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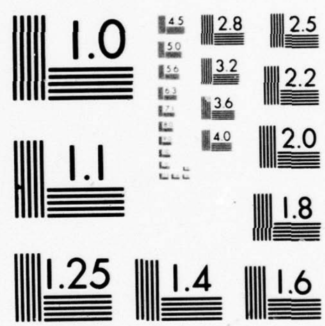

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DISCONTINUOUS DEFORMATION GRADIENTS IN THE FINITE TWISTING OF AN INCOMPRESSIBLE ELASTIC TUBE

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Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube*

by

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Summary

It is known that the type of the system of partial differential equations governing finite elastostatics can change from elliptic to non-elliptic at sufficiently large deformations for certain materials. This introduces the possibility that the elastostatic field may exhibit certain discontinuities. Some aspects of the general theory associated with these issues were examined in a recent series of studies by Knowles and Sternberg. In this paper we illustrate the occurrence of elastostatic fields with discontinuous deformation gradients in a physical problem. The body is assumed to be composed of a material which belongs to a particular class of isotropic, incompressible, elastic materials which allow for a loss of ellipticity. It is shown that no solution which is smooth in the classical sense exists to this problem for certain ranges of the applied loading. Next, we admit solutions involving elastostatic shocks into the discussion and find that the problem may then be solved completely. When this is done, however, there results a lack of uniqueness of solutions to the boundary-value problem. In order to resolve this non-uniqueness, the dissipativity and stability of the solutions are investigated.

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1.1 Introduction

In the course of investigating some crack problems [1], [2], [3], Knowles and Sternberg encountered certain difficulties which suggested that the problem may not admit a classically smooth solution. In order to clarify this situation, a series of preliminary studies were undertaken (References [4] - [7]) in which these authors looked at the question of the change of type of the displacement equations of equilibrium from elliptic to non-elliptic, and the related issue of the existence of solutions possessing certain discontinuities – referred to as elastostatic shocks. The presence of such elastostatic shocks is found to affect the energy balance of the field. This led Knowles and Sternberg [6], [7] to propose a notion of dissipativity associated with such fields. Subsequently, Abeyaratne [8] examined the corresponding issues in the case of incompressible materials.

In order to illustrate the occurrence of elastostatic shocks in a boundary-value problem, we consider a problem in finite plane strain for a hollow circular cylinder. Specifically, we examine the case in which the outer surface of the cylinder is held fixed while the inner surface is twisted circumferentially. The cylinder is presumed to be composed of a homogeneous, isotropic, incompressible elastic material. Although this problem has been considered before[1], our interest centers on those materials whose strain energy density permits a failure of ellipticity of the displacement equations of equilibrium at sufficiently severe deformations.

We demonstrate for our choice of material that, while for both sufficiently large and small values of the prescribed twist the problem

[1] See Rivlin [9] as well as Green and Zerna [10] page 95. The problem has been reconsidered more recently by Ogden, Chadwick and Haddon [11].

admits a unique smooth solution, there are certain intermediate ranges of the prescribed twist at which no classically smooth solutions exist. We then show that there are however, an infinite number of weak solutions involving elastostatic shocks in these ranges of the applied twist.

We then consider the quasi-static problem in which the prescribed twist is gradually changed in time, and explore the consequences of the dissipation inequality. It turns out that enforcing this inequality fails to single out a unique weak solution.

In an attempt to clarify this issue of non-uniqueness, we examine the stability of the various equilibrium solutions against purely circumferential perturbations. It is found that the classical energy criterion for stability, without reference to the dissipation inequality, picks out a unique solution to the boundary-value problem at every value of the prescribed twist. In the discussion of the various issues outlined above, we restrict attention to configurations involving not more than one elastostatic shock. As a consequence of the stability criterion, we find that an equilibrium solution involving more than one shock is, in fact, unstable.

Ericksen [12] has discussed the equilibrium of a bar composed of a material whose stress response in uniaxial tension is qualitatively similar to the shear stress response in simple shear of the class of materials considered here. There is a striking similarity between his results and ours; in fact, certain aspects of our study of the stability of weak solutions were suggested by the arguments in [12].

In Section 2 we set up the classical problem governing the twisting of a hollow cylinder composed of an arbitrary homogeneous, incompressible, isotropic, elastic solid. We then discuss the particular class of materials with which we will be concerned. In Section 3 we determine the solutions of this problem and construct the associated torque-twist curves. For sufficiently small and large values of the prescribed twist, we have a unique smooth solution at which the displacement equations of equilibrium are elliptic. Depending on the details of the geometry and constitutive law, it is also possible to have a unique, non-elliptic, smooth solution at certain — but not all — values of the twist in the intermediate range. In all cases there are ranges of values of the prescribed twist for which we find no solution. We then prove that, in fact, no smooth solutions exist in these ranges of the prescribed twist.

We next set up and solve, in Section 4 the problem in its weak formulation. We now find a solution corresponding to every value of the prescribed twist, but unfortunately there are many solutions corresponding to certain twist values.

In Section 5 we make use of the dissipation inequality in an unsuccessful attempt to extract a unique solution from among the many solutions to the boundary value problem. Finally, in Section 6 we examine the stability of each of the available solutions against purely circumferential perturbations. We find that at every value of the prescribed twist there is precisely one stable solution to the boundary-value problem in its weak formulation. For sufficiently small and large values of the applied twist, this unique stable solution is smooth and elliptic. For all intermediate values, the unique stable solution involves an elastostatic shock and is elliptic.

2.1 Formulation of Problem

Suppose that the open region $\mathcal{R}$ occupied by the interior of a body in its undeformed configuration is a hollow right circular cylinder of internal and external radii a and b, respectively. Let Π be the open middle cross-section of the cylinder $\mathcal{R}$, and let O be the center of the annular region Π.

Suppose the inner surface of the cylinder is rotated circumferentially through an angle ϕ_0, while the outer lateral surface is held fixed. We assume that the resulting deformation maps the point with cylindrical coordinates (r, θ, z) in the undeformed configuration onto the point with cylindrical coordinates (ρ, ψ, ζ), where

$$\rho = \hat{\rho}(r,\theta,z) = r \ , \quad \psi = \hat{\psi}(r,\theta,z) = \theta + \phi(r) \ , \quad \zeta = \hat{\zeta}(r,\theta,z) = z \quad . \tag{2.1}$$

This describes a plane deformation in which each particle moves circumferentially through an angle $\phi(r)$. Suitable tractions are presumed to be applied on the ends of the cylinder so as to maintain such a state of plane strain.

The deformation (2.1) may be equivalently expressed as follows. Let X be a fixed rectangular cartesian coordinate frame with its origin at O and orthonormal base vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$, such that $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ are in the plane of Π and $\underset{\sim}{e}_3$ is normal to Π. If $\underset{\sim}{x}$ is the position vector after deformation of the particle which was located at $\underset{\sim}{x}$ in the undeformed configuration, we can write (2.1) as

$$\left.\begin{aligned} y_1 &= x_1 \cos\phi(r) - x_2 \sin\phi(r) , \\ y_2 &= x_2 \cos\phi(r) + x_1 \sin\phi(r) , \\ y_3 &= x_3 , \end{aligned}\right\} \tag{2.2}$$

where

$$r = \sqrt{x_1^2 + x_2^2} . \tag{2.3}$$

Here y_i and x_i, $i = 1, 2, 3$, are the components of the vectors $\underset{\sim}{y}$ and $\underset{\sim}{x}$ in the frame X. We will temporarily assume that the local angle of twist $\phi(r)$ is twice continuously differentiable on (a, b).

It is convenient to express the field quantities at any point (r, θ, z) in terms of components in the rectangular cartesian coordinate frame X' which is obtained by rotating the frame X through an angle $\hat{\psi}(r, \theta, z)$ about the $\underset{\sim}{e}_3$-axis. The matrix of components of the deformation gradient tensor $\underset{\sim}{F} = \nabla \underset{\sim}{y}$ in the frame X', $\underset{\sim}{F}^{X'}$, is easily computed from (2.2), (2.3) and the change-of-frame formula for tensors to be

$$\underset{\sim}{F}^{X'} = \begin{bmatrix} \cos\phi(r) & -\sin\phi(r) & 0 \\ \sin\phi(r) + r\phi'(r)\cos\phi(r) & \cos\phi(r) - r\phi'(r)\sin\phi(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.4}$$

Note that the matrix $\underset{\sim}{F}^{X'}$ may be decomposed as follows;

$$\underset{\sim}{F}^{X'} = \begin{bmatrix} 1 & 0 & 0 \\ r\phi'(r) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi(r) & -\sin\phi(r) & 0 \\ \sin\phi(r) & \cos\phi(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{2.5}$$

which implies that locally the deformation (2.1) is composed of a rigid rotation through an angle ϕ about the $\underset{\sim}{e}_3$-axis followed by a simple shear parallel to the circumferential direction with an amount of shear $r\phi'(r)$. Set

$$k(r) = r\phi'(r) , \tag{2.6}$$

so that $k(r)$ is the local amount of shear.

Equation (2.4) indicates that $\det \underset{\sim}{F} = 1$, so that the deformation (2.1) is locally volume preserving. From (2.4) and (2.6) we have the components of the left Cauchy-Green tensor $\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T$:

$$\underset{\sim}{G}^{X'} = \begin{bmatrix} 1 & k(r) & 0 \\ k(r) & 1 + k^2(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.7}$$

The principal invariants of $\underset{\sim}{G}$ are found from (2.7) to be

$$\left. \begin{aligned} I_1 &= \operatorname{tr} \underset{\sim}{G} = 3 + k^2(r) , \\ I_2 &= \tfrac{1}{2}\left\{(\operatorname{tr} \underset{\sim}{G})^2 - \operatorname{tr} \underset{\sim}{G}^2\right\} = 3 + k^2(r) , \\ I_3 &= \det \underset{\sim}{G} = 1 . \end{aligned} \right\} \tag{2.8}$$

Suppose that the body is composed of a homogeneous, isotropic, incompressible, elastic solid which possesses an elastic potential $W = \overset{*}{W}(I_1, I_2)$. W represents the strain energy density per unit undeformed volume. The constitutive law for the Cauchy stress tensor $\underset{\sim}{\tau}$ is then

$$\underset{\sim}{\tau} = 2\left(\frac{\partial \overset{*}{W}}{\partial I_1} + I_1 \frac{\partial \overset{*}{W}}{\partial I_2}\right)\underset{\sim}{G} - 2\,\frac{\partial \overset{*}{W}}{\partial I_2}\,\underset{\sim}{G}^2 - p\underset{\sim}{1} \ , \qquad (2.9)^1$$

where $p(\underset{\sim}{y})$ is a pressure field arising because of the constraint of incompressibility. We suppose for the moment that $p(\underset{\sim}{y})$ is continuously differentiable on $\mathcal{R}$. Using (2.7), (2.8) and (2.9) we find that the stresses induced by the deformation (2.1) are given by

$$\left.\begin{aligned}
&\tau_{11}^{X'} = 2W'(I) - q \ , \\
&\tau_{12}^{X'} = \tau_{21}^{X'} = 2kW'(I) \ , \\
&\tau_{22}^{X'} = 2(1 + k^2)W'(I) - q \ , \\
&\tau_{13}^{X'} = \tau_{31}^{X'} = \tau_{23}^{X'} = \tau_{32}^{X'} = 0 \ , \\
&\tau_{33}^{X'} = 2W'(I) + 2k^2\,\frac{\partial \overset{*}{W}}{\partial I_2}(I+1,\ I+1) - q \ ,
\end{aligned}\right\} \qquad (2.10)$$

where we have set

$$I = 2 + k^2(r) \ , \ \ W(I) = \overset{*}{W}(I+1,\ I+1) \quad \text{for } I \geq 2 \qquad (2.11)$$

and

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[1]See Truesdell and Noll [13], page 319.

$$q = p - 2\frac{\partial \overset{*}{W}}{\partial I_2} . \tag{2.12}$$

Since the pressure p might depend on the coordinates ψ and ξ, in addition to ρ, it follows that the Cauchy stress tensor $\underset{\sim}{\tau}$ might depend on all three of ρ, ψ, and ξ.

It is suggestive to introduce the notation

$$\left.\begin{aligned} \tau_{\rho\rho} &= \tau^{X'}_{11} , \\ \tau_{\psi\psi} &= \tau^{X'}_{22} , \\ \tau_{\rho\psi} &= \tau_{\psi\rho} = \tau^{X'}_{12} . \end{aligned}\right\} \tag{2.13}$$

The equilibrium equation[1] in the axial direction is easily shown to be satisfied if and only if q does not depend on y_3 (and hence ξ). It follows from (2.10), that the Cauchy stress tensor $\underset{\sim}{\tau}$ is also independent of the axial coordinate ξ. The remaining two equilibrium equations now take the form

$$\frac{\partial \tau_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\rho\psi}}{\partial \psi} + \frac{1}{\rho}(\tau_{\rho\rho} - \tau_{\psi\psi}) = 0 , \tag{2.14}$$

$$\frac{\partial \tau_{\rho\psi}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\psi\psi}}{\partial \psi} + \frac{2}{\rho}\tau_{\rho\psi} = 0 . \tag{2.15}$$

From (2.1), (2.6), (2.10), (2.11) and (2.13) we see that $\tau_{\rho\psi}$ is independent of the coordinate ψ, whence (2.14) and (2.15) specialize to

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[1]Body forces are presumed to be absent.

$$\frac{\partial \tau_{\rho\rho}}{\partial \rho} + \frac{\tau_{\rho\rho} - \tau_{\psi\psi}}{\rho} = 0 \ , \tag{2.16}$$

$$\frac{d}{d\rho}\left(\rho^2 \tau_{\rho\psi}\right) + \rho \frac{\partial \tau_{\psi\psi}}{\partial \psi} = 0 \ . \tag{2.17}$$

Integration of (2.17) with respect to ψ leads to

$$\tau_{\psi\psi} = -\left\{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho^2 \tau_{\rho\psi}\right)\right\}\psi + c(\rho) \ , \tag{2.18}$$

where $c(\rho)$ is a "constant" of integration depending on ρ alone. It is apparent from (2.18) that $\tau_{\psi\psi}$ is single-valued only if

$$\frac{d}{d\rho}\left(\rho^2 \tau_{\rho\psi}\right) = 0 \ . \tag{2.19}$$

It now follows from (2.18) and (2.19) that $\tau_{\psi\psi}$, and hence q, p and $\tau_{\rho\rho}$ as well, are independent of the angular coordinate ψ. Using (2.10) and (2.13) in (2.16) and (2.19) we obtain the governing system of ordinary differential equations for $\phi(\rho)$ and $q(\rho)$:

$$\frac{d}{d\rho}\left\{\rho^3 \phi'(\rho) W'(2 + \rho^2 \phi'^2(\rho))\right\} = 0 \ , \tag{2.20}$$

$$2\frac{d}{d\rho} W'(2 + \rho^2 \phi'^2(\rho)) - 2\rho\phi'^2(\rho) W'(2 + \rho^2 \phi'^2(\rho)) = \frac{dq}{d\rho} \ . \tag{2.21}$$

On integrating (2.20) with respect to ρ we find that

$$\rho^3 \phi'(\rho) W'(2 + \rho^2 \phi'^2(\rho)) = -\frac{T}{4\pi} \ , \tag{2.22}$$

where T is a constant of integration. Likewise, integration of (2.21) with respect to ρ and making use of (2.22) gives

$$q(\rho) = 2W'(2+\rho^2\phi'^2(\rho)) + \frac{T}{2\pi}\int_a^\rho \frac{\phi'(\xi)}{\xi^2}\,d\xi + q_0 \ , \tag{2.23}$$

where q_0 is a constant.

It is convenient to define the scalar valued function f by

$$f(k) = 2kW'(2+k^2) \quad \text{for} \quad -\infty<k<\infty \ . \tag{2.24}$$

It is readily seen that, if an incompressible, isotropic, elastic solid is subjected to a simple shear deformation, the shear stress corresponding to an amount of shear k is $f(k)$. Accordingly the function f may be interpreted as the <u>shear-stress response function in simple shear</u>.

Equation (2.22) can now be written as

$$f(\rho\phi'(\rho)) = -\frac{T}{2\pi\rho^2} \quad \text{on} \quad (a,b) \ , \tag{2.25}$$

which, together with the boundary conditions

$$\phi(a) = \phi_0 \ , \tag{2.26}$$

$$\phi(b) = 0 \ , \tag{2.27}$$

constitutes the boundary value problem for $\phi(\rho)$. We wish to find a function $\phi(\rho)$, continuous on $[a,b]$ and twice continuously differentiable on (a,b), and a real number T such that (2.25) - (2.27) hold. We will refer to such a solution as a <u>smooth solution</u>. Note that once $\phi(\rho)$

has been so determined, (2.23) gives $q(\rho)$ directly.

