AD-A071 662 SALFORD UNIV (ENGLAND) DEPT OF AERONAUTICAL AND MECH--ETC F/G 9/2

MULTIVARIABLE DIGITAL CONTROL SYSTEMS.(U)

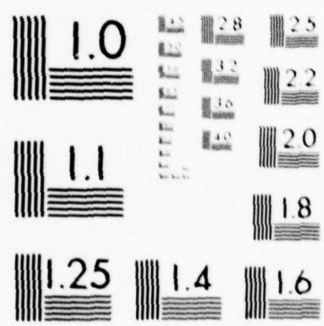
JUN 79 B PORTER AFOSR-76-3005

UNCLASSIFIED USAME/DC/101/79 NL

1 OF 2

AD

A071662



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

USAME/DC/101/79

June 1979

LEVEL II

UNIVERSITY OF SALFORD

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING

MULTIVARIABLE DIGITAL CONTROL SYSTEMS

PROFESSOR B PORTER

FINAL REPORT FOR PERIOD 1 JUNE 1976 - 31 MAY 1979

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. Report Number	2. Govt Accession No.	3. Recipient's Catalog Number
4. Title (and Subtitle) 6 MULTIVARIABLE DIGITAL CONTROL SYSTEMS.		5. Type of Report & Period Covered 9 Final Scientific Report. 1 Jun[redacted]76-31 May [redacted]79 6. Performing Org. Report Number 14 USAME/DC/101/79
7. Author(s) 10 B. PORTER		8. Contract or Grant Number 15 AFOSR-76-3005
9. Performing Organization Name and Address Dept of Aeronautical & Mechanical Engg University of Salford Salford M5 4WT, England		10. Program Element, Project, Task Area & Work Unit Numbers 61102F 2307/03 16 17 12 18φp.
11. Controlling Office Name and Address Air Force Flight Dynamics Laboratory(FGL) Wright-Patterson AFB, Ohio 45433, USA		12. Report Date 11 Jun[redacted]79 13. Number of Pages 174
14. Monitoring Agency Name and Address EOARD (Box 14) FPO NY 09510		15.

16. & 17. Distribution Statement

Approved for public release; distribution unlimited.

18. Supplementary Notes

19. Key Words

Digital Control Systems - Multivariable Control System Design - Discrete-Time Dynamical Systems

20. Abstract

The fundamental system-theoretic research and the parallel development of design techniques which have lead to the production of the comprehensive software package EIGENFORTRAC are outlined. The capability of EIGENFORTRAC in relation to the computer-aided design of high-performance digital control systems whose functions are simultaneously to reject the unmeasurable disturbances and to track multiple command inputs is described. Numerous references are provided to the system-theoretic research and to the computer algorithms embodied in EIGENFORTRAC.

FORM 1473

411 302

79 07 23 209

mt

USAME/DC/101/79 June 1979

MULTIVARIABLE DIGITAL CONTROL SYSTEMS

PROFESSOR B PORTER
DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

FINAL REPORT FOR PERIOD 1 JUNE 1976 - 31 MAY 1979

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

PREPARED FOR

E ROBERT LEMBLE
AFFDL/FG
WRIGHT-PATTERSON AIR FORCE BASE
OHIO 45433
USA

EOARD/LNE
DEPARTMENT OF THE AIR FORCE
223-231 MARYLEBONE ROAD
LONDON NW1 5TH

FOREWORD

This work was supported by the United States Air Force under Grant AFOSR-76-3005 and was undertaken at the University of Salford during the period 1 June 1976-31 May 1979. The major technical assistance was provided by the research associate Dr D Daintith but important conceptual contributions were made by Dr A Bradshaw and Professor J J D'Azzo. This final report was expertly typed and assembled by Mrs D Millward.

CONTENTS

1. INTRODUCTION

There has long been a crucial need for the development of techniques sufficiently powerful for the routine computer-aided design of on-board digital controllers for airborne systems. In order to fulfil this need, however, it was essential that the controllability, observability, time-optimality, and eigenstructure-assignability properties of multivariable discrete-time dynamical systems be clarified by fundamental system-theoretic research since these properties cannot be adequately characterised by classical z-transform techniques.

This report outlines both the fundamental system-theoretic research and the parallel development of design techniques which have lead to the production of the comprehensive software package EIGENFORTRAC(1). The use of EIGENFORTRAC greatly facilitates the design of high-performance multivariable digital control systems for a wide range of flight-control applications.

2. SYSTEM-THEORETIC RESEARCH

2.1 State-Feedback Regulators and Observers

The results obtained by Porter(2) (see Appendix 1) completely characterise the entire range of possible finite settling-time state-feedback regulators and observers by specifying the permissible[3] Smith canonical forms[4] of the closed-loop plant matrices of discrete-time multivariable systems. The design of such state-feedback regulators and observers can be readily effected by the method of entire eigenstructure assignment[5][6] and, in particular, by the algorithms developed by Porter and D'Azzo[7][8][9] (see Appendices 2, 3, and 4). These algorithms readily yield the vectors which are required for the simultaneous assignment of Jordan canonical forms, eigenvectors, and generalised eigenvectors to the plant matrices of closed-loop multivariable linear systems.

2.2 Output-Feedback Regulators

In most practical cases it is, of course, impossible to implement state-feedback regulators since the state of the plant is inaccessible and only the plant output is available for control purposes. However, the results obtained by Porter and Bradshaw[10][11] (see Appendices 5 and 6) characterise the closed-loop eigenstructure assignable by output-feedback regulators in terms of two families of well-defined subspaces. In the case of self-conjugate distinct eigenvalue spectra, for example, the eigenvectors and reciprocal eigenvectors of the closed-loop plant matrix must lie in two such families of subspaces and simultaneously satisfy appropriate orthogonality conditions. In contrast, the closed-loop eigenstructure assignable by state feedback[6] is constrained only by the requirement that the eigenvectors of the closed-loop plant matrix must lie in just one family of well-defined subspaces.

2.3 Dynamic Compensators

The severe constraints on the closed-loop eigenstructure assignable by output feedback imply that it is frequently impossible to achieve satisfactory closed-loop behaviour by means of static output-feedback regulators, and that it is consequently necessary to introduce dynamic compensators[5][12]. However, the results obtained by Porter and Bradshaw[13][14] (see Appendices 7 and 8) indicate that the design of such dynamic compensators can be effected by applying the method of entire eigenstructure assignment to appropriately augmented[5] systems. In this way, the use of observers can be avoided in the design of error-actuated multivariable tracking systems even when the special conditions previously established by Bradshaw and Porter[15] (see Appendix 9) for the existence of such error-actuated tracking systems are violated.

Indeed, in view of these fundamental new insights into the structure of linear multivariable systems, the design of dynamic compensators is in general reduced to the selection of pairwise-orthogonal eigenvectors and reciprocal eigenvectors for two families of well-defined subspaces which are parametrised by associated self-conjugate eigenvalue spectra. This selection can be effected by the use of a powerful new algorithm[16][17] (see Appendices 10 and 11) which requires the performance of restricted elementary row and column operations on matrices formed from the spanning vectors of these subspaces. The principal computational attraction of this algorithm is that no operations with polynomial matrices are involved, so that dynamic compensators for large-scale systems can be readily designed.

3. SOFTWARE PACKAGE DEVELOPMENT

3.1 Capability of EIGENFORTRAC Software

The EIGENFORTRAC software package is essentially an updated version of FORTRAC(18) based solely on the unifying method of entire eigenstructure assignment and, in particular, on the powerful algorithm for the design of dynamic compensators(16)(17). Synthesis techniques for state-feedback regulators, observers, output-feedback regulators, and dynamic compensators are embodied in EIGENFORTRAC. These techniques have been applied to the design of controllers for a variety of aircraft in a number of flight modes. Thus, for example, digital flight control systems have been designed by D'Azzo and Porter(19) for the F-4 fighter aircraft and by D'Azzo and Kennedy(20) for the C-141 transport aircraft.

3.2 Configuration of EIGENFORTRAC Software

The EIGENFORTRAC program configuration has been described by Porter, Bradshaw, and Daintith[1], together with a detailed description of all the EIGENFORTRAC subroutines. Detailed listings of the computer output for a simple example illustrating the design of discrete-time tracking systems incorporating error-actuated dynamic compensators have also been provided[1].

3.3 Operation of EIGENFORTRAC Software

The basic requirements of EIGENFORTRAC are the plant, input, and output matrices (A,B,C) of the uncontrolled system described in the continuous-time domain. The class of controller required is then specified (eg, state-feedback regulator, output-feedback regulator, error-actuated dynamic compensator) together with the sampling interval, T. The plant, input, and output matrices (A(T),B(T),C) of the sampled uncontrolled system are then computed, and appropriate augmentation[5] is automatically introduced. The closed-loop eigenvector and reciprocal eigenvector subspaces are then computed, and pairwise-orthogonal sets of closed-loop eigenvectors and reciprocal eigenvectors are then selected from these subspaces. Finally, these sets of eigenvectors and reciprocal eigenvectors are used in the computation of the compensator matrices. The performance of the resulting controller is checked by performing a discrete-time simulation which is followed by a continuous-time simulation using a Kutta-Merson routine in the case of promising designs.

4. CONCLUSIONS

Fundamental new insights into the structure of linear multivariable systems have been obtained by developing a unified theory of entire eigenstructure assignment. These system-theoretic results have been implemented in the comprehensive software package EIGENFORTRAC[1] which is currently available for the routine computer-aided design of on-board digital controllers for a wide range of flight-control applications.

REFERENCES

1. PORTER B, BRADSHAW A, and DAINTITH D, 1979, "EIGENFORTRAC: a software package for the design of multivariable digital control systems", Report USAME/DC/102/79, University of Salford.

2. PORTER B, 1976, "Direct synthesis of discrete-time feedback systems by equivalence transformations of polynomial matrices", Electronics Letters, 12 , pp 400-401.

3. DICKINSON B W, 1974, "On the fundamental theorem of linear state-variable feedback", IEEE Trans Autom Control, 19, pp 577-579.

4. BARNETT S, 1971, "Matrices in Control Theory" (London: Van Nostrand-Reinhold).

5. KIMURA H, 1975, "Pole assignment by gain output feedback", IEEE Trans Autom Control, 20, pp 509-516.

6. MOORE B C, 1976, "On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment", IEEE Trans Autom Control, 21, pp 689-692.

7. PORTER B and D'AZZO J J, 1977, "Algorithm for the synthesis of state-feedback regulators by entire eigenstructure assignment", Electronics Letters, 13, pp 230-231.

8. PORTER B and D'AZZO J J, 1978, "Closed-loop eigenstructure assignment by state feedback in multivariable linear systems", Int J Control, 27, pp 487-492.

9. PORTER B and D'AZZO J J, 1978, "Algorithm for closed-loop eigenstructure assignment by state feedback in multivariable linear systems", Int J Control, 27, pp 943-947.

10. PORTER B and BRADSHAW A, 1978, "Design of linear multivariable continuous-time output-feedback regulators", Int J Systems Sci, 9, pp 445-450.

11. PORTER B and BRADSHAW A, 1978, "Design of linear multivariable discrete-time output-feedback regulators", Int J Systems Sci, 9, pp 857-863.

12. BRASCH F M and PEARSON J B, 1970, "Pole-placement using dynamic compensators", IEEE Trans Autom Control, 15, pp 34-43.

13. PORTER B and BRADSHAW A, 1978, "Design of linear multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers", Int J Systems Sci, 9, pp 627-637.

14. BRADSHAW A and PORTER B, 1978, "Design of linear multivariable discrete-time tracking systems incorporating error-actuated dynamic controllers", Int J Systems Sci, 9, pp 1079-1090.

15. BRADSHAW A and PORTER B, 1978, "Design of linear multivariable discrete-time tracking systems incorporating error-actuated controllers", Int J Systems Sci, 9, pp 185-191.

16. BRADSHAW A, FLETCHER L R, and PORTER B, 1978, "Synthesis of output-feedback control laws for linear multivariable continuous-time systems", Int J Systems Sci, 9, pp 1331-1340.

17. PORTER B and BRADSHAW A, 1979, "Computer-aided design of dynamic compensators for linear multivariable continuous-time systems", Proc IFAC Symposium on Computer Aided Design of Control Systems, Zurich, Switzerland.

18. PORTER B, 1977, "FORTRAC: a software package for the design of multivariable digital control systems", Report USAME/DC/102/77, University of Salford.

19. D'AZZO J J and PORTER B, 1978, "Synthesis of digital flight control systems by the method of entire eigenstructure assignment", Proc IEEE National Aerospace and Electronics Conference, Dayton, USA.

20. D'AZZO J J and KENNEDY T A, 1979, "Synthesis of digital flight control tracking systems by the method of entire eigenstructure assignment", Proc IEEE National Aerospace and Electronics Conference, Dayton, USA.

APPENDIX 1

DIRECT SYNTHESIS OF DISCRETE-TIME FEEDBACK SYSTEMS BY EQUIVALENCE TRANSFORMATIONS OF POLYNOMIAL MATRICES

PROFESSOR B PORTER

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(ELECTRONICS LETTERS, VOL 12, PP400-401, 1976)

ABSTRACT

It is shown that the synthesis of closed-loop linear multivariable discrete-time systems can be directly effected by performing equivalence transformations on appropriate polynomial matrices. These polynomial matrices are the Smith canonical forms of the closed-loop characteristic matrices of such systems subject to the constraints imposed by the fundamental theorem of linear state-variable feedback.

1. Introduction

In this paper it is shown that the synthesis of closed-loop linear multivariable discrete-time systems governed by state and feedback equations of the respective forms

$$\underset{\sim}{x}(k+1) = \underset{\sim}{A}\underset{\sim}{x}(k) + \underset{\sim}{B}\underset{\sim}{u}(k) \tag{1a}$$

and

$$\underset{\sim}{u}(k) = \underset{\sim}{G}\underset{\sim}{x}(k) \tag{1b}$$

where $\underset{\sim}{x}(k) \in R^n$, $\underset{\sim}{u}(k) \in R^m$, and $(\underset{\sim}{A},\underset{\sim}{B})$ is a reachable pair can be directly effected by performing equivalence transformations on appropriate polynomial matrices. This synthesis procedure consists essentially in the specification of the Smith canonical form[1] $\underset{\sim}{S}(\lambda)$ of the closed-loop characteristic matrix $(\lambda\underset{\sim}{I}_n-\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{G})$ subject to the constraints on $\underset{\sim}{S}(\lambda)$ imposed by the fundamental theorem of linear state-variable feedback[2][3].

