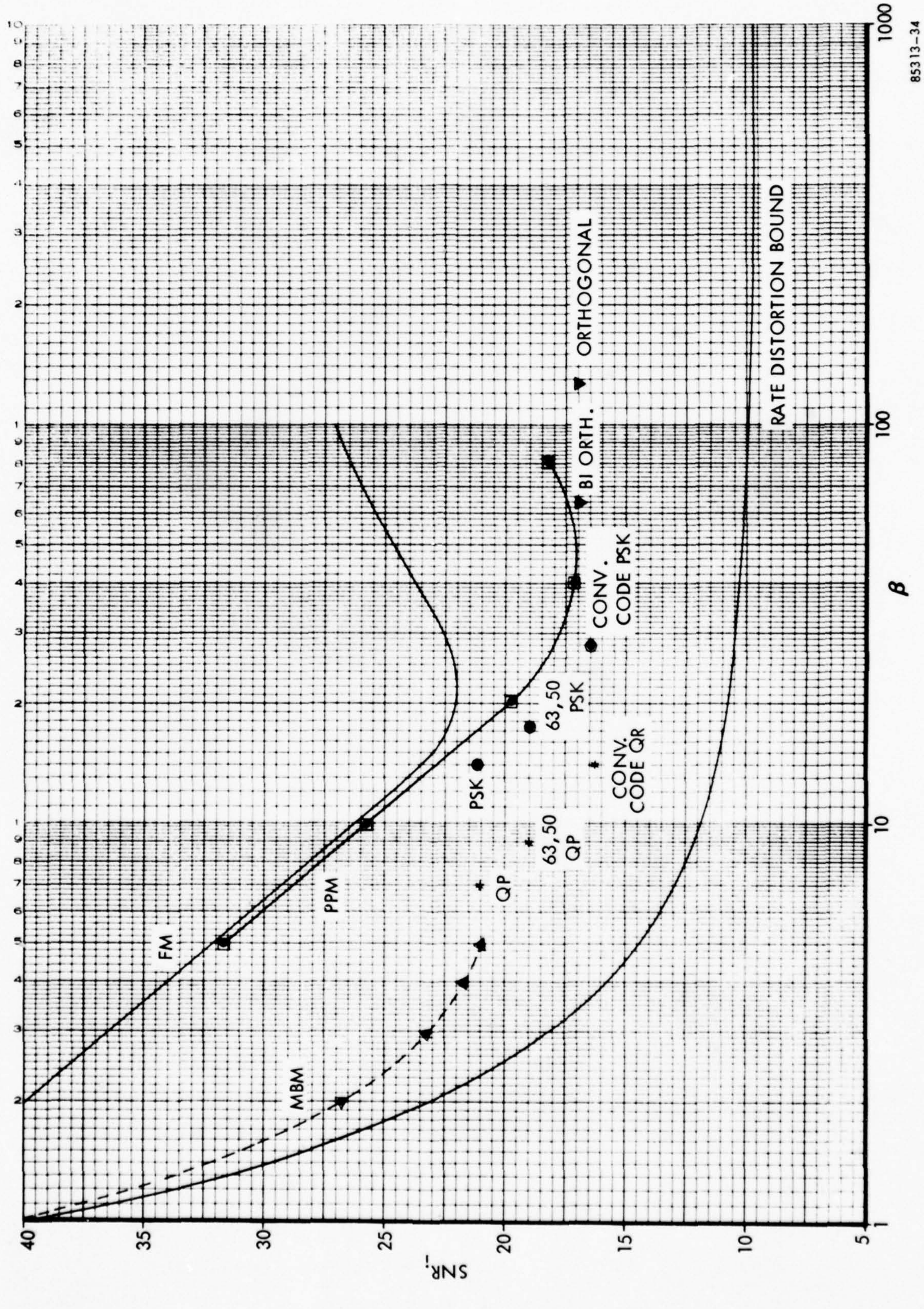

FIGURE 24 SYSTEM COMPARISON

APPENDIX I

DERIVATION OF THE OPTIMUM MEMORYLESS OPERATORS

Assume the two operators to be $0_1(m)$ and $0_2(r)$ as shown in Figure 2. The optimum choice of $0_2(r)$ is:

$$\hat{m} = 0_2(r) = E[m|r]$$

$$\hat{m} = \int m\, f(m|r)\, dm = \int \frac{m\, f(r|m)\, f(m)\, dm}{f(r)}$$

$$\hat{m} = \frac{\int m\, f(r|m)\, f(m)\, dm}{\int f(r|m)\, f(m)\, dm}$$

However, $f(r|m) = f_n[r - 0_1(m)] = \frac{e^{-[r-0_1(m)]^2/2\sigma_n^2}}{\sqrt{2\pi}\,\sigma_n}$

where σ_n^2 is the variance of the noise:

Thus, the optimum receiver for any $0_1(m)$ is one that chooses $\hat{m}$ to be:

$$\hat{m} = \frac{\int m\, e^{-[r-a(m)]^2/2\sigma_n^2}\, f(m)\, dm}{\int e^{-[r-0_1(m)]^2/2\sigma_n^2}} = \frac{N(r)}{D(r)} \qquad 1\text{-}1$$

For such a receiver the error is uncorrelated with the estimate. Thus:

$$E[\hat{m}(\hat{m}-m)] = 0$$

$$E[m\hat{m}] = E[\hat{m}^2]$$

and if $E[m^2] = 1$

$$\epsilon = [(\hat{m}-m)^2] = E[m^2] - E[\hat{m}^2]$$

Since $\hat{m}$ is dependent upon the random variable, r, the expected value of $\hat{m}^2$ is:

$$E[\hat{m}^2] = \int \hat{m}^2(r)\, f(r)\, dr = \int \frac{\left[\int m\, f(r|m)\, f(m)\, dm\right]^2 f(r)\, dr}{f^2(r)}$$

$$E[\hat{m}^2] = \int \frac{\left[\int m\, f(r|m)\, f(m)\, dm\right]^2}{\int f(r|m)\, f(m)\, dm}\, dr$$

Hence:

$$\epsilon = \int m^2\, f(m)\, dm - \frac{1}{\sqrt{2\pi}\,\sigma_n} \int \frac{N^2(r)}{D(r)}\, dr \qquad 1\text{-}2$$

By use of standard variational techniques under the constraint that:

$$P_s = \int 0_1^2(m)\, f(m)\, dm \qquad 1\text{-}3$$

we may determine that the optimum choice of $0_1(m)$ satisfies:

$$2\lambda\, 0_1(m) = \frac{1}{\sqrt{2\pi}\,\sigma_n^3} \int \frac{N(r)}{D(r)} [r - 0_1(m)] \left[2m - \frac{N(r)}{D(r)}\right] \cdot$$

$$e^{-[r-0_1(m)]^2/2\sigma_n^2}\, dr \qquad 1\text{-}4$$

where:

$$N(r) = \int m\, f(m)\, e^{-[r - 0_1(m)]^2 / 2\sigma_n^2}\, dm$$

$$D(r) = \int f(m)\, e^{-[r - 0_1(m)]^2 / 2\sigma_n^2}\, dm$$

and λ is a constant which is chosen to satisfy the power constraint of 1-3. If three memoryless operators are as illustrated in Figure 3, the optimum operator, $0_3(r_1, r_2)$ is:

$$\hat{m} = \frac{\int m\, f(r_1, r_2 | m)\, f(m)\, dm}{f(r_1, r_2)}$$

$$\hat{m} = \frac{\int m\, e^{([r_1 - 0_1(m)]^2 + [r_2 - 0_2(m)]^2)/2\sigma_n^2}\, f(m)\, dm}{\int e^{([r_1 - 0_1(m)]^2 + [r_2 - 0_2(m)]/2\sigma_n^2}\, f(m)\, dm} = \frac{N(r_1, r_2)}{D(r_1, r_2)} \qquad \text{1-5}$$

Using the same procedure as before:

$$\epsilon = E[(m - \hat{m})^2] = E[m^2] - E[\hat{m}^2]$$

$$\epsilon = \int m^2\, f(m)\, dm - \frac{1}{\sqrt{2\pi}\,\sigma_n} \int\int \frac{N^2(r_1, r_2)}{D(r_1, r_2)}\, dr_1\, dr_2 \qquad \text{1-6}$$

This procedure easily generalizes to N memoryless operators to yield:

$$\hat{m} = 0_3(r_1, r_2, --r_n) = \frac{N(r_1, r_2, --r_n)}{D(r_1, r_2, --r_n)} \qquad \text{1-7}$$

where:

$$N(r_1, r_2 --- r_n) = \int mf(m)\, e^{-\frac{1}{2\sigma_n^2}\left[\sum_{i-1}^{N} (r_1 - 0_i (m))^2\right]} dm$$

$$D(r_1, r_2 --- r_n) = \int f(m)\, e^{-\frac{1}{2\sigma_n^2}\left[\sum_{i=1}^{N} (r_1 - 0_i(m))^2\right]} dm$$

APPENDIX 2

DERIVATION OF SNR_o EQUATION FOR ONE DISCRETE PULSE

Let us analyze the output signal-to-noise ratio of an MBM system with bandspread by two and mapping as shown in Figure 6. Assume that the data is uniformly distributed between $\pm$ A and that this range is mapped into M lines. The two dimensions are transmitted as two pulses, the first with uniform distribution over a range $\pm a_o A$ and width T_o and the second an M-ary pulse with M equally likely levels ranging from $\pm a_1 A$ and width T_1.

The receiver is assumed to make a hard decision with the second received pulse as to the line corresponding to the second transmitted pulse and then accept the first pulse directly. If the received value of the first pulse is h_o and the discrete decision value of the second pulse is k, then the estimated transmitted value, $\hat{m}$, is

$$\hat{m} = k\left(\frac{2A}{M}\right) + \frac{1}{a_o}\left[\frac{h_o}{M}\right] - A \qquad 0 \le k \le M-1 \qquad \text{2-1}$$

If the actual transmitted value was m:

$$m = j\left(\frac{2A}{M}\right) + m_o - A \qquad \text{2-2}$$

then the error, ϵ, is:

$$\epsilon = (m-\hat{m}) = (k-j)\left(\frac{2A}{M}\right) + \left(\frac{h_o}{a_o M} - m_o\right) \qquad \text{2-3}$$

However,

$$h_o = a_o M m_o + n_o \qquad \text{2-4}$$

where n_o is the noise associated with the continuous pulse.