Finally, note from (2.6), (2.10), (2.13) and (2.22) that $\tau_{\rho\phi}(\rho) = -T/2\pi\rho^2$ so that T is the torque per unit axial length of the cylinder acting on the inner surface, measured positive in the counter-clockwise sense.

2.2 A Particular Class of Constitutive Laws

We now describe the particular class of homogeneous, isotropic, incompressible, elastic materials to which we will restrict attention in this study. It is adequate for our purposes to specify the response of the material in simple shear alone. Observe from (2.24) that $W(I)$ is completely determined by the function f. Consequently, one can show that the response in simple shear determines completely the in-plane response in all plane deformations for such materials.

Equation (2.24) implies that f is an odd function, i.e.,

$$f(k) = -f(-k) \quad \text{for} \quad -\infty < k < \infty \ . \tag{2.28}$$

We assume that

(i) f is continuously differentiable on $(-\infty, \infty)$,

(ii) f is positive on $(0, \infty)$, whence it follows from (2.28) that

$$kf(k) > 0 \quad \text{for} \quad k \neq 0 \ , \tag{2.29}$$

(iii) there exist real numbers k_1 and k_2 $(0 < k_1 < k_2 < \infty)$ such that

$$\left.\begin{aligned} &f'(k_1) = f'(k_2) = 0 \ , \\ &f'(k) > 0 \quad \text{for} \quad 0 \le k < k_1 \ , \ k_2 < k < \infty \ , \\ &f'(k) < 0 \quad \text{for} \quad k_1 < k < k_2 \ , \end{aligned}\right\} \tag{2.30}$$

(iv) $f(k) \to \infty$ as $k \to \infty$.

Since f is an odd function it now follows that $f'(-k_1) = f'(-k_2) = 0$, $f'(k) > 0$ for $-\infty < k < -k_2$, $-k_1 < k \leq 0$, $f'(k) < 0$ on $-k_2 < k < -k_1$ and $f(k) \to -\infty$ as $k \to -\infty$. Therefore, the function $f(k)$ has local maxima at $k = k_1$, $-k_2$ and local minima at $k = k_2$, $-k_1$ and is monotone in between. A graph of such a function f is shown in Fig. 1, where we have set

$$f(k_1) = \tau_{max}, \quad f(k_2) = \tau_{min} . \tag{2.31}$$

Note that necessarily

$$\left.\begin{array}{rl} |f(k)| \leq \tau_{max} & \text{for } |k| \leq k_1 , \\ \tau_{min} \leq |f(k)| \leq \tau_{max} & \text{for } k_1 \leq |k| \leq k_2 , \\ |f(k)| \geq \tau_{min} & \text{for } |k| \geq k_2 . \end{array}\right\} \tag{2.32}$$

An isotropic, incompressible, elastic solid subjected to plane deformations conforms to the <u>in-plane Baker-Ericksen inequality</u> if and only if $W'(I) > 0$ for $I > 2$. By virtue of (2.24) this is equivalent to $kf(k) > 0$ for $k \neq 0$. Because of (2.29), the class of materials under consideration satisfies this condition.

Moreover, we have from Section 3.2 of [8] that in <u>any</u> plane deformation, the plane strain displacement equations of equilibrium are elliptic at a point if and only if the associated local amount of shear[1] is less than k_1 or greater than k_2 . In the context of the problem considered here, we have from (2.6) and (2.11) that <u>the</u>

[1]See Section 2.2 of [8].

displacement equations of equilibrium are elliptic, at a solution corresponding to $\phi(r)$ and a point, if and only if $|r\phi'(r)|$ is less than k_1 or greater than k_2 at that point.

It is clear that for the particular class of materials just described, f has no single-valued inverse. The restrictions of f to certain subintervals of $(-\infty,\infty)$, on the other hand, do have unique inverses. Let F_1, F_2 and F_3 be the functions defined by

$$\left.\begin{aligned} F_1(f(k)) &= k \quad \text{for } |k| \le k_1 , \\ F_2(f(k)) &= k \quad \text{for } k_1 \le |k| \le k_2 , \\ F_3(f(k)) &= k \quad \text{for } k_2 \le |k| < \infty . \end{aligned}\right\} \qquad (2.33)$$

By virtue of (2.32), it follows that F_1, F_2 and F_3 are defined on $[-\tau_{max}, \tau_{max}]$, $[-\tau_{max}, -\tau_{min}] \cup [\tau_{min}, \tau_{max}]$ and $(-\infty, -\tau_{min}] \cup [\tau_{min}, \infty)$ respectively, and that they are continuously differentiable on the corresponding open intervals.

The following properties of the inverses F_i $(i = 1, 2, 3)$ can be easily verified; we list them here for subsequent reference.

$$\left.\begin{aligned} f(F_1(\tau)) &= \tau \quad \text{for } |\tau| \le \tau_{max} , \\ f(F_2(\tau)) &= \tau \quad \text{for } \tau_{min} \le |\tau| \le \tau_{max} , \\ f(F_3(\tau)) &= \tau \quad \text{for } \tau_{min} \le |\tau| < \infty , \end{aligned}\right\} \qquad (2.34)$$

$$\left.\begin{aligned} &F_1'(\tau)>0 \quad \text{for } |\tau|<\tau_{max} , \\ &F_2'(\tau)<0 \quad \text{for } \tau_{min}<|\tau|<\tau_{max} , \\ &F_3'(\tau)>0 \quad \text{for } \tau_{min}<|\tau|<\infty , \end{aligned}\right\} \tag{2.35}$$

$$\left.\begin{aligned} F_1(\tau_{max}) &= F_2(\tau_{max}) = k_1 , \\ F_2(\tau_{max}) &= F_3(\tau_{max}) = k_2 , \end{aligned}\right\} \tag{2.36}$$

$$\left.\begin{aligned} |F_1(\tau)| \le k_1 &\quad \text{for } |\tau| \le \tau_{max} , \\ k_1 \le |F_2(\tau)| \le k_2 &\quad \text{for } \tau_{min} \le |\tau| \le \tau_{max} , \\ |F_3(\tau)| \ge k_3 &\quad \text{for } |\tau| \ge \tau_{min} , \end{aligned}\right\} \tag{2.37}$$

$$F_3(|\tau|)>F_2(|\tau|)>F_1(|\tau|) \quad \text{for } \tau_{min}<|\tau|<\tau_{max} , \tag{2.38}$$

$$F_i(-\tau) = -F_i(\tau) , \tag{2.39}$$

$$\tau F_i(\tau)>0, (\tau \neq 0) , \tag{2.40}$$

$$|F_i(\tau)| = F_i(|\tau|) , \tag{2.41}$$

$i = 1, 2, 3$ and for τ in the appropriate interval

$$F_3(\tau) \to \pm\infty \quad \text{as } \tau \to \pm\infty . \tag{2.42}$$

3.1 Smooth Solutions

We now return to the task of solving (2.25) - (2.27) for the special class of materials described in the previous section. To this end, we first establish the following preliminary result.

Lemma: There does not exist a solution $\phi(r)$ in the class $C^2(a,b)$ to the differential equation

$$f(r\phi'(r)) = -T/2\pi r^2 \tag{3.1}$$

where f is a continuously differentiable function conforming to (2.28) - (2.30) and T is a constant, such that at some radius s, $a<s<b$,

$$s\phi'(s) = \pm k_1 \text{ or } \pm k_2 . \tag{3.2}$$

Proof: Suppose that there is such a solution $\phi(r)$. Differentiation of (3.1) with respect to r and setting $r = s$ leads to

$$f'(s\phi'(s))\left\{\phi'(s) + s\phi''(s)\right\} = T/\pi s^3 ,$$

which because of (2.28), (2.30) and (3.2), yields

$$T = 0 . \tag{3.3}$$

The differential equation (3.1) now reads $f(r\phi'(r)) = 0$, which because of (2.28) and (2.29) implies that

$$\phi'(r) = 0 \quad \text{on } (a,b) \ . \tag{3.4}$$

Equation (3.4), however, contradicts the assumption (3.2). This establishes the lemma.

Now suppose that the prescribed twist ϕ_0 is a number in the interval

$$-\int_a^b \frac{1}{\xi} F_1\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi \le \phi_0 \le \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi \ . \tag{3.5}$$

On using (2.26), (2.27), (2.35) and (2.37) in (3.5) we have

$$-\int_a^b \frac{k_1}{\xi} d\xi < -\int_a^b \phi'(\xi) d\xi < \int_a^b \frac{k_1}{\xi} d\xi \ , \tag{3.6}$$

whence

$$\int_a^b \frac{\xi\phi'(\xi) + k_1}{\xi} d\xi > 0 \ , \ \int_a^b \frac{\xi\phi'(\xi) - k_1}{\xi} d\xi < 0 \ . \tag{3.7}$$

From the preceding lemma we know that $r\phi'(r) \ne \pm k_1$ on (a, b), so that (3.7) implies

$$k_1 > r\phi'(r) > -k_1 \quad \text{on } (a,b) \ , \tag{3.8}$$

since the integrands in (3.7) are continuous on (a, b). Therefore, if ϕ_0 is a number such that (3.5) holds, then necessarily the solution to (2.25) - (2.27) must satisfy (3.8) But because of (2.33) and (2.34), we see that, (2.25) and (3.8) hold if and only if

$$r\phi'(r) = F_1\left(-\frac{T}{2\pi r^2}\right) . \tag{3.9}$$

Integrating this and using the boundary condition (2.27) together with (2.39) leads to

$$\phi(r) = \int_r^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi . \tag{3.10}$$

On enforcing the boundary condition (2.26), we have from (3.10) that

$$\phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi . \tag{3.11}$$

Finally, we verify that (3.11) determines a unique number T for every given number ϕ_0 in the interval defined by (3.5). To this end, define the function Φ_1 by

$$\Phi_1(T) = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{for} \quad |T| \leq 2\pi a^2 \tau_{max} . \tag{3.12}$$

On differentiating (3.12) with respect to T and making use of (2.35), we find that

$$\Phi_1'(T) > 0 \quad \text{for} \quad |T| \leq 2\pi a^2 \tau_{max} , \tag{3.13}$$

whence Φ_1 is monotonically increasing on $[-2\pi a^2 \tau_{max}, 2\pi a^2 \tau_{max}]$. Thus, if ϕ_0 is a number such that

$$\Phi_1(-2\pi a^2 \tau_{max}) \leq \phi_0 \leq \Phi_1(2\pi a^2 \tau_{max}) , \tag{3.14}$$

then, $\phi_0 = \Phi_1(T)$ defines a unique number T . Note that (3.14) because of (3.12), is identical to (3.5).

Therefore, we conclude that, if the prescribed twist ϕ_0 is in the interval defined by (3.5) , Equation (3.11) determines a unique real number T , which together with (3.10) gives the corresponding unique smooth solution to (2.25) - (2.27).

In an entirely analogous manner, we can show that, if the prescribed twist ϕ_0 satisfies

$$\int_a^b \frac{1}{\xi} F_2\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \le |\phi_0| \le \int_a^b \frac{1}{\xi} F_2\left(\frac{b^2 \tau_{min}}{\xi^2}\right) d\xi \ , \tag{3.15}$$

then, the relation

$$\phi_0 = \int_a^b \frac{1}{\xi} F_2\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{3.16}$$

determines a unique real number T , which together with

$$\phi(r) = \int_r^b \frac{1}{\xi} F_2\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{3.17}$$

is the corresponding unique smooth solution to (2.25) - (2.27).

Similarly, if the prescribed twist ϕ_0 obeys

$$|\phi_0| \ge \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2 \tau_{min}}{\xi^2}\right) d\xi \ , \tag{3.18}$$

the relation

$$\phi_0 = \int_a^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{3.19}$$

determines a unique real number T, which together with

$$\phi(r) = \int_r^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{3.20}$$

is the corresponding unique smooth solution to (2.25) - (2.27).

We will refer to (3.10), (3.17) and (3.20) as (smooth) <u>Solution</u> 1, <u>Solution</u> 2 and <u>Solution</u> 3 respectively. Equations (3.11), (3.16) and (3.19) are the corresponding <u>torque-twist relations</u>. One sees readily from (3.10), (3.17), (3.20) and the discussion of ellipticity in Section 2.2, that the displacement equations of equilibrium are elliptic everywhere in Π at Solution 1 and Solution 3, and that they are non-elliptic at Solution 2.

Because of (2.35) one has

$$\int_a^b \frac{1}{\xi} F_2\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi \le \int_a^b \frac{1}{\xi} F_2\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi \ , \tag{3.21}$$

if and only if

$$b^2\tau_{min} \le a^2\tau_{max} \ . \tag{3.22}$$

Accordingly, it is only when (3.22) holds that there are values of ϕ_0 in the interval (3.15), and consequently that Solution 2 exists. In this paper, we will consider in detail the case when the dimensions of the tube and the constitutive law of the material are such that

$$\frac{a^2}{b^2} > \frac{\tau_{min}}{\tau_{max}} , \tag{3.23}$$

and trace the other cases (which are in fact less complicated) through footnotes. The end result turns out to be the same in all cases. For a given material, one could view (3.23) as requiring the thickness of the tube to be sufficiently small. Since $\tau_{\rho\psi}(\rho) = -T/2\pi\rho^2$, we have $\tau_{\rho\psi}(a)/\tau_{\rho\psi}(b) = b^2/a^2$ in any equilibrium configuration of the body irrespective of the magnitude of the applied twist. Thus (3.23) can be written as

$$\frac{\tau_{\rho\psi}(a)}{\tau_{\rho\psi}(b)} < \frac{\tau_{max}}{\tau_{min}} .$$

The torque-twist relations (3.11), (3.16) and (3.19) are sketched in Fig. 2. Clearly, these curves are anti-symmetric with respect to the axes.

We observe from the preceding calculations, and also from Fig. 2, that we have not as yet found any solutions to (2.25) - (2.27) if the prescribed twist lies in one of the intervals

$$\left.\begin{aligned} \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi < |\phi_0| < \int_a^b \frac{1}{\xi} F_2\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi , \\ \int_a^b \frac{1}{\xi} F_2\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi < |\phi_0| < \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi . \end{aligned}\right\} \tag{3.24}$$

We show in the next section that there are, in fact, <u>no</u> smooth solutions when the prescribed twist lies in these ranges.

3.2 Non-existence of Smooth Solutions

We now establish a sequence of lemmas leading to a result which is in fact stronger than the one claimed at the end of the last section. We will show that there is no solution $\phi(r)$ to (2.25) - (2.27) which is continuously differentiable, if the prescribed twist ϕ_0 is in one of the intervals defined by (3.24).

> Lemma 1: There is no continuously differentiable solution $\phi(r)$ to the differential equation (3.1), where T is a constant and f is a continuously differentiable function conforming to (2.28) - (2.30), for which (3.2) holds at some radius s, $a<s<b$.

Proof: Assume that there exists such a solution $\phi(r)$ and suppose that

$$k(s) = s\phi'(s) = +k_1 \ . \tag{3.25}$$

By hypothesis $k(r) = r\phi'(r)$ is continuous on (a, b) so that, in particular, it is continuous at $r = s$. Therefore, given any number $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon) > 0$ such that $|k(s) - k(r)| < \epsilon$ for all r such that $|r - s| < \delta(\epsilon)$. Using (3.25) we may write this as

$$|k_1 - k(r)| < \epsilon \quad \text{for all } |r - s| < \delta(\epsilon) \ . \tag{3.26}$$

Recall that $f(k)$ has a local maximum at $k = k_1$, so that there is a number $\eta > 0$ such that

$$f(k_1) \geq f(k) \quad \text{for } |k_1 - k| < \eta \ . \tag{3.27}$$

Combining (3.26) with (3.27), we have

$$f(k_1) \geq f(k(r)) \quad \text{for } |r - s| < \delta(\eta) \ ,$$

which on using (2.6), (3.1) and (3.25) leads to

$$\frac{-T}{2\pi s^2} \geq -\frac{T}{2\pi r^2} \quad \text{for} \quad |r-s| < \delta(\eta) \ . \tag{3.28}$$

Note from (3.1) and (3.25) that $T = -2\pi s^2 f(k_1)$, whence $T < 0$. Equation (3.28) now requires that

$$r^2 \geq s^2 \quad \text{for} \quad s - \delta(\eta) < r < s + \delta(\eta)\ , \ \delta(\eta) > 0 \ , \tag{3.29}$$

which is impossible. Consequently there cannot exist a solution $\phi(r)$ with the properties we assumed.