2. Synthesis Procedure

Thus, if

$$\underset{\sim}{S}(\lambda) = \mathrm{diag}(1,1,\ldots.,\psi_q(\lambda),\psi_{q-1}(\lambda),\ldots.,\psi_2(\lambda),\psi_1(\lambda)) \tag{2}$$

where the $\psi_i(\lambda)$ $(i=1,2,\ldots.,q)$ are any monic polynomials in $R[\lambda]$ such that

$$\psi_{i+1}(\lambda)\,|\,\psi_i(\lambda) \qquad (1 \le i \le q-1 \le m-1) \tag{3}$$

and

$$\sum_{i=1}^{q} \deg\psi_i(\lambda) = n \quad , \tag{4}$$

then there exists a matrix $\underset{\sim}{G} \in R^{m\times n}$ so that $\underset{\sim}{S}(\lambda)$ is the Smith canonical form of $(\lambda\underset{\sim}{I}_n-\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{G})$ provided that

$$\sum_{i=1}^{p} \deg\psi_i(\lambda) \geq \sum_{i=1}^{p} \kappa_i \qquad (1 \le p \le q) \tag{5}$$

where $\kappa_1 \geq \kappa_2 \geq \ldots . \geq \kappa_m$ are the ordered Kronecker invariants[2] of the pair $(\underset{\sim}{A},\underset{\sim}{B})$. It is accordingly evident that closed-loop discrete-time systems governed by state and feedback equations of the form (1) can be synthesised by the following procedure:

(i) Determine the Kronecker invariants $\kappa_i (i=1,2,\ldots.,m)$ of the pair $(\underset{\sim}{A},\underset{\sim}{B})$;

(ii) Prescribe an admissible matrix $\underset{\sim}{S}(\lambda)$ on the basis of the values of the $\kappa_i (i=1,2,\ldots.,m)$;

(iii) Transform $\underset{\sim}{S}(\lambda)$ by elementary row and column operations into an equivalent polynomial matrix of the form

$$\underset{\sim}{\Sigma}(\lambda) = \lambda\underset{\sim}{I}_n - \underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{G} \quad ; \tag{6}$$

(iv) Determine the set of linear simultaneous equations satisfied by the elements of the feedback matrix $\underset{\sim}{G}$ by inspection of $\underset{\sim}{\Sigma}(\lambda)$;

(v) Solve the set of linear simultaneous equations for the elements of the feedback matrix $\underset{\sim}{G}$.

3. Illustrative Example

This procedure can be conveniently illustrated by synthesising a closed-loop system governed by the state and feedback equations

$$\underset{\sim}{x}(k+1) = \begin{bmatrix} 0 , & 1 , & 2 \\ -2 , & 3 , & 0 \\ -2 , & -1 , & 0 \end{bmatrix} \underset{\sim}{x}(k) + \begin{bmatrix} 1 , & 2 \\ 1 , & 0 \\ 0 , & 0 \end{bmatrix} \underset{\sim}{u}(k) \tag{7a}$$

and

$$\underset{\sim}{u}(k) = \underset{\sim}{G}\underset{\sim}{x}(k) = \begin{bmatrix} g_{11} , & g_{12} , & g_{13} \\ g_{21} , & g_{22} , & g_{23} \end{bmatrix} \underset{\sim}{x}(k) \tag{7b}$$

so that the eigenvalues of the closed-loop plant matrix are all equal to zero. In this case it is evident from equations (7) that

$$\underset{\sim}{\Sigma}(\lambda) = \begin{bmatrix} \lambda-g_{11}-2g_{21} , & -1-g_{12}-2g_{22} , & -2-g_{13}-2g_{23} \\ 2-g_{11} , & \lambda-3-g_{12} , & -g_{13} \\ 2 , & 1 , & \lambda \end{bmatrix} \tag{8}$$

and that the associated Kronecker invariants are $\kappa_1=2$, $\kappa_2=1$. The conditions (3), (4), and (5) therefore indicate that

$$\underset{\sim}{S}_1(\lambda) = \text{diag}(1,\lambda,\lambda^2) \tag{9}$$

and

$$\underset{\sim}{S}_2(\lambda) = \text{diag}(1,1,\lambda^3) \tag{10}$$

are the only admissible forms of the Smith canonical form $\underset{\sim}{S}(\lambda)$ of the characteristic matrix of the closed-loop system governed by equations (7).

In the case of $\underset{\sim}{S}_1(\lambda)$ it is readily found that

$$\begin{bmatrix} 0 , & 1 , & 0 \\ \lambda , & -2 , & -1 \\ 1 , & 0 , & 0 \end{bmatrix} \begin{bmatrix} 1 , & 0 , & 0 \\ 0 , & \lambda , & 0 \\ 0 , & 0 , & \lambda^2 \end{bmatrix} \begin{bmatrix} 2 , & 1 , & \lambda \\ 1 , & 0 , & 0 \\ 0 , & 0 , & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda , & 0 , & 0 \\ 0 , & \lambda , & 0 \\ 2 , & 1 , & \lambda \end{bmatrix} = \underset{\sim}{\Sigma}_1(\lambda) \tag{11}$$

so that comparison of equations (8) and (11) indicates that the corresponding feedback matrix in equation (7b) is

$$\underset{\sim}{G}_1 = \begin{bmatrix} 2 , & -3 , & 0 \\ -1 , & 1 , & -1 \end{bmatrix} . \tag{12}$$

Similarly, in the case of $\underset{\sim}{S}_2(\lambda)$, it is readily found that

$$\begin{bmatrix} 1 , & 0 , & 0 \\ -2 , & \lambda , & 1 \\ \lambda , & 1 , & 0 \end{bmatrix} \begin{bmatrix} 1 , & 0 , & 0 \\ 0 , & 1 , & 0 \\ 0 , & 0 , & \lambda^3 \end{bmatrix} \begin{bmatrix} \lambda & , 0 , & 1 \\ 2-\lambda^2 & , 1 , & 0 \\ 1 & , 0 , & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda , & 0 , & 1 \\ 0 , & \lambda , & -2 \\ 2 , & 1 , & \lambda \end{bmatrix} = \underset{\sim}{\Sigma}_2(\lambda) \quad (13)$$

so that comparison of equations (8) and (13) indicates that the corresponding feedback matrix in equation (7b) is

$$\underset{\sim}{G}_2 = \begin{bmatrix} 2 , & -3 , & 2 \\ -1 , & 1 , & -5/2 \end{bmatrix} . \quad (14)$$

It is clear that, as desired, the characteristic polynomial of the closed-loop plant matrix is

$$c(\lambda) = \lambda^3 \quad (15)$$

in both cases but that the minimum polynomials associated with the feedback matrices $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ are respectively

$$m_1(\lambda) = \lambda^2 \quad (16)$$

and

$$m_2(\lambda) = \lambda^3 . \quad (17)$$

4. Conclusion

This procedure for the synthesis of linear multivariable discrete-time feedback systems constitutes a generalised eigenvalue-assignment procedure in that both the cyclic structure and the eigenvalues of the closed-loop plant matrices are synthesised. Moreover, the fact that the synthesis of such systems is directly effected by performing equivalence transformations on $\underset{\sim}{S}(\lambda)$ ensures that only those cyclic structures which are conformable with the constraints imposed by the fundamental theorem of linear state-variable feedback are considered. In particular, the synthesis procedure facilitates the assignment of both closed-loop characteristic polynomials and admissible closed-loop minimum polynomials. This facility is particularly important in the case of discrete-time systems since it obviously provides a basis for the design of time-optimal linear multivariable control systems[4]. It is evident, however, that the generalised eigenvalue-assignment procedure is equally applicable to the synthesis of linear multivariable continuous-time feedback systems.

ACKNOWLEDGEMENT

This research was sponsored by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005.

REFERENCES

1. BARNETT, S.: "Matrices in control theory" (Van Nostrand Reinhold, 1971).

2. KALMAN, R. E.: "Kronecker invariants and feedback", Proc. Conf. Ordinary Differential Equations, Washington D.C., 1971.

3. DICKINSON, B. W.: "On the fundamental theorem of linear state variable feedback", I.E.E.E. Trans., 1974, AC-19, pp 577-579.

4. PORTER, B.: "Design of time-optimal regulators for linear multivariable discrete-time plants", Electron. Lett., 1976, 12, pp 196-197.

APPENDIX 2

ALGORITHM FOR THE SYNTHESIS OF STATE-FEEDBACK REGULATORS BY ENTIRE EIGENSTRUCTURE ASSIGNMENT

Professor B Porter

Department of Aeronautical and Mechanical Engineering
University of Salford
Salford M5 4WT
England

and

Professor J J D'Azzo

Department of Electrical Engineering
Air Force Institute of Technology
Wright-Patterson Air Force Base
Ohio 45433
USA

(Electronics Letters, vol 13, pp230-231, 1977)

Abstract

The algorithm for the computation of a basis for ker $[\underset{\sim}{A}-\lambda_o \underset{\sim}{I}_n \text{ , } \underset{\sim}{B}]$ presented in this paper greatly facilitates the synthesis of state-feedback regulators by entire eigen-structure assignment. It is ideally suited for digital computer implementation and can be readily dualised for use in the synthesis of full-order observers by entire eigen-structure assignment.

1. INTRODUCTION

In view of the recent results obtained by Kimura[1] and Moore[2], it is evident that an efficient algorithm for the computation of a basis for

$$\ker \underset{\sim}{S}(\lambda_o) \;=\; \ker\, [\underset{\sim}{A}-\lambda_o \underset{\sim}{I}_n \;,\; \underset{\sim}{B}] \tag{1}$$

where $\lambda_o \in C$ and $[\underset{\sim}{A}-\lambda_o \underset{\sim}{I}_n \;,\; \underset{\sim}{B}] \in C^{nx(n+m)}$ is essential for the synthesis by entire eigenstructure assignment of state-feedback regulators for multivariable linear systems governed by state, output, and control-law equations of the respective forms

$$\dot{\underset{\sim}{x}}(t) \;=\; \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t) \quad , \tag{2}$$

$$\underset{\sim}{y}(t) \;=\; \underset{\sim}{C}\underset{\sim}{x}(t) \quad , \tag{3}$$

and

$$\underset{\sim}{u}(t) \;=\; \underset{\sim}{K}\underset{\sim}{x}(t) \quad . \tag{4}$$

Indeed, the real state-feedback matrix[1][2]

$$\underset{\sim}{K} \;=\; [\underset{\sim}{\omega}_1 \;,\; \underset{\sim}{\omega}_2 \;,\; \ldots \;,\; \underset{\sim}{\omega}_n]\, [\underset{\sim}{\chi}_1 \;,\; \underset{\sim}{\chi}_2 \;,\; \ldots \;,\; \underset{\sim}{\chi}_n]^{-1} \tag{5}$$

simultaneously assigns the self-conjugate distinct eigenvalue spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the corresponding eigenvector set $\{\underset{\sim}{\chi}_1, \underset{\sim}{\chi}_2, \ldots, \underset{\sim}{\chi}_n\}$ to the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K}) \in R^{nxn}$ just in case

$$[\underset{\sim}{\chi}_i' \;,\; \underset{\sim}{\omega}_i']' \in \ker \underset{\sim}{S}(\lambda_i) \qquad (i=1,2,\ldots,n) \quad . \tag{6}$$

2. Algorithm

The following algorithm provides an efficient means for the computation of a basis for ker $[A-\lambda_o I_n \, , \, B]$:

(i) Form the matrix

$$\hat{S}(\lambda_o) = \begin{bmatrix} A-\lambda_o I_n \, , & B \\ I_{n+m} & \end{bmatrix} \tag{7}$$

where $\lambda_o \in C$;

(ii) Perform elementary column operations on $\hat{S}(\lambda_o)$ until

$$\hat{S}(\lambda_o) \rightarrow \begin{bmatrix} \tilde{S}_{11} \, , & O \\ \tilde{S}_{21} \, , & \tilde{S}_{22} \end{bmatrix} = \tilde{S}(\lambda_o) \tag{8}$$

where $\tilde{S}_{11} \in C^{n \times r}$, rank $\tilde{S}_{11} = r =$ rank $[A-\lambda_o I_n \, , \, B]$, $\tilde{S}_{21} \in C^{(n+m) \times r}$, and $\tilde{S}_{22} \in C^{(n+m) \times (n+m-r)}$.

The required basis vectors for ker $[A-\lambda_o I_n \, , \, B]$ are then given by the $(n+m-r)$ columns of $\tilde{S}_{22}$, where obviously $r = n$ in case λ_o is not an input-decoupling zero of the system. This follows from the fact that equations (7) and (8) imply that

$$\begin{aligned} \tilde{S}(\lambda_o) &= \hat{S}(\lambda_o) E = \hat{S}(\lambda_o) [E_1 \, , \, E_2] \\ &= \begin{bmatrix} [A-\lambda_o I_n \, , \, B] E_1 \, , & [A-\lambda_o I_n \, , \, B] E_2 \\ E_1 \, , & E_2 \end{bmatrix} \end{aligned} \tag{9}$$

where $\underset{\sim}{E} \in C^{(n+m)\times(n+m)}$ is a product of elementary matrices, rank $\underset{\sim}{E}_1 = r$, rank $\underset{\sim}{E}_2 = n + m - r$, $[\underset{\sim}{A} - \lambda_o \underset{\sim}{I}_n \, , \, \underset{\sim}{B}]\underset{\sim}{E}_1 = \underset{\sim}{\tilde{S}}_{11}$, $[\underset{\sim}{A} - \lambda_o \underset{\sim}{I}_n \, , \, \underset{\sim}{B}]\underset{\sim}{E}_2 = \underset{\sim}{C}$, $\underset{\sim}{E}_1 = \underset{\sim}{\tilde{S}}_{21}$, and $\underset{\sim}{E}_2 = \underset{\sim}{\tilde{S}}_{22}$.