Thus:

$$\epsilon = (k-j)\left(\frac{2A}{M}\right) + \left(\frac{a_o M m_o + n_o}{a_o M} - m_o\right)$$

$$\epsilon = (k-j)\left(\frac{2A}{M}\right) + \frac{n_o}{a_o M} \qquad 2\text{-}5$$

The two error terms are caused by the additive noise on the two channels and are independent. Hence:

$$E[\epsilon^2] = E[(k-j)^2]\left(\frac{2A}{M}\right)^2 + \frac{\sigma_{no}^2}{a_o^2 M^2} \qquad 2\text{-}6$$

To evaluate the variance of the error in the discrete pulse it is necessary to evaluate $E[(k-j)^2]$

$$E[(k-j)^2] = \sum_{j=1}^{M} \sum_{k=1}^{M} (j-k)^2 \, P(j|k)\, P(k)$$

$$E[(k-j)^2] = \frac{1}{M} \sum_{j=1}^{M} \sum_{k=1}^{M} (j-k)^2 \, P(j|k) \qquad 2\text{-}7$$

From Figure 7 we can deduce:

$$P(1|k) = Q\left\{\frac{(|1-k| - \frac{1}{2})\, 2Aa_1}{\sigma_{n1}(M-1)}\right\} \qquad j = 1$$

$$P(M|k) = Q\left\{\frac{(|M-k| - \frac{1}{2})\, 2Aa_1}{\sigma_{n1}(M-1)}\right\} \qquad j = M$$

$$P(j|k) = Q\left\{\frac{(|j-k|-\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\} - Q\left\{\frac{(|j-k|+\frac{1}{2})Aa_1}{\sigma_{n1}(M-1)}\right\} \qquad 2 \le j \le M-1 \tag{2-8}$$

where

$$Q(x) = \int_x^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$ and σ_{n1} is the standard deviation of the noise on the discrete pulse.

Since $E[(k-j)^2]$ is a function of the absolute value of j-k only:

$$E[(k-j)^2] = \frac{2}{M}\sum_{j=2}^{M}\sum_{k=1}^{j-1}(j-k)^2\, P(j|k)$$

We have dropped the terms where $j = k$.

$$E[k-j)^2] = \frac{2}{M}\sum_{k=1}^{M-1}(M-k)^2\quad Q\left\{\frac{(M-k-\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}$$

$$+\frac{2}{M}\sum_{j=2}^{M-1}\sum_{k=1}^{j-1}(j-k)^2\left[Q\left\{\frac{(j-k-\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}\right.$$

$$\left. - Q\left\{\frac{(j-k+\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}\right] \tag{2-9}$$

If $M-k = m$ in the first summation and $j-k = m$ in the second:

$$E[(k-j)^2] = \frac{2}{M}\sum_{m=1}^{M-1} m^2 \quad Q\left\{\frac{(m-\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}$$

$$+\frac{2}{M}\sum_{m=1}^{M-1}\sum_{k=1}^{M-1-m} m^2\left[Q\left\{\frac{(m-\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}\right.$$

$$\left.-Q\left\{\frac{(m+\frac{1}{2})2Aa_1}{\sigma_{n1}(M-1)}\right\}\right]$$

$$E[(k-j)^2] = \frac{2}{M}\sum_{m=1}^{M-1} m^2 \quad Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n1}(M-1)}\right\}$$

$$+\frac{2}{M}\sum_{m=1}^{M-2}(M-1-m)\,m^2\left[Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n1}(M-1)}\right\}\right]$$

$$-\frac{2}{M}\sum_{m=1}^{M-2}(M-1-m)\,m^2\left[Q\left\{\frac{(2m+1)Aa_1}{\sigma_{n1}(M-1)}\right\}\right]$$

Letting m = n-1 the last summation and:

$$Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n1}(M-1)}\right\} = Q_m$$

$$E[(k-j)^2] = \frac{2}{M}\left\{\sum_{m=1}^{M-1} m^2 Q_m + \sum_{m=1}^{M-2}(M-1-m)m^2\, Q_m\right.$$

$$\left.-\sum_{n=2}^{M-1}(M-n)(n-1)^2\, Q_n\right\}$$

$$= \frac{2}{M} \Big\{ Q_1 + (M-1)^2 Q_{M-1} + (M-2)Q_1 - (M-2)^2 Q_{M-1}$$

$$+ \sum_{m=2}^{M-2} \left[m^2 + (M-1-m)^2 - (M-m)(m-1)^2\right] Q_n \Big\}$$

$$= \frac{2}{M} \Big\{ (M-1)Q_1 + (2M-3)Q_{M-1} + \sum_{m=2}^{M-2} (2m-1)(M-m)Q_m \Big\}$$

$$E[(k-j)^2] = \frac{2}{M} \sum_{m=1}^{M-1} (2m-1)(N-m)\, Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n_1}(M-1)}\right\} \qquad \text{2-10}$$

Thus the mean square error is:

$$E[\epsilon^2] = \frac{\sigma_{no}^2}{a_o^2 M^2} + \frac{8A^2}{M^3} \sum_{m=1}^{M-1} (2m-1)(M-m)\, Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n_1}(M-1)}\right\}$$

The signal component of $\hat{m}$ is equal to m. Hence the mean square output signal to noise ratio is:

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{A^2}{3E[\epsilon]} = \frac{1}{3\sigma_{no}^2 / a_o^2 M^2 A^2 + 24/M^3 \sum_{m=1}^{M-1} (2m-1)(M-m)\, Q\left\{\frac{(2m-1)Aa_1}{\sigma_{n_1}(M-1)}\right\}} \qquad \text{2-11}$$

The power in the first pulse is:

$$P_o = \frac{a_o^2 A^2}{3}$$

and the energy is:

$$E_o = \frac{a_o A^2 T_o}{3}$$

The variance of the noise adding to the first pulse is:

$$\sigma_{no}^2 = \frac{N_o}{2T_o}$$

The variance of the second pulse is:

$$P_1 = \frac{1}{M} \sum_{k=o}^{M-1} k^2 \left(\frac{2Aa_1}{M-1}\right) - (Aa_1)^2$$

$$P_1 = \frac{4A^2 a_1^2}{M(M-1)} \left[\frac{(M-1)M(2M-1)}{6}\right] - (Aa_1)^2$$

$$P_1 = \frac{2A^2 a_1^2 (2M-1)}{3(M-1)} - A^2 a_1^2$$

$$P_1 = \frac{a_1^2 A^2}{3} \left(\frac{M+1}{M-1}\right) \qquad 2\text{-}12$$

The energy is:

$$E_1 = P_1 T_1 = \frac{a_1^2 A^2 T_1}{3} \left(\frac{M+1}{M-1}\right) \qquad 2\text{-}13$$

The variance of the noise in the second pulse is:

$$\sigma_{n1}^2 = \frac{N_o}{2T_1}$$

The energy in the total signal is:

$$E_s = E_o + E_1 = \frac{a_1^2 A^2 T_1}{3} \left(\frac{M+1}{M-1}\right) + \frac{a_o^2 A^2 T_o}{3}$$

$$E_s = \frac{A^2}{3} \left[a_1^2 T_1 \left(\frac{M+1}{M-1}\right) + a_o^2 T_o\right] \qquad 2\text{-}14$$

If we define:

$$R_o = \frac{E_o}{E_s} = \frac{a_o^2 T_o}{a_1^2 T_1 \left(\frac{M+1}{M-1}\right) + a_o^2 T_o} \qquad 2\text{-}15$$

and

$$R_1 = \frac{E_1}{E_s} = \frac{a_1^2 T_1 \left(\frac{M+1}{M-1}\right)}{a_1^2 T_1 \left(\frac{M+1}{M-1}\right) + a_o^2 T_o} \qquad 2\text{-}16$$

then:

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{1}{1/M^2 \left(\frac{2E_o}{N_o}\right) + 24/M^3 \sum_{m=1}^{M-1} (2m-1)(M-m)Q\left\{(2m-1)\left[\frac{6E_1}{N_o}\left(\frac{1}{M^2-1}\right)\right]^{\frac{1}{2}}\right\}}$$

if $2\,E_s/N_o = Y$

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{M^2 R_o Y}{1 + \frac{24 R_o Y}{M} \sum_{m=1}^{M-1} (2m-1)(M-m)Q\left\{(2m-1)(3R_1\,Y/M^2-1)^{\frac{1}{2}}\right\}} \qquad 2\text{-}17$$

The case where multiple discrete pulses are utilized may be analyzed in a similar manner. Let us assume that J discrete pulses are transmitted with M_j levels for the j^{th} such pulse. The peak values of the j^{th} pulse are $\pm Aa_j$. As before the peak values of the continuous pulse is Aa_o. The first discrete pulse is used to divide the total range into M_1 segments; the second divides each of the M_1 segments into M_2 smaller segments, etc. If, as before, decisions are made in the usual manner on the discrete pulses, the error at the receiver is:

$$\epsilon = \frac{n_o}{a_o \prod_{j=1}^{J} M_j} + \frac{2A}{M_1}(k_1 - j_1) + \frac{2A}{M_1 M_2} + \cdots \frac{2A}{\prod_{j=1}^{J} M_j}(k_J - j_J)$$

$$\epsilon = \frac{n_o}{a_o \prod_{j=1}^{J} M_j} + \sum_{n=1}^{J} \frac{2A}{\prod_{j=1}^{J} M_j}(k_n - j_n) \quad \text{2-18}$$

where as before the j^{th} level of the pulse was transmitted and the k^{th} level received. The mean square error is:

$$E[\epsilon^2] = \frac{\sigma_{no}^2}{a_o^2 [\prod_{j=1}^{J} M_j]^2} + 4A^2 \sum_{n=1}^{J} \frac{E[(k_n - j_n)^2]^2}{(\prod_{j=1}^{n} M_j)^2} \quad \text{2-19}$$

Using equation 2-10 to express the $E[(k_n - j_n)^2]$ *we have:*

$$E[\epsilon^2] = \frac{\sigma_{no}^2}{a_o^2 [\prod_{j=1}^{J} M_j]^2} + 8A^2 \sum_{n=1}^{J} 1/M_n (\prod_{j=1}^{n} M_j)^2 \sum_{m-1}^{M_n - 1} (2m-1)(M_n - m) Q\left\{\frac{(2m-1)Aa_n}{\sigma_{nj}(M_n - 1)}\right\} \quad \text{2-20}$$

The signal to noise output is then:

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{1}{3\sigma_{no}^2 / A^2 a_o^2 [\prod_{j=1}^{J} M_j]^2 + 24 \sum_{n=1}^{J} (1/M_n (\prod_{j=1}^{n} M_j)^2) \sum_{m=1}^{M_n - 1} (2m-1)(M_n - m) Q_{mn}}$$

where: $$Q_{mn} = Q\left\{\frac{(2m-1)(Aa_n}{\sigma_{nj}(M_n - 1)}\right\} \quad \text{2-21}$$