The cases $s\phi'(s) = -k_1$, $\pm k_2$ can be dealt with similarly.

<u>Lemma 2:</u> Suppose that there exists a continuously differentiable solution $\phi(r)$ to (2.25) - (2.27), where T is a constant and f is as in Lemma 1. Then

(i) $|r\phi'(r)| < k_1$ on (a, b) if and only if ϕ_0 is in the interval (3.5).

(ii) $k_1 < |r\phi'(r)| < k_2$ on (a, b) if and only if ϕ_0 is in the interval (3.29)

(iii) $|r\phi'(r)| > k_2$ on (a, b) if and only if ϕ_0 is in the interval (3.18)

<u>Proof:</u> Considering part (i), suppose that ϕ_0 is in the interval (3.5). By virtue of Lemma 1, the steps leading from (3.5) to (3.8), go through even when ϕ is merely continuously differentiable. Thus necessarily $|r\phi'(r)| < k_1$ on (a, b).

Conversely, suppose that $|r\phi'(r)| < k_1$ on (a, b). It follows from (2.25) and (3.25) that

$$\frac{|T|}{2\pi r^2} \leq \tau_{max} \quad \text{on } (a, b) \ , \tag{3.30}$$

whence

$$|T| \leq 2\pi a^2 \tau_{max} \; . \tag{3.31}$$

Since $|r\phi'(r)| < k_1$ on (a, b), we have because of (2.33) that (2.25) holds only if

$$r\phi'(r) = F_1\left(-\frac{T}{2\pi r^2}\right) \; . \tag{3.32}$$

Integrating (3.32) and using (2.26), (2.27) and (2.39) gives

$$\phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi \xi^2}\right) d\xi \; , \tag{3.33}$$

which by virtue of (2.40) and (2.41) leads to

$$|\phi_0| = \int_a^b \frac{1}{\xi} F_1\left(\frac{|T|}{2\pi \xi^2}\right) d\xi \; . \tag{3.34}$$

Since by (2.35) F_1 is a monotone increasing function, it follows from (3.31) and (3.34) that

$$|\phi_0| \leq \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \; . \tag{3.35}$$

which completes the proof of part (i) of the lemma. Parts (ii) and (iii) can be similarly established.

> <u>Lemma 3:</u> There does <u>not</u> exist a continuously differentiable solution $\phi(r)$ to (2.25) - (2.27), where T is a constant and f is as in Lemma 1, if the prescribed twist ϕ_0 is in one of the intervals (3.24).

Proof: This result follows immediately from Lemmas 1 and 2. For, suppose that there is such a solution $\phi(r)$. It follows from Lemma 2 that we must have

$$r\phi'(r) = \pm k_1 \text{ or } \pm k_2 \quad \text{at some } r,\ a<r<b \ . \tag{3.36}$$

But Lemma 1 says that this is impossible.

We have thus shown that for certain ranges of the prescribed twist, there is no solution in the classical sense to the problem under consideration.

4.1 Weak Formulation of Problem

There are some problems of considerable physical interest in which the field quantities do not vary smoothly through the body. Rice gives some examples of such problems in [14]. We have observed that the problem under consideration here has no smooth solution for certain ranges of the applied loading. One possibility, which we shall not consider, is that the tube buckles, possibly into some unsymmetric state of plane strain at such a loading. An alternative possibility is that the tube remains in a configuration of axisymmetric plane strain, but that now the field quantities are no longer smooth and exhibit certain discontinuities. This latter possibility is suggested by the observation in Section 2.2 that the displacement equations of equilibrium may suffer a loss of ellipticity at certain deformations for the material at hand. In particular, in view of known results in the theory of transonic gas flows, one would anticipate the occurrence of curves across which the first derivatives of the displacement field suffer jump discontinuities, while the displacement field itself remains continuous.

General questions concerning the possibility of the change of type of the differential equations governing finite plane elastostatics and the related issue of the existence of equilibirum fields with discontinuous deformation gradients have been investigated in [4] - [8]. Elastostatic fields possessing discontinuities of this type are referred to as "elastostatic shocks".

We now relax the smoothness demanded of the local twist $\phi(r)$

and the pressure field $q(r)$, in the hope that this will enable us to explain what happens when the prescribed twist ϕ_0 is in one of the intervals (3.24).

To this end, let $\bar{r}$ be a number in the interval $[a, b]$. If in fact $a < \bar{r} < b$, we will now require that $\phi(r)$ be merely twice continuously differentiable on the intervals $(a, \bar{r})$ and $(\bar{r}, b)$ and continuous on $[a, b]$. The stress field and pressure field induced by the deformation (2.1) are to be continuously differentiable on $(a, \bar{r})$ and $(\bar{r}, b)$ while the traction is presumed to be continuous at $r = \bar{r}$. Accordingly, we have admitted the possibility of the existence of a cylindrical elastostatic shock[1] of radius $\bar{r}$ co-axial with the cylindrical region $\mathcal{R}$.

The global balance laws, which continue to be meaningful, now reduce to the same differential equations obtained in Section 2.1 on $(a, \bar{r})$ and $(\bar{r}, b)$, together with jump conditions at $r = \bar{r}$. Accordingly we now have

$$\left.\begin{aligned} &\frac{d}{dr}\left\{r^2 f(r\phi'(r))\right\} = 0, &(4.1)\\ &2\frac{d}{dr}W'(2 + r^2\phi'^2(r)) - 2r\phi'^2(r)W'(2 + r^2\phi'^2(r)) = \frac{dq}{dr}, &(4.2)\end{aligned}\right\} \text{ on } a < r < \bar{r},\ \bar{r} < r < b,$$

instead of (2.20) and (2.21). On integrating (4.1) we have

$$f(r\phi'(r)) = -\frac{\overset{-}{T}}{2\pi r^2} \text{ on } (a, \bar{r}), \quad f(r\phi'(r)) = -\frac{\overset{+}{T}}{2\pi r^2} \text{ on } (\bar{r}, b), \tag{4.3}$$

where $\overset{+}{T}$ and $\overset{-}{T}$ are (not necessarily equal) constants. Because of

[1] We formulate the problem in the case when a single elastostatic shock exists. We will find that this suffices for our purposes, and more importantly, that a configuration involving more than one shock is necessarily unstable (in a sense to be made precise).

(2.10), (2.13), (2.24) and (4.3) we have that

$$\tau_{\rho\psi}(r) = -\frac{\overset{-}{T}}{2\pi r^2} \text{ on } (a,\bar{r}) , \quad \tau_{\rho\psi}(r) = -\frac{\overset{+}{T}}{2\pi r^2} \text{ on } (\bar{r},b) . \tag{4.4}$$

At $r=\bar{r}$, equilibrium considerations require that the tractions be continuous. Clearly, this is equivalent to

$$\tau_{\rho\psi}(\bar{r}-) = \tau_{\rho\psi}(\bar{r}+) , \tag{4.5}$$

$$\tau_{\rho\rho}(\bar{r}-) = \tau_{\rho\rho}(\bar{r}+) . \tag{4.6}$$

Equations (4.4) and (4.5) lead to

$$\overset{-}{T} = \overset{+}{T} \equiv T . \tag{4.7}$$

We therefore have the following problem governing the local twist $\phi(r)$. Given a number ϕ_0, find a function $\phi(r)$ which is continuous on $[a,b]$ and twice continuously differentiable on $(a,\bar{r})$ and $(\bar{r},b)$, and numbers T, $\bar{r}$ with $a \le \bar{r} \le b$, such that

$$f(r\phi'(r)) = -\frac{T}{2\pi r^2} \quad \text{on } a<r<\bar{r} , \ \bar{r}<r<b , \tag{4.8}$$

$$\phi(a) = \phi_0 , \tag{4.9}$$

$$\phi(b) = 0 . \tag{4.10}$$

Integration of (4.2) leads to

$$q(r) = \begin{cases} 2W'(2 + r^2\phi'^2(r)) + \int_r^{\bar{r}} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi + \bar{q}_0 & \text{on } (a, \bar{r}) \\ 2W'(2 + r^2\phi'^2(r)) - \int_{\bar{r}}^{r} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi + \overset{+}{q}_0 & \text{on } (\bar{r}, b) \end{cases} \tag{4.11}$$

which, together with (2.10) and (2.13), gives

$$\tau_{\rho\rho}(r) = \begin{cases} -\int_r^{\bar{r}} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi - \bar{q}_0 & \text{on } (a, \bar{r}) , \quad (4.12) \\ \int_{\bar{r}}^{r} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi - \overset{+}{q}_0 & \text{on } (\bar{r}, b) . \quad (4.13) \end{cases}$$

We see from (4.12) and (4.13) that the traction continuity condition (4.6) holds if and only if

$$\overset{+}{q}_0 = \bar{q}_0 . \tag{4.14}$$

Once (4.8) - (4.10) has been solved for the function $\phi(r)$, equation (4.11) together with (4.14) directly gives the pressure field $q(r)$.

4.2 Weak Solutions

We first observe that the Lemma at the beginning of Section 3.1 continues to hold if we replace (a,b) by (r_1, r_2) where r_1 and r_2 are any two numbers such that $a \le r_1 < r_2 \le b$. This result, with the particular choices $r_1 = a$, $r_2 = \bar{r}$ and $r_1 = \bar{r}$, $r_2 = b$, leads to the conclusion that all admissible solutions of (4.8) are necessarily such that

$$r\phi'(r) \neq \pm k_1, \pm k_2 \quad \text{on } a < r < \bar{r}, \ \bar{r} < r < b . \tag{4.15}$$

Since $\phi'(r)$ is continuous on $(a, \bar{r})$ it now follows that any admissible solution to (4.8) must be such that $r\phi'(r)$ takes on values exclusively in one of the intervals $(-\infty, -k_2)$, $(-k_2, -k_1)$, $(-k_1, k_1)$, (k_1, k_2) or (k_1, ∞), at all points in $(a, \bar{r})$. The same must be true on $(\bar{r}, b)$. Therefore we see, because of (2.33) and (2.39), that (4.8) holds if and only if

$$r\phi'(r) = \begin{cases} -F_i\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (a, \bar{r}) , \\ \\ -F_j\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (\bar{r}, b) , \end{cases} \tag{4.16}$$

for some fixed $i, j = 1, 2, 3$.

Integrating (4.16) and using the boundary conditions (4.9) and (4.10) leads to

$$\phi(r) = \begin{cases} \phi_0 - \displaystyle\int_a^r \frac{1}{\xi} F_i\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } [a, \bar{r}) , \\ \\ \displaystyle\int_r^b \frac{1}{\xi} F_j\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } (\bar{r}, b] . \end{cases} \tag{4.17}$$

Finally, we require that

$$\phi_0 = \int_a^{\bar{r}} \frac{1}{\xi} F_i\left(\frac{T}{2\pi\xi^2}\right) d\xi + \int_{\bar{r}}^b \frac{1}{\xi} F_j\left(\frac{T}{2\pi\xi^2}\right) d\xi , \tag{4.18}$$

since the local twist $\phi(r)$ given by (4.17) is supposed to be continuous at $r = \bar{r}$.

Collecting the preceeding results, we come to the following conclusion. Given a real number ϕ_0 , if there exist real numbers T and $\bar{r}$, $a \le \bar{r} \le b$, such that (4.18) holds for some fixed choice of the subscripts $i, j = 1, 2, 3$, then (4.17) with this choice of T , $\bar{r}$, i and j is a solution to (4.8) - (4.10) at the given ϕ_0 .

Clearly in the case when $i = j = 1, 2, 3$, (4.17) and (4.18) describe the smooth solutions we obtained in Section 3.1. This is not surprising, since any smooth solution of (2.25) - (2.27) is also a solution of the problem in its weak formulation. Likewise, in the particular cases when $\bar{r} = a$ and $\bar{r} = b$, (4.17) and (4.18) are readily seen to reduce again to these same smooth solutions. A solution defined by (4.17) and (4.18) is therefore <u>not</u> smooth only if $i \neq j$ and $a < \bar{r} < b$.

The existence of a solution (4.17) corresponding to the prescribed value of the twist ϕ_0 is contingent upon the existence of numbers T and $\bar{r}$, $a \le \bar{r} \le b$, such that (4.18) holds. We now examine this latter issue. First note that since (4.18) furnishes only one scalar restriction on the two numbers T and $\bar{r}$, we expect that if there are values of T and $\bar{r}$ conforming to (4.18), then there would in fact be many such values. If, therefore, we <u>momentarily</u> imagine specifying <u>both</u> ϕ_0 and T , we may pose the following question: at each fixed choice of the subscripts $i, j = 1, 2, 3$, $i \neq j$, for what values of the pair (ϕ_0, T) will (4.18) determine a value for $\bar{r}$, $a \le \bar{r} \le b$? We will, with no loss of generality, restrict attention to the first quadrant of the (ϕ_0, T)-plane. We now show that for each fixed choice of the subscripts $i, j = 1, 2, 3$, $i \neq j$, there is a <u>simply connected closed region</u> A_{ij} <u>in the</u>

<u>first quadrant of the $(\phi_0 - T)$ plane such that (4.18) determines a value for $\bar{r}$ if and only if (ϕ_0, T) is in A_{ij}. Furthermore, this value of $\bar{r}$ is unique</u>.

To this end, define the functions ϕ_{ij}, $i, j = 1, 2, 3$, $i \neq j$, by

$$\phi_{ij}(\bar{r}, T) = \int_a^{\bar{r}} \frac{1}{\xi} F_i\left(\frac{T}{2\pi\xi^2}\right) d\xi + \int_{\bar{r}}^{b} \frac{1}{\xi} F_j\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } B_{ij}, \tag{4.19}$$

where the domains of definition B_{ij} of the functions ϕ_{ij} on the $(\bar{r}, T)$-plane are given by

$$\begin{aligned}
B_{31} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi\bar{r}^2\tau_{min} \le T \le 2\pi\bar{r}^2\tau_{max}\right\}, \\
B_{13} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi b^2\tau_{min} \le T \le 2\pi a^2\tau_{max}\right\}, \\
B_{21} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi\bar{r}^2\tau_{min} \le T \le 2\pi a^2\tau_{max}\right\}, \\
B_{12} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi b^2\tau_{min} \le T \le 2\pi a^2\tau_{max}\right\}, \\
B_{32} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi b^2\tau_{min} \le T \le 2\pi\bar{r}^2\tau_{max}\right\}, \\
B_{23} &= \left\{(\bar{r}, T) \mid a \le \bar{r} \le b,\ 2\pi b^2\tau_{min} \le T \le 2\pi a^2\tau_{max}\right\}.
\end{aligned} \tag{4.20}$$[1]

We now consider the <u>case</u> $i = 3$, $j = 1$ in detail. For each fixed value of $\bar{r}$ in $[a, b]$, it follows from (4.19), (4.20) that $\phi_0 = \phi_{31}(\bar{r}, T)$ defines a segment of a smooth curve on the (ϕ_0, T)-plane for $2\pi\bar{r}^2\tau_{min} \le T \le 2\pi\bar{r}^2\tau_{max}$. Therefore, we have a family of such curves on the (ϕ_0, T)-plane, each corresponding to a different value of $\bar{r}$ in

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[1]Because of (2.32), (2.33) one sees that these are the largest possible domains of definition of the functions ϕ_{ij}. In the case when $a^2\tau_{max} < b^2\tau_{min}$ — so that (3.23) does not hold — we see that B_{13}, B_{12} and B_{23} are empty. In this case, therefore, solutions (4.17) with $(i, j) = (1, 3)$, $(1, 2)$ and $(2, 3)$ do not exist.