3. Illustrative Example

This algorithm can be conveniently illustrated by synthesising a state-feedback regulator for a multivariable linear system characterised by the matrices

$$\underset{\sim}{A} = \begin{bmatrix} -1 , & 0 , & 0 \\ 0 , & -2 , & 0 \\ 1 , & 1 , & -3 \end{bmatrix} , \qquad (10)$$

$$\underset{\sim}{B} = \begin{bmatrix} 1 , & 0 \\ 0 , & 1 \\ 0 , & 1 \end{bmatrix} , \qquad (11)$$

and

$$\underset{\sim}{C} = \begin{bmatrix} 1 , & 2 , & 0 \\ 1 , & 2 , & 1 \end{bmatrix} , \qquad (12)$$

which is such that

$$\sigma(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K}) = \{-1,-2,-3\} = \sigma(\underset{\sim}{A}) \qquad (13)$$

but such that the 'slow' mode corresponding to the eigenvalue $\lambda_1 = -1$ is eliminated from the output. Hence, in accordance with the algorithm, it is found that

$$\hat{\underset{\sim}{S}}(-1) = \left[\begin{array}{ccccc} 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ \hdashline 0, & -1, & 1, & -2, & 2 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \\ 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 1, & 0 \end{array}\right] , \tag{14}$$

$$\hat{\underset{\sim}{S}}(-2) = \left[\begin{array}{ccccc} 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 1, & 1, & 0, & 0 \\ \hdashline 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 1, & 0, & 1 \\ 0, & 0, & 0, & 1, & 1 \\ 1, & 0, & 0, & -1, & 0 \\ 0, & 1, & 0, & 0, & 0 \end{array}\right] , \tag{15}$$

and

$$\hat{\underset{\sim}{S}}(-3) = \left[\begin{array}{ccccc} 2, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 1, & 0, & 1, & 0, & 0 \\ \hdashline 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & -1, & 0 \end{array}\right] . \tag{16}$$

In view of the equivalences (14), (15), and (16) it therefore follows from the algorithm that

$$\ker \underset{\sim}{S}(-1) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\\0\end{bmatrix}\right\}, \tag{17}$$

$$\ker \underset{\sim}{S}(-2) = \text{span}\left\{\begin{bmatrix}1\\0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}\right\}, \tag{18}$$

and

$$\ker \underset{\sim}{S}(-3) = \text{span}\left\{\begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right\}. \tag{19}$$

It is evident from ker $\underset{\sim}{S}(-1)$, ker $\underset{\sim}{S}(-2)$, and ker $\underset{\sim}{S}(-3)$ that the closed-loop eigenvectors corresponding to the eigenvalue spectrum $\{-1,-2,-3\} = \sigma(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ can be assigned to the respective subspaces

$$\Sigma(-1) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}, \tag{20}$$

$$\Sigma(-2) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\;\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\;, \tag{21}$$

and

$$\Sigma(-3) \;=\; \text{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\;\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \tag{22}$$

subject only to the requirement that the resulting set of eigenvectors be linearly independent. In case

$$[\underset{\sim}{X}_1' \;,\; \underset{\sim}{\omega}_1']' \;=\; [-2 \;,\; 1 \;,\; 0 \;,\; 0 \;,\; 1]' \;, \tag{23}$$

$$[\underset{\sim}{X}_2' \;,\; \underset{\sim}{\omega}_2']' \;=\; [1 \;,\; 0 \;,\; 1 \;,\; -1 \;,\; 0]' \;, \tag{24}$$

and

$$[\underset{\sim}{X}_3' \;,\; \underset{\sim}{\omega}_3']' \;=\; [0 \;,\; 1 \;,\; 0 \;,\; 0 \;,\; -1]' \;, \tag{25}$$

it follows from equation (5) that

$$\underset{\sim}{K} \;=\; \begin{bmatrix}0 \;,& -1 \;,& 0\\ 1 \;,& 0 \;,& -1\end{bmatrix}\begin{bmatrix}-2 \;,& 1 \;,& 0\\ 1 \;,& 0 \;,& 1\\ 0 \;,& 1 \;,& 0\end{bmatrix}^{-1} \;=\; \begin{bmatrix}0 \;,& 0 \;,& -1\\ -1 \;,& -1 \;,& 1\end{bmatrix} \tag{26}$$

and therefore from equations (10), (11), and (26) that

$$(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K}) \;=\; \begin{bmatrix}-1 \;,& 0 \;,& -1\\ -1 \;,& -3 \;,& 1\\ 0 \;,& 0 \;,& -2\end{bmatrix}\;. \tag{27}$$

The eigenvalues have accordingly been unaltered by state feedback, as required, but the corresponding eigenvectors have become

$$\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{28}$$

where

$$\underset{\sim}{x}_1 = [-2 , 1 , 0]' \in \ker \underset{\sim}{C} , \tag{29}$$

as required. This elimination of the 'slow' mode corresponding to the eigenvalue $\lambda_1 = -1$ is possible because -1 is an invariant zero and $\underset{\sim}{x}_1$ is a corresponding state zero-direction of the system[3].

4. Conclusion

This algorithm for the computation of a basis for $\ker [\underset{\sim}{A} - \lambda_o \underset{\sim}{I}_n , \underset{\sim}{B}]$ greatly facilitates the synthesis of state-feedback regulators by entire eigenstructure assignment since it is ideally suited to digital computer implementation. In addition, it is evident that the same algorithm also greatly facilitates the synthesis of full-order observers by entire eigenstructure assignment since it can clearly be used for the computation of a basis for $\ker [\underset{\sim}{A}' - \lambda_o \underset{\sim}{I}_n , \underset{\sim}{C}']$.

Acknowledgements

This research was supported in part by the Air Force

Wright Aeronautical Laboratories, United States Air Force, under Grant AFOSR-76-3005 and the computer implementation of the algorithm was expertly undertaken by David Daintith at the University of Salford.

References

1. KIMURA, H.: "Pole assignment by gain output feedback", I.E.E.E. Trans., 1975, AC-20, pp. 509-516.

2. MOORE, B. C.: "On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment", I.E.E.E. Trans., AC-21, pp. 689-692.

3. SHAKED, U. and KARCANIAS, N.: "The use of zeros and zero-directions in model reduction", Int. J. Control, 1976, 23, pp. 113-135.

APPENDIX 3

CLOSED-LOOP EIGENSTRUCTURE ASSIGNMENT BY STATE FEEDBACK IN MULTIVARIABLE LINEAR SYSTEMS

PROFESSOR B PORTER
DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

AND

PROFESSOR J J D'AZZO
DEPARTMENT OF ELECTRICAL ENGINEERING
AIR FORCE INSTITUTE OF TECHNOLOGY
WRIGHT-PATTERSON AIR FORCE BASE
OHIO 45433
USA

(INTERNATIONAL JOURNAL OF CONTROL, VOL 27, PP 487-492, 1978)

Abstract

In this paper, results are presented which facilitate the complete exploitation of state feedback in the assignment of the entire closed-loop eigenstructure of multivariable linear systems. These results include an algorithm for the direct computation of the state-feedback matrix which assigns prescribed Jordan canonical forms, eigenvectors, and generalised eigenvectors to the plant matrices of closed-loop systems. This algorithm is illustrated by assigning the entire closed-loop eigenstructure of a third-order two-input discrete-time system in such a way that the resulting closed-loop system exhibits finite settling time behaviour.

1. Introduction

It is well known that, except in the case of single-input systems, specification of closed-loop eigenvalues does not define a unique closed-loop system. This non-uniqueness has, however, been only partially exploited in only a few instances by algorithms which permit the specification of a number of components of the closed-loop eigenvectors (Srinathkumar and Rhoten 1975, Shah et al 1975) and by algorithms which avoid large feedback gains (Porter and Crossley 1972, Lee 1975). The results presented in this paper facilitate the complete exploitation of state feedback in the assignment of the entire closed-loop eigenstructure of multivariable linear systems. These results include an algorithm for the direct computation of the state-feedback matrix which assigns prescribed Jordan canonical forms, eigenvectors, and generalised eigenvectors to the plant matrices of closed-loop systems. The expression for this state-feedback matrix assumes a simple form which is equivalent to that obtained by Kimura (1975) in the context of eigenvalue assignment by output feedback and by Moore (1976) in the context of output regulation for the special case of distinct prescribed eigenvalues.

2. Theory

The sequences of equations

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I} \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(1,j)} \\ \underset{\sim}{w}_{\lambda_i}^{(1,j)} \end{bmatrix} = \underset{\sim}{0} \quad , \quad (1a)$$

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I} \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(2,j)} \\ \\ \underset{\sim}{w}_{\lambda_i}^{(2,j)} \end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(1,j)} \qquad (1b)$$

....

....

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I} \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(m_{ji},j)} \\ \\ \underset{\sim}{w}_{\lambda_i}^{(m_{ji},j)} \end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(m_{ji}-1,j)} \quad , \qquad (1m_{ji})$$

$$(j=1,2,\ldots,k_i \ ; \ i=1,2,\ldots,p)$$

generate k_i strings of vectors associated with the eigenvalue λ_i, where $\underset{\sim}{v}_{\lambda_i}^{(\ell,j)}$ is the ℓth vector in the jth string of length m_{ji} associated with the eigenvalue λ_i. The vectors $\underset{\sim}{v}_{\lambda_i}^{(1,j)}$ $(j=1,2,\ldots,k_i)$ are the k_i eigenvectors associated with the eigenvalue λ_i, whilst the remaining vectors in each of the k_i strings of vectors generated by equations (1) are generalised eigenvectors associated with the eigenvalue λ_i. The total number of vectors associated with the eigenvalue λ_i is evidently

$$m_i = \sum_{j=1}^{k_i} m_{ji} \qquad (i=1,2,\ldots,p) \qquad (2)$$

and the entire set of vectors associated with the eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$ will accordingly serve as a basis for n-dimensional state space only if

$$n = \sum_{i=1}^{p} m_i \quad . \qquad (3)$$

In case the eigenvalues λ_i and the integers m_{ji} and k_i are chosen so that this entire set of vectors not only satisfies (3) but is also linearly independent and self-conjugate, then the real state-feedback matrix

$$\underset{\sim}{K} = [\underset{\sim}{w}_{\lambda_1}^{(1,1)},\ldots,\underset{\sim}{w}_{\lambda_p}^{(m_{k_p p},k_p)}]\ [\underset{\sim}{v}_{\lambda_1}^{(1,1)},\ldots,\underset{\sim}{v}_{\lambda_p}^{(m_{k_p p},k_p)}]^{-1} \tag{4}$$

is such that the Jordan canonical form of the nxn closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ contains the eigenvalue λ_i $(i=1,2,\ldots,p)$ with geometric multiplicity k_i and algebraic multiplicity m_i. This follows from the fact that if the real state-feedback matrix $\underset{\sim}{K}$ is such that the Jordan canonical form of the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ contains the eigenvalue λ_i $(i=1,2,\ldots,p)$ with geometric multiplicity k_i, algebraic multiplicity m_i, and associated eigenvectors $\underset{\sim}{v}_{\lambda_i}^{(1,j)}$ $(j=1,2,\ldots,k_i)$ then

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}\ ,\ \underset{\sim}{B}]\begin{bmatrix}\underset{\sim}{v}_{\lambda_i}^{(1,j)}\\ \underset{\sim}{K}\underset{\sim}{v}_{\lambda_i}^{(1,j)}\end{bmatrix} = \underset{\sim}{0}\ , \tag{5a}$$

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}\ ,\ \underset{\sim}{B}]\begin{bmatrix}\underset{\sim}{v}_{\lambda_i}^{(2,j)}\\ \underset{\sim}{K}\underset{\sim}{v}_{\lambda_i}^{(2,j)}\end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(1,j)}\ , \tag{5b}$$

....

....

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}\ ,\ \underset{\sim}{B}]\begin{bmatrix}\underset{\sim}{v}_{\lambda_i}^{(m_{ji},j)}\\ \underset{\sim}{K}\underset{\sim}{v}_{\lambda_i}^{(m_{ji},j)}\end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(m_{ji}-1,j)}\ , \tag{5m_{ji}}$$

where the m_{ji} satisfy (2) and (3).

It is evident that, in the special case when $p = n$ and $k_i = m_i = 1$ $(i=1,2,\ldots,n)$, then $j = 1$ and $m_{ji} = 1$ $(i=1,2,\ldots,n)$. Each of the sequences of equations (1) accordingly reduces to just a single equation, and there are clearly n such equations

$$\left[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}\ ,\ \underset{\sim}{B}\right] \begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(1,1)} \\ \\ \underset{\sim}{w}_{\lambda_i}^{(1,1)} \end{bmatrix} = \underset{\sim}{O} \qquad (i=1,2,\ldots,n) \quad (6)$$

for the eigenvectors $\underset{\sim}{v}_{\lambda_i}^{(1,1)}$ $(i=1,2,\ldots,n)$ of the nxn closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$. In this special case, the expression (4) for the state-feedback matrix assumes the simple form

$$\underset{\sim}{K} = \left[\underset{\sim}{w}_{\lambda_1}^{(1,1)}, \underset{\sim}{w}_{\lambda_2}^{(1,1)}, \ldots, \underset{\sim}{w}_{\lambda_n}^{(1,1)}\right] \left[\underset{\sim}{v}_{\lambda_1}^{(1,1)}, \underset{\sim}{v}_{\lambda_2}^{(1,1)}, \ldots, \underset{\sim}{v}_{\lambda_n}^{(1,1)}\right]^{-1} \quad (7)$$

which is equivalent to that obtained by Kimura (1975) and Moore (1976). The computation of $\underset{\sim}{K}$ in the case of distinct eigenvalues thus reduces to the determination of the kernels of each of the n matrices.

$$\underset{\sim}{S}_{\lambda_i} = \left[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}\ ,\ \underset{\sim}{B}\right] \qquad (i=1,2,\ldots,n)\ . \quad (8)$$

3. Illustrative Example

These results can be conveniently illustrated by assigning the entire closed-loop eigenstructure of the discrete-time system governed by the state and feedback equations (Porter 1976)

$$\underset{\sim}{x}(k+1) = \begin{bmatrix} 0 , & 1 , & 2 \\ -2 , & 3 , & 0 \\ -2 , & -1 , & 0 \end{bmatrix} \underset{\sim}{x}(k) + \begin{bmatrix} 1 , & 2 \\ 1 , & 0 \\ 0 , & 0 \end{bmatrix} \underset{\sim}{u}(k) \tag{9a}$$

and

$$\underset{\sim}{u}(k) = \underset{\sim}{K}\underset{\sim}{x}(k) \tag{9b}$$

in such a way that the resulting closed-loop system exhibits finite settling time behaviour. Such an assignment clearly requires that $p = 1$, $m_1 = 3$, and $\lambda_1 = 0$ but, in consonance with the fundamental theorem of linear state-variable feedback (Dickinson 1974), it is possible further to require either that $k_1 = 2$, $m_{11} = 2$, and $m_{21} = 1$ or that $k_1 = 1$ and $m_{11} = 3$. In the former case, equations (1) indicate that

$$\{\underset{\sim}{v}_{\lambda_1}^{(1,1)} , \underset{\sim}{v}_{\lambda_1}^{(2,1)} , \underset{\sim}{v}_{\lambda_1}^{(1,2)}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} \tag{10}$$

constitutes an admissible set of closed-loop eigenvectors and generalised eigenvectors whilst, in the latter case, equations (1) indicate that

$$\{\underset{\sim}{v}_{\lambda_1}^{(1,1)} , \underset{\sim}{v}_{\lambda_1}^{(2,1)} , \underset{\sim}{v}_{\lambda_1}^{(3,1)}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} , \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \right\} \tag{11}$$

constitutes an admissible set of closed-loop eigenvectors and generalised eigenvectors since also

$$\underset{\sim}{S}_{\lambda_1} = \begin{bmatrix} 0, & 1, & 2, & 1, & 2 \\ -2, & 3, & 0, & 1, & 0 \\ -2, & -1, & 0, & 0, & 0 \end{bmatrix} \tag{12}$$

and therefore

$$\text{Ker } \underset{\sim}{S}_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 8 \\ -3 \end{bmatrix} \right\} . \tag{13}$$

In the former case,

$$\{\underset{\sim}{w}_{\lambda_1}^{(1,1)}, \underset{\sim}{w}_{\lambda_1}^{(2,1)}, \underset{\sim}{w}_{\lambda_1}^{(1,2)}\} = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ -6 \end{bmatrix}, \begin{bmatrix} 8 \\ -3 \end{bmatrix} \right\} \tag{14}$$

so that in view of equation (4)

$$\underset{\sim}{K} = \begin{bmatrix} 0, & 11, & 8 \\ -1, & -6, & -3 \end{bmatrix} \begin{bmatrix} 0, & 1, & 1 \\ 0, & -3, & -2 \\ 1, & 2, & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2, & -3, & 0 \\ -1, & 1, & -1 \end{bmatrix} \tag{15}$$

and therefore

$$(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K}) = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ -2, & -1, & 0 \end{bmatrix} \tag{16}$$

which has the Jordan canonical form

$$\begin{bmatrix} 0 , & 1 , & 0 \\ 0 , & 0 , & 0 \\ 0 , & 0 , & 0 \end{bmatrix} \tag{17}$$

together with the eigenvectors $\underset{\sim}{v}_{\lambda_1}^{(1,1)}$ and $\underset{\sim}{v}_{\lambda_1}^{(1,2)}$ and the generalised eigenvector $\underset{\sim}{v}_{\lambda_1}^{(2,1)}$ prescribed in equation (10), as required. In the latter case,

$$\{\underset{\sim}{w}_{\lambda_1}^{(1,1)} , \underset{\sim}{w}_{\lambda_1}^{(2,1)} , \underset{\sim}{w}_{\lambda_1}^{(3,1)}\} = \left\{\begin{bmatrix} 0 \\ -1 \end{bmatrix} , \begin{bmatrix} 11 \\ -6 \end{bmatrix} , \begin{bmatrix} 11 \\ -3 \end{bmatrix}\right\} \tag{18}$$

so that in view of equation (4)

$$\underset{\sim}{K} = \begin{bmatrix} 0 , & 11 , & 11 \\ -1 , & -6 , & -3 \end{bmatrix} \begin{bmatrix} 0 , & 1 , & 1 \\ 0 , & -3 , & -4 \\ 1 , & 2 , & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 11 , & 0 , & 0 \\ -7 , & -1 , & -1 \end{bmatrix} \tag{19}$$

and therefore

$$(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K}) = \begin{bmatrix} -3 , & -1 , & 0 \\ 9 , & 3 , & 0 \\ -2 , & -1 , & 0 \end{bmatrix} \tag{20}$$

which has the Jordan canonical form

$$\begin{bmatrix} 0 , & 1 , & 0 \\ 0 , & 0 , & 1 \\ 0 , & 0 , & 0 \end{bmatrix} \tag{21}$$

together with the eigenvector $\underset{\sim}{v}_{\lambda_1}^{(1,1)}$ and the generalised eigenvectors $\underset{\sim}{v}_{\lambda_1}^{(2,1)}$ and $\underset{\sim}{v}_{\lambda_1}^{(3,1)}$ prescribed in equation (11), as required.