As before the energy in the continuous pulse is:

$$E_o = \frac{a_o^2 A^2 T_o}{3}$$

and the j^{th} discrete pulse is:

$$E_j = \frac{a_j^2 A^2 T_j}{3} \left(\frac{M_j + 1}{M_j - 1} \right)$$

The total energy is:

$$E_s = \frac{A^2}{3} \left[a_o^2 T_o + \sum_{j=1}^{J} a_j^2 T_j \left(\frac{M_j + 1}{M_j - 1} \right) \right]$$

As before we will define:

$$R_j = \frac{E_j}{E_s}$$

and:

$$\sigma_{nj}^2 = \frac{N_o}{2T_j}$$

Substitution into equation 2-21 yields:

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{\left[\prod_{j=1}^{J} M_j \right]^2 R_o}{1 + 24 R_o Y \sum_{n=1}^{J} \left[\left(\prod_{j=1}^{J} M_j \right)^2 / M_n \left(\prod_{j=1}^{n} M_j \right)^2 \right] \sum_{m=1}^{M_n - 1} (2m-1)(M_n - m)\, Q_{mn}}$$

where: $$Q_{mn} = Q \left\{ \left[\frac{(2m-1)^2 R_n Y}{M_n^2 - 1} \right]^{\frac{1}{2}} \right\} \qquad \text{2-22}$$

where, as before, $Y = 2E_s/N_o$

In the case where:

$$M_j = M$$

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{M^2 J R_o Y}{1 + 24R_o Y \sum_{n=1}^{J} (M^2 J / M^{2n+1}) \sum_{m=1}^{M-1} (2m-1)(M-m) Q\left\{\left[\frac{(2m-1)^2 R_n Y}{M^2 - 1}\right]^{\frac{1}{2}}\right\}} \qquad 2\text{-}23$$

If in addition:

$$R_n = R_1 \qquad n \neq o$$

$$\frac{\sigma_o^2}{\sigma_n^2} = \frac{M^2 J R_o Y}{1 + 24R_o Y \left[\frac{M^2 J_{-1}}{M(M^2 - 1)}\right] \sum_{m=1}^{M-1} (2m-1)(M-m) Q\left\{\left[\frac{(2m-1)^2 R_1 Y}{M^2 - 1}\right]^{\frac{1}{2}}\right\}} \qquad 2\text{-}24$$

If:

$$T_j = T$$

$$a_j = a$$

and:

$$E_s = \frac{a^2 A^2 T}{3} \left[1 + J \frac{(M+1)}{(M-1)}\right]$$

$$R_o = \frac{M-1}{(M-1) + J(M+1)}$$

$$R_1 = \frac{(M+1)}{(M-1) + J(M+1)}$$

APPENDIX 3

PERFORMANCE OF THE MINIMUM MEAN SQUARE ERROR RECEIVER FOR MBM

The equation for the MMSE receiver for the general N + 1 dimensional memoryless mapping given in Appendix 1 is rather cumbersome. The MBM form of mapping, however, reduces the problem considerably. This can be argued in the simplest fashion by noting that the input is in the form of:

$$m = \sum_{i=0}^{N} B_i h_i \qquad 3\text{-}1$$

The received signal set, r_o, r----r_n, are related to the transmitted signals, h_o, h_1----h_N, in that:

$$r_j = h_j + n_j$$

The random variables h_j and n_j are jointly independent. Thus:

$$f(h_o, h_1, ----h_N, r_o, r_1 ----r_n) = f(h_o, r_o)\, f(h_1, r_1)----f(h_N, r_N)$$

and hence:

$$f(h_o, h_1, ----h_N | r_o, r_1, ----r_N) = f(h_o | r_o)\, f(h_1 | r_1)----f(h_N | r_N) \qquad 3\text{-}2$$

The optimum receiver is one that outputs:

$$\hat{m} = E[m | r_o, r_1, ----r_N) = \int m\, f(m | r_o, r_1 ----r_N) dm$$

$$\hat{m} = \int \cdots \iiint \left(\sum_{i=0}^{N} B_i h_i \right) f(h_o, h_1 \cdots h_N \mid r_o, r_1 \cdots r_N) d\,h_o dh_1 \cdots dh_N$$

$$\hat{m} = \int \cdots \iiint \left(\sum_{i=0}^{N} B_i h_i \right) f(h_o \mid r_o)\, f(h_1 \mid r_1) \cdots f(h_N \mid r_N)\, dh_o\, dh_1 \cdots dh_N$$

which yields:

$$\hat{m} = \sum_{i=0}^{N} B_i \int h_i\, f(h_i \mid r_i)\, dn_i = \sum_{i=0}^{N} B_i\, E(h_i \mid r_i) = \sum_{i=0}^{N} B_i \hat{h}_i \qquad 3\text{-}3$$

The equation, 3-3, states that the minimum mean square estimate of m is a weighted sum of the minimum mean-square estimate of each transmitted pulse. Further, the error in the estimate is:

$$m - \hat{m} = \sum_{i=0}^{N} B_i (h_i - \hat{h}_i) \qquad 3\text{-}4$$

and the mean squared error is:

$$\epsilon\ \mathbf{E}(m^2) = E[(m-\hat{m})^2] = \sum_{i=0}^{N} \sum_{j=0}^{N} B_i B_j E\left\{(h_i - \hat{h}_i)(h_j - \hat{h}_j)\right\}$$

Since:

$$E[h_i h_j] = 0 \text{ and } E[h_i \hat{h}_j] = 0 \qquad i \neq j$$

$$\epsilon\ E(m^2) = E[(m-m)^2] = \sum_{i=0}^{N} B_i^2 E\left\{(h_i - \hat{h})^2\right\} = \sum_{i=0}^{N} B_i^2 e_i^2 \qquad 3\text{-}5$$

Thus, the mean square error associated with the MMSE receiver is the weighted sum of the mean square error associated with the optimum estimate of the transmitted values.

In the case of the continuous pulse:

$$f(h_o) = \frac{1}{2a_oA} \qquad -a_oA < h_i < +a_oA$$

From Appendix 1

$$\hat{h}_o = \int h_o f(h_o|r_o)\, dh_o = \frac{\int_{-a_oA}^{+a_oA} h_o e^{-(r_o-h_o)^2/2\sigma_n^2}\, dh_o}{\int_{-a_oA}^{+a_oA} e^{-(r_o-h_o)^2/2\sigma_n^2}\, dh_o}$$

Performing the integration yields:

$$\hat{h}_o = r_o - \frac{\sigma_n}{\sqrt{2\pi}} \frac{\left[e^{-(a_oA-r_o)^2/2\sigma_n^2} - e^{-(a_oA+r_o)^2/2\sigma_n^2}\right]}{Q\left(\frac{a_oA-r_o}{\sigma_n}\right) - Q\left(\frac{-a_oA-r_o}{\sigma_n}\right)}$$

Since: $r_o = h_o + n_o$

$$\hat{h}_o = h_o + n_o - \frac{\sigma_n}{\sqrt{2\pi}} \frac{\left[e^{-(a_oA-h_o-n_o)^2/2\sigma_n^2} - e^{-(a_oA+h_o+n_o)^2/2\sigma_n^2}\right]}{Q\left(\frac{a_oA-h_o-n_o}{\sigma_n}\right) - Q\left(\frac{-a_oA-h_o-n_o}{\sigma_n}\right)}$$

$$\hat{h}_o = h_o + n_o - F(h_o, r_o)$$

The mean square error, e_o^2, is:

$$e_o^2 = E[(h_o-\hat{h}_o)^2] = E[n_o^2] - 2E[n_oF(h_o,n_o)] + E[F^2(h_o,n_o)]$$

$$e_o^2 = \sigma_n^2 - \frac{1}{2_oA} \int_{-\infty}^{+\infty} [2n_oF(h_o,n_o) - F^2(h_o,n_o)] \frac{e^{-n_o^2/2}}{\sqrt{2\pi}\sigma_n}\, dh_o\, dn_o \qquad 3\text{-}6$$

where:

$$F(h_o, n_o) = \frac{\sigma_n}{\sqrt{2\pi}} \quad \frac{\left[e^{-(a_oA-h_o-n_o)^2/2\sigma_n^2} - e^{-(a_oA-h_o+n_o)^2/2\sigma_n^2} \right]}{Q\left(\frac{a_oA-h_o-n_o}{\sigma_n}\right) - Q\left(\frac{-a_oA-h_o-n_o}{\sigma_n}\right)}$$

Although in equation 3-6 it is desired to average over the value of h_o as shown, it is interesting to observe the value of e_o^2 for a fixed value of h_o. If the ratio of peak signal to rms noise, a_oA/σ_n, is reasonably large (≥ 5) then the calculatable ratio of e_o^2 to σ_n^2 depends only upon the number of noise variance intervals that h_o differs from plus or minus a_oA. The reduction in e_o^2 is plotted as a function of σ_n^2 in Figure 3-1. The MMSE receiver based on a uniform density does not increase the error for any transmitted value of h_o. This fact can be verified from equation 3-4. Therefore, such a MMSE receiver will provide results with the continuous pulse which are at least as good as those provided by the simpler receiver for any density function which is zero outside the range a_oA and $-a_oA$. The average reduction in error for values of h_o such that $|a_oA-h_o| \leq 5\sigma_n$ is $0.181\ \sigma_n^2$. Since the reduction when h_o is outside this region is negligible we may deduce that:

$$e_o^2 = \sigma_n^2 \left[1 - .181\left(\frac{5\sigma_n}{a_oA}\right)\right] \qquad a_oA \geq 8\sigma_n$$

If $Y_o = R_o\ Y = \dfrac{a_o^2 A^2}{12\sigma_n^2}$

then: $e_o^2 = \sigma_n^2 \left[1 - .905/(12Y_o)^{\frac{1}{2}}\right] = \sigma_n^2 \left[1 - .262/Y_o^{\frac{1}{2}}\right] \quad Y_o \geq 5.33(7.3\text{dB})$

Figure 3-2 is a plot of e_o^2 versus Y_o for a MMSE receiver and the simple receiver described in Section V.