[a, b] , and all of them having their end points on the curves $\phi_0 = \phi_{31}(\sqrt{T/2\pi\tau_{min}}, T)$ and $\phi_0 = \phi_{31}(\sqrt{T/2\pi\tau_{max}}, T)$. Since by (2.38) and (4.19) we have

$$\frac{\partial \phi_{31}(\bar{r}, T)}{\partial \bar{r}} = \frac{1}{\bar{r}}\left\{F_3\left(\frac{T}{2\pi\bar{r}^2}\right) - F_1\left(\frac{T}{2\pi\bar{r}^2}\right)\right\} > 0 \quad \text{on } B_{31} \ , \qquad (4.21)$$

it follows that the different members of this family of curves do not intersect each other. Furthermore, a curve corresponding to a larger value of $\bar{r}$ lies to the right of a curve corresponding to a smaller value of $\bar{r}$. And finally, since ϕ_{31} depends continuously on $\bar{r}$, these curves span a simply connected region, A_{31} , in the (ϕ_0, T)-plane. From the above discussion it follows that A_{31} is the closed region bounded by the curves $\phi_0 = \phi_{31}(a, T)$, $\phi_0 = \phi_{31}(b, T)$, $\phi_0 = \phi_{31}(\sqrt{T/2\pi\tau_{min}}, T)$ and $\phi_0 = \phi_{31}(\sqrt{T/2\pi\tau_{max}}, T)$. A sketch of this region, together with the spanning family of curves, is shown in Fig. 3(i) . The fact that a curve corresponding to a larger value of $\bar{r}$ is to the right of a curve associated with a smaller value of $\bar{r}$ is indicated in Fig. 3(i) by the arrow labelled "direction of increasing $\bar{r}$". Since there is exactly one of these curves passing through any point in A_{31} , it follows that there is a unique number $\bar{r}$ associated with every point (ϕ_0, T) in A_{31} , such that $\phi_0 = \phi_{31}(\bar{r}, T)$. This is what we set out to establish. We may express this analytically as follows: there exists a function $\hat{r}_{31}$, defined on A_{31}, such that $\bar{r}$ determined by

$$\bar{r} = \hat{r}_{31}(\phi_0, T) \qquad (4.22)$$

conforms to $\phi_0 = \phi_{31}(\bar{r}, T)$, i.e. $\phi_0 = \phi_{31}\left(\hat{r}_{31}(\phi_0, T), T\right)$ on A_{31} .

Summarizing the results for this case, we have that, if ϕ_0 and T are numbers such that (ϕ_0, T) is in A_{31}, (the region PQRS in Fig. 3(i)), then there is a unique number $\bar{r}$ $(a \le \bar{r} \le b)$ such that (4.18) holds (with $i = 3$, $j = 1$). Equation (4.17), with these values of T, $\bar{r}$, i and j, is a solution to (4.8) - (4.10) at that value of ϕ_0.

The other cases — corresponding to the remaining choices of the subscripts i, j — may be likewise examined. In each case we find a simply connected closed region A_{ij}, shown in Figs. 3-5, such that, if ϕ_0 and T are numbers with (ϕ_0, T) in A_{ij}, then there is a unique number $\bar{r}$ $(a \le \bar{r} \le b)$ such that (4.18) holds for that choice of i, j. Equation (4.17) then provides the corresponding solution $\phi(r)$. Accordingly, in each case there exist functions $\hat{r}_{ij}$ defined on A_{ij}, such that

$$\bar{r} = \hat{r}_{ij}(\phi_0, T) \quad \text{on } A_{ij} \tag{4.23}$$

conforms to $\phi_0 = \phi_{ij}(\bar{r}, T)$.

The composite torque-twist diagram, wherein all of these admissible regions A_{ij} together with the torque-twist curves for the smooth solutions are sketched on one figure, is shown in Fig. 6. We observe that the admissible regions A_{ij} "fit" appropriately between the torque-twist curves associated with the smooth solutions (Fig. 2). Therefore corresponding to <u>any</u> given value of the twist ϕ_0 we now have a solution. However, we are now faced with the unsatisfactory situation in which there is an infinite number of admissible solutions at certain values of the prescribed twist ϕ_0.

We observe from Fig. 6 that at sufficiently small twists ϕ_0 $(\le \phi_s)$ and at sufficiently large twists ϕ_0 $(\ge \phi_Q)$ we have a unique solution, which is smooth. When the prescribed twist ϕ_0 is in one of the intervals $\phi_P < \phi_0 < \phi_M$, $\phi_N < \phi_0 < \phi_R$, we have an infinite number of solutions, all of which are weak solutions. In the remaining intervals, we have an infinite number of solutions, one of which is smooth, all the rest being weak solutions.

Observe from Figure 6 that even a knowledge of both ϕ_0 and T may be insufficient, in some cases, to determine a unique solution. For example, there are four solutions corresponding to any point in PMNK, one for each of the pairs, $(i,j) = (1,2)$, $(2,1)$, $(1,3)$ and $(3,1)$. We remark that at any point (ϕ_0, T) on PS or RQ there is in fact only one solution — the smooth one. One sees this from (4.17), (4.18), since all of the weak solutions at such a point have either $\bar{r} = a$ or $\bar{r} = b$ (see Figs. 3-5). Likewise, at any point on MN we only have smooth Solution 2 or the weak solutions $(i,j) = (1,3)$, $(3,1)$.

Finally, we observe that it is convenient to visualize the various solutions as follows. Consider for example a weak solution with $i = 3$, $j = 1$. Let $\bar{r}$ denote the radius of the associated shock. Let A, B, C and D be points on a radial line in the cross-section Π of the tube in the undeformed configuration, see Fig. 7(a), such that A and D are at the inside and outside boundaries respectively, while B and C are points just inside and outside the shock-line. The solution at hand is given by (4.17) with $i = 3$, $j = 1$. If we use this to compute $r\phi'(r)$ and then plot the points with coordinates $(|r\phi'(r)|, T/2\pi r^2)$ (suppose $T > 0$) for each r in the intervals $a \le r < \bar{r}$, $\bar{r} < r \le b$, we obtain the curves A_1B_1 and C_1D_1 (typically) shown in Fig. 7(b). The graph of f(k) has been superimposed on this diagram. The abscissa of any point on A_1B_1 or C_1D_1 gives the value of the local amount of shear $|r\phi'(r)|$ at the corresponding point in the tube, while the ordinate gives the corresponding shear stress $|\tau_{\rho\psi}|$. Observe from Fig. 7(b) how the local amount of shear varies continuously on either side of the shock but suffers a jump discontinuity across it. The shear stress $\tau_{\rho\psi}$, on the other hand, is seen to vary continuously throughout the tube. If we refer to the portions of the curve f(k) vs.k between $0 \le k \le k_1$, $k_1 \le k \le k_2$ and $k_2 \le k < \infty$ as the first, second and third branches of f(k) respectively, we see that this solution $(i=3, j=1)$ is associated with the third and first branches of f(k), with the region inside the shock-line associated with the former branch. In general, the weak solution (i, j) is associated with the i^{th} and j^{th} branches of f(k), with the part of the tube inside the shock-line corresponding to the i^{th} branch.

We see from this and Section 2.2 that the type of these weak solutions is mixed, in general. The displacement equations of equilibrium are elliptic on that part of Π for which $a < r < \bar{r}$ and non-elliptic where $\bar{r} < r < b$, at solutions with $(i, j) = (1, 2)$, $(3, 2)$, while they are elliptic where $\bar{r} < r < b$ and non-elliptic where $a < r < \bar{r}$, at the solutions $(i, j) = (2, 1)$, $(2, 3)$. In the case of the solutions corresponding to $(i, j) = (1, 3)$, $(3, 1)$, the displacement equations of equilibrium are elliptic everywhere in Π where $r \neq \bar{r}$.

5.1 Dissipativity

The lack of uniqueness encountered in the preceeding section is not unexpected since we had enlarged the admissible class of solutions there. In such circumstances, it is usually the case that not all of the solutions admitted by the differential equations are physically reasonable. In gas dynamics, for example, there are problems in which the differential equations admit solutions which are unacceptable since they are associated with a decrease of entropy. It is essential therefore to introduce additional criteria which will single out a physically admissible solution.

Knowles and Sternberg proposed such a criterion in [6], in the context of finite elastostatics, which they referred to as the dissipativity inequality. A thermodynamic motivation for this inequality, stemming from the Clausius-Duhem inequality, was given by Knowles in [7]. The dissipativity inequality is essentially an expression of the physically reasonable idea that the rate at which elastic energy is being stored in any part of the body, in some quasi-static process, cannot exceed the rate at which work is being done on that part.

We now examine the implications of the dissipativity inequality in the context of the present problem. While we could specialize the general dissipativity inequality given in [7] for our problem, it is illustrative (and equally easy) to derive it from first principles.

We now consider a quasi-static time-dependent family of equilibrium solutions. The time t merely plays the role of a history parameter and no inertia effects are accounted for. Accordingly, we are concerned with a one-parameter family of functions $\phi(r,t)$, depending on the parameter t in some time interval $\mathcal{T}$, such that at each t in $\mathcal{T}$, $\phi(r,t)$ is a solution to (4.8) - (4.10). The torque, twist and shock radius are all time dependent now, and we write $T(t)$, $\phi_0(t)$ and $\bar{r}(t)$. It is convenient to set

$$\left.\begin{aligned} \overset{+}{\Pi}(t) &= \{r \,|\, \bar{r}(t) < r < b\} \ , \\ \overset{-}{\Pi}(t) &= \{r \,|\, a < r < \bar{r}(t)\} \ , \end{aligned}\right\} \text{ for } t \text{ in } \mathcal{T} \ . \tag{5.1}$$

Then $\phi(\cdot,t)$ is continuous on $[a,b]$ and twice continuously differentiable on $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ at each t in $\mathcal{T}$. Furthermore

$$f(r\phi_r(r,t)) = -\frac{T(t)}{2\pi r^2} \quad \text{on } \overset{+}{\Pi}(t) \text{ and } \overset{-}{\Pi}(t) \ , \tag{5.2}$$ [1]

$$\phi(a,t) = \phi_0(t) \ , \tag{5.3}$$

$$\phi(b,t) = 0 \ , \tag{5.4}$$

at each t in $\mathcal{T}$. Here $\phi_0(t)$ is the prescribed twist, and we suppose it to be continuous and piecewise continuously differentiable on $\mathcal{T}$. In certain discussions, as we observed previously, it will be temporarily necessary to imagine that $T(t)$ is also specified. In such circumstances, we presume $T(t)$ to possess the same smoothness as $\phi_0(t)$ on $\mathcal{T}$.

It is convenient to set

$$k(r,t) = r\phi_r(r,t) \ , \tag{5.5}$$

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[1] We use the notation $\phi_r = \frac{\partial\phi(r,t)}{\partial r}$ and $\phi_t = \frac{\partial\phi(r,t)}{\partial t}$.

$$\overset{+}{k}(t) = \bar{r}(t)\phi_r(\bar{r}(t)+, t) \quad , \tag{5.6}$$

$$\bar{k}(t) = \bar{r}(t)\phi_r(\bar{r}(t)-, t) \quad , \tag{5.7}$$

with the understanding that when $\bar{r}(t) = b$ we take[1] $\bar{r}(t)+ = b$ and when $\bar{r}(t) = a$ we take $\bar{r}(t)- = a$. $\overset{+}{k}$ and $\bar{k}$ represent the instantaneous local amounts of shear at points just outside and inside the shock, respectively.

We will now require that at each instant in $\mathcal{T}$, the rate at which the external forces on the tube are doing work should not be less than the rate of increase of the stored energy, i.e. we demand that

$$T(t)\frac{d}{dt}\phi_0(t) \geq \frac{d}{dt}\int_a^b W(2 + k^2(r,t))2\pi r dr \quad \text{for all } t \text{ in } \mathcal{T} \ . \tag{5.8}$$

We may evaluate the right hand side of (5.8), using (5.6) and (5.7), as follows:

$$\frac{d}{dt}\int_a^b W(2 + k^2(r,t))2\pi r dr$$

$$= \frac{d}{dt}\int_a^{\bar{r}(t)} W(2 + k^2(r,t))2\pi r dr + \frac{d}{dt}\int_{\bar{r}(t)}^b W(2 + k^2(r,t))2\pi r dr \quad ,$$

$$= \{ W(2 + \bar{k}^2) - W(2 + \overset{+}{k}{}^2)\} 2\pi\bar{r}(t)\frac{d\bar{r}}{dt}(t) + \int_a^b 4\pi r k \frac{\partial k}{\partial t} W'(2 + k^2) dr \ . \tag{5.9}$$

Using (2.24), (5.2) and (5.5) in (5.9), gives

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[1]This is admissible since we observed in Section 4.2 that when $\bar{r} = a$ or b , the solution is in fact smooth. Thus ϕ_r exists there.

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr$$

$$=\{W(2+\bar{k}^2)-W(2+\overset{+}{k}{}^2)\}2\pi\bar{r}(t)\frac{d\bar{r}}{dt}-T(t)\int_a^b \phi_{rt}(r,t)dr \quad . \tag{5.10}$$

However, because of (5.3) and (5.4), we have

$$\int_a^b \phi_{rt}(r,t)dr=\phi_t(b,t)-\phi_t(\bar{r}(t)+,t)+\phi_t(\bar{r}(t)-,t)-\phi_t(a,t) \quad ,$$

$$=-\frac{d}{dt}\phi_0(t)-\phi_t(\bar{r}(t)+,t)+\phi_t(\bar{r}(t)-,t) \quad , \tag{5.11}$$

so that we may write (5.10) as

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr=\{W(2+\bar{k}^2)-W(2+\overset{+}{k}{}^2)\}\,2\pi\bar{r}(t)\frac{d\bar{r}}{dt}+T(t)\frac{d}{dt}\phi_0(t)$$

$$+T(t)\{\phi_t(\bar{r}(t)+,t)-\phi_t(\bar{r}(t)-,t)\} \quad . \tag{5.12}$$

Since the displacements are continuous across the shock, we have

$$\phi(\bar{r}(t)+,t)=\phi(\bar{r}(t)-,t) \quad \text{for } t \text{ in } \mathcal{T} \quad , \tag{5.13}$$

which when differentiated with respect to t leads to

$$\phi_r(\bar{r}(t)+,t)\frac{d\bar{r}}{dt}(t)+\phi_t(\bar{r}(t)+,t)=\phi_r(\bar{r}(t)-,t)\frac{d\bar{r}}{dt}(t)+\phi_t(\bar{r}(t)-,t) \quad . \tag{5.14}$$

Using (5.6), (5.7) and (5.14) in (5.12) gives

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr = \{W(2+\bar{k}^2) - W(2+\overset{+}{k})^2\}2\pi\bar{r}\frac{d\bar{r}}{dt} + T(t)\frac{d}{dt}\phi_0(t)$$

$$+\frac{1}{\bar{r}(t)}T(t)\{\bar{k}(t) - \overset{+}{k}(t)\}\frac{d\bar{r}}{dt}(t) \ , \qquad (5.15)$$

which because of (5.2) , (5.6) can be written as

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr = T(t)\frac{d}{dt}\phi_0(t) + \{W(2+\bar{k}^2) - W(2+\overset{+}{k}{}^2)\}2\pi\bar{r}\frac{d\bar{r}}{dt}$$

$$+ 2\pi\bar{r}f(\overset{+}{k})(\overset{+}{k} - \bar{k})\frac{d\bar{r}}{dt}(t) \ . \qquad (5.16)^1$$

The dissipativity requirement (5.8) can now be written as

$$\{W(2+\overset{+}{k}{}^2) - W(2+\bar{k}^2) - f(\overset{+}{k})(\overset{+}{k} - \bar{k})\}2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{on } \mathcal{T} \ ,$$

or alternatively, because of (2.24), as

$$\left\{\int_{\bar{k}}^{\overset{+}{k}} f(\xi)d\xi - (\overset{+}{k} - \bar{k})f(\overset{+}{k})\right\}2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for all } t \text{ in } \mathcal{T} \ . \qquad (5.17)$$

This is the form of the <u>dissipativity inequality</u> that we shall find useful.

It follows from the results of Section 4.2 that all admissible quasi-static families of equilibrium solutions are of the form

[1] Note from (5.2) that $f(\overset{+}{k}) = f(\bar{k})$.

$$\phi(r,t) = \begin{cases} \phi_0(t) - \int_a^r \frac{1}{\xi} F_i\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi & \text{on } \bar{\Pi}(t) , \\ \\ \int_r^b \frac{1}{\xi} F_j\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi & \text{on } \overset{+}{\Pi}(t) , \end{cases} \tag{5.18}$$

subject to the restriction

$$\phi_0(t) = \int_a^{\bar{r}(t)} \frac{1}{\xi} F_i\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi + \int_{\bar{r}(t)}^b \frac{1}{\xi} F_j\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi , \tag{5.19}$$

for $i, j = 1, 2, 3$ and for t in $\mathcal{T}$. We now proceed to apply the dissipativity inequality (5.17) to the various families of solutions represented by (5.18), (5.19).

We first note that, if at some instant t we have a smooth solution, then (5.17) holds at that instant by virtue of the continuity of ϕ_r, i.e. since $\bar{k} = \overset{+}{k}$. Therefore, we may restrict attention to the cases for which $i \neq j$ in (5.18), (5.19), and to times in $\mathcal{T}$ for which

$$a < \bar{r}(t) < b . \tag{5.20}$$[1]

Equations (5.6), (5.7) and (5.18) now give

$$\left.\begin{aligned} \overset{+}{k}(t) &= -F_j\left(\frac{T(t)}{2\pi\bar{r}^2(t)}\right) , \\ \bar{k}(t) &= -F_i\left(\frac{T(t)}{2\pi\bar{r}^2(t)}\right) , \end{aligned}\right\} \tag{5.21}$$

[1] See discussion following (4.18).

so that we may use (2.28), (2.39), (2.40), (2.41), (5.2), (5.6) and (5.21) to write (5.17) as

$$2\pi\bar{r}(t)\frac{d\bar{r}}{dt}(t)\int_{F_i[\eta(t)]}^{F_j[\eta(t)]}\{f(\xi)-\eta(t)\}\,d\xi \geq 0 \ , \quad \eta(t)=\frac{|T(t)|}{2\pi\bar{r}^2(t)} \ , \tag{5.22}$$

for all t in $\mathcal{T}$ for which (5.20) holds.