4. Conclusion

These results facilitate the complete exploitation of state feedback in the assignment of the entire closed-loop eigenstructure of multivariable linear systems and are clearly equally applicable to both continuous-time and discrete-time systems. It is evident that, even in the case of systems for which the pair $(\underset{\sim}{A},\underset{\sim}{B})$ is uncontrollable, certain prescribed eigenvectors of $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ can be assigned by state feedback. In the case of systems with asymptotically stable but uncontrollable modes, it is therefore frequently possible to achieve significant improvements in the dynamical behaviour of such systems by the introduction of appropriate state-feedback controllers.

Acknowledgement

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005.

References

SRINATHKUMAR, S., and RHOTEN, R. P., 1975, Electron. Lett., 11, 124.

SHAH, S. L., FISHER, D. G., and SEBORG, D. E., 1975, Electron. Let., 11, 388.

PORTER, B., and CROSSLEY, T. R., 1972, "Modal control-theory and applications" (London: Taylor and Francis).

LEE, G., 1975, Proc. I.E.E.E. Conf. Decision and Control, 188.

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509.

MOORE, B. C., 1976, I.E.E.E. Trans. autom. Control, 21, 689.

PORTER, B., 1976, Electron. Lett., 12, 400.

DICKINSON, B. W., 1974, I.E.E.E. Trans. autom. Control, 19, 577.

APPENDIX 4

ALGORITHM FOR CLOSED-LOOP EIGENSTRUCTURE ASSIGNMENT BY STATE FEEDBACK IN MULTIVARIABLE LINEAR SYSTEMS

PROFESSOR B PORTER

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

AND

PROFESSOR J J D'AZZO

DEPARTMENT OF ELECTRICAL ENGINEERING
AIR FORCE INSTITUTE OF TECHNOLOGY
WRIGHT-PATTERSON AIR FORCE BASE
OHIO 45433
USA

(INTERNATIONAL JOURNAL OF CONTROL, VOL 27, PP943-947, 1978)

ABSTRACT

In this paper, an algorithm is presented which greatly facilitates the complete exploitation of state feedback in the assignment of the entire closed-loop eigenstructure of controllable multi-input systems. This algorithm is a generalisation of the algorithm of MacLane and Birkhoff (1968) for the computation of a basis for the null space of a matrix and is ideally suited to digital computer implementation. The algorithm readily yields the vectors which are required (Porter and D'Azzo 1977) for the simultaneous assignment of Jordan canonical forms, eigenvectors, and generalised eigenvectors to the plant matrices of closed-loop controllable multivariable linear systems. The effectiveness of the algorithm is illustrated by assigning the entire closed-loop eigenstructure of a third-order two-input discrete-time system in such a way that the resulting closed-loop system exhibits time-optimal behaviour.

1. INTRODUCTION

The algorithm presented in this paper readily yields the vectors which are required (Porter and D'Azzo 1977) for the determination of the state-feedback matrix which simultaneously assigns Jordan canonical forms, eigenvectors, and generalised eigenvectors to the plant matrices of closed-loop controllable multi-input linear systems. These vectors satisfy the sequences of equations (Porter and D'Azzo 1977)

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}]\begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(1,j)} \\ \underset{\sim}{w}_{\lambda_i}^{(1,j)} \end{bmatrix} = \underset{\sim}{0} \quad , \qquad (1a)$$

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}]\begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(2,j)} \\ \underset{\sim}{w}_{\lambda_i}^{(2,j)} \end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(1,j)} \quad , \qquad (1b)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}]\begin{bmatrix} \underset{\sim}{v}_{\lambda_i}^{(m_{ji},j)} \\ \underset{\sim}{w}_{\lambda_i}^{(m_{ji},j)} \end{bmatrix} = \underset{\sim}{v}_{\lambda_i}^{(m_{ji}-1,j)} \quad , \qquad (1m_{ji})$$

$$(j=1,2,\ldots,k_i \ ; i=1,2,\ldots,p)$$

which together generate k_i strings of eigenvectors and generalised eigenvectors associated with the eigenvalue λ_i, where $\underset{\sim}{v}_{\lambda_i}^{(\ell,j)}$ is the ℓth vector in the jth string of length m_{ji} associated with the eigenvalue λ_i. In case the eigenvalues

λ_i $(i=1,2,\ldots,p)$ and the integers m_{ji} and k_i are chosen so that this entire set of eigenvectors and generalised eigenvectors is linearly independent and self-conjugate, then the real state-feedback matrix (Porter and D'Azzo 1977)

$$\underset{\sim}{K} = \left[\underset{\sim}{w}_{\lambda_1}^{(1,1)},\ldots,\underset{\sim}{w}_{\lambda_p}^{(m_{k_p p},k_p)}\right]\left[\underset{\sim}{v}_{\lambda_1}^{(1,1)},\ldots,\underset{\sim}{v}_{\lambda_p}^{(m_{k_p p},k_p)}\right]^{-1} \tag{2}$$

is such that the Jordan canonical form of the nxn closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ contains the eigenvalue λ_i $(i=1,2,\ldots,p)$ with geometric multiplicity k_i and algebraic multiplicity

$$m_i = \sum_{j=1}^{k_i} m_{ji} \qquad (i=1,2,\ldots,p) \quad . \tag{3}$$

It is evident that, in the special case when $p = n$ and $k_i = m_i = 1$ $(i=1,2,\ldots,n)$, then $j = 1$, $m_{ji} = 1$ $(i=1,2,\ldots,n)$, and each of the sequences of equations (1) reduces to just a single equation for the eigenvector $\underset{\sim}{v}_{\lambda_i}^{(1,1)}$ of the nxn closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{K})$ associated with the eigenvalue λ_i. In this special case of self-conjugate distinct eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$, the general expression (2) for the state-feedback matrix $\underset{\sim}{K}$ assumes the simple form obtained by Kimura (1975) and Moore (1976).

2. ALGORITHM

The vectors $\{\underset{\sim}{v}_{\lambda_1}^{(1,1)}, \ldots, \underset{\sim}{v}_{\lambda_p}^{(m_{k_p p},k_p)}\}$ and

$\{\underset{\sim}{w}_{\lambda_1}^{(1,1)},\ldots,\underset{\sim}{w}_{\lambda_p}^{(m_{k_p p},k_p)}\}$ required for the determination of the state-feedback matrix $\underset{\sim}{K}$ expressed by equation (2) can be readily computed by the following algorithm which is a generalisation of the algorithm of MacLane and Birkhoff (1968) for the computation of a basis for the null-space of a matrix:

(i) Form the matrix

$$\hat{\underset{\sim}{S}}(\lambda_i) = \begin{bmatrix} \underset{\sim}{A}-\lambda_i\underset{\sim}{I}_n & , & \underset{\sim}{B} \\ & \underset{\sim}{I}_{n+m} & \end{bmatrix} \tag{4}$$

for λ_i $(i=1,2,\ldots,p)$;

(ii) Perform elementary column operations on $\hat{\underset{\sim}{S}}(\lambda_i)$ until

$$\hat{\underset{\sim}{S}}(\lambda_i) \sim \begin{bmatrix} \tilde{\underset{\sim}{S}}_{11}^{(1,j)} & , & \tilde{\underset{\sim}{S}}_{12}^{(1,j)} \\ \tilde{\underset{\sim}{S}}_{21}^{(1,j)} & , & \tilde{\underset{\sim}{S}}_{22}^{(1,j)} \end{bmatrix} = \tilde{\underset{\sim}{S}}^{(1,j)}(\lambda_i) \tag{5}$$

where $\tilde{\underset{\sim}{S}}_{11}^{(1,j)} \in C^{nxn}$, $\tilde{\underset{\sim}{S}}_{12}^{(1,j)} = \underset{\sim}{0}$, and rank $\tilde{\underset{\sim}{S}}_{11}^{(1,j)}$ = n = rank $[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}_n , \underset{\sim}{B}]$ since $(\underset{\sim}{A},\underset{\sim}{B})$ is a controllable pair;

(iii) Perform successive elementary column operations on $\tilde{\underset{\sim}{S}}^{(1,j)}(\lambda_i)$ until

$$\tilde{\underset{\sim}{S}}^{(m_{ji}-1,j)} \sim \begin{bmatrix} \tilde{\underset{\sim}{S}}_{11}^{(1,j)} & , & \underset{\sim}{V}_{\lambda_i}^{(m_{ji}-1,j)} \\ \tilde{\underset{\sim}{S}}_{21}^{(1,j)} & , & \tilde{\underset{\sim}{S}}_{22}^{(m_{ji},j)} \end{bmatrix} = \tilde{\underset{\sim}{S}}^{(m_{ji},j)} \tag{6}$$

where

$$\begin{bmatrix} \underset{\sim}{V}_{\lambda_i}^{(m_{ji},j)} \\ \\ \underset{\sim}{W}_{\lambda_i}^{(m_{ji},j)} \end{bmatrix} = \underset{\sim}{\tilde{S}}_{22}^{(m_{ji},j)} \quad . \tag{7}$$

The matrices $\{\underset{\sim}{V}_{\lambda_i}^{(1,j)}, \underset{\sim}{V}_{\lambda_i}^{(2,j)}, \ldots, \underset{\sim}{V}_{\lambda_i}^{(m_{ji},j)}\}$ and $\{\underset{\sim}{W}_{\lambda_i}^{(1,j)}, \underset{\sim}{W}_{\lambda_i}^{(2,j)}, \ldots, \underset{\sim}{W}_{\lambda_i}^{(m_{ji},j)}\}$ thus generated are clearly such that

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{V}_{\lambda_i}^{(1,j)} \\ \\ \underset{\sim}{W}_{\lambda_i}^{(1,j)} \end{bmatrix} = \underset{\sim}{O} \quad , \tag{8a}$$

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{V}_{\lambda_i}^{(2,j)} \\ \\ \underset{\sim}{W}_{\lambda_i}^{(2,j)} \end{bmatrix} = \underset{\sim}{V}_{\lambda_i}^{(1,j)} \quad , \tag{8b}$$

....

$$[\underset{\sim}{A}-\lambda_i \underset{\sim}{I}_n \ , \ \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{V}_{\lambda_i}^{(m_{ji},j)} \\ \\ \underset{\sim}{W}_{\lambda_i}^{(m_{ji},j)} \end{bmatrix} = \underset{\sim}{V}_{\lambda_i}^{(m_{ji}-1,j)} \quad . \tag{8m_{ji}}$$

and are therefore such that the vectors $\left[\underset{\sim}{V}_{\lambda_i}^{(n_{ji},j)'}, \underset{\sim}{W}_{\lambda_i}^{(n_{ji},j)'}\right]'$, $(n_{ji}=1,2,\ldots,m_{ji})$ required for use in equation (2) are linear combinations of corresponding columns of successive members of the entire sequence of matrices $\left[\underset{\sim}{V}_{\lambda_i}^{(n_{ji},j)'}, \underset{\sim}{W}_{\lambda_i}^{(n_{ji},j)'}\right]'$, $(n_{ji}=1,2,\ldots,m_{ji})$.

3. Illustrative Example

This algorithm can be conveniently illustrated by assigning the entire closed-loop eigenstructure of the discrete-time system governed by the state and feedback equations (Porter 1976a,b)

$$\underset{\sim}{x}(k+1) = \begin{bmatrix} 0 , & 1 , & 2 \\ -2 , & 3 , & 0 \\ -2 , & -1 , & 0 \end{bmatrix} \underset{\sim}{x}(k) + \begin{bmatrix} 1 , & 2 \\ 1 , & 0 \\ 0 , & 0 \end{bmatrix} \underset{\sim}{u}(k) \qquad (9a)$$

and

$$\underset{\sim}{u}(k) = \underset{\sim}{K}\underset{\sim}{x}(k) \qquad (9b)$$

in such a way that the resulting closed-loop system exhibits time-optimal behaviour. Such an assignment clearly requires that $p = 1$, $k_1 = 2$, $m_{11} = 2$, $m_{21} = 1$, and $\lambda_1 = 0$. In order to compute a suitable state-feedback matrix $\underset{\sim}{K}$ it is therefore only necessary to perform the following sequence of elementary column operations in accordance with the algorithm:

$$\begin{bmatrix} 0 , & 1 , & 2 , & 1 , & 2 \\ -2 , & 3 , & 0 , & 1 , & 0 \\ -2 , & -1 , & 0 , & 0 , & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 , & 0 , & 0 , & 0 , & 0 \\ 0 , & 1 , & 0 , & 0 , & 0 \\ 0 , & 0 , & 1 , & 0 , & 0 \\ 0 , & 0 , & 0 , & 1 , & 0 \\ 0 , & 0 , & 0 , & 0 , & 1 \end{bmatrix} \sim \begin{bmatrix} 1 , & 2 , & 1 , & 0 , & 0 \\ 3 , & 0 , & 1 , & 0 , & 0 \\ -1 , & 0 , & 0 , & 0 , & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 , & 0 , & 0 , & 0 , & 1 \\ 1 , & 0 , & 0 , & 0 , & -2 \\ 0 , & 1 , & 0 , & 1 , & 0 \\ 0 , & 0 , & 1 , & 0 , & 8 \\ 0 , & 0 , & 0 , & -1 , & -3 \end{bmatrix} \qquad (10a)$$

and

$$\left[\begin{array}{ccccc} 1 , & 2 , & 1 , & 0 , & 0 \\ 3 , & 0 , & 1 , & 0 , & 0 \\ -1 , & 0 , & 0 , & 0 , & 0 \\ \hline 0 , & 0 , & 0 , & 0 , & 1 \\ 1 , & 0 , & 0 , & 0 , & -2 \\ 0 , & 1 , & 0 , & 1 , & 0 \\ 0 , & 0 , & 1 , & 0 , & 8 \\ 0 , & 0 , & 0 , & -1 , & -3 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 , & 2 , & 1 , & 0 , & 1 \\ 3 , & 0 , & 1 , & 0 , & -2 \\ -1 , & 0 , & 0 , & 1 , & 0 \\ \hline 0 , & 0 , & 0 , & 1 , & 0 \\ 1 , & 0 , & 0 , & -3 , & 0 \\ 0 , & 1 , & 0 , & 2 , & 3/2 \\ 0 , & 0 , & 1 , & 11 , & -2 \\ 0 , & 0 , & 0 , & -6 , & 0 \end{array}\right] \qquad (10b)$$