In the case of the discrete pulse, h, with M levels ranging from plus to minus $a_1 A$, the MMSE receiver is:

$$\hat{h}(r) = E(h|r) = \sum_{j=1}^{M} h \quad P(h_j|r) \qquad \text{where } h_j = \frac{a_1 A}{M}\left(j - \frac{M+1}{2}\right)$$

$$\hat{h}(r) = \frac{\sum_{j=1}^{M} h_j \, P(h_j|r)}{\sum_{j=1}^{M} f(r|h_j)\, P(h_j)}$$

Since:

$$f(r|h_j) = \frac{e^{-(r-h_j)^2/2\sigma_n^2}}{\sqrt{2\pi}\,\sigma_n}$$

and

$$P(h_j) = \frac{1}{M}$$

then:

$$\hat{h}(r) = \frac{\sum_{j=1}^{M} h_j \, e^{-(r-h_j)^2/2\sigma_n^2}}{\sum_{j=1}^{M} e^{-(r-h_j)^2/2\sigma_n^2}} \qquad 3\text{-}7$$

If the actual transmitted value was h_k then $r = h_k + n$ and:

$$\hat{h}(r) = \frac{\sum_{j=1}^{M} h_j \, e^{-[(h_k-h_j)+n]^2/2\sigma_n^2}}{\sum_{j=1}^{M} e^{-[(h_k-h_j)+n]^2/2\sigma_n^2}} \qquad 3\text{-}8$$

If the kth term in each summation is taken out of equation 3-6

$$\hat{h}(r) = \frac{h_k e^{-n^2/2\sigma_n^2} + \sum_{j=1\neq k}^{M} h_j e^{-[(h_k-h_j)+n]^2/2\sigma_n^2}}{e^{-n^2/2\sigma_n^2} + \sum_{j=1\neq k}^{M} e^{-[(h_k-k_j)+n]^2/2\sigma_n^2}}$$

This is in the form:

$$\hat{h}(r) = \frac{a+b}{c+d} = \frac{a}{c} + \frac{b-ad/c}{c+d}$$

Thus:

$$\hat{h}(r) = h_k + \frac{\sum_{j=1}^{M} (h_j-h_k)\, e^{-[(h_k-h_j)+n]^2/2\sigma_n^2}}{\sum_{j=1}^{M} e^{-[(h_k-h_j)+n]^2/2\sigma_n^2}}$$

$$\hat{h}(r) = h_k + F(k,n)$$

The second term is obviously the error. The mean square value of this error if all values of k are equally likely is:

$$e^2 = \frac{1}{M} \sum_{k=1}^{M} \int F^2(k,n)\, dn$$

$$\text{where } F(k,n) = \frac{\left(\frac{aA^2}{M}\right) \sum_{j=1} (j-k)\, e^{-\left[\frac{aA}{M}(j-k)+n\right]^2/2\sigma_n^2}}{\sum_{j=1}^{M} e^{-\left[\frac{aA}{M}(j-k)+n\right]^2/2\sigma_n^2}} \qquad 3\text{-}9$$

Equation 3-7 was numerically integrated to produce curves of e^2 versus Y_1 for values of M from 1 to 10 where:

$$Y_1 = \frac{a^2 \; A^2}{3\sigma_n^{\;2}}$$

These are plotted in Figure 3-3 along with the equivalent error produced by the simple receiver. As was the case with the continued pulse error evaluation the error for the MMSE receiver given a particular transmitted value, h_k, is equal to or less than the error with the simple receiver for all values of k.

Equation 3-9 can be used in combination with the data in Figures 3-2 and 3-3 to calculate SNR_i versus SNR_o for MBM with a MMSE receiver and a specified energy ratio for the pulses.

APPENDIX 4

OPTIMUM ENERGY RATIOS

The mean square error for MBM transmission can be expressed in the form:

$$e^2 = \sum_{i=0}^{J} B_i^2 \, e_i^2 \, (Y_i) \qquad 4\text{-}1$$

where: J + 1 is the bandspread factor, β, or the number of output pulses per input pulse.

B_i is the weighting or basis for the ith pulse

$Y_i = \frac{2E_i}{N_o}$ = average pulse energy to noise spectral height ratio for the ith pulse.

Equation 4-1 applies to both the case of the simple receiver and the MMSE receiver. If the total energy to noise ratio is:

$$Y = \frac{2E_s}{N_o} = \sum_{i=0}^{J} Y_i \qquad 4\text{-}2$$

then the optimum problem is one of minimizing e^2 in equation 4-1 under the constraint of equation 4-2. Fortunately, this problem can be handled as a sequence of two-dimensional optimization problems rather than a J + 1 dimensional problem. If one starts with the first two terms:

$$e_{mi}^2 = B_o^2 \, e_o^2 \, (Y_o) + B_1^2 \, e_1^2 \, (Y_1)$$

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and determines the minimum achievable $e_{m_1}^2$ for a given $Y_1' = Y_o + Y_1$, then a new curve exists such that:

$$e_{m_1}^2 = e_1'(Y_1')$$

If one then adds the next term:

$$e_{m_2}^2 = e_{m_1}^2 + B_2^2 e_2^2 (Y_2) = e_1'(Y_1') + B_2^2 e_2^2 (Y_2)$$

and minimizes for all $Y_2' = Y_2 + Y_1'$ this produces a new curve:

$$e_{m_2}^2 = e_2'(Y_2')$$

which corresponds to the optimum split between the first three pulses. This procedure can be continued until all J + 1 pulses are included.

Plots of $e_1^2(Y_i)$ are given for both the simple and MMSE receiver in Figures 3-2 and 3-3 in the previous Appendix. Since the analytical forms for these plots were cumbersome it was found easiest to produce the combined optimums by hand and replot the new curve. The resulting optimum energy ratios are given in Figures 4-1 through 4-4. The resulting SNR_o versus SNR_i for optimum energy split are plotted in Section IV.

APPENDIX 5

SYSTEM PERFORMANCE

SINGLE SIDEBAND (SS)

A single sideband system transmits data of bandwidth, B_v. The sampling rate is $2B_v$ and sample time, T_s is:

$$T_s = \frac{1}{2B_v} \tag{5-1}$$

If the pulse has average power, P_s, the energy, E_s is:

$$E_s = P_s T_s = \frac{P_s}{2B_v} \tag{5-2}$$

If the pulse is transmitted in the presence of white noise with two sided height of $N_o/2$, and a matched filter with impulse response:

$$\begin{aligned} H(t) &= \frac{1}{T_s} && 0 \le t \le T_s \\ &= 0 && \text{otherwise} \end{aligned} \tag{5-3}$$

is used then the output due to signal is equal to the input and has variance, σ_o^2,

$$\sigma_o^2 = P_s$$

The variance of the noise, σ_n^2 is:

$$\sigma_n^2 = \frac{N_o}{2} \int_o^{T_s} \left(\frac{1}{T_s}\right)^2 dt = \frac{N_o}{2T_s} \tag{5-4}$$

Thus, the output signal to noise ratio is:

$$SNR_o = \sigma_o^2/\sigma_n^2 = \frac{2P_sT_s}{N_o} = \frac{2E_s}{N_o} = \frac{P_s}{N_oB_v} = SNR_i \qquad 5\text{-}5$$

Since the output sampling rate equals the input sampling rate the bandspreading is:

$\beta = 1$

OTHER FORMS OF AMPLITUDE MODULATION

Suppressed carrier A.M. differs from single-sideband only in the method of frequency translation. The effect is to leave the relationship between SNR_o and SNR_i the same but double the bandspread ratio. That is for SCAM:

$SCAM| SNR_o = SNR_i$

$\beta = 2$

Ordinary non-suppressed carrier A.M. is the same as SCAM except that additional power is added to the signal in the form of carrier. This signal increases the required input power without increasing the SNR_o. For the case of a uniformly distributed modulating signal and 100% modulation index:

$AM| SNR_o = .25\ SNR_i$

$\beta = 2$

FREQUENCY MODULATION (FM)

The general equation used to calculate SNR_o versus SNR_i for frequency modulation is:

$$SNR_o = \frac{12\rho\, fd^2 B_{if}/B_v (12B_s^2 + B_v^2)}{1 + \rho\, F(\rho) \cdot 3B_{if}(B_{if} - B_v)/(12B_s^2 + B_v^2)} \qquad 5\text{-}6$$

where:

$\rho = \frac{B_v}{B_{if}}$ SNR_i = IF signal to noise ratio

fd = rms frequency deviation

B_v = bandwidth of video filter

B_s = center frequency of video filter

B_{if} = bandwidth of I. F. filter

and:

$$F(\rho) = \frac{B_{if}}{2[3(1+z)]^{\frac{1}{2}}} \Big\{ (1+z)\, Q(\sqrt{2\rho}(1+z))\, Q(-\sqrt{2\rho}(1+z)) \qquad 5\text{-}7$$

$$+ [z(1+z)]^{\frac{1}{2}} e^{-\rho} [Q(-\sqrt{2\rho}z) - Q(+\sqrt{2\rho}z)][Q(-\sqrt{2\rho}(1+z)-Q(\sqrt{2\rho}(1+z)]$$

$$+ ze^{-2\rho}\, Q(\sqrt{2\rho}z)\, Q(-\sqrt{2\rho}z)$$

where:

$$Q(x) = \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt$$

$$z = \frac{12 f^2_{dc}}{B_{if}^2}$$

f_{dc} = explained in the next paragraph.

Equation 5-6 is taken from reference [20] where the term representing the spectral height of the FM output noise at the frequency origin has been replaced by $F(\rho)$. The expression for $F(\rho)$ is taken from reference [21] and is the spectral height of the noise at the origin based on the noise model of that report. This spectral height is derived assuming an unmodulated but not necessarily tuned carrier. The term, f_{dc}, represents the detuning of the carrier from the center of the IF.

Equation 7-1 applies to cases where a bandpass video filter is used as well as the case where a lowpass filter is used. Figure 5-1 is a plot of normalized SNR_o versus SNR_i with thresholding noise based upon no frequency offset. These curves for the case of the lowpass video filter correspond very closely to curves based on Stumpers [22] equations (see Akima [18]). Figure 5-2 is similar except that the threshold noise is based upon a frequency offset of half the IF bandwidth.

Figure 19 in Section VII is taken from Figure 5-1 where $f_d = B_{if}/\sqrt{12}$ and $B_s = B_v/2$. For uniformly distributed, m, this corresponds to setting the peak-to-peak deviation equal to the IF bandwidth. For FM we call $B_{if}/B_v = \beta$.

DIGITAL TECHNIQUES

Phase Shift Keying (PSK)

The bit error probability, p_e, for PSK is well known to be:

$$p_e = Q\left[\frac{2E_b}{N_o}\right]$$

where:

E_b = energy per bit

$N_o/2$ = noise spectral density

If the message value, m, is quantized to n bits then:

$$SNR_i = \frac{2E_s}{N_o} = \frac{2nE_b}{N_o}$$

$$p_e = Q\left[\frac{SNR_i}{n}\right]$$

Also if the distribution of m is uniform, the quantization error normalized to the rms value of the data will be:

$$E_q^2 = 1/2^n$$

The normalized link error is:

$$E_\ell^2 = 4p_e\left[1 - \frac{1}{2^{2n}}\right] \quad 5\text{-}8$$

$$SNR_o = \frac{1}{E_q^2 + E_\ell^2} = \frac{1}{1/2^n + 4[1 - 1/2^{2n}]\, Q\left[\frac{SNR_i}{n}\right]} \quad 5\text{-}9$$

The pulse shapes can in theory be made so that the channel bandwidth is equal to the bit rate which is n times the sample rates and 2n times B_v.

$$\beta = 2n$$

Quadraphase (QP)

The only difference between quadraphase and PSK is in the channel bandwidth which is one-half that of PSK. Thus:

$$\beta = n$$

and SNR_o versus SNR_i is given by equation 5-5.