It is convenient to define the functions A_1 and A_2 by

$$A_1(\tau)=\int_{F_1(\tau)}^{F_2(\tau)}\left\{f(\xi)-\tau\right\}d\xi \quad \text{for } \tau_{min}\leq\tau\leq\tau_{max} \ , \tag{5.23}$$

$$A_2(\tau)=\int_{F_2(\tau)}^{F_3(\tau)}\left\{\tau-f(\xi)\right\}d\xi \quad \text{for } \tau_{min}\leq\tau\leq\tau_{max} \ . \tag{5.24}$$

These functions A_1 and A_2 have the following geometrical interpretation. If in Fig. 1 we draw a line parallel to the k-axis at a distance τ above it ($\tau_{min}\leq\tau\leq\tau_{max}$), then $A_1(\tau)$ and $A_2(\tau)$ are the areas of the two loops formed. Clearly

$$A_1(\tau_{max})=A_2(\tau_{min})=0 \ , \tag{5.25}$$

$$A_1(\tau)>0 \quad \text{for } \tau_{min}\leq\tau<\tau_{max} \ , \tag{5.26}$$

$$A_2(\tau)>0 \quad \text{for } \tau_{min}<\tau\leq\tau_{max} \ . \tag{5.27}$$

Using (5.23) and (5.24) in (5.22) leads to

$$\left\{A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) - A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right)\right\} 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (3,1) \ , \tag{5.28}$$

$$-\left\{A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) - A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right)\right\} 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (1,3) \ , \tag{5.29}$$

$$-A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (2,1) \ , \tag{5.30}$$

$$A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (1,2) \ , \tag{5.31}$$

$$A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (3,2) \ , \tag{5.32}$$

$$-A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for } (i,j) = (2,3) \ , \tag{5.33}$$

in each of the different cases.

Now consider, for example, the <u>case</u> $(i,j) = (2,1)$, i.e. suppose that for all times sufficiently close to some t_1 in $\mathcal{T}$, the quasi-static family of solutions (5.18) has $i = 2$, $j = 1$. We then have from (2.32), (2.33), (5.1), (5.18), (5.26) and (5.30) that the dissipativity inequality is satisfied at a time t_1 for which (5.20) holds if and only if

$$\frac{d\bar{r}}{dt}(t_1) \leq 0 \ . \tag{5.34}$$

As previously observed, in the event that (5.20) does not hold, so that

$$\bar{r}(t_1) = a \text{ or } b \ , \tag{5.35}$$

the dissipativity inequality holds without need for any restrictions such as (5.34). The meaning of these restrictions is most transparent when viewed in the torque-twist diagram (see Fig. 8). With no loss of generality we suppose that $T(t)$ and $\phi_0(t)$ are non-negative for all times in $\mathcal{T}$. We shall refer to the piecewise smooth oriented[1] curve Γ in the torque-twist plane defined by $\phi_0 = \phi_0(t)$, $T = T(t)$ for t in $\mathcal{T}$ as the <u>loading path</u>. By hypothesis, for all values of t sufficiently close to t_1, the loading path Γ lies in A_{21}. Let $Z = (\phi_0(t_1), T(t_1))$ be the point on Γ corresponding to $t = t_1$.

Recall[2] that the region A_{21} is spanned by a one-parameter family of curves $\phi_0 = \phi_{21}(\bar{r}, T)$, $a \leq \bar{r} \leq b$, and that a member of this family of curves corresponding to a larger value of the parameter $\bar{r}$ lies to the right of a curve corresponding to a smaller value. Let C be the particular member of this family with equation $\phi_0 = \phi_{21}(\bar{r}(t_1), T)$, so that C passes through Z, (see Fig. 8). It follows that the shock radius $\bar{r}$ corresponding to any point in A_{21} to the right of C is greater than $\bar{r}(t_1)$, while at a point to the left of C, it is less than $\bar{r}(t_1)$. Therefore, dissipativity — (5.34) — requires that the loading path Γ should be oriented at Z in such a way that it does not point to the right of C, provided Z is not a point on PS or MN. This is shown in Fig. 4(i) as well, in which the cluster of arrows indicates the admissible orientations of a loading path through a typical point. This is true for all points in A_{21} except for those which lie on PS and MN. At a point on these curves the loading path may be arbitrarily

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[1] Γ is oriented in the direction of increasing time.

[2] See Fig. 4(i).

oriented, by virtue of (5.35).

Clearly we can analyze the other cases in an entirely analogous manner. We find that dissipativity is essentially equivalent to

$$\frac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad (i,j) = (1,2) \ , \tag{5.36}$$

$$\frac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad (i,j) = (3,2) \ , \tag{5.37}$$

$$\frac{d\bar{r}}{dt} \leq 0 \quad \text{if} \quad (i,j) = (2,3) \ , \tag{5.38}$$

and these are geometrically interpreted in Figs. 4 and 5 as before. The only exceptions to (5.36) - (5.38) are respectively at points on the curves PK , MN and MN, RQ and MN , LR , whereat the orientation is arbitrary.

Equations (5.28) and (5.29) — i.e. the cases $(i,j) = (3,1)$ and $(1,3)$ — can also be similarly examined, taking care now to note that $\{A_2(\tau) - A_1(\tau)\}$ is not always of the same sign. If we set

$$A(\tau) = A_2(\tau) - A_1(\tau) \quad \text{for} \quad \tau_{min} \leq \tau \leq \tau_{max} \ , \tag{5.39}$$

where A_1 and A_2 are as defined previously, we find because of (5.23) - (5.27) and (5.39) that

$$A(\tau_{min}) < 0 \ , \ A(\tau_{max}) > 0 \ , \tag{5.40}$$

$$\frac{dA}{d\tau}(\tau) > 0 \quad \text{for} \quad \tau_{min} \leq \tau \leq \tau_{max} \ . \tag{5.41}$$

Since $A(\tau)$ is continuous, it follows from (5.40) and (5.41) that there is a unique number τ_c in (τ_{min}, τ_{max}) such that

$$A(\tau_c) = 0 \ , \tag{5.42}$$

$$A(\tau) > 0 \ \text{ on } \ (\tau_c, \tau_{max}] \ , \ A(\tau) < 0 \ \text{ on } \ [\tau_{min}, \tau_c) \ . \tag{5.43}$$

The number τ_c is shown in Fig. 1; since $A_1(\tau_c) = A_2(\tau_c)$, the two hatched regions are of equal area. The dissipativity conditions (5.28) and (5.29), because of (5.39), (5.42) and (5.43), are equivalent to

$$\left.\begin{array}{l} \dfrac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad \dfrac{|T|}{2\pi\bar{r}^2} > \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \leq 0 \quad \text{if} \quad \dfrac{|T|}{2\pi\bar{r}^2} < \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \quad \text{is arbitrary if} \quad \dfrac{|T|}{2\pi\bar{r}^2} = \tau_c \ , \end{array}\right\} (i,j) = (3,1) \ , \tag{5.44}$$

and

$$\left.\begin{array}{l} \dfrac{d\bar{r}}{dt} \leq 0 \quad \text{if} \quad \dfrac{|T|}{2\pi\bar{r}^2} > \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad \dfrac{|T|}{2\pi\bar{r}^2} < \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \quad \text{is arbitrary if} \quad \dfrac{|T|}{2\pi\bar{r}^2} = \tau_c \ , \end{array}\right\} (i,j) = (1,3) \ . \tag{5.45}$$

Consider the case $(i,j) = (3,1)$. One shows easily that $\phi_0 = \phi_{31}(\sqrt{T/2\pi\tau_c}, T)$ is a curve in A_{31} which qualitatively looks as shown in Fig. 3(i) — curve XY. Corresponding to any point on this curve, we have $T/2\pi\bar{r}^2 = \tau_c$ while it is readily seen that at any point above or below XY $T/2\pi\bar{r}^2$ is respectively, greater than or less than τ_c. Accordingly, the dissipativity inequality is equivalent to the first, second and third of (5.44) at points in A_{31}, respectively, above, below and on the curve XY. The arrows in Fig. 3(i) indicate the admissible orientations of a loading path at some typical points in A_{31}. As before, the orientation at points on PS and QR is arbitrary. The solution $(i, j) = (1, 3)$ may be similarly interpreted, as shown[1] in Fig. 3(ii).

5.2 Consequences of Dissipativity

The dissipativity inequality was introduced in the hope that it would single out a physically admissible solution from among the many available equilibrium solutions. We now demonstrate that, if we require the local twist $\phi(r, \cdot)$ to be continuous[2] on $\mathcal{T}$ at each r in $[a, b]$, and if we suppose that the body was in an undeformed configuration at some time, then a configuration corresponding to solutions $(i, j) = (1, 2)$, $(2, 1)$, $(2, 3)$ $(3, 2)$ or smooth Solution 2 cannot be attained at any subsequent time.

First, omit the weak solutions $(i, j) = (1, 3)$ and $(3, 1)$ from discussion. We observe from Fig. 4 that any loading path in Fig. 6 conforming with the dissipativity inequality and starting from O is necessarily confined to the curve OP for all subsequent time. Note similarly,

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[1] An examination of the details of the curve $\phi_0 = \phi_{13}(\sqrt{T/2\pi\tau_c}, T)$ show that it is possible for this curve, depending on the specific geometry and constitutive law, to intersect a different pair of boundaries of A_{13} than shown in Fig. 3(ii). The figure is drawn for $b^2/a^2\tau_{min} < \tau_c < a^2/b^2\tau_{max}$.

[2] Note that despite the presumed continuity of $\phi_0(t)$ and $T(t)$, $\phi(r, \cdot)$ defined by (5.18) is not necessarily continuous on $\mathcal{T}$, since the subscripts i and j may change values at certain times.

from Figs. 5 and 6, that any admissible loading path starting from O' is likewise restricted to $O'R$ for all subsequent time. The only possible way of achieving a solution $(i,j) = (1,2)$, $(2,3)$, $(3,2)$, $(2,1)$ or Solution 2 is then, by virtue of a loading path which is associated with one of the solutions $(i,j) = (3,1)$, $(1,3)$ for some time interval less than some time t_1, and with one of these solutions after time t_1. One sees readily from (5.18) however, that this involves a discontinuity in $\phi(r,\cdot)$ at the time t_1. Since we have disallowed this possibility, we now conclude that a configuration corresponding to <u>any solution associated with the second branch of the graph of $f(k)$ vs. k cannot be attained through a dissipative quasi-static deformation process</u>. These are, incidentally, the solutions at which the displacement equations of equilibrium are non-elliptic somewhere in Π.

However, even if we now discard the solutions associated with the second branch of f, we would not have overcome our troubles with non-uniqueness. For example, consider the Solutions 1, 3 and $(i,j) = (3,1)$. The appropriate torque-twist diagram is shown in Fig. 9. If we imagine gradually <u>increasing</u> the applied twist ϕ_0 from zero, the only available loading path initially is OS. During the next stage, $\phi_s < \phi_0 < \phi_x$, dissipativity — see Fig. 3(i) — disallows all loading paths except SX. Once the applied twist ϕ_0 exceeds the value ϕ_x, however, the loading path could lie anywhere in PQYX, and we have no criterion for deciding which path to follow. Eventually, for $\phi_0 > \phi_Q$, we are restricted to the path QO'. Likewise, during a steady <u>decrease</u> of the applied twist the loading path would be restricted to $O'QY$, then allowed to follow an arbitrary path (consistent with dissipativity) in XYRS and finally restricted to SO. It is interesting to note that if in either case the loading path lies on the curve $OXYO'$, then the quasi-static process is dissipationless in the sense that (5.8) would hold with equality at every instant t .

It is therefore imperative that we seek an additional — or possibly an alternative — physical criterion, to the dissipativity inequality, that would sort out more completely the issue of non-uniqueness.

6.1 Preliminaries on Stability

We now look into the possibility of using a stability criterion instead of the dissipativity inequality, in order to single out a physically admissible solution to the boundary-value problem under consideration. We draw attention to the fact that we will not make use of the partial success achieved through the dissipativity inequality, since we are at present examining the possibility of an alternative — rather than additional — criterion.

The notion of stability that we will use is a static one based on the energy criterion.[1] According to this, an equilibrium configuration of a body is stable if the work done by the external loads in every sufficiently small kinematically possible virtual displacement from this equilibrium configuration is less than the corresponding increase in the stored energy.

In order to mathematically formulate this criterion, it is necessary to decide on a measure for the virtual displacements and to specify the behavior of the applied loading during a virtual displacement. We will consider two possibilities — stability under dead loading and stability with fixed boundaries.

First consider the case in which we have dead loading on the inner surface of the tube while the outer surface remains fixed. The torque T then remains constant during a virtual displacement. Let

[1] See page 195 of [15] for a discussion of this criterion.

$\phi(r)$ be the equilibrium solution whose stability we wish to investigate. Define the potential energy functional $V\{\psi\}$ by

$$V\{\psi\} = \int_a^b \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2)\} 2\pi r dr - T\{\psi(a) - \phi(a)\} \tag{6.1}$$

for all functions $\psi(r)$ in some set χ. In order to interpret $\psi(r)$ as a virtual twist[1] from the undeformed configuration, we suppose that χ is the set of all functions which are continuous and have piecewise continuous first and second derivatives on $[a, b]$, and for which $\psi(b) = 0$. Since this limited degree of smoothness is all that is required of an equilibrium solution $\phi(r)$, it seems reasonable not to impose more severe smoothness requirements on the virtual displacement. Finally, we will measure the departure of a virtual twist ψ from the solution ϕ by

$$||\psi - \phi|| = \int_a^b 2\pi r^3 [\psi' - \phi']^2 dr \quad . \tag{6.2}$$

We now say that an equilibrium solution $\phi(r)$ is stable if there exists some number $\epsilon > 0$ such that

$$V\{\psi\} > 0 \quad \text{for all functions } \psi \text{ in } \chi \text{ for which } ||\psi - \phi|| < \epsilon\ ,\ \psi \not\equiv \phi\ . \tag{6.3}$$

A solution which is not stable is unstable, i.e. if, given any number $\epsilon > 0$, there exists a function $\psi(r)$ such that

[1] We restrict attention to purely circumferential virtual displacements.

$$V\{\psi\} \leq 0 \ , \ ||\psi - \phi|| < \epsilon \ , \ \psi \not\equiv \phi \ , \tag{6.4}$$

the equilibrium solution $\phi(r)$ is <u>unstable</u>.

We now determine from (6.1), (6.3) a sufficient condition for stability which will be useful for our purposes. After making use of (2.25), (2.27) and $\psi(b) = 0$, we can rewrite (6.1) as

$$V\{\psi\} = \int_a^b \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi')\} 2\pi r dr \tag{6.5}$$

for any ψ in X. It follows that a sufficient condition for the stability of $\phi(r)$ is that for every ψ in X, $\psi \not\equiv \phi$,

$$W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi') \geq 0 \tag{6.6}$$

at each r in (a, b) where the left hand side exists, and

$$W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi') > 0 \tag{6.7}$$

at each r in some sub-interval of (a, b) where the left hand side exists. On the other hand, if at each r in (a, b) where $k(r) = r\phi'(r)$ exists, we have

$$W(2 + \kappa^2) - W(2 + k^2(r)) - f(k(r))(\kappa - k(r)) > 0 \tag{6.8}$$

for all <u>numbers</u> $\kappa \neq k(r)$, it follows that (6.6) and (6.7) hold. Equation (6.8) is thus a <u>sufficient condition for the stability</u> of the solution $\phi(r)$.

Now consider the case in which both the inner and outer surfaces of the tube are held <u>fixed</u> during a virtual displacement. The potential energy functional $V\{\psi\}$ is now defined by

$$V\{\psi\} = \int_a^b \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2)\} 2\pi r dr \tag{6.9}$$

for all functions ψ in some set χ. In this case we take χ to be the subset of the previous set of admissible virtual twists which conforms to $\psi(a) = \phi(a) = \phi_0$. Stability is defined as before. By virtue of (2.25) and (2.27) we can again write $V\{\psi\}$ in the form given by (6.5), whence (6.8) continues to be a sufficient condition for stability.