It is evident from the equivalences (10a) and (10b) that

$$\{\underset{\sim}{v}_{\lambda_1}^{(1,1)}, \underset{\sim}{v}_{\lambda_1}^{(2,1)}, \underset{\sim}{v}_{\lambda_1}^{(1,2)}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} \qquad (11)$$

constitutes an admissible set of closed-loop eigenvectors and generalised eigenvectors and, correspondingly, that

$$\{\underset{\sim}{w}_{\lambda_1}^{(1,1)}, \underset{\sim}{w}_{\lambda_1}^{(2,1)}, \underset{\sim}{w}_{\lambda_1}^{(1,2)}\} = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} , \begin{bmatrix} 11 \\ -6 \end{bmatrix} , \begin{bmatrix} 8 \\ -3 \end{bmatrix} \right\}. \qquad (12)$$

The required state-feedback matrix determined by equation (2) is therefore

$$\underset{\sim}{K} = \begin{bmatrix} 0 , & 11 , & 8 \\ -1 , & -6 , & -3 \end{bmatrix} \begin{bmatrix} 0 , & 1 & 1 \\ 0 , & -3 , & -2 \\ 1 , & 2 , & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 , & -3 , & 0 \\ -1 , & 1 , & -1 \end{bmatrix}$$

(13)

so that the plant matrix of the closed-loop system governed by equations (9) and (13) is

$$\underset{\sim}{A} + \underset{\sim\sim}{BK} = \begin{bmatrix} 0 , & 0 , & 0 \\ 0 , & 0 , & 0 \\ -2 , & -1 , & 0 \end{bmatrix} . \tag{14}$$

This plant matrix is clearly nilpotent of index two, as required, and furthermore possesses the eigenvectors and generalised eigenvectors prescribed in equation (11).

4. CONCLUSION

In this paper, an algorithm has been presented which greatly facilitates the synthesis of state-feedback regulators by entire eigenstructure assignment. This algorithm, which is equally applicable to both continuous-time and discrete-time systems, has been illustrated by assigning the entire closed-loop eigenstructure of a third-order two-input discrete-time system in such a way that the closed-loop system exhibits time-optimal behaviour. In view of the simple elementary column operations involved, it is evident that the algorithm is ideally suited to digital computer implementation.

ACKNOWLEDGEMENTS

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force, under Grant AFOSR-76-3005 and the computer implementation of the algorithm was expertly undertaken by David Daintith at the University of Salford.

References

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509.

MACLANE, S. and BIRKHOFF, G., 1968, "Algebra" (New York: MacMillan), 248.

MOORE, B. C., 1976, I.E.E.E. Trans. autom. Control, 21, 689.

PORTER, B., 1976a Electron. Lett., 12, 1976b, ibid, 12, 400.

PORTER, B. and D'AZZO, J. J., 1977, Int. J. Control (in the press).

APPENDIX 5

DESIGN OF LINEAR MULTIVARIABLE CONTINUOUS-TIME OUTPUT-FEEDBACK REGULATORS

PROFESSOR B PORTER AND DR A BRADSHAW

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, VOL 9, PP445-450, 1978)

ABSTRACT

In this paper, the method of entire eigenstructure assignment (Kimura 1975, Moore 1976, Porter and D'Azzo 1977) is applied to the design of linear multivariable continuous-time output-feedback regulators. It is shown that, in the case of self-conjugate distinct eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the eigenvectors and reciprocal eigenvectors lie in well-defined subspaces. The method is illustrated by designing an output-feedback regulator for a third-order continuous-time system.

1. INTRODUCTION

In most practical cases, it is impossible to implement state-feedback control laws since the state of the plant is inaccessible and only the plant output is available for control purposes. Much effort (see, for example, Davison and Wang (1975)) has accordingly been expended on the investigation of the closed-loop dynamics achievable by the implementation of output-feedback control laws. However, apart from the partial results obtained by Kimura (1975), this effort has led to results concerned only with closed-loop eigenvalues and not with closed-loop eigenvectors. In this paper, the method of entire eigenstructure assignment (Kimura 1975, Moore 1976, Porter and D'Azzo 1977) is therefore applied to the design of output-feedback regulators for multivariable linear continuous-time systems governed by state and output equations of the respective forms

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t) \tag{1}$$

and

$$\underset{\sim}{y}(t) = \underset{\sim}{C}\underset{\sim}{x}(t) \quad , \tag{2}$$

where $\underset{\sim}{A} \in R^{nxn}$, $\underset{\sim}{B} \in R^{nxm}$, $\underset{\sim}{C} \in R^{pxn}$, rank $\underset{\sim}{B}$ = m, and rank $\underset{\sim}{C}$ = p.

2. THEORY

Thus, if output feedback is applied to the system governed by the state equation (1) in accordance with the control-law equation

$$\underset{\sim}{u}(t) = \underset{\sim}{G}\underset{\sim}{y}(t) \tag{3}$$

and the output-feedback matrix $\underset{\sim}{G} \in R^{m\times p}$ is such that the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})$ has a self-conjugate spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ of distinct eigenvalues and corresponding eigenvector and reciprocal eigenvector sets $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_n\}$, then obviously

$$(\underset{\sim}{A}-\lambda_i\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})\underset{\sim}{\chi}_i = \underset{\sim}{O} \qquad (i=1,2,\ldots,n) \tag{4}$$

and

$$\underset{\sim}{\phi}_j'(\underset{\sim}{A}-\lambda_j\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C}) = \underset{\sim}{O} \qquad (j=1,2,\ldots,n) \tag{5}$$

so that

$$\left[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}\ ,\ \underset{\sim}{B}\right]\begin{bmatrix}\underset{\sim}{\chi}_i\\ \underset{\sim}{\omega}_i\end{bmatrix} = \underset{\sim}{O} \qquad (i=1,2,\ldots,n) \tag{6}$$

and

$$\left[\underset{\sim}{A}'-\lambda_j\underset{\sim}{I}\ ,\ \underset{\sim}{C}'\right]\begin{bmatrix}\underset{\sim}{\phi}_j\\ \underset{\sim}{\zeta}_j\end{bmatrix} = \underset{\sim}{O} \qquad (j=1,2,\ldots,n) \tag{7}$$

where

$$\underset{\sim}{\omega}_i = \underset{\sim}{G}\underset{\sim}{C}\underset{\sim}{\chi}_i \qquad (i=1,2,\ldots,n) \tag{8}$$

$$\underset{\sim}{\zeta}_j = \underset{\sim}{G}'\underset{\sim}{B}'\underset{\sim}{\phi}_j \qquad (j=1,2,\ldots,n) \tag{9}$$

and

$$\underset{\sim}{\phi}_j'\underset{\sim}{\chi}_i = \delta_{ij} \qquad (i,j=1,2,\ldots,n)\ . \tag{10}$$

Conversely, if equations (6), (7), and (10) are satisfied by a self-conjugate set $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ of distinct complex numbers and corresponding self-conjugate sets $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_n\}$ of linearly independent vectors, then equations (8) and (9) are satisfied by a matrix $\underset{\sim}{G} \in R^{mxp}$ such that $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is the eigenvalue spectrum of the matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})$ and $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_n\}$ constitute corresponding eigenvector and reciprocal eigenvector sets. It accordingly follows from equations (8) and (9), respectively, that the real output-feedback matrix

$$\underset{\sim}{G} = [\underset{\sim}{\omega}_1 , \underset{\sim}{\omega}_2 , \cdots , \underset{\sim}{\omega}_p][\underset{\sim}{C}\underset{\sim}{\chi}_1 , \underset{\sim}{C}\underset{\sim}{\chi}_2 , \cdots , \underset{\sim}{C}\underset{\sim}{\chi}_p]^{-1} \tag{11}$$

and the real transposed output-feedback matrix

$$\underset{\sim}{G}' = [\underset{\sim}{\zeta}_1 , \underset{\sim}{\zeta}_2 , \cdots , \underset{\sim}{\zeta}_m][\underset{\sim}{B}'\underset{\sim}{\phi}_1 , \underset{\sim}{B}'\underset{\sim}{\phi}_2 , \cdots , \underset{\sim}{B}'\underset{\sim}{\phi}_m]^{-1} \tag{12}$$

assign the self-conjugate distinct eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ and the corresponding eigenvector and reciprocal eigenvector sets $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_n\}$ to the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})$ in case $\{\underset{\sim}{C}\underset{\sim}{\chi}_1,\underset{\sim}{C}\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{C}\underset{\sim}{\chi}_p\}$ is a set of p linearly independent vectors and $\{\underset{\sim}{B}'\underset{\sim}{\phi}_1,\underset{\sim}{B}'\underset{\sim}{\phi}_2, \ldots,\underset{\sim}{B}'\underset{\sim}{\phi}_m\}$ is a set of m linearly independent vectors. Such sets $\{\underset{\sim}{C}\underset{\sim}{\chi}_1,\underset{\sim}{C}\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{C}\underset{\sim}{\chi}_p\}$ and $\{\underset{\sim}{B}'\underset{\sim}{\phi}_1,\underset{\sim}{B}'\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{B}'\underset{\sim}{\phi}_m\}$ clearly exist when rank $\underset{\sim}{C} = p$, rank $\underset{\sim}{B} = m$, and $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_n\}$ are sets of linearly independent vectors.

It is thus evident that, in the case of self-conjugate distinct eigenvalue spectra, closed-loop eigenstructure is assignable by output feedback just in case the eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is such that the corresponding eigenvector and reciprocal eigenvector sets $\{\chi_1,\chi_2,\ldots,\chi_n\}$ and $\{\phi_1,\phi_2,\ldots,\phi_n\}$ lie in the subspaces determined (Porter and D'Azzo 1977) in accordance with equations (6) and (7), respectively, by the kernels of each of the n matrices

$$S(\lambda_i) = [A-\lambda_i I \ , \ B] \qquad (i=1,2,\ldots,n) \qquad (13)$$

together with the kernels of each of the n matrices

$$T'(\lambda_j) = [A'-\lambda_j I \ , \ C'] \qquad (j=1,2,\ldots,n) \quad . \qquad (14)$$

3. Illustrative Example

These results can be conveniently illustrated by designing an output-feedback regulator for the system governed by the respective state and output equations (Davison and Wang 1975)

$$\dot{x}(t) = \begin{bmatrix} 0 \ , & 1 \ , & 0 \\ 0 \ , & 0 \ , & 1 \\ 0 \ , & 0 \ , & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \ , & 0 \\ 1 \ , & 0 \\ 1 \ , & 1 \end{bmatrix} u(t) \qquad (15)$$

and

$$y(t) = \begin{bmatrix} 1 \ , & 0 \ , & 0 \\ 0 \ , & 1 \ , & 0 \end{bmatrix} x(t) \qquad (16)$$

such that the eigenvalue spectrum of the closed-loop plant matrix is

$$\sigma(A+BGC) = \{\lambda_1,\lambda_2,\lambda_3\} = \{-1,-2,-5\} \quad . \qquad (17)$$

Indeed, it is evident from equations (13), (14), (15), and (16) that

$$\underset{\sim}{S}(\lambda) = \begin{bmatrix} -\lambda , & 1 , & 0 , & 1 , & 0 \\ 0 , & -\lambda , & 1 , & 1 , & 0 \\ 0 , & 0 , & -\lambda , & 1 , & 1 \end{bmatrix} \tag{18}$$

and

$$\underset{\sim}{T}'(\lambda) = \begin{bmatrix} -\lambda , & 0 , & 0 , & 1 , & 0 \\ 1 , & -\lambda , & 0 , & 0 , & 1 \\ 0 , & 1 , & -\lambda , & 0 , & 0 \end{bmatrix} . \tag{19}$$

It therefore follows immediately (Porter and D'Azzo 1977) from equation (18) that

$$\ker \underset{\sim}{S}(-1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \tag{20}$$

$$\ker \underset{\sim}{S}(-2) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \\ -2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \right\} \tag{21}$$

and

$$\ker \underset{\sim}{S}(-5) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \\ -5 \\ -20 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \\ 21 \end{bmatrix} \right\} , \tag{22}$$

and similarly (Porter and D'Azzo 1977) from equation (19) that

$$\ker \underset{\sim}{T}'(-1) = \text{span}\left\{\begin{bmatrix}1\\0\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\\-1\end{bmatrix}\right\} \qquad (23)$$

$$\ker \underset{\sim}{T}'(-2) = \text{span}\left\{\begin{bmatrix}1\\0\\0\\-2\\-1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\\4\end{bmatrix}\right\} \qquad (24)$$

and

$$\ker \underset{\sim}{T}'(-5) = \text{span}\left\{\begin{bmatrix}1\\0\\0\\-5\\-1\end{bmatrix}, \begin{bmatrix}0\\-5\\1\\0\\25\end{bmatrix}\right\} . \qquad (25)$$

It is thus evident from equations (20), (21), and (22) that the closed-loop eigenvectors corresponding to the eigenvalue spectrum (17) must be assigned to the respective subspaces

$$\Sigma(-1) = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \qquad (26)$$

$$\Sigma(-2) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} \tag{27}$$

and

$$\Sigma(-5) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\5\end{bmatrix},\begin{bmatrix}0\\1\\-4\end{bmatrix}\right\}\;, \tag{28}$$

and it is similarly evident from equations (23), (24), and (25) that the closed-loop reciprocal eigenvectors corresponding to the eigenvalue spectrum (17) must be assigned to the respective subspaces

$$\Gamma(-1) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} \tag{29}$$

$$\Gamma(-2) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-2\\1\end{bmatrix}\right\} \tag{30}$$

and

$$\Gamma(-5) \;=\; \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-5\\1\end{bmatrix}\right\}\;. \tag{31}$$