Coded PSK Systems

The calculation of bit error rates as a function of E_b/N_o for several error correction codes is shown in Figure 5-3. As before the relationship between E_b/N_o and SNR_i is:

$$SNR_i = \frac{2nE_b}{N_o} \qquad 5\text{-}9$$

where 2n is the number of quantization increments of the code. A fact not widely known is that the relationship between bit error probability and rms link error for most coded-systems is the same as that given in equation 7-4. The necessary and sufficient requirements for this to be true are:

(a) The probability that a particular information bit is in error is the same as that of any other bit being in error.

(b) The error in a particular bit is equally likely to be in either polarity.

The argument to justify the statement is to observe that the mean square error when several bits in a single word are in error is the same as that when the bits occur in separate words. This is a consequence of (b). If the absolute error in a particular word is e_1 and the error due to an additional bit error in the same word is e_2 which is equally probable to add or subtract from e_1 then the mean squared error is:

$$E(e^2) = .5(e_1 + e_2)^2 + .5(e_1 - e_2)^2 = e_1^2 + e_2^2 \qquad 5\text{-}10$$

This is the same squared error that would occur if the additional error had been in a separate word. If all bits in the n bit word are in error with probability p_e and the mean squared error is the same as that which would occur if they were in separate words then the total mean squared error is:

$$mse = p_e \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{2j} = \frac{p_e}{3} [1 - 1/2^{2n}] \qquad 5\text{-}11$$

This error is normalized to full scale. Normalizing to the rms value of a uniform density required multiplication by 12 which produces equation 5-8.

Although both block and convolution codes tend to cause clustered errors, the usual decoding techniques still meet requirement (a) even in the case where blocks are an integer number of words.

Thus, the equation for SNR_o for coded PSK is:

$$SNR_o = \frac{1}{1/2^{2n} + 4\, p_e\, (SNR_i)} \qquad 5\text{-}12$$

A plot of $p_e(E_b/N_o)$ is given in Figure 5-3 for several coding-decoding techniques.

The bandspreading is the bandspreading caused by the coding, α, times the bandspreading of the PSK system. Thus:

$$\beta = 2\alpha n \quad \text{for coded PSK}$$
$$\beta = \alpha n \quad \text{for coded QP} \qquad 5\text{-}13$$

<u>Orthogonal and Biorthogonal Systems</u>

The symbol error probability, p_s, as a function of E_b/N_o for orthogonal and biorthogonal systems is well known and has been tabulated by Viterbi [23]. Totty [24] has shown that the mean square error normalized to full scale of orthogonal signals is:

$$\text{mse} = P_s/6 \qquad (5\text{-}14)$$

and that for biorthogonal signals the mean square error is very nearly the same. Actually the argument associating bit error probability with mean square error given in the preceding section can also be applied to orthogonal and biorthogonal systems. This would allow exact calculation bor biorthogonal systems since the bit error probability as well as the symbol error probability was tabulated by Viterbi. Unfortunately the tabulated bit error probabilities for the biorthogonal case are in error so that we must settle for use of equation 5-14 and Totty's bounding arguments. Normalizing equation 5-14 to the mean square output for uniform data we have:

Orthogonal and Biorthogonal $$SNR_o \approx \frac{1}{1/2^{2n} + 2\,P_s(SNR_i)} \qquad (5\text{-}15)$$

where as before:

2^n = number of quantizing increments

$SNR_i = 2nE_b/N_o$

The bandspread factor is:

Orthogonal $\beta = 2^n$

Biorthogonal $\beta = 2^{n-1}$

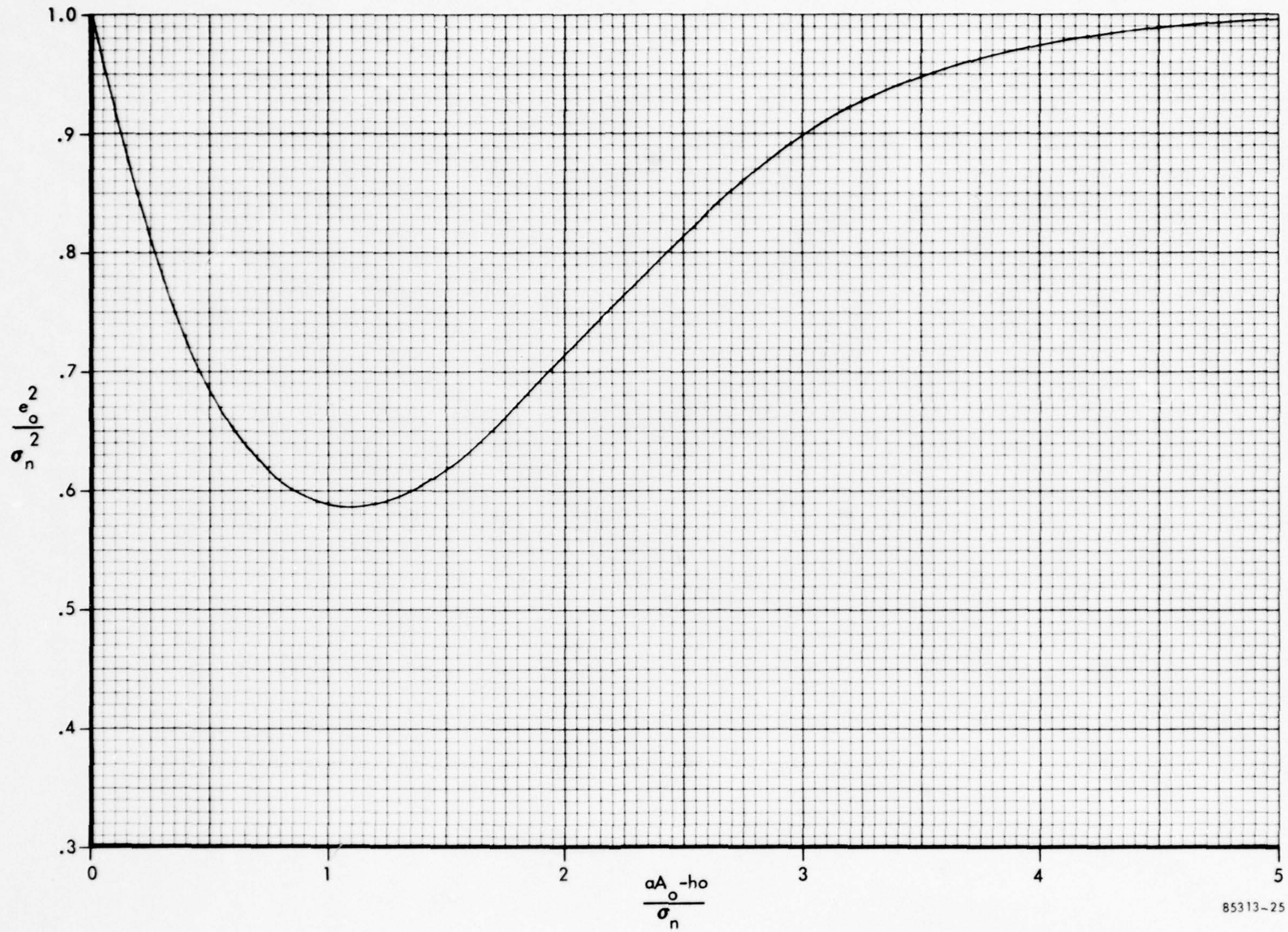

FIGURE 3-1 REDUCTION IN ERROR OF CONTINUOUS PULSE BY USE OF MMSE RECEIVER

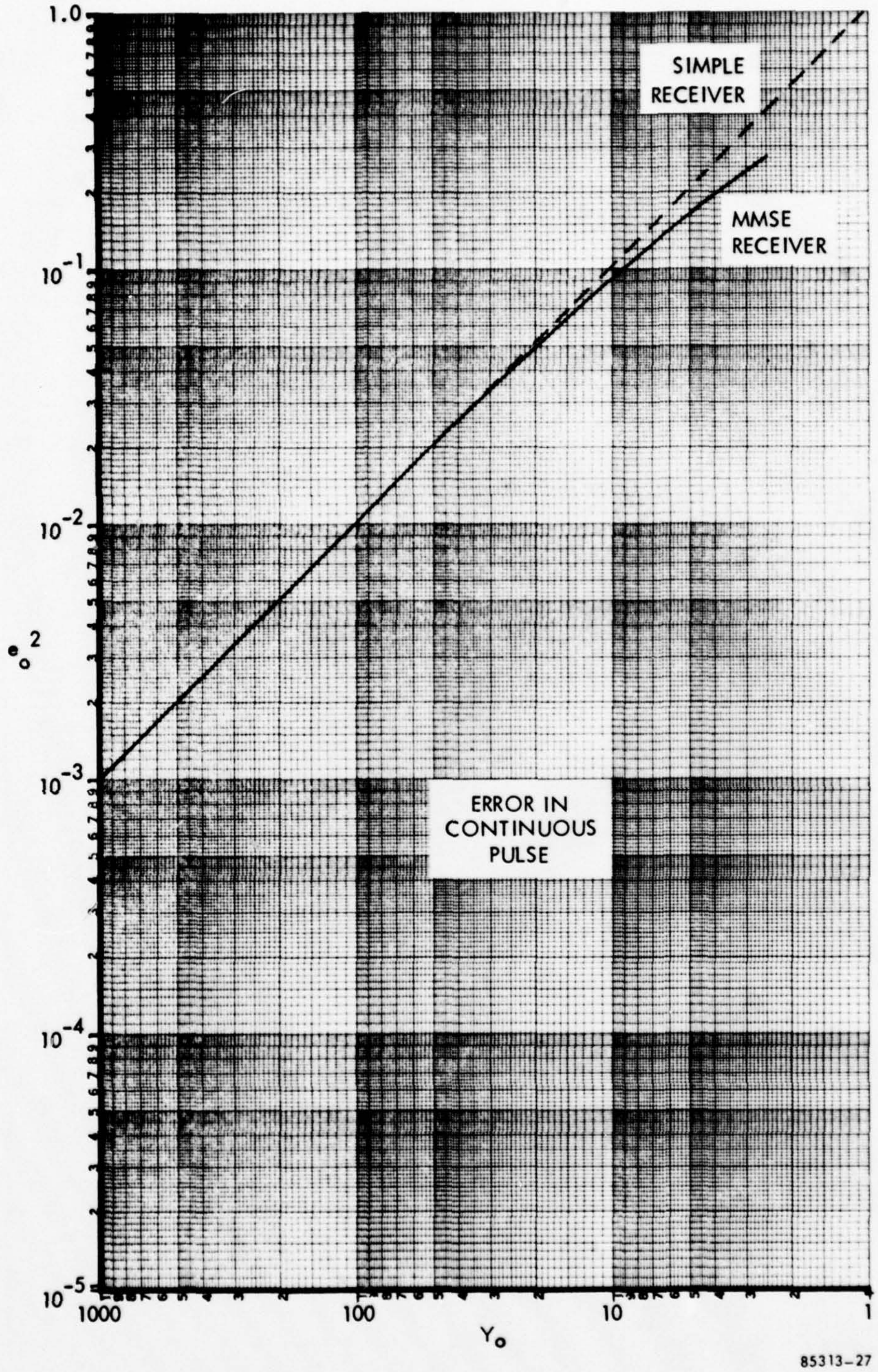

FIGURE 3-2 ERROR IN CONTINUOUS PULSE

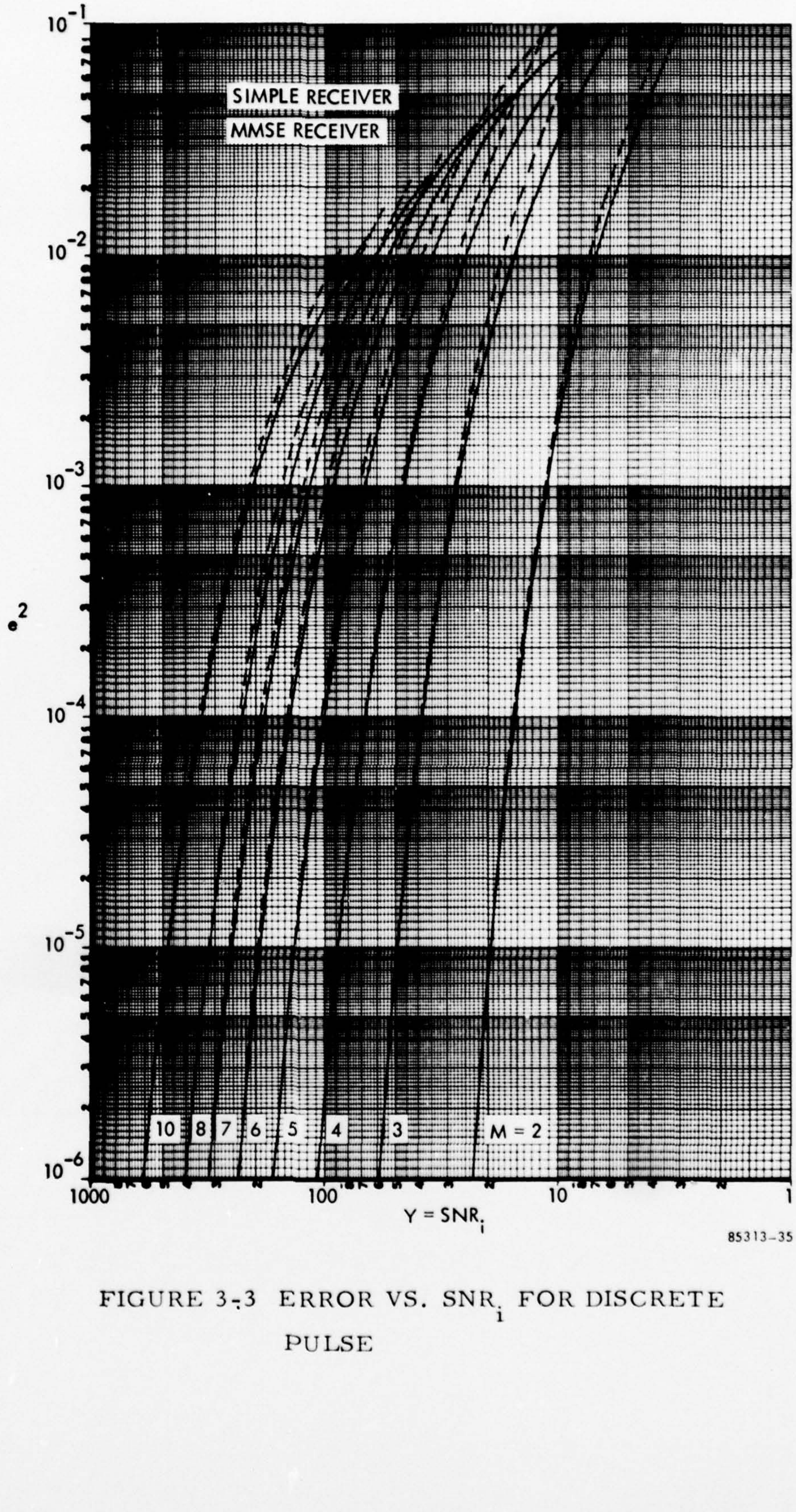

FIGURE 3-3 ERROR VS. SNR$_i$ FOR DISCRETE PULSE