6.2 Consequences of Stability

Following Ericksen [12], we first make note of a geometric property of the response curve in shear. Recall the functions $A_1(\tau)$ and $A_2(\tau)$ defined by (5.23) and (5.24), representing the areas of the loops formed by drawing a line in Fig. 1 parallel to the k-axis at a distance τ, $\tau_{min} \le \tau \le \tau_{max}$, above it. Recall also that

$$A_1(\tau_c) = A_2(\tau_c) ,$$

$$A_1(\tau) > A_2(\tau) \text{ for } \tau_{min} \le \tau < \tau_c , \; A_1(\tau) < A_2(\tau) \text{ for } \tau_c < \tau \le \tau_{max} . \tag{6.10}$$

Keeping this in mind, one observes the following properties of $f(k)$ upon examining its graph (Fig. 1). If we set

$$k_3 = F_1(\tau_c) , \; k_4 = F_3(\tau_c) , \tag{6.11}$$

then we may observe first that

(i) if k is any number such that either

$$|k| < k_3 \text{ or } |k| > k_4 , \tag{6.12}$$

then

$$\int_k^{\varkappa} f(\xi)d\xi > f(k)(\varkappa - k) \quad \text{for all } \varkappa \neq k \ . \tag{6.13}$$

Equation (6.13) is a statement of the geometric observation that, provided (6.12) holds, the area under the response curve from k to $\varkappa$, for any $\varkappa \neq k$ is greater than the area of the rectangle of the same width and of height $f(k)$. By virtue of (2.24), we can write (6.13) as

$$W(2+\varkappa^2) - W(2+k^2) - f(k)(\varkappa - k) > 0 \quad \text{for all } \varkappa \neq k \ . \tag{6.14}$$

Next, it may be noted that

(ii) if k is any number such that

$$k_3 < |k| < k_4 \ , \tag{6.15}$$

then there exists some sub-interval $\mathcal{J}$ of $(-\infty, \infty)$ such that

$$\int_k^{\varkappa} f(\xi)d\xi < f(k)(\varkappa - k) \quad \text{for } \varkappa \text{ in } \mathcal{J} \ , \tag{6.16}$$

whence by (2.24), we have

$$W(2+\varkappa^2) - W(2+k^2) - f(k)(\varkappa - k) < 0 \quad \text{for } \varkappa \text{ in } \mathcal{J} \ . \tag{6.17}$$

Alternatively, (i) and (ii) can be established analytically.

We now conclude, by virtue of (6.8), (6.12) and (6.14), that any equilibrium solution $\phi(r)$ for which either

$$|k(r)| = |r\phi'(r)| < k_3 \quad \text{or} \quad |k(r)| = |r\phi'(r)| > k_4 \ , \tag{6.18}$$

at each r in (a,b) where ϕ' exists, is stable. It is a trivial exercise to examine all the available equilibrium solutions $\phi(r)$ — given by (4.17) —

and to determine those that conform to (6.18). One finds that only the following do:

(i) Smooth Solution 1 with $|\phi_0| \leq \int_a^b \frac{1}{\xi} F_1(a^2\tau_c/\xi^2)d\xi$,

(ii) Smooth Solution 3 with $|\phi_0| \geq \int_a^b \frac{1}{\xi} F_3(b^2\tau_c/\xi^2)d\xi$,

(iii) Weak Solution (3, 1) with $|T| = 2\pi\bar{r}^2\tau_c$, i.e. the solution (3, 1) with the torque given by $|\phi_0| = \phi_{31}(\sqrt{|T|/2\pi\tau_c}\,,\ |T|)$ and with shock radius $\bar{r} = \sqrt{|T|/2\pi\tau_c}$.

These solutions therefore are stable. On Fig. 9, these refer to the solutions associated with points on the curves OX, YO′ and XY respectively.

We will now show that all the other solutions are unstable. We do this by exhibiting particular admissible functions $\psi(r)$ which render $V\{\psi\}$ negative. It is readily shown that these remaining solutions – i.e. solutions (4.17) which do not conform to (6.18) – all have

$$k_3 < |r\phi'(r)| < k_4 \ , \tag{6.19}$$

on some sub-interval of $[a,b]$. In each case $\psi(r)$ is chosen to take advantage of (6.15), (6.17) and (6.19). We first consider the case of dead loading.

Consider for example Solution 1 with

$$\int_a^b \frac{1}{\xi} F_1(a^2\tau_c/\xi^2)d\xi < |\phi_0| \leq \int_a^b \frac{1}{\xi} F_1(a^2\tau_{max}/\xi^2)d\xi \ . \tag{6.20}$$

Recall that in this case

$$r\phi'(r) = -F_1\left(\frac{T}{2\pi r^2}\right), \quad \phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \ . \tag{6.21}$$

It follows from (6.20), (6.21) because of the monotonicity of the function F_1 that

$$2\pi a^2 \tau_c < |T| \le 2\pi a^2 \tau_{max} \ . \tag{6.22}$$

By virtue of this, there is a number s, $a<s<b$, such that

$$|T| > 2\pi r^2 \tau_c \quad \text{for } a \le r < s \ . \tag{6.23}$$

It is readily shown that $k_4 > |r\phi'(r)| > k_3$ on (a, s). Let δ be any number, $0 < \delta \le s-a$. We now choose the function $\psi(r)$ in χ such that

$$r\psi'(r) = \begin{cases} -F_3\left(\frac{T}{2\pi r^2}\right) & \text{for } a<r<a+\delta \ , \\ \\ -F_1\left(\frac{T}{2\pi r^2}\right) & \text{for } a+\delta<r<b \ . \end{cases} \tag{6.24}$$

Observe from (6.2), (6.21) and (6.24) that

$$\|\psi - \phi\| = \int_a^{a+\delta} 2\pi r\left[F_3\left(\frac{T}{2\pi r^2}\right) - F_1\left(\frac{T}{2\pi r^2}\right)\right]^2 dr \ ,$$

$$= 2\pi a\left[F_3\left(\frac{T}{2\pi a^2}\right) - F_1\left(\frac{T}{2\pi a^2}\right)\right]^2 \delta + o(\delta) \quad \text{as } \delta \to 0 \ . \tag{6.25}$$

Therefore, given any number $\epsilon > 0$, we can choose δ sufficiently small, $0 < \delta < s-a$, such that $||\psi - \phi|| < \epsilon$.

On using (6.21) and (6.24) in (6.5) we find by virtue of (2.24), (2.28), (2.34), (2.40) and (2.41) that

$$V\{\psi\} = \int_a^{a+\delta} 2\pi r \int_{F_1(\eta)}^{F_3(\eta)} \{f(\xi) - \eta\} d\xi dr \; , \; \eta = \frac{|T|}{2\pi r^2} \; . \tag{6.26}$$

This can in turn be written as

$$V\{\psi\} = -\int_a^{a+\delta} A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr \tag{6.27}$$

because of (5.23), (5.24) and (5.39). Since (6.22) and (6.23) imply that

$$\tau_{max} \geq \frac{|T|}{2\pi r^2} > \tau_c \quad \text{for } a \leq r < a + \delta \; , \tag{6.28}$$

it now follows from (5.43), (6.27) and (6.28) that

$$V\{\psi\} < 0 \; . \tag{6.29}$$

Therefore Solution 1 with (6.20) in effect is unstable. On Fig. 9 these solutions are associated with points on the curve XP (excluding X).

The instability of the other solutions may be established in an entirely analogous manner, i.e. taking advantage of (6.16), (6.19) to choose $\psi(r)$ arbitrarily close to $\phi(r)$ such that $V\{\psi\} < 0$.

Instability in the case when both boundaries are held fixed may be established in a similar manner, taking care now to satisfy the boundary condition $\psi(a) = \phi(a) = \phi_0$. For example, consider Solution 1 with (6.20) in effect. Let $c(\delta)$ be the function defined implicitly by

$$\phi_0 = \int_a^{c(\delta)} \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi + \int_{c(\delta)}^{b} \frac{1}{\xi} F_1\left(\frac{T-\delta}{2\pi\xi^2}\right) d\xi \quad . \tag{6.30}$$

By virtue of (2.38), (6.21) and the implicit function theorem, one can show that (6.30) does in fact define a function $c(\delta)$ which is twice[1] continuously differentiable in a neighborhood of $\delta = 0$, and that

$$c(0) = a \ , \ c'(0) > 0 \quad . \tag{6.31}$$

Thus $c(\delta) > a$ for sufficiently small positive δ.

Now consider the virtual twist $\psi(r)$ defined by

$$\psi(r) = \begin{cases} \phi_0 - \int_a^r \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } [a, c(\delta)] \ , \\ \\ \int_r^b \frac{1}{\xi} F_1\left(\frac{T-\delta}{2\pi\xi^2}\right) d\xi & \text{on } [c(\delta), b] \ , \end{cases} \tag{6.32}$$

for a sufficiently small $\delta > 0$. Note that ψ is in χ by virtue of (6.30). Observe from (6.2), (6.21) and (6.32) that

$$||\psi - \phi|| = \int_a^{c(\delta)} 2\pi r \left[F_3\left(\frac{T}{2\pi r^2}\right) - F_1\left(\frac{T}{2\pi r^2}\right)\right]^2 dr + \int_{c(\delta)}^{b} 2\pi r \left[F_1\left(\frac{T-\delta}{2\pi r^2}\right) - F_1\left(\frac{T}{2\pi r^2}\right)\right]^2 dr$$

$$= 2\pi a \left[F_3\left(\frac{T}{2\pi a^2}\right) - F_1\left(\frac{T}{2\pi a^2}\right)\right]^2 c'(o)\delta + o(\delta) \quad \text{as } \delta \to 0 \tag{6.33}$$

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[1] Twice continuously differentiable when $T < 2\pi a^2 \tau_{max}$. The argument presented here can be readily modified in the case $T = 2\pi a^2 \tau_{max}$.

where we have also used (6.31) in the second equation. Therefore, given any number $\epsilon > 0$ we can choose $\delta > 0$ sufficiently small such that $||\psi - \phi|| < \epsilon$.

If we now set

$$\left.\begin{aligned} V_1(\delta) &= \int_a^{c(\delta)} \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi')\}2\pi r dr \quad , \\ V_2(\delta) &= \int_{c(\delta)}^{b} \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi')\}2\pi r dr \quad , \end{aligned}\right\} \qquad (6.34)$$

with ψ given by (6.32), we may write (6.5) as

$$V\{\psi\} = V_1(\delta) + V_2(\delta) \quad . \qquad (6.35)$$

We find from (6.21), (6.31), (6.32) and (6.34) that

$$V_2(0) = 0 \quad , \quad V_2'(0) = 0 \quad , \qquad (6.36)$$

because of (2.24) and (2.28). Likewise we find

$$V_1(0) = 0 \quad , \quad V_1'(0) = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0) \quad , \qquad (6.37)$$

where we have also used (2.34), (2.40), (2.41), (5.23), (5.24), (5.39) and (6.31). Note because of (6.20), (6.21) and the monotonicity of F_1 that $\tau_c < |T|/2\pi a^2 \le \tau_{max}$, whence by (5.43), (6.31) and (6.37) we have

$$V_1'(0) = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0) < 0 \quad . \qquad (6.38)$$

On using (6.36), (6.37) in (6.35) we find

$$V\{\psi\} = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0)\delta + o(\delta) \quad \text{as } \delta \to 0 \ , \tag{6.39}$$

so that by (6.38) $V\{\psi\} < 0$ for sufficiently small $\delta > 0$. This establishes the instability of Solution 1 with (6.20) in effect in this case. The instability of the other solutions may be likewise established. This completes our instability analysis.

The preceding results lead to the conclusion that the only stable solutions[1] are the ones given by (i), (ii), (iii) following Equation (6.18). Recall that on the torque-twist diagram, Fig. 9, these are the solutions associated with the curve OXYO′. We therefore have that there is a unique stable solution $\phi(r)$ to the boundary value problem in its weak formulation corresponding to every value of the applied twist ϕ_0, i.e. there is a unique solution $\phi(r)$ to (4.8) - (4.10) which conforms to (6.2), (6.3), (6.5). Note that at every value of ϕ_0, the displacement equations of equilibrium are elliptic on Π $(r \neq \bar{r})$ at this unique solution.

We now refer to a remark made in Section 4.1 that a configuration involving more than one elastostatic shock is unstable. In the case of a solution with a single shock we showed instability whenever (6.19) held. Clearly, it is (6.19) and not the number of shocks that is important in that argument.[2] It is readily established that (6.19) holds for every weak solution involving more than one elastostatic shock. This is most easily seen from a visualization of such a solution in the manner explained in

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[1] These solutions exist irrespective of the geometric and constitutive details, i.e. even in the cases when (3.23) does not hold these are the only stable solutions.

[2] As remarked previously, the importance of (6.19) for instability is related to the property (6.15), (6.17).

association with Fig. 7. This leads to the instability of such a solution.

Now consider a quasi-static loading of the body. If the loading is performed in a manner which involves disturbances of the type we have allowed for, one would expect that at each instant, the tube seeks out the configuration which is stable against such disturbances. On increasing the applied twist we might then expect the loading path to follow the curve OXYO′ in Fig. 9. We would first observe a smooth configuration of the tube. An elastostatic shock would then emerge at the inner boundary and gradually move outwards, dissappearing upon reaching the outer boundary and giving way to a smooth configuration. On decreasing the applied twist, we would observe this process in reverse. Note from Section 5.2 that this loading path conforms to the dissipation inequality, even though we did not demand – here – that it should be so. In fact this is the dissipation-free path referred to previously.

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References

[1] J.K. Knowles and Eli Sternberg, An asymptotic finite-deformation analysis of the elastostatic field near the tip of a crack, Journal of Elasticity, 3 (1973), 2, p. 67.

[2] J.K. Knowles and Eli Sternberg, Finite-deformation analysis of the elastostatic field near the tip of a crack: Reconsideration and higher-order results, Journal of Elasticity, 4 (1974), 3, p. 201.

[3] J.K. Knowles, The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids, International Journal of Fracture, 13 (1977), 5, p. 611.

[4] J.K. Knowles and Eli Sternberg, On the ellipticity of the equations of nonlinear elastostatics for a special material, Journal of Elasticity, 5 (1975), 3-4, p. 341.

[5] J.K. Knowles and Eli Sternberg, On the failure of ellipticity of the equations for finite elastostatic plane strain, Archive for Rational Mechanics and Analysis, 63 (1977), 4, p. 321.

[6] J.K. Knowles and Eli Sternberg, On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics, Journal of Elasticity, 8 (1978), 4, p. 329.

[7] J.K. Knowles, On the dissipation associated with equilibrium shocks in finite elasticity, Journal of Elasticity, 9 (1979), 2.

[8] R.C. Abeyaratne, Discontinuous deformation gradients in plane finite elastostatics of incompressible materials, Technical Report No. 41, Office of Naval Research, Contract N00014-75-C-0196, California Institute of Technology, Pasadena, California, May 1979. To appear in Journal of Elasticity.

[9] R.S. Rivlin, Large elastic deformations of isotropic materials VI. Further results in the theory of torsion, shear and flexure, Philosophical Transactions of the Royal Society of London, A242 (1949), p. 173.

[10] A.E. Green and W. Zerna, Theoretical Elasticity, Clarendon Press, Oxford, 1954.

[11] R.W. Ogden, P. Chadwick and E.W. Haddon, Combined axial and torsional shear of a tube of incompressible isotropic elastic material, 26 (1973), 1, p. 23.

[12] J.L. Ericksen, Equilibrium of Bars, Journal of Elasticity, 5 (1975), 3-4, p. 191.

[13] C. Truesdell and W. Noll, The nonlinear field theories of mechanics, Handbuch der Physik, Vol. III/1, Springer, Berlin, 1965.

[14] J.R. Rice, The localization of plastic deformation, Proceedings of the 14th IUTAM Congress, (W. T. Koiter, editor), Delft, Netherlands, August 1976, p. 207.

[15] R. J. Knops and E. W. Wilkes, Theory of elastic stability, Handbuch der Physik, Vol. VIa/3, Springer, Berlin, (1973).