Since the vectors

$$\underset{\sim}{\chi}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \Sigma(-1) \tag{32}$$

$$\underset{\sim}{\chi}_2 = \begin{bmatrix} -3 \\ 2 \\ -8 \end{bmatrix} \in \Sigma(-2) \tag{33}$$

$$\underset{\sim}{\chi}_3 = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix} \in \Sigma(-5) \tag{34}$$

$$\underset{\sim}{\phi}_1 = \begin{bmatrix} 5/2 \\ 3/4 \\ -3/4 \end{bmatrix} \in \Gamma(-1) \tag{35}$$

$$\underset{\sim}{\phi}_2 = \begin{bmatrix} 1 \\ 2/3 \\ -1/3 \end{bmatrix} \in \Gamma(-2) \tag{36}$$

and

$$\underset{\sim}{\phi}_3 = \begin{bmatrix} 1/2 \\ 5/12 \\ -1/12 \end{bmatrix} \in \Gamma(-5) \tag{37}$$

are clearly such that $\{\underset{\sim}{\chi}_1, \underset{\sim}{\chi}_2, \underset{\sim}{\chi}_3\}$ and $\{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3\}$ constitute sets of linearly independent vectors with the property that

$$\underset{\sim}{\phi}_j' \underset{\sim}{\chi}_i = \delta_{ij} \qquad (i,j=1,2,3) \quad , \tag{38}$$

it follows from equations (11) and (12) that equation (17) is satisfied by the output-feedback matrix

$$\underset{\sim}{G} = \begin{bmatrix} -4 \ , & -4 \\ -10 \ , & -9 \end{bmatrix} \quad . \tag{39}$$

The corresponding output-feedback regulator is accordingly governed by the control-law equation

$$\underset{\sim}{u}(t) = \begin{bmatrix} -4 \ , & -4 \\ -10 \ , & -9 \end{bmatrix} \underset{\sim}{y}(t) \quad . \tag{40}$$

4. CONCLUSION

In this paper, the method of entire eigenstructure assignment has been applied to the design of linear multivariable continuous-time output-feedback regulators. It has been shown that, in the case of self-conjugate distinct eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the elements of the sets of linearly independent self-conjugate vectors $\{\underset{\sim}{\chi}_1, \underset{\sim}{\chi}_2, \ldots, \underset{\sim}{\chi}_n\}$ and $\{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \ldots, \underset{\sim}{\phi}_n\}$ lie in subspaces determined by the kernels of $\underset{\sim}{S}(\lambda_i)$ $(i=1,2,\ldots,n)$ and $\underset{\sim}{T}'(\lambda_j)$ $(j=1,2,\ldots,n)$, respectively, and satisfy the orthogonality conditions (10). In constrast, the closed-loop eigenstructure assignable by state feedback is constrained only by the requirement that the elements of the set of linearly independent self-conjugate vectors $\{\underset{\sim}{\chi}_1, \underset{\sim}{\chi}_2, \ldots, \underset{\sim}{\chi}_n\}$ lie in subspaces determined by the kernels of $\underset{\sim}{S}(\lambda_i)$ $(i=1,2,\ldots,n)$. It is because of the severe constraints on the closed-loop eigenstructure assignable by output feedback that it is frequently impossible to achieve satisfactory closed-loop behaviour by means of static continuous-time output-feedback regulators, and that it is consequently necessary to introduce dynamic compensators

(Brasch and Pearson 1970). However, the design of such dynamic continuous-time output-feedback regulators can be effected by applying the method of entire eigenstructure assignment in the manner of Section 2 to appropriately augmented (Brasch and Pearson 1970, Kimura 1975) continuous-time systems.

ACKNOWLEDGEMENT

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005A.

REFERENCES

BRASCH, F. M. and PEARSON, J. B., 1970, I.E.E.E. Trans. autom. Control, 15, 34.

DAVISON, E. J. and WANG, S. H., 1975, I.E.E.E. Trans. autom. Control, 20, 516.

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509.

MOORE, B. C., 1976, I.E.E.E. Trans. autom. Control, 21, 689.

PORTER, B. and D'AZZO, J. J., 1977, Electron. Lett., 13, 230.

APPENDIX 6

DESIGN OF LINEAR MULTIVARIABLE DISCRETE-TIME OUTPUT-FEEDBACK REGULATORS

Dr A Bradshaw and Professor B Porter

Department of Aeronautical and Mechanical Engineering
University of Salford
Salford M5 4WT
England

(International Journal of Systems Science, vol 9, pp857-863, 1978)

ABSTRACT

In this paper, the method of entire eigenstructure assignment (Porter and D'Azzo 1978a,b) is applied to the design of linear multivariable discrete-time output-feedback regulators. It is shown that, in the case of self-conjugate eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the eigenvectors and generalised eigenvectors and the reciprocal eigenvectors and generalised reciprocal eigenvectors lie in well-defined subspaces. The method is illustrated by designing an output-feedback regulator for a third-order discrete-time system.

1. Introduction

In this paper, the method of entire eigenstructure assignment (Porter and D'Azzo 1978a,b) is applied to the design of output-feedback regulators for multivariable linear discrete-time systems governed by state and output equations of the respective forms

$$\underset{\sim}{x}(k+1) = \underset{\sim}{A}\underset{\sim}{x}(k) + \underset{\sim}{B}\underset{\sim}{u}(k) \tag{1}$$

and

$$\underset{\sim}{y}(k) = \underset{\sim}{C}\underset{\sim}{x}(k) \quad , \tag{2}$$

where $\underset{\sim}{A} \in R^{nxn}$, $\underset{\sim}{B} \in R^{nxm}$, $\underset{\sim}{C} \in R^{pxn}$, rank $\underset{\sim}{B}$ = m, and rank $\underset{\sim}{C}$ = p. The theory is analogous to that developed by Porter and Bradshaw (1978) for continuous-time regulators but is significantly extended in order to allow the assignment of confluent eigenvalues to the plant matrix of the closed-loop system. It is therefore possible, for example, to apply this theory to the design of output-feedback regulators with finite settling times.

2. Theory

Thus, if output feedback is applied to the system governed by the state equation (1) in accordance with the control-law equation

$$\underset{\sim}{u}(k) = \underset{\sim}{G}\underset{\sim}{y}(k) \tag{3}$$

and the output-feedback matrix $\underset{\sim}{G} \in R^{mxp}$ is such that the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})$ has a self-conjugate eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$, a corresponding eigenvector and generalised eigenvector set $\{\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}:h=1,2,\ldots,m_{ji};$ $j=1,2,\ldots,k_i;i=1,2,\ldots,t\}$, and a corresponding reciprocal eigenvector and generalised reciprocal eigenvector set $\{\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\ldots,m_{ba};b=1,2,\ldots,k_a;a=1,2,\ldots,t\}$, then

$$(\underset{\sim}{A}-\lambda_i\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})\underset{\sim}{\chi}_{\lambda_i}^{(1,j)} = \underset{\sim}{O} \tag{4a}$$

$$(\underset{\sim}{A}-\lambda_i\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})\underset{\sim}{\chi}_{\lambda_i}^{(2,j)} = \underset{\sim}{\chi}_{\lambda_i}^{(1,j)} \tag{4b}$$

....

....

$$(\underset{\sim}{A}-\lambda_i\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})\underset{\sim}{\chi}_{\lambda_i}^{(m_{ji},j)} = \underset{\sim}{\chi}_{\lambda_i}^{(m_{ji}-1,j)} \tag{$4m_{ji}$}$$

$$(j=1,2,\ldots,k_i;i=1,2,\ldots,t) \quad ,$$

and

$$\underset{\sim}{\phi}_{\lambda_a}^{(1,b)\prime}(\underset{\sim}{A}-\lambda_a\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C}) = \underset{\sim}{\phi}_{\lambda_a}^{(2,b)\prime} \tag{5a}$$

....

....

$$\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba}-1,b)\prime}(\underset{\sim}{A}-\lambda_a\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C}) = \underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)\prime} \tag{$5m_{ba}-1$}$$

$$\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)}(\underset{\sim}{A}-\lambda_a\underset{\sim}{I}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C}) = \underset{\sim}{O}' \tag{$5m_{ba}$}$$

$$(b=1,2,\ldots,k_a;a=1,2,\ldots,t) \quad ,$$

where $\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}$ is the hth vector in the jth string of length m_{ji} associated with the eigenvalue λ_i, and where $\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}$ is the cth vector in the bth string of length m_{ba} associated with the eigenvalue λ_a. The vectors $\underset{\sim}{\chi}_{\lambda_i}^{(1,j)}$ $(j=1,2,\ldots,k_i)$

are the k_i eigenvectors associated with the eigenvalue λ_i, whilst the remaining vectors in each of the k_i strings of vectors satisfying equations (4) are generalised eigenvectors associated with the eigenvalue λ_i. Similarly, the vectors $\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)}$ $(b=1,2,\ldots,k_a)$ are the k_a reciprocal eigenvectors associated with the eigenvalue λ_a, whilst the remaining vectors in each of the k_a strings of vectors satisfying equations (5) are reciprocal generalised eigenvectors associated with the eigenvalue λ_a. The total number of vectors associated with the eigenvalue λ_f in each set is evidently

$$m_f = \sum_{g=1}^{k_f} m_{gf} \qquad (f=1,2,\ldots,t) \tag{6}$$

and

$$n = \sum_{f=1}^{t} m_f \quad . \tag{7}$$

Equations (4) and (5) can be written in the form

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I} \; , \; \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{\chi}_{\lambda_i}^{(1,j)} \\ \underset{\sim}{\omega}_{\lambda_i}^{(1,j)} \end{bmatrix} = \underset{\sim}{0} \tag{8a}$$

$$[\underset{\sim}{A}-\lambda_i\underset{\sim}{I} \; , \; \underset{\sim}{B}] \begin{bmatrix} \underset{\sim}{\chi}_{\lambda_i}^{(2,j)} \\ \underset{\sim}{\omega}_{\lambda_i}^{(2,j)} \end{bmatrix} = \underset{\sim}{\chi}_{\lambda_i}^{(1,j)} \tag{8b}$$

....

....

$$\left[\underset{\sim}{A}-\lambda_i\underset{\sim}{I}\ ,\ \underset{\sim}{B}\right]\begin{bmatrix}\underset{\sim}{\chi}_{\lambda_i}^{(m_{ji},j)} \\ \underset{\sim}{\omega}_{\lambda_i}^{(m_{ji},j)}\end{bmatrix} = \underset{\sim}{\chi}_{\lambda_i}^{(m_{ji}-1,j)} \tag{8m_{ji}}$$

$$(j=1,2,\ldots,k_i;i=1,2,\ldots,t)$$

and

$$\left[\underset{\sim}{A}'-\lambda_a\underset{\sim}{I}\ ,\ \underset{\sim}{C}'\right]\begin{bmatrix}\underset{\sim}{\phi}_{\lambda_a}^{(1,b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(1,b)}\end{bmatrix} = \underset{\sim}{\phi}_{\lambda_a}^{(2,b)} \tag{9a}$$

....

....

$$\left[\underset{\sim}{A}'-\lambda_a\underset{\sim}{I}\ ,\ \underset{\sim}{C}'\right]\begin{bmatrix}\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba}-1,b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(m_{ba}-1,b)}\end{bmatrix} = \underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)} \tag{9m_{ba}-1}$$

$$\left[\underset{\sim}{A}'-\lambda_a\underset{\sim}{I}\ ,\ \underset{\sim}{C}'\right]\begin{bmatrix}\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(m_{ba},b)}\end{bmatrix} = \underset{\sim}{O} \tag{9m_{ba}}$$

$$(b=1,2,\ldots,k_a;a=1,2,\ldots,t)$$

where

$$\underset{\sim}{\omega}_{\lambda_i}^{(h,j)} = \underset{\sim}{G}\underset{\sim}{C}\underset{\sim}{\chi}_{\lambda_i}^{(h,j)} \tag{10}$$

$$\underset{\sim}{\zeta}_{\lambda}^{(c,b)} = \underset{\sim}{G}'\underset{\sim}{B}'\underset{\sim}{\phi}_{\lambda_a}^{(c,b)} \tag{11}$$

and

$$\boldsymbol{\phi}_{\lambda_a}^{(c,b)\prime}\,\boldsymbol{\chi}_{\lambda_i}^{(h,j)} = \delta_{ai}\delta_{bj}\delta_{ch} \qquad (12)$$

$$(h=1,2,\ldots,m_{ji};\, j=1,2,\ldots,k_i;\, i=1,2,\ldots,t)$$

$$(c=1,2,\ldots,m_{ba};\, b=1,2,\ldots,k_a;\, a=1,2,.q.,t) \;.$$

Conversely, if equations (8), (9), and (12) are satisfied by a self-conjugate set $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$ of complex numbers and corresponding self-conjugate sets $\{\boldsymbol{\chi}_{\lambda_i}^{(h,j)}: h=1,2,\ldots,m_{ji};\, j=1,2,\ldots,k_i;\, i=1,2,\ldots,t\}$ and $\{\boldsymbol{\phi}_{\lambda_a}^{(c,b)}: c=1,2,\ldots,m_{ba};\, b=1,2,\ldots,k_a;\, a=1,2,\ldots,t\}$ of linearly independent vectors, then equations (10) and (11) are satisfied by an mxn matrix $\mathbf{G}$ such that $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$ is the eigenvalue spectrum of the matrix $(\mathbf{A}+\mathbf{BGC})$, $\{\boldsymbol{\chi}_{\lambda_i}^{(h,j)}: k=1,2,\ldots,m_{ji};\, j=1,2,\ldots,k_i;\, i=1,2,\ldots,t\}$ constitutes a corresponding eigenvector and generalised eigenvector set, and $\{\boldsymbol{\phi}_{\lambda_a}^{(c,b)}: c=1,2,\ldots,m_{ba};\, b=1,2,\ldots,k_a;\, a=1,2,\ldots,t\}$ constitutes a corresponding reciprocal eigenvector and reciprocal generalised eigenvector set. It accordingly follows from equations (10) and (11) respectively that the real output-feedback matrix

$$\mathbf{G} = [\boldsymbol{\omega}_1\,,\,\boldsymbol{\omega}_2\,,\,\ldots\,,\,\boldsymbol{\omega}_p][\mathbf{C}\boldsymbol{\chi}_1\,,\,\mathbf{C}\boldsymbol{\chi}_2\,,\,\ldots\,,\,\mathbf{C}\boldsymbol{\chi}_p]^{-1} \qquad (13)$$

and the real transposed output-feedback matrix

$$\mathbf{G}' = [\boldsymbol{\zeta}_1\,,\,\boldsymbol{\zeta}_2\,,\,\ldots\,,\,\boldsymbol{\zeta}_m][\mathbf{B}'\boldsymbol{\phi}_1\,,\,\mathbf{B}'\boldsymbol{\phi}_2\,,\,\ldots\,,\,\mathbf{B}'\boldsymbol{\phi}_m]^{-1} \qquad (14)$$

Indeed, it is evident from equations (13), (14), (15), and (16) that

assign the self-conjugate eigenvalue spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_t\}$, the corresponding eigenvector and generalised eigenvector set $\{\mathbf{x}_{\lambda_i}^{(h,j)} : h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$, and the corresponding reciprocal eigenvector and reciprocal generalised eigenvector set $\{\boldsymbol{\phi}_{\lambda_a}^{(c,b)} : c=1,2,\ldots,m_{ba}; b=1,2,\ldots,k_a; a=1,2,\ldots,t\}$ to the closed-loop plant matrix $(\mathbf{A}+\mathbf{BGC})$ in case $\{\mathbf{Cx}_1, \mathbf{Cx}_2, \ldots, \mathbf{Cx}_p\}$ is a subset of p linearly independent members of the set $\{\mathbf{Cx}_{\lambda_i}^{(h,j)} : h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$ and $\{\mathbf{B}'\boldsymbol{\phi}_1, \mathbf{B}'\boldsymbol{\phi}_2, \ldots, \mathbf{B}'\boldsymbol{\phi}_m\}$ is a subset of m linearly independent members of the set $\{\mathbf{B}'\boldsymbol{\phi}_{\lambda_a}^{(c,b)} : c=1,2,\ldots,m_{ba}; b=1,2,\ldots,k_a; a=1,2,\ldots,t\}$.