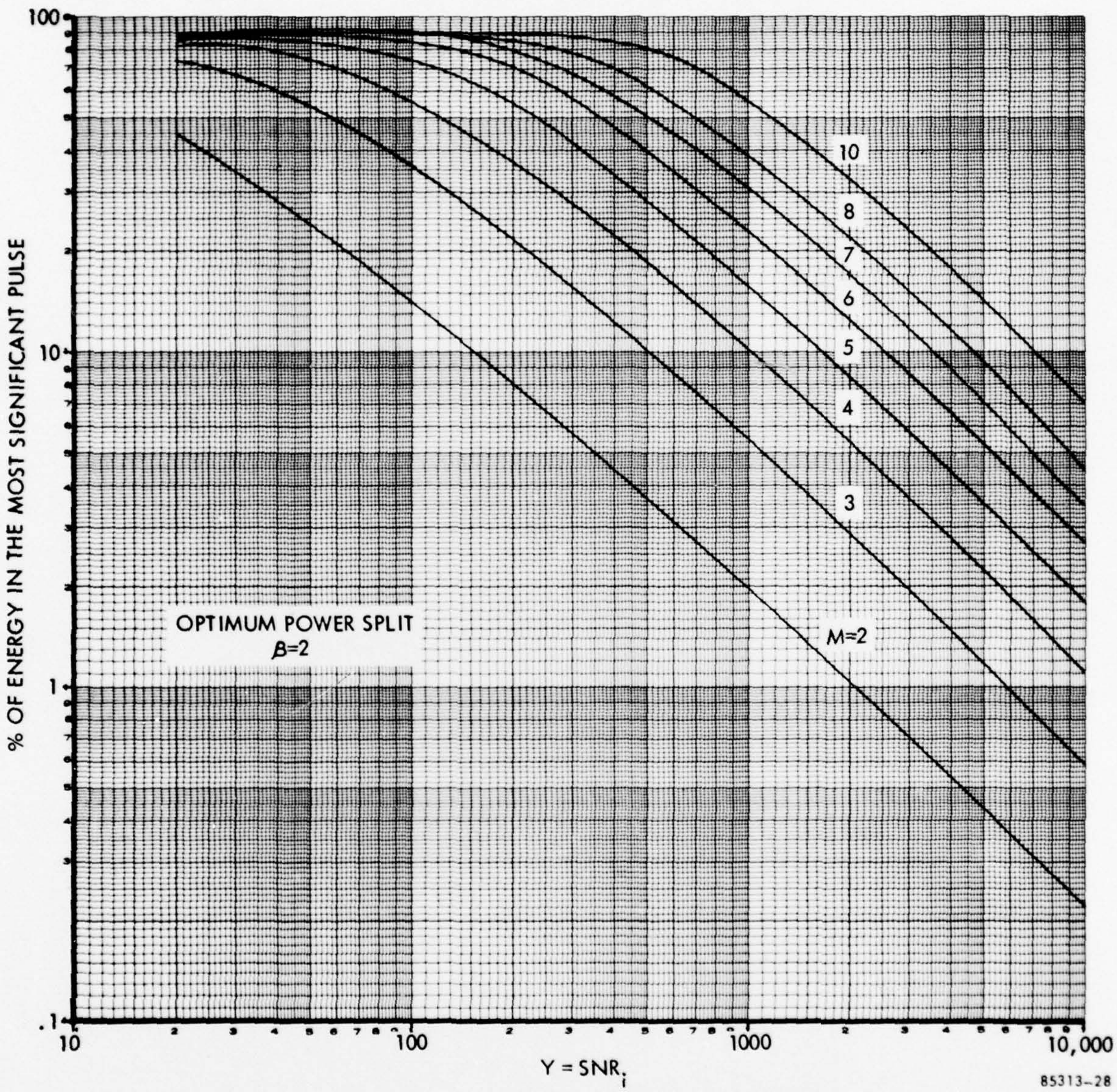

FIGURE 4-1 OPTIMUM POWER SPLIT β = 2

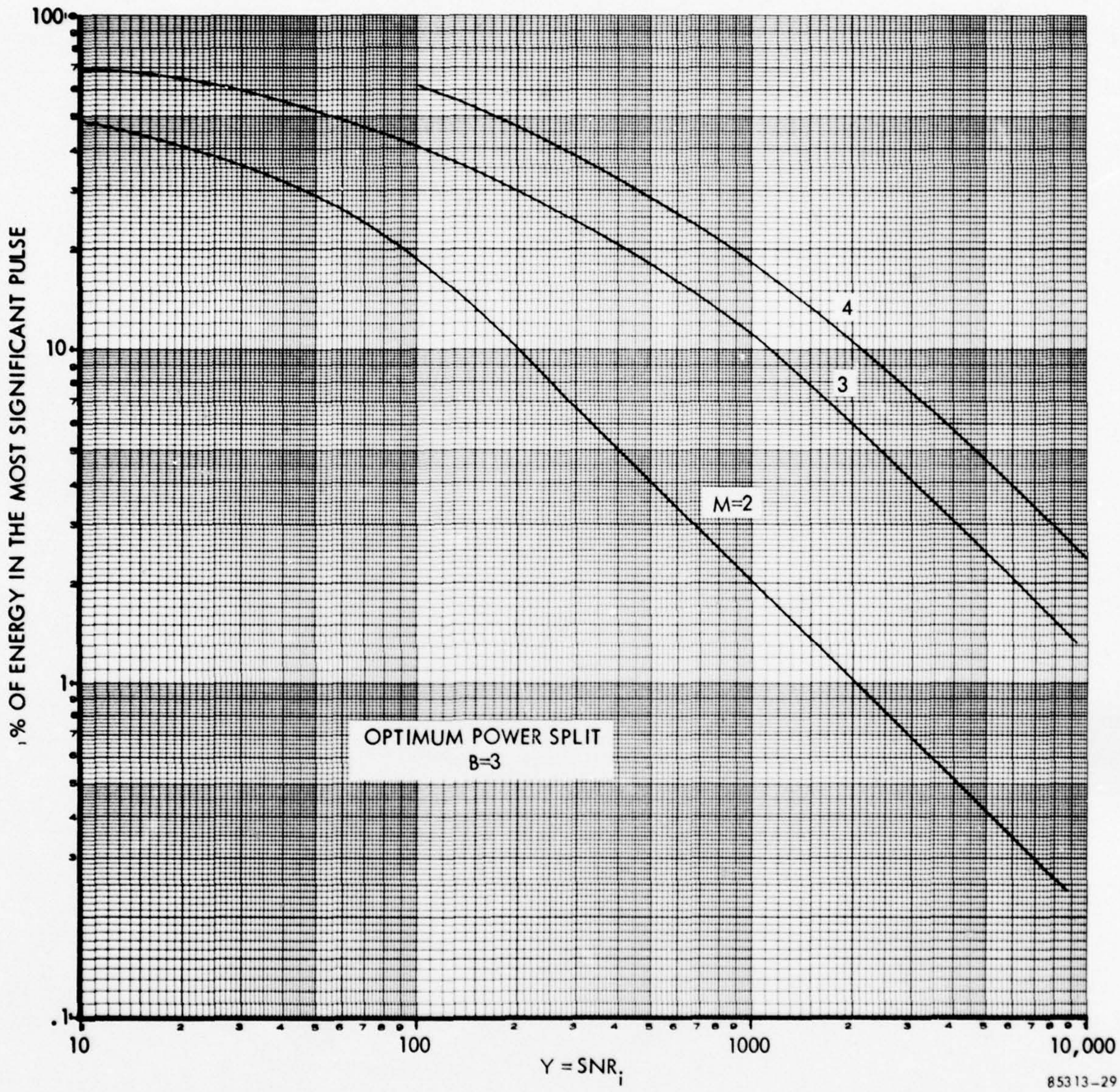

FIGURE 4-2 OPTIMUM POWER SPLIT β = 3

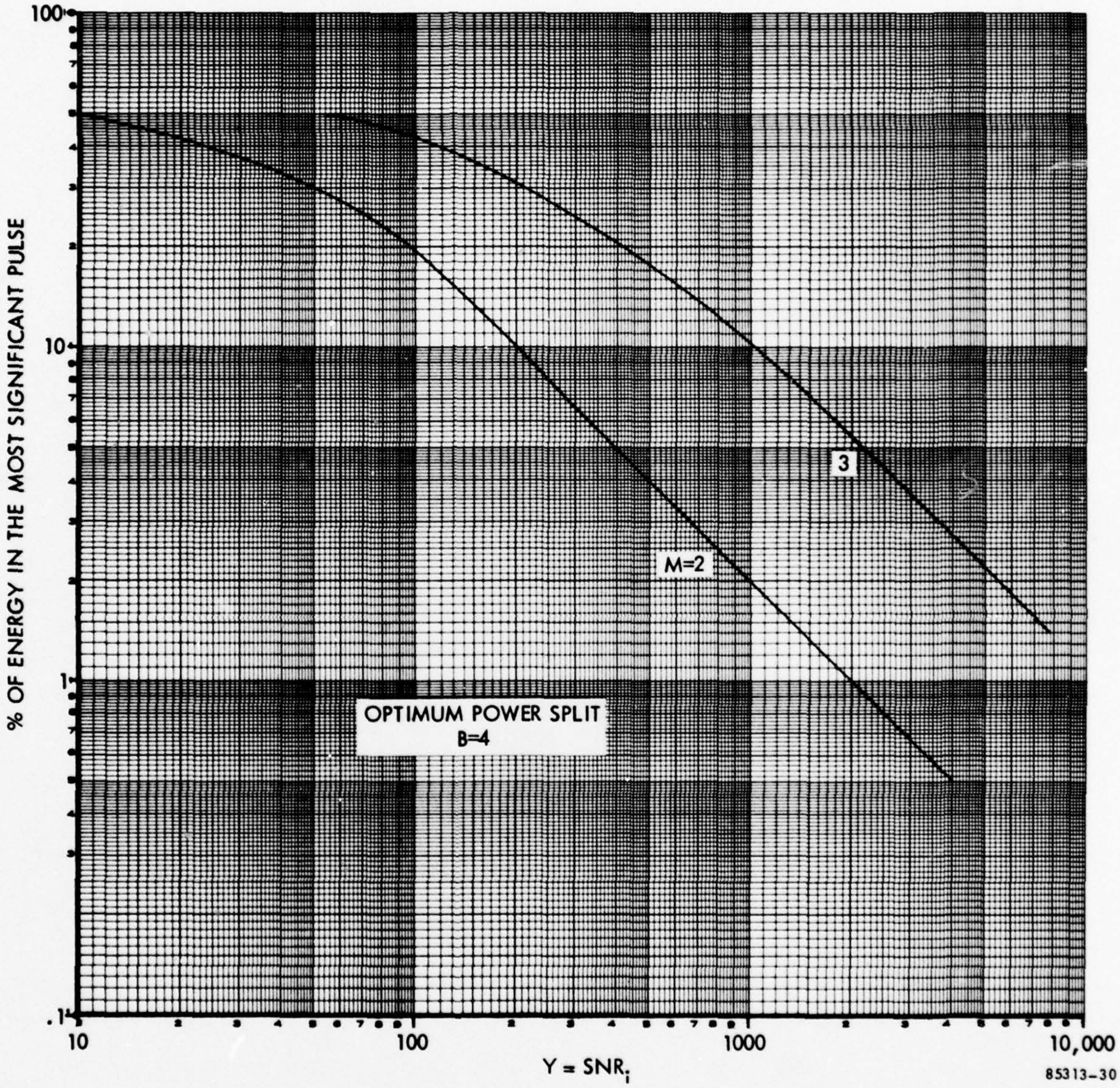

FIGURE 4-3 OPTIMUM POWER SPLIT β = 4

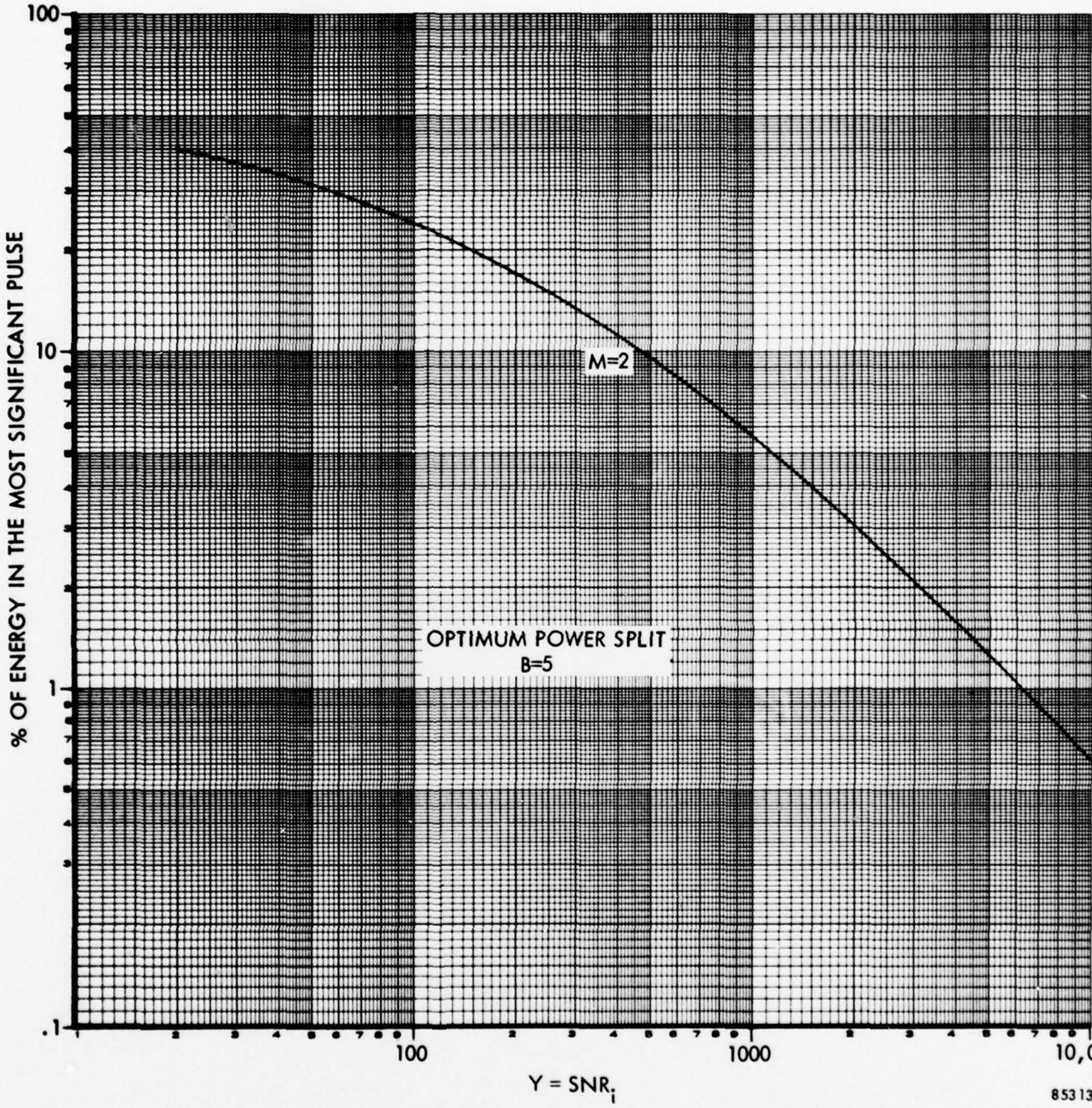

FIGURE 4-4 OPTIMUM POWER SPLIT β = 5