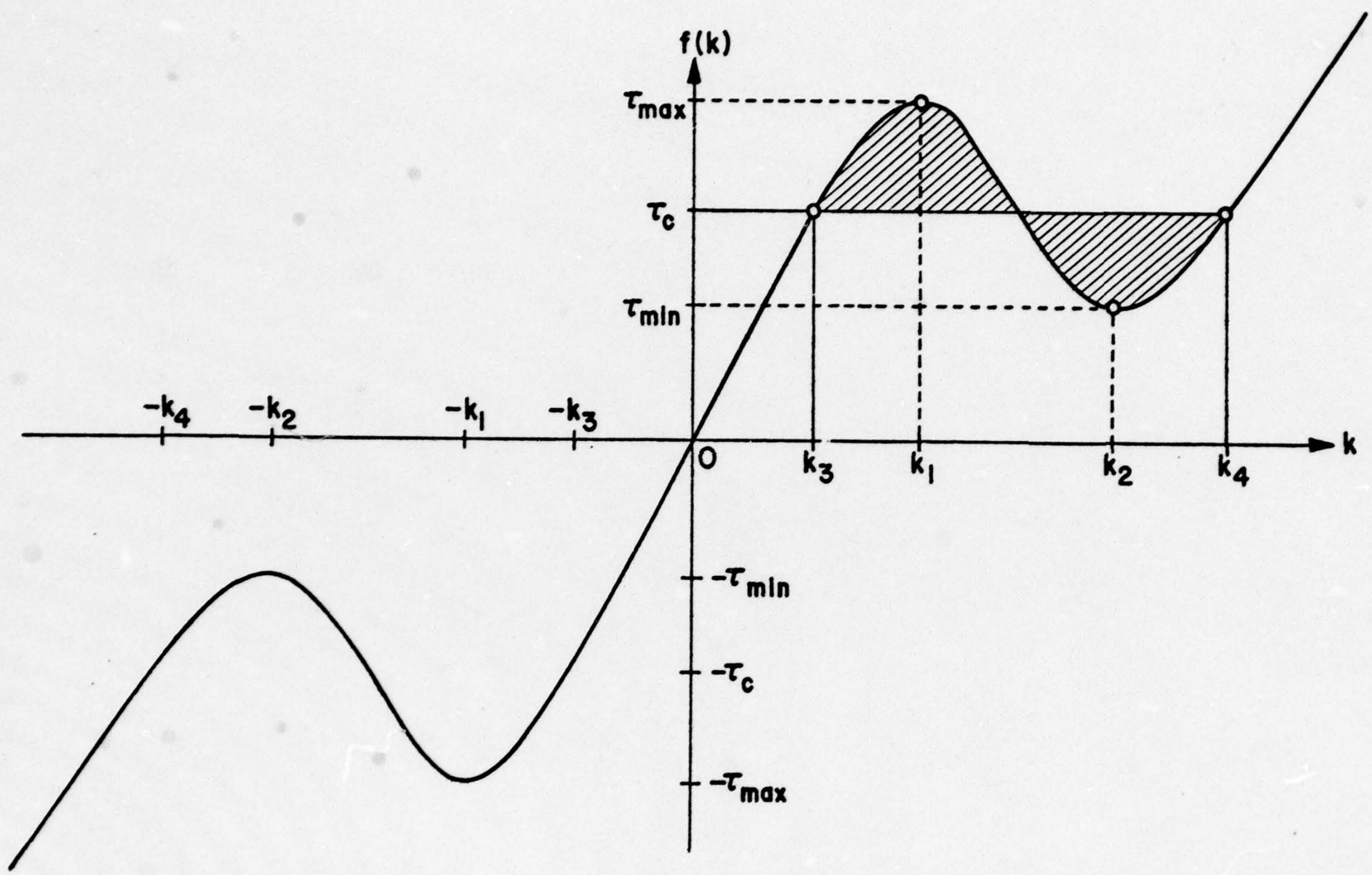
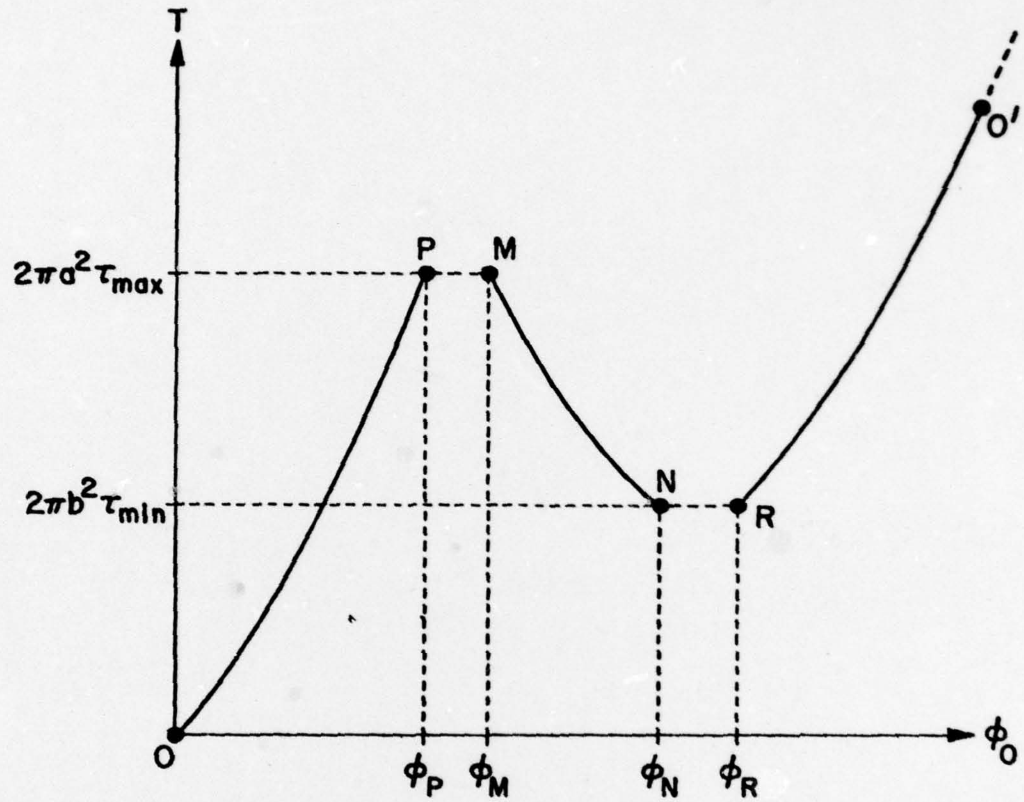


FIGURE 1. RESPONSE CURVE IN SIMPLE SHEAR FOR A HYPOTHETICAL MATERIAL. SHEAR STRESS VS. AMOUNT OF SHEAR

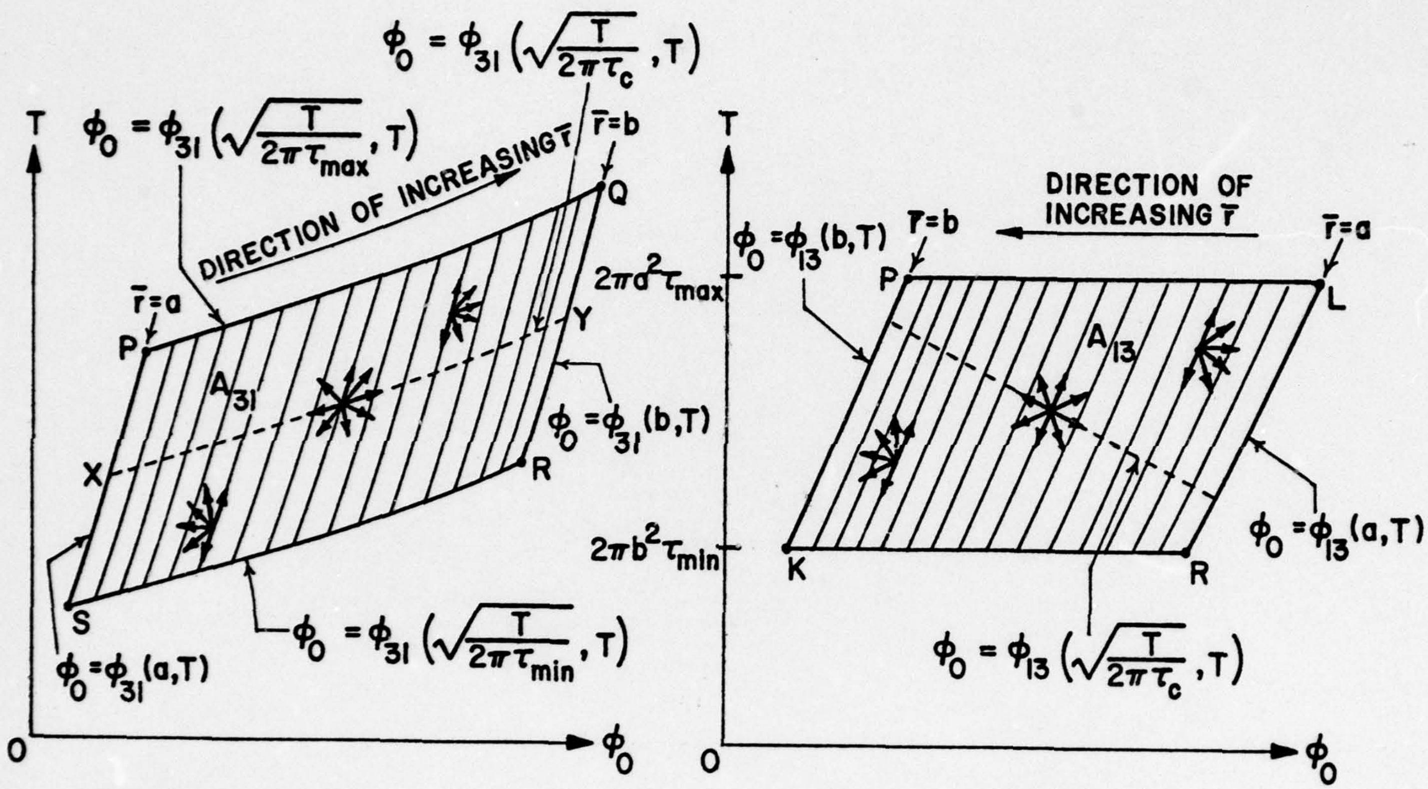


SOLUTION 1. OP : $\phi_O = \int_a^b \frac{1}{\zeta} F_1(T/2\pi\zeta^2)\, d\zeta$

SOLUTION 2. MN : $\phi_O = \int_a^b \frac{1}{\zeta} F_2(T/2\pi\zeta^2)\, d\zeta$

SOLUTION 3. RO′: $\phi_O = \int_a^b \frac{1}{\zeta} F_3(T/2\pi\zeta^2)\, d\zeta$

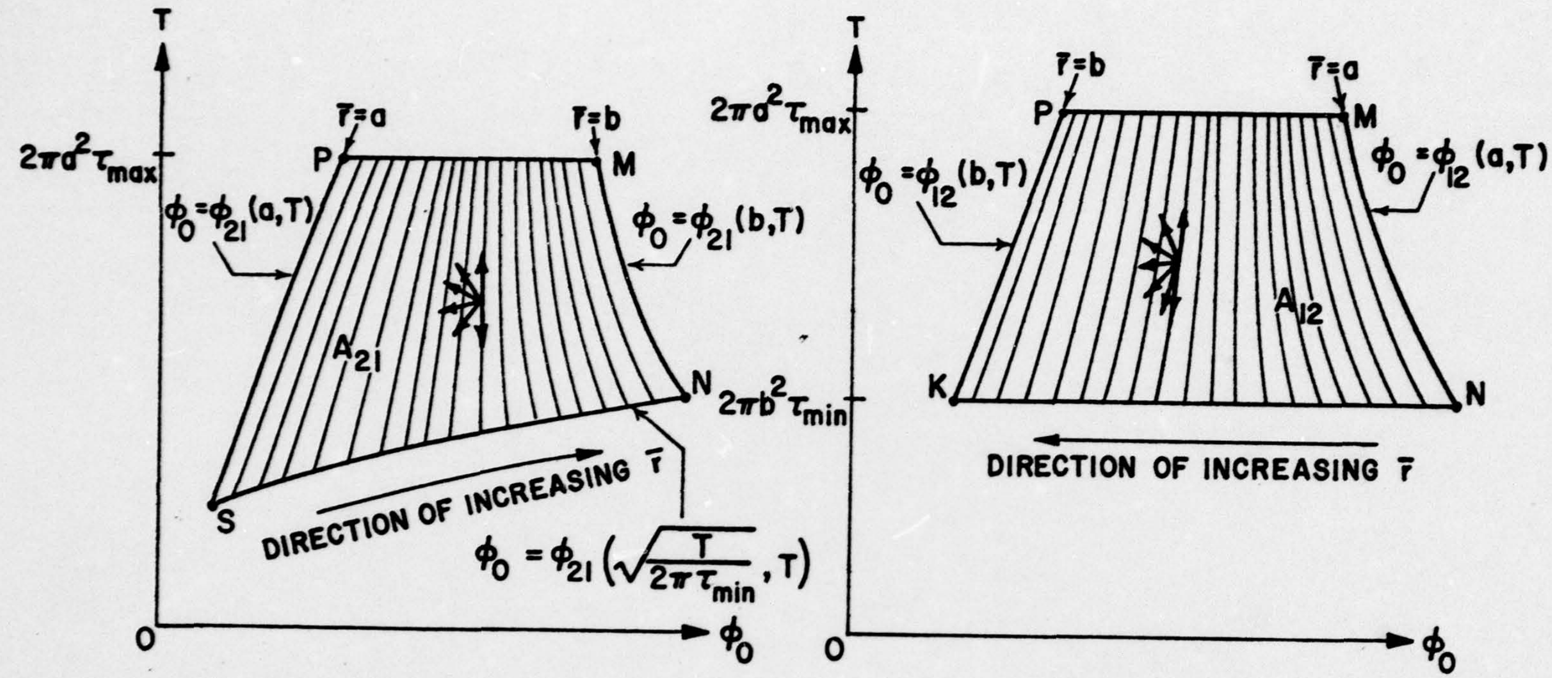
FIGURE 2. TORQUE VS. TWIST CURVES FOR SMOOTH SOLUTIONS



(i) SOLUTION i=3, j=1

(ii) SOLUTION i=1, j=3

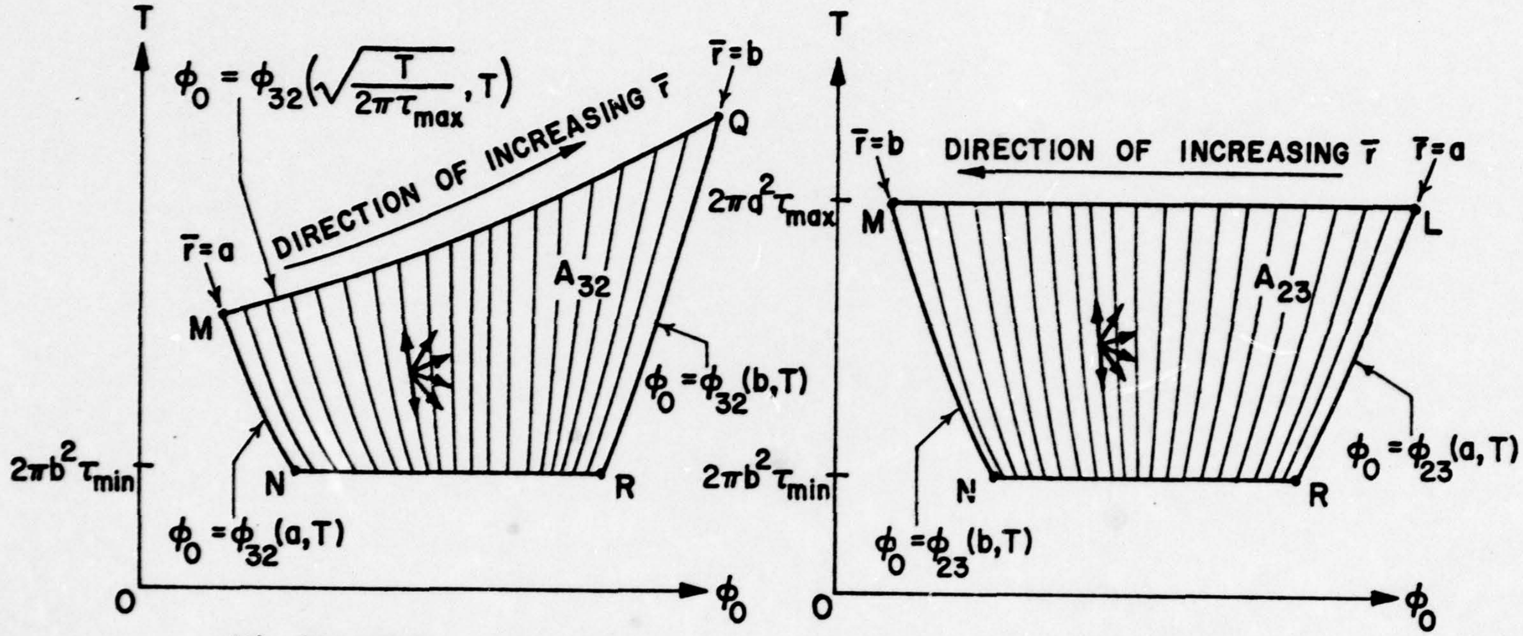
FIGURE 3. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