It is thus evident that closed-loop eigenstructure is assignable by output feedback just in case the self-conjugate eigenvalue spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ is such that the corresponding eigenvector/generalised eigenvector and reciprocal eigenvector/reciprocal generalised eigenvector sets $\{\mathbf{x}_{\lambda_i}^{(h,j)} : h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$ and $\{\boldsymbol{\phi}_{\lambda_a}^{(c,b)} : c=1,2,\ldots,m_{ba}; b=1,2,\ldots,k_a; a=1,2,\ldots,t\}$ lie in the subspaces determined (Porter and D'Azzo 1978b) in accordance with equations (8) and (9), respectively, by the kernels and generalised kernels of each of the t matrices

$$\mathbf{S}(\lambda_i) = [\mathbf{A}-\lambda_i\mathbf{I} \; , \; \mathbf{B}] \qquad (i=1,2,\ldots,t) \qquad (15)$$

together with the kernels and generalised kernels of each of the t matrices

$$\mathbf{T}'(\lambda_a) = [\mathbf{A}'-\lambda_a\mathbf{I} \; , \; \mathbf{C}'] \qquad (a=1,2,\ldots,t) \; . \qquad (16)$$

3. Illustrative Example

These results can be conveniently illustrated by designing an output-feedback regulator for the system governed by the respective state and output equations

$$\underset{\sim}{x}(k+1) = \begin{bmatrix} 0 , 1 , 0 \\ 1 , 1 , 0 \\ 0 , 0 , 1 \end{bmatrix} \underset{\sim}{x}(k) + \begin{bmatrix} 0 , 0 \\ 1 , 0 \\ 0 , 1 \end{bmatrix} \underset{\sim}{u}(k) \tag{17}$$

and

$$\underset{\sim}{y}(k) = \begin{bmatrix} 1 , 0 , 0 \\ 0 , 0 , 1 \end{bmatrix} \underset{\sim}{x}(k) \tag{18}$$

such that $\underset{\sim}{x}(k) = 0$ after a finite number of discrete-time intervals. Indeed, it is evident from equations (15), (16), (17), and (18) that

$$\underset{\sim}{S}(\lambda) = \begin{bmatrix} -\lambda , & 1 , & 0 , & 0 , & 0 \\ 1 , & 1-\lambda , & 0 , & 1 , & 0 \\ 0 , & 0 , & 1-\lambda , & 0 , & 1 \end{bmatrix} \tag{19}$$

and

$$\underset{\sim}{T}'(\lambda) = \begin{bmatrix} -\lambda , & 1 , & 0 , & 1 , & 0 \\ 1 , & 1-\lambda , & 0 , & 0 , & 0 \\ 0 , & 0 , & 1-\lambda , & 0 , & 1 \end{bmatrix} . \tag{20}$$

In this case, it is necessary to assign the value zero to all the eigenvalues of the closed-loop plant matrix. Such an assignment clearly requires that $t = 1$, $m_1 = 3$, and $\lambda_1 = 0$ and therefore, in consonance with the results of Rosenbrock and Hayton (1977), that $k = 1$ and $m_{11} = 3$. It therefore follows immediately (Porter and D'Azzo 1978b) from equation (19) that

$$\ker \underset{\sim}{S}(0) = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}\right\} \tag{21}$$

and similarly (Porter and D'Azzo 1978b) from equation (20) that

$$\ker \underset{\sim}{T}'(0) = \operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}\right\} . \tag{22}$$

It is evident from equation (21) that the closed-loop eigenvector $\underset{\sim}{\chi}_0^{(1,1)}$ corresponding to the eigenvalue $\lambda_1 = 0$ must be assigned to the subspace

$$\Sigma(0) = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \tag{23}$$

whilst the associated string of generalised eigenvectors $\underset{\sim}{\chi}_0^{(2,1)}$ and $\underset{\sim}{\chi}_0^{(3,1)}$ must be generated in accordance with equations (8), and it is similarly evident from equation (22) that the closed-loop reciprocal eigenvector $\underset{\sim}{\phi}_0^{(3,1)}$ corresponding to the eigenvalue $\lambda_1 = 0$ must be assigned to the subspace

$$\Gamma(0) = \text{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \tag{24}$$

whilst the associated string of generalised reciprocal eigenvectors $\underset{\sim}{\phi}_o^{(2,1)}$ and $\underset{\sim}{\phi}_o^{(1,1)}$ must be generated in accordance with equations (9). Since the vectors

$$\begin{bmatrix}\underset{\sim}{\chi}_o^{(1,1)}\\ \underset{\sim}{\omega}_o^{(1,1)}\end{bmatrix} = \begin{bmatrix}1\\0\\1\\-1\\-1\end{bmatrix} \tag{25}$$

$$\begin{bmatrix}\underset{\sim}{\chi}_o^{(2,1)}\\ \underset{\sim}{\omega}_o^{(2,1)}\end{bmatrix} = \begin{bmatrix}1\\1\\0\\-2\\1\end{bmatrix} \tag{26}$$

$$\begin{bmatrix}\underset{\sim}{\chi}_o^{(3,1)}\\ \underset{\sim}{\omega}_o^{(3,1)}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} \tag{27}$$

$$\begin{bmatrix}\underset{\sim}{\phi}_o^{(3,1)}\\ \underset{\sim}{\zeta}_o^{(3,1)}\end{bmatrix} = \begin{bmatrix}-1\\1\\1\\-1\\-1\end{bmatrix} \tag{28}$$

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(2,1)} \\ \underset{\sim}{\zeta}_o^{(2,1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 2 \end{bmatrix} \tag{29}$$

and

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(1,1)} \\ \underset{\sim}{\zeta}_o^{(1,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \tag{30}$$

are clearly such that

$$\underset{\sim}{\phi}_o^{(c,1)'} \underset{\sim}{\chi}_o^{(h,1)} = \delta_{ch} \qquad (c,h=1,2,3) \tag{31}$$

it follows from equations (13) and (14) that the required eigenstructure assignment is achieved by the output feedback matrix

$$\underset{\sim}{G} = \begin{bmatrix} -2 , & 1 \\ 1 , & -2 \end{bmatrix} . \tag{32}$$

The corresponding output-feedback regulator is accordingly governed by the control-law equation

$$\underset{\sim}{u}(k) = \begin{bmatrix} -2 , & 1 \\ 1 , & -2 \end{bmatrix} \underset{\sim}{y}(k) . \tag{33}$$

It can be readily verified that the state of the closed-loop system governed by equations (17), (18), and (33) is reduced from any initial value to zero in at most three discrete-time intervals, as required.

4. CONCLUSION

In this paper, the method of entire eigenstructure assignment has been applied to the design of linear multivariable discrete-time output feedback regulators. It has been shown that the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the elements of the sets of linearly independent self-conjugate vectors $\{\chi_{\lambda_i}^{(h,j)}: h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$ and $\{\phi_{\lambda_a}^{(c,b)}: c=1,2,\ldots,m_{ba}; b=1,2,\ldots,k_a; a=1,2,\ldots,t\}$ lie in subspaces determined by the kernels and generalised kernels of $S(\lambda_i)$ $(i=1,2,\ldots,t)$ and $T'(\lambda_a)$ $(a=1,2,\ldots,t)$, respectively, and satisfy the orthogonality conditions (12). In contrast, the closed-loop eigenstructure assignable by state feedback is constrained only by the requirement that the elements of the set of linearly independent self-conjugate vectors $\{\chi_{\lambda_i}^{(h,j)}: h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$ lie in subspaces determined by the kernels and generalised kernels of $S(\lambda_i)$ $(i=1,2,\ldots,t)$. It is because of the severe constraints on the closed-loop eigenstructure assignable by output feedback that it is frequently impossible to achieve satisfactory closed-loop behaviour by means of static discrete-time output-feedback regulators, and that it is consequently necessary to introduce dynamic compensators (Brasch and Pearson 1970). However, the design of such dynamic discrete-time output-feedback regulators can be effected by applying the method of entire eigenstructure assignment in the manner of Section 2 to appropriately augmented (Brasch and Pearson 1970) discrete-time systems.

Acknowledgement

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005A.

References

BRASCH, F. M. and PEARSON, J. B., 1970, I.E.E.E. Trans. autom. Control, 15, 34.

PORTER, B. and BRADSHAW, A., 1978, Int. J. Systems Sci., (in the press).

PORTER, B. and D'AZZO, J. J., 1978a, Int. J. Control (in the press); 1978b, Ibid. (in the press).

ROSENBROCK, H. H. and HAYTON, G. E., 1977, Report No. 288, Control Systems Centre, University of Manchester Institute of Science and Technology, Manchester, England.

APPENDIX 7

DESIGN OF LINEAR MULTIVARIABLE CONTINUOUS-TIME TRACKING SYSTEMS INCORPORATING ERROR-ACTUATED DYNAMIC CONTROLLERS

PROFESSOR B PORTER AND DR A BRADSHAW

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, VOL 9, PP627-637, 1978)

ABSTRACT

In this paper, the method of entire eigenstructure assignment (Porter and Bradshaw 1978) is applied to the design of linear multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. The method is illustrated by designing an error-actuated dynamic controller which causes the output of a second-order continuous-time plant to track a constant command input in the presence of an unmeasurable constant disturbance input.

1. INTRODUCTION

In this paper, the method of entire eigenstructure assignment (Porter and Bradshaw 1978) is applied to the design of linear multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. Such tracking systems consist of a controllable and observable nth-order linear multivariable plant governed by state and output equations of the respective forms

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t) + \underset{\sim}{D}\underset{\sim}{d}(t) \tag{1}$$

and

$$\underset{\sim}{y}(t) = \underset{\sim}{C}\underset{\sim}{x}(t) \tag{2}$$

where $\underset{\sim}{B}$ and $\underset{\sim}{C}$ have full rank, together with an error-actuated dynamic controller which is required to cause the pxl output vector $\underset{\sim}{y}(t)$ of the plant to track a pxl command input vector $\underset{\sim}{r}(t)$ in the sense that

$$\lim_{t\to\infty} \underset{\sim}{e}(t) = \lim_{t\to\infty}\{\underset{\sim}{r}(t)-\underset{\sim}{y}(t)\} = \underset{\sim}{0} \tag{3}$$

for polynomial command and disturbance inputs of the respective forms

$$\underset{\sim}{r}(t) = \sum_{i=1}^{r} \underset{\sim}{\alpha}_{i-1} t^{i-1} \tag{4}$$

and

$$\underset{\sim}{d}(t) = \sum_{i=1}^{s} \underset{\sim}{\beta}_{i-1} t^{i-1} \quad . \tag{5}$$

It is important to note that tracking systems incorporating error-actuated dynamic controllers can be designed for a much larger class of plants than tracking systems incorporating error-actuated static controllers (Porter and Bradshaw 1976) in view of the fact that eigenstructure assignment by error-actuated static controllers and by output-feedback controllers (Kimura 1975,1977) are essentially equivalent.

2. THEORY

The first stage in the design of the required error-actuated dynamic controller for the plant governed by equations (1) and (2) involves the introduction (Porter and Bradshaw 1974) of a vector comparator and a series of $q = \max(r,s)$ vector integrators in order to generate the q vectors defined by the equations

$$\left.\begin{array}{lll} \dot{\underset{\sim}{z}}_1(t) & = & \underset{\sim}{e}(t) \quad , \\ \dot{\underset{\sim}{z}}_2(t) & = & \underset{\sim}{z}_1(t) \quad , \\ \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots \\ \dot{\underset{\sim}{z}}_q(t) & = & \underset{\sim}{z}_{q-1}(t) \quad . \end{array}\right\} \qquad (6)$$

It is then evident from equations (1), (2), and (5) that the open-loop tracking system is governed by state and output equations of the respective forms

$$\begin{bmatrix} \dot{\underset{\sim}{x}}(t) \\ \dot{\underset{\sim}{z}}_1(t) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \dot{\underset{\sim}{z}}_{q-1}(t) \\ \dot{\underset{\sim}{z}}_q(t) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ -\underset{\sim}{C}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{I}_p, & \underset{\sim}{O} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \underset{\sim}{z}_{q-1}(t) \\ \underset{\sim}{z}_q(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{B} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{u}(t) + \begin{bmatrix} \underset{\sim}{O} \\ \underset{\sim}{I}_p \\ \cdot\cdot \\ \cdot\cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{r}(t) + \begin{bmatrix} \underset{\sim}{D} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{d}(t) \tag{7}$$

and

$$\begin{bmatrix} \underset{\sim}{y}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \underset{\sim}{z}_{q-1}(t) \\ \underset{\sim}{z}_q(t) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{C}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{I}_p, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{I}_p, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{I}_p \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \underset{\sim}{z}_{q-1}(t) \\ \underset{\sim}{z}_q(t) \end{bmatrix} . \tag{8}$$

The open-loop tracking system governed by equations (7) and (8) is controllable if and only if (Porter and Bradshaw 1974)

$$\text{rank} \begin{bmatrix} \underset{\sim}{B}, & \underset{\sim}{A} \\ \underset{\sim}{O}, & -\underset{\sim}{C} \end{bmatrix} = n + p \tag{9}$$

since $(\underset{\sim}{A},\underset{\sim}{B})$ and $(\underset{\sim}{C},\underset{\sim}{A})$ are respectively controllable and observable pairs.