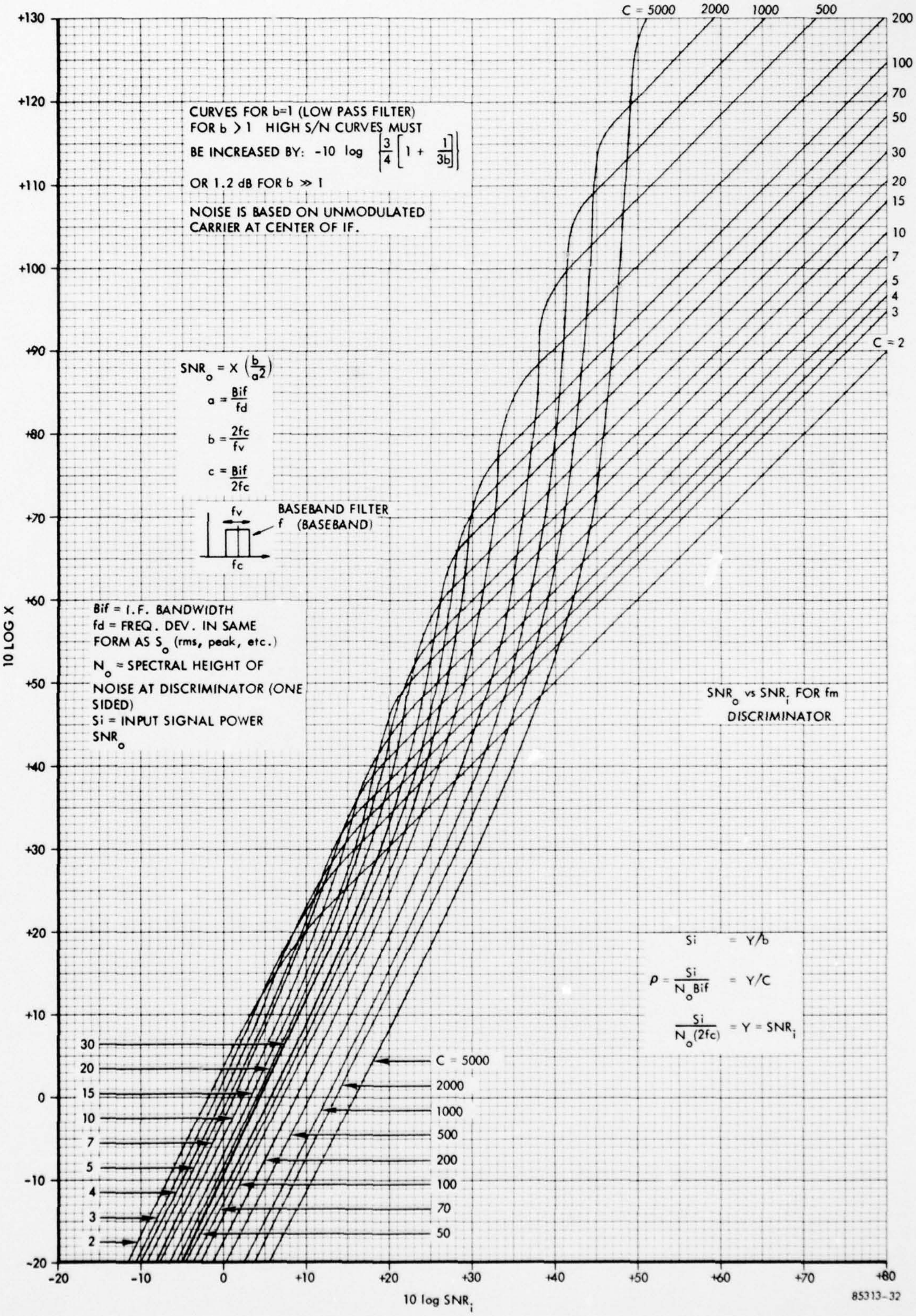

FIGURE 5-1 SNR_o VS. SNR_i FOR FM DISCRIMINATOR

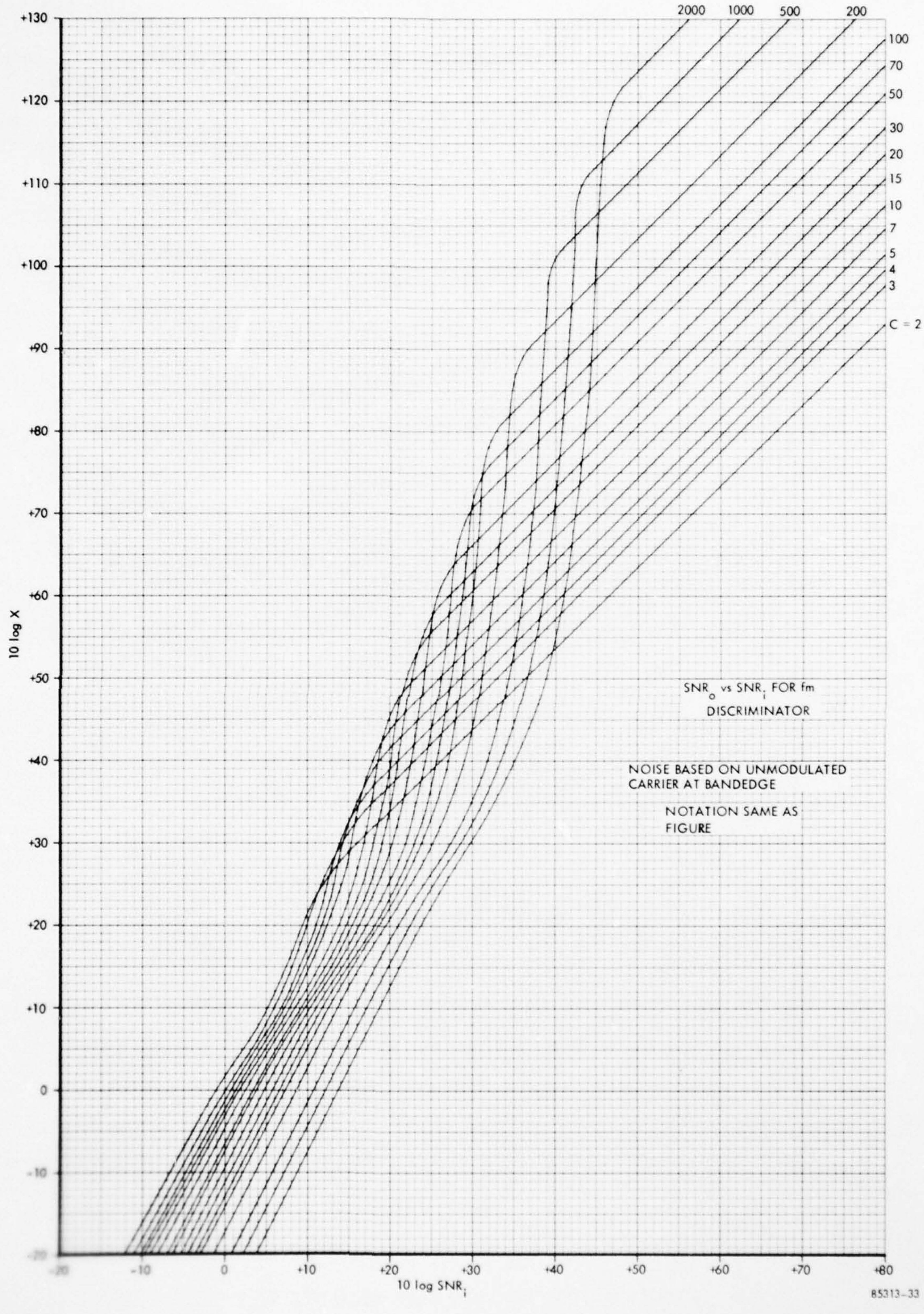

FIGURE 5-2 SNR_o VS. SNR_i FOR FM DISCRIMINATOR

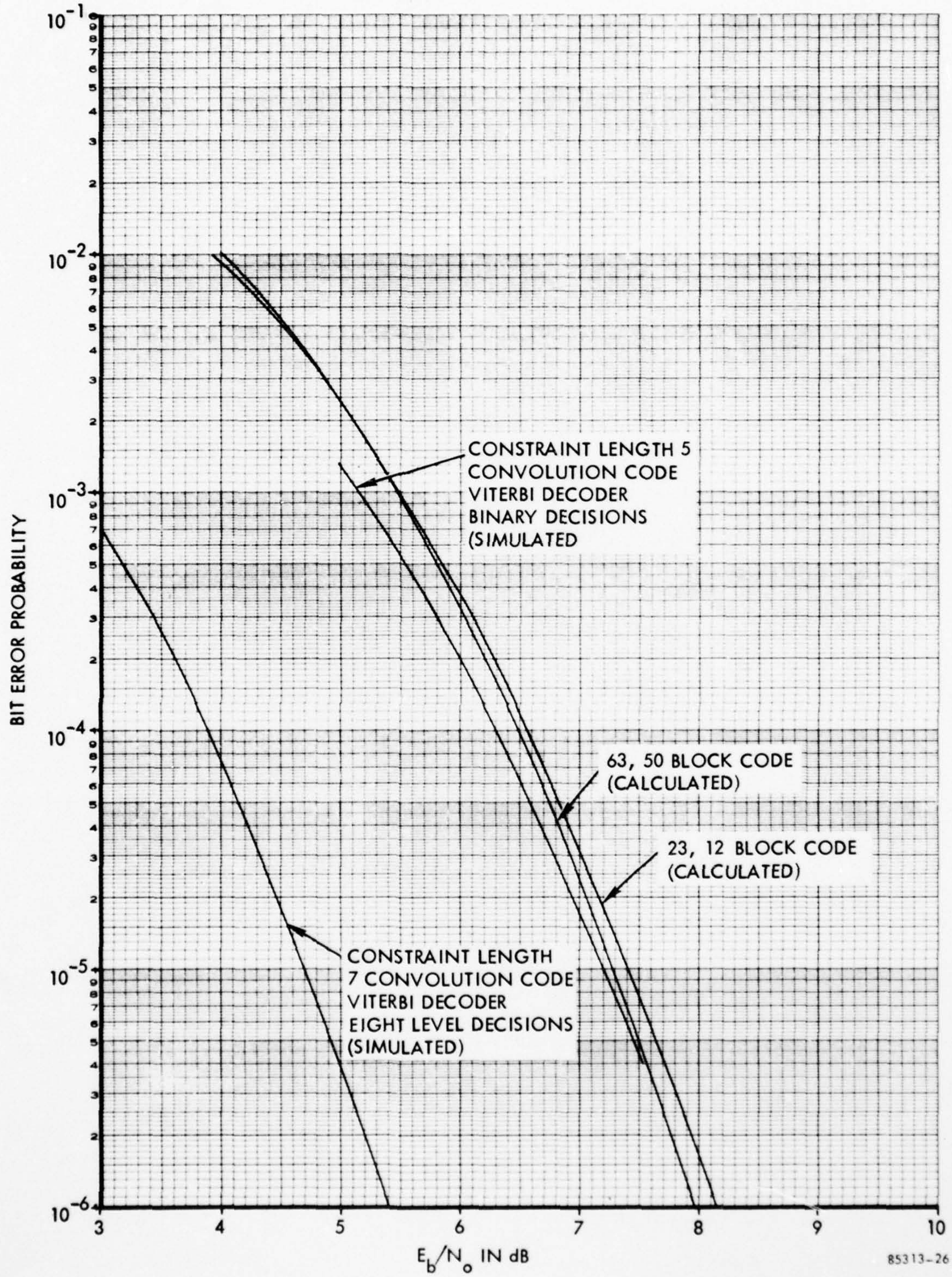

FIGURE 5-3 BIT ERROR RATE VS. E_b/N_o FOR CODED SYSTEMS

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