(i) SOLUTION $i=2,\ j=1$

(ii) SOLUTION $i=1,\ j=2$

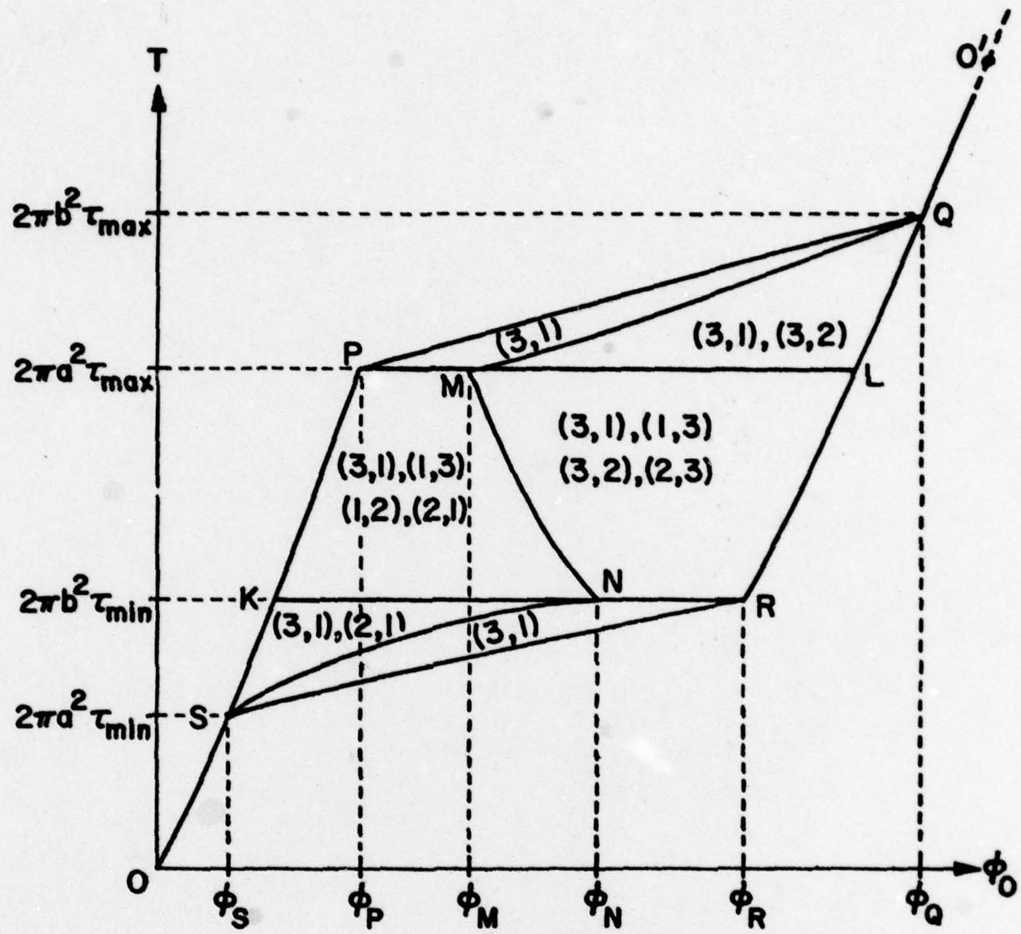
FIGURE 4. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



(i) SOLUTION $i=3,\ j=2$

(ii) SOLUTION $i=2,\ j=3$

FIGURE 5. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



SOLUTION (i, j)	ADMISSIBLE REGION A_{ij}
(1, 2)	PMNK
(2, 1)	PMNS
(2,3)	MLRN
(3,2)	MQRN
(3,1)	PQRS
(1,3)	PLRK

FIGURE 6. ADMISSIBLE REGIONS OF TORQUE-TWIST PLANE

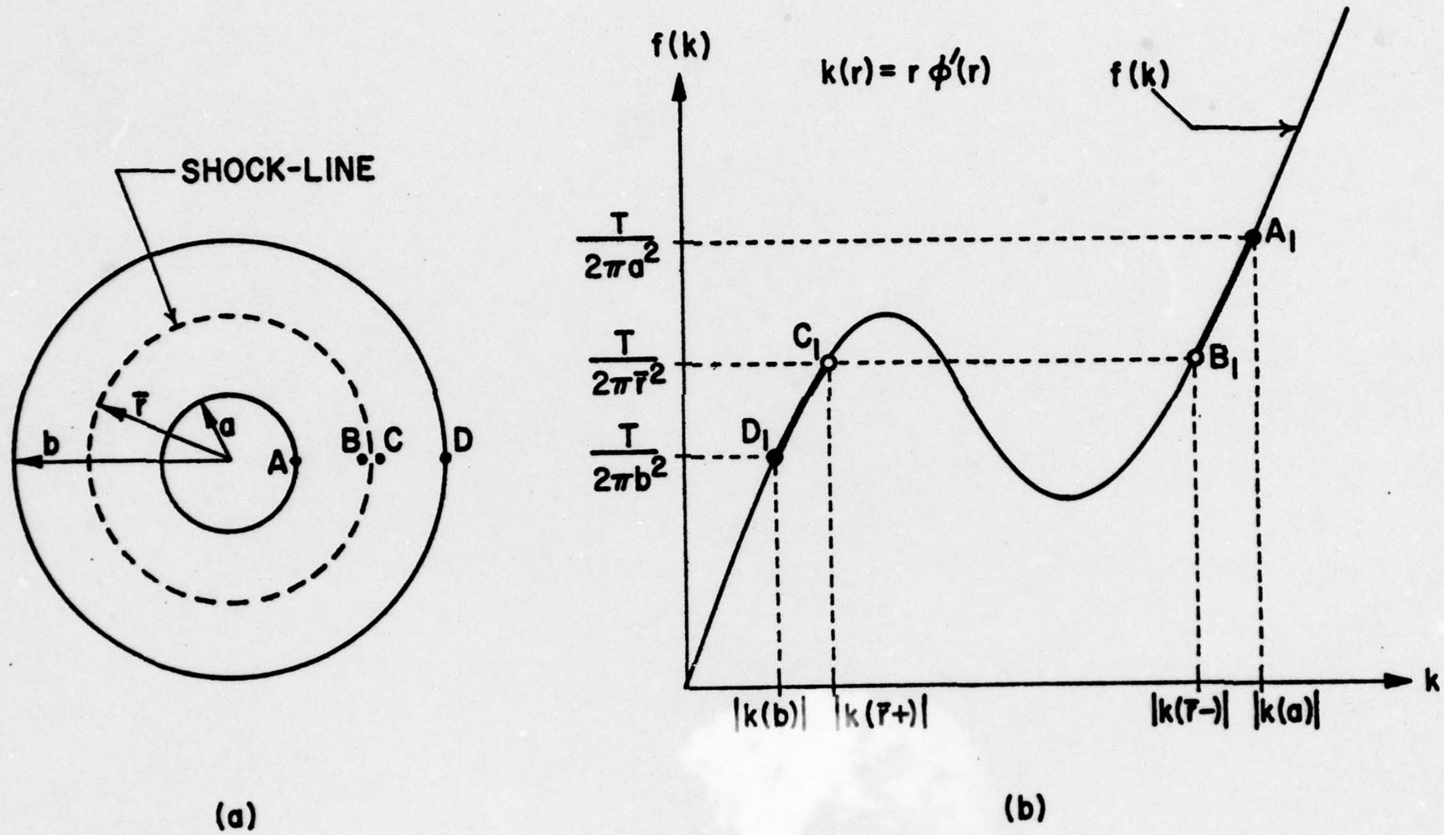


FIGURE 7. SHEAR STRESS VS. AMOUNT OF LOCAL SHEAR THROUGH THE TUBE. SOLUTION $i=3$, $j=1$

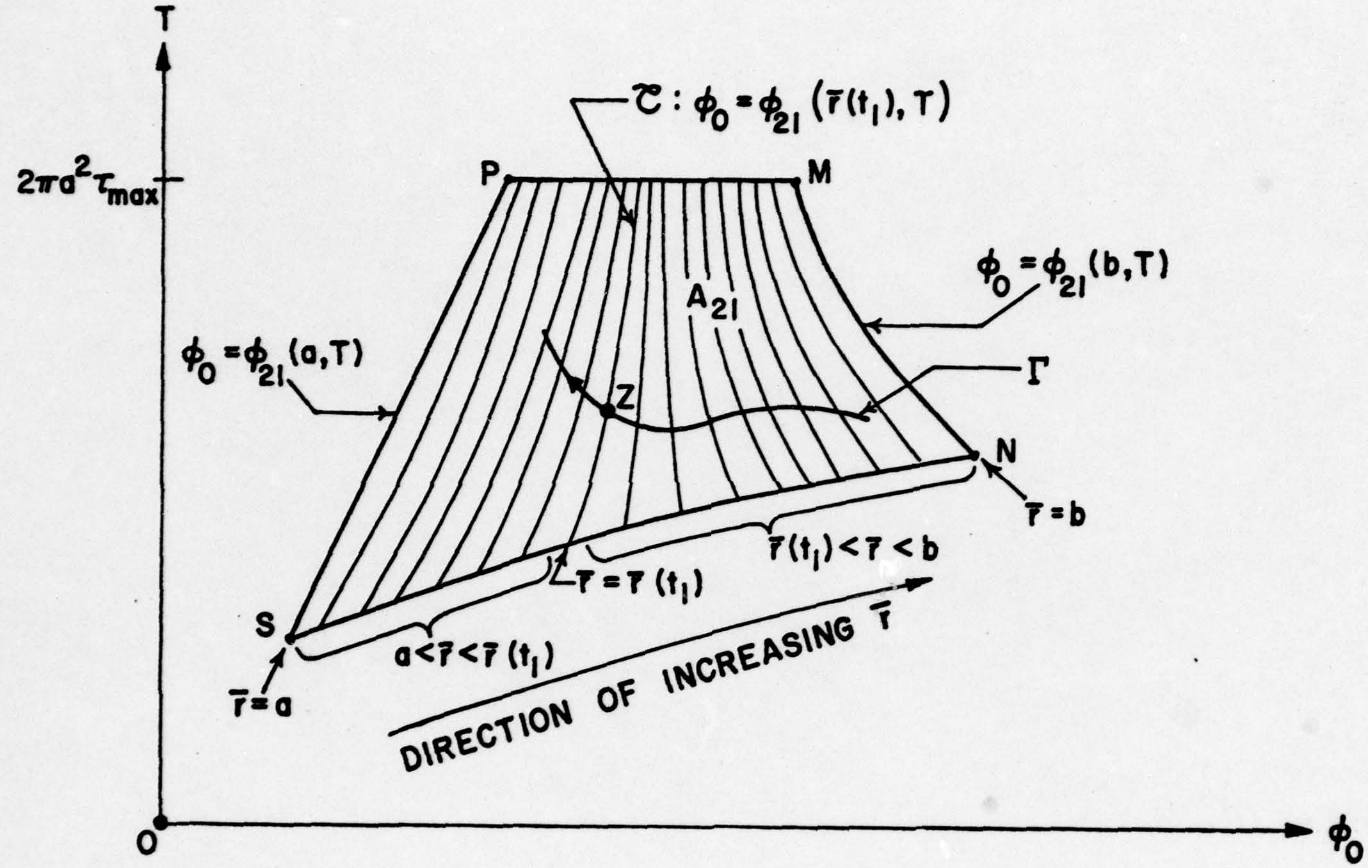
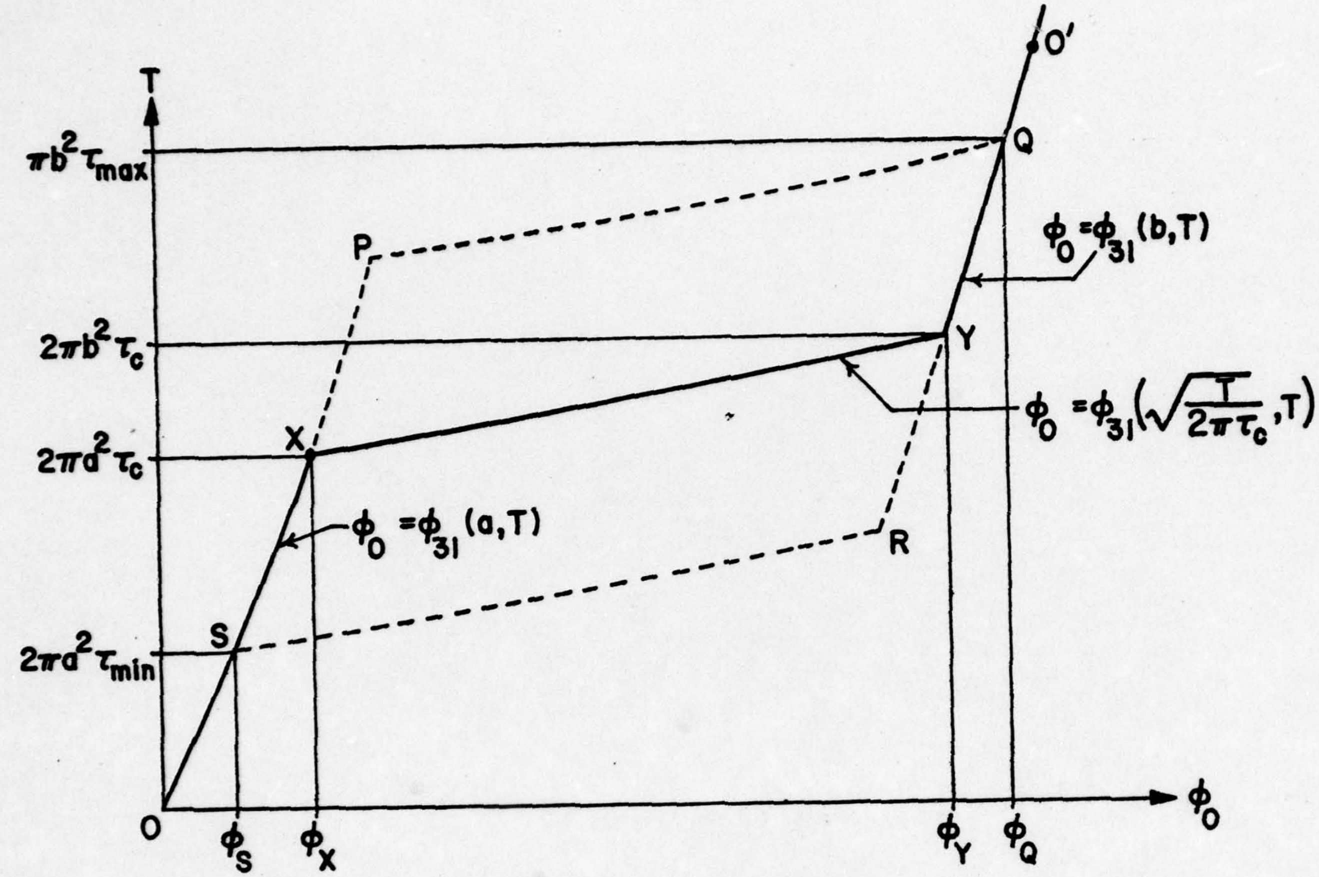


FIGURE 8. ADMISSIBLE ORIENTATIONS OF A LOADING PATH FOR SOLUTION $i=2, j=1$



OXYO'-FINITELY STABLE TORQUE-TWIST CURVE

FIGURE 9. TORQUE-TWIST PLANE. SOLUTIONS 1, 3, (3, 1).

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It is known that the type of the system of partial differential equations governing finite elastostatics can change from elliptic to non-elliptic at sufficiently large deformations for certain materials. This introduces the possibility that the elastostatic field may exhibit certain discontinuities. Some aspects of the general theory associated with these issues were examined in a recent series of studies by Knowles and Sternberg. In this paper we illustrate the occurrence of elastostatic fields with discontinuous deformation gradients in a

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physical problem. The body is assumed to be composed of a material which belongs to a particular class of isotropic, incompressible, elastic materials which allow for a loss of ellipticity. It is shown that no solution which is smooth in the classical sense exists to this problem for certain ranges of the applied loading. Next, we admit solutions involving elastostatic shocks into the discussion and find that the problem may then be solved completely. When this is done, however, there results a lack of uniqueness of solutions to the boundary-value problem. In order to resolve this non-iniqueness, the dissipativity and stability of the solutions are investigated.