In the case of such controllable and observable open-loop tracking systems, the second stage in the design of the error-actuated dynamic controller involves the introduction of an ℓth-order dynamic compensator (Brasch and Pearson 1970) governed by state and output equations of the respective forms

$$\dot{\underset{\sim}{w}}(t) = \underset{\sim}{F}\underset{\sim}{w}(t) + \underset{\sim}{G}\underset{\sim}{e}(t) + \sum_{i=1}^{q} \underset{\sim}{H}_i \underset{\sim}{z}_i(t) \tag{10}$$

and

$$\underset{\sim}{u}(t) = \underset{\sim}{K}\underset{\sim}{w}(t) + \underset{\sim}{L}\underset{\sim}{e}(t) + \sum_{i=1}^{q} \underset{\sim}{M}_i \underset{\sim}{z}_i(t) \tag{11}$$

where

$$\ell = \min(\nu_c - 1, \nu_o - 1) \tag{12}$$

and ν_c and ν_o are respectively the controllability and observability indices of the open-loop tracking system governed by equations (7) and (8). It is then evident from equations (7), (8), (10), and (11) that the closed-loop tracking system is governed by state and output equations of the respective forms

$$\begin{bmatrix} \dot{\underset{\sim}{x}}(t) \\ \dot{\underset{\sim}{z}}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \dot{\underset{\sim}{z}}_q(t) \\ \dot{\underset{\sim}{w}}(t) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{L}\underset{\sim}{C} , & \underset{\sim}{B}\underset{\sim}{M}_1 , & \ldots , & \underset{\sim}{B}\underset{\sim}{M}_q , & \underset{\sim}{B}\underset{\sim}{K} \\ -\underset{\sim}{C} , & \underset{\sim}{O} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ -\underset{\sim}{G}\underset{\sim}{C} , & \underset{\sim}{H}_1 , & \ldots , & \underset{\sim}{H}_q , & \underset{\sim}{F} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(t) \\ \underset{\sim}{w}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{B}\underset{\sim}{L} \\ \underset{\sim}{I}_p \\ \cdot\cdot \\ \cdot\cdot \\ \underset{\sim}{O} \\ \underset{\sim}{G} \end{bmatrix} \underset{\sim}{r}(t) + \begin{bmatrix} \underset{\sim}{D} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{d}(t) \tag{13}$$

and

$$\begin{bmatrix} \underset{\sim}{y}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(t) \\ \underset{\sim}{w}(t) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{C}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{I}_p, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{I}_p, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{I}_\ell \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(t) \\ \underset{\sim}{w}(t) \end{bmatrix} . \tag{14}$$

In view of equations (4) and (5), it is clear by differentiating equation (13) (q-1) times that the closed-loop tracking system will behave so that

$$\lim_{t\to\infty} \frac{d^q \underset{\sim}{z}_q}{dt^q} = \underset{\sim}{O} \tag{15}$$

and therefore, in view of equations (6), so that equation (3) will be satisfied if the error-actuated dynamic compensator governed by equations (10) and (11) is designed such that all the eigenvalues of the plant matrix of the closed-loop tracking system governed by equations (13) and (14) are assigned to the open left-half of the complex plane.

3. Compensator Design

It is evident from equations (7), (8), (10), and (11) that such a compensator can be designed by the synthesis of an appropriate output-feedback control law of the form

$$\underset{\sim}{u}_\ell(t) = \underset{\sim}{G}_\ell \underset{\sim}{y}_\ell(t) \tag{16}$$

for the augmented open-loop system governed by state and output equations of the respective forms

$$\dot{\underset{\sim}{x}}_\ell(t) = \underset{\sim}{A}_\ell \underset{\sim}{x}_\ell(t) + \underset{\sim}{B}_\ell \underset{\sim}{u}_\ell(t) \tag{17}$$

and

$$\underset{\sim}{y}_\ell(t) = \underset{\sim}{C}_\ell \underset{\sim}{x}_\ell(t) \quad , \tag{18}$$

where

$$\underset{\sim}{u}_\ell(t) = \begin{bmatrix} \underset{\sim}{u}(t) \\ \underset{\sim}{v}(t) \end{bmatrix} \tag{19}$$

$$\underset{\sim}{x}_\ell(t) = \begin{bmatrix} \underset{\sim}{x}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(t) \\ \underset{\sim}{w}(t) \end{bmatrix} \tag{20}$$

$$\underset{\sim}{y}_\ell(t) = \begin{bmatrix} \underset{\sim}{y}(t) \\ \underset{\sim}{z}_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(t) \\ \underset{\sim}{w}(t) \end{bmatrix} \tag{21}$$

$$\underset{\sim}{G}_\ell = \begin{bmatrix} -\underset{\sim}{L} \; , & \underset{\sim}{M}_1 \; , & \ldots \; , & \underset{\sim}{M}_q \; , & \underset{\sim}{K} \\ -\underset{\sim}{G} \; , & \underset{\sim}{H}_1 \; , & \ldots \; , & \underset{\sim}{H}_q \; , & \underset{\sim}{F} \end{bmatrix} \tag{22}$$

$$\underset{\sim}{A}_\ell = \begin{bmatrix} \underset{\sim}{A} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ -\underset{\sim}{C} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \end{bmatrix} \qquad (23)$$

$$\underset{\sim}{B}_\ell = \begin{bmatrix} \underset{\sim}{B} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots \\ \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{I}_\ell \end{bmatrix} \qquad (24)$$

and

$$\underset{\sim}{C}_\ell = \begin{bmatrix} \underset{\sim}{C} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{I}_p , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{I}_p , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{I}_\ell \end{bmatrix} . \qquad (25)$$

Thus, if the $(m+\ell)\times(p+pq+\ell)$ output-feedback matrix $\underset{\sim}{G}_\ell$ is such that the closed-loop plant matrix $(\underset{\sim}{A}_\ell+\underset{\sim}{B}_\ell\underset{\sim}{G}_\ell\underset{\sim}{C}_\ell)$ has a self-conjugate spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$ of distinct eigenvalues and corresponding eigenvector and reciprocal eigenvector sets $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_{n+pq+\ell}\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_{n+pq+\ell}\}$, then obviously

$$(\underset{\sim}{A}_\ell-\lambda_i\underset{\sim}{I}+\underset{\sim}{B}_\ell\underset{\sim}{G}_\ell\underset{\sim}{C}_\ell)\underset{\sim}{\chi}_i = \underset{\sim}{O} \qquad (i=1,2,\ldots,n+pq+\ell) \qquad (26)$$

and

$$\phi_j'(A_\ell-\lambda_j I+B_\ell G_\ell C_\ell) = 0 \qquad (j=1,2,\ldots,n+pq+\ell) \qquad (27)$$

so that

$$\left[A_\ell-\lambda_i I \ , \ B_\ell\right]\begin{bmatrix} \chi_i \\ \omega_i \end{bmatrix} = 0 \qquad (i=1,2,\ldots,n+pq+\ell) \qquad (28)$$

and

$$\left[A_\ell'-\lambda_j I \ , \ C_\ell'\right]\begin{bmatrix} \phi_j \\ \zeta_j \end{bmatrix} = 0 \qquad (j=1,2,\ldots,n+pq+\ell) \qquad (29)$$

where

$$\omega_i = G_\ell C_\ell \chi_i \qquad (i=1,2,\ldots,n+pq+\ell) \qquad (30)$$

$$\zeta_j = G_\ell' B_\ell' \phi_j \qquad (j=1,2,\ldots,n+pq+\ell) \qquad (31)$$

and

$$\phi_j' \chi_i = \delta_{ij} \qquad (i,j=1,2,\ldots,n+pq+\ell) \ . \qquad (32)$$

Conversely, if equations (28), (29), and (32) are satisfied by a self-conjugate set $\{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$ of distinct complex members and corresponding self-conjugate sets $\{\chi_1,\chi_2,\ldots,\chi_{n+pq+\ell}\}$ and $\{\phi_1,\phi_2,\ldots,\phi_{n+pq+\ell}\}$ of linearly independent vectors, then equations (30) and (31) are satisfied by an $(m+\ell) \times (n+pq+\ell)$ matrix G_ℓ such that $\{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$ is the eigenvalue spectrum of the matrix $(A_\ell+B_\ell G_\ell C_\ell)$ and $\{\chi_1,\chi_2,\ldots,\chi_{n+pq+\ell}\}$ and $\{\phi_1,\phi_2,\ldots,\phi_{n+pq+\ell}\}$ constitute corresponding eigenvector and reciprocal eigenvector sets. It accordingly follows from equations (30) and (31) respectively that the real output-feedback matrix

$$\underset{\sim}{G}_\ell = [\underset{\sim}{\omega}_1 , \underset{\sim}{\omega}_2 , \cdots , \underset{\sim}{\omega}_{p+pq+\ell}][\underset{\sim}{C}_\ell \underset{\sim}{\chi}_1 , \underset{\sim}{C}_\ell \underset{\sim}{\chi}_2 , \cdots , \underset{\sim}{C}_\ell \underset{\sim}{\chi}_{p+pq+\ell}]^{-1} \tag{33}$$

and the real transposed output-feedback matrix

$$\underset{\sim}{G}'_\ell = [\underset{\sim}{\zeta}_1 , \underset{\sim}{\zeta}_2 , \cdots , \underset{\sim}{\zeta}_{m+\ell}][\underset{\sim}{B}'_\ell \underset{\sim}{\phi}_1 , \underset{\sim}{B}'_\ell \underset{\sim}{\phi}_2 , \cdots , \underset{\sim}{B}'_\ell \underset{\sim}{\phi}_{m+\ell}]^{-1} \tag{34}$$

assign the self-conjugate distinct eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$ and the corresponding eigenvector and reciprocal eigenvector sets $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_{n+pq+\ell}\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_{n+pq+\ell}\}$ to the closed-loop plant matrix $(\underset{\sim}{A}_\ell+\underset{\sim}{B}_\ell\underset{\sim}{G}_\ell\underset{\sim}{C}_\ell)$ in case $\{\underset{\sim}{C}_\ell\underset{\sim}{\chi}_1,\underset{\sim}{C}_\ell\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{C}_\ell\underset{\sim}{\chi}_{p+pq+\ell}\}$ is a set of $(p+pq+\ell)$ linearly independent vectors and $\{\underset{\sim}{B}'_\ell\underset{\sim}{\phi}_1,\underset{\sim}{B}'_\ell\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{B}'_\ell\underset{\sim}{\phi}_{m+\ell}\}$ is a set of $(m+\ell)$ linearly independent vectors, respectively.

In view of equations (28), (29), (33), and (34), the computation of $\underset{\sim}{G}_\ell$ is thus reduced to the determination (Porter and D'Azzo 1977) of the kernels of each of the n matrices

$$\underset{\sim}{S}_\ell(\lambda_i) = [\underset{\sim}{A}_\ell-\lambda_i\underset{\sim}{I} , \underset{\sim}{B}_\ell] \qquad (i=1,2,\ldots,n+pq+\ell) \tag{35}$$

together with the kernels of each of the n matrices

$$\underset{\sim}{T}'_\ell(\lambda_j) = [\underset{\sim}{A}'_\ell-\lambda_j\underset{\sim}{I} , \underset{\sim}{C}'_\ell] \qquad (j=1,2,\ldots,n+pq+\ell) \tag{36}$$

followed by the selection of sets of linearly independent self-conjugate vectors $\{\underset{\sim}{\chi}_1,\underset{\sim}{\chi}_2,\ldots,\underset{\sim}{\chi}_{n+pq+\ell}\}$ and $\{\underset{\sim}{\phi}_1,\underset{\sim}{\phi}_2,\ldots,\underset{\sim}{\phi}_{n+pq+\ell}\}$ from subspaces determined by the kernels of $\underset{\sim}{S}_\ell(\lambda_i)\,(i=1,2,\ldots,n+pq+\ell)$ and $\underset{\sim}{T}'_\ell(\lambda_j)\,(j=1,2,\ldots,n+pq+\ell)$, respectively, such that the orthogonality conditions (32) are satisfied. It is finally evident from equations (10), (11), and (22) that the matrices in the respective state and output equations of the required ℓth-order dynamic compensator are determined by the sub-matrices of the output-feedback matrix $\underset{\sim}{G}_{\ell}$.

4. ILLUSTRATIVE EXAMPLE

The results presented in Sections 2 and 3 can be conveniently illustrated by designing an error-actuated dynamic controller which will cause the output of the controllable and observable linear plant governed by the respective state and output equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 , & 1 \\ 1 , & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(t) \tag{37}$$

and

$$y(t) = [1 , 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{38}$$

to track any constant command input $r(t)$ in the presence of any unmeasurable constant disturbance input $d(t)$. In this case it is clear that $r = s = q = 1$, so that the open-loop tracking system is governed by the respective state and output equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 , & 1 , & 0 \\ 1 , & 1 , & 0 \\ -1 , & 0 , & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} d(t) \tag{39}$$

and

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 , 0 , 0 \\ 0 , 0 , 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{bmatrix} . \qquad (40)$$

This system is clearly controllable and observable with $\nu_c = 3$ and $\nu_o = 2$ so that (Brasch and Pearson 1970) equation (12) indicates that $\ell = 1$. Furthermore, in the notation of equations (23), (24), and (25), it follows from equations (39) and (40) that

$$\underset{\sim}{A}_1 = \begin{bmatrix} 0 , 1 , 0 , 0 \\ 1 , 1 , 0 , 0 \\ -1 , 0 , 0 , 0 \\ 0 , 0 , 0 , 0 \end{bmatrix} \qquad (41)$$

$$\underset{\sim}{B}_1 = \begin{bmatrix} 0 , 0 \\ 1 , 0 \\ 0 , 0 \\ 0 , 1 \end{bmatrix} \qquad (42)$$

and

$$\underset{\sim}{C}_1 = \begin{bmatrix} 1 , 0 , 0 , 0 \\ 0 , 0 , 1 , 0 \\ 0 , 0 , 0 , 1 \end{bmatrix} . \qquad (43)$$

It is thus evident from equations (35), (36), (41), (42), and (43) that

$$\underset{\sim}{S}_1(\lambda) = \begin{bmatrix} -\lambda , 1 , 0 , 0 , 0 , 0 \\ 1 , 1-\lambda , 0 , 0 , 1 , 0 \\ -1 , 0 , -\lambda , 0 , 0 , 0 \\ 0 , 0 , 0 , -\lambda , 0 , 1 \end{bmatrix} \qquad (44)$$

and

$$\underset{\sim}{T}_1'(\lambda) = \begin{bmatrix} -\lambda, & 1, & -1, & 0, & 1, & 0, & 0 \\ 1, & 1-\lambda, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & -\lambda, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & -\lambda, & 0, & 0, & 1 \end{bmatrix} \quad . \tag{45}$$

In order to design an error-actuated dynamic compensator for the open-loop tracking system governed by equations (39) and (40) such that the eigenvalues of the plant matrix of the closed-loop tracking system are

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-1,-2,-3,-4\} \tag{46}$$

the design method described in Section 3 can be readily used to compute an output-feedback matrix $\underset{\sim}{G}_1$ such that

$$\sigma(\underset{\sim}{A}_1 + \underset{\sim}{B}_1 \underset{\sim}{G}_1 \underset{\sim}{C}_1) = \{-1,-2,-3,-4\} \quad . \tag{47}$$

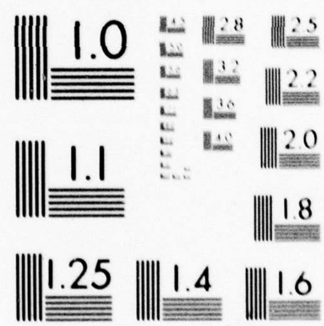
Indeed, it follows immediately (Porter and D'Azzo 1977) from equation (44) that

$$\ker \underset{\sim}{S}_1(-1) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \tag{48}$$

AD-A071 662 SALFORD UNIV (ENGLAND) DEPT OF AERONAUTICAL AND MECH--ETC F/G 9/2
MULTIVARIABLE DIGITAL CONTROL SYSTEMS.(U)
JUN 79 B PORTER AFOSR-76-3005
UNCLASSIFIED USAME/DC/101/79 NL

2 OF 2
AD
A071662

END
DATE
FILMED
8-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

$$\ker \underset{\sim}{S}_1(-2) = \text{span}\left\{ \begin{bmatrix} -2 \\ 4 \\ -1 \\ 0 \\ -10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\} \tag{49}$$

$$\ker \underset{\sim}{S}_1(-3) = \text{span}\left\{ \begin{bmatrix} -3 \\ 9 \\ -1 \\ 0 \\ -33 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right\} \tag{50}$$

and

$$\ker \underset{\sim}{S}_1(-4) = \text{span}\left\{ \begin{bmatrix} -4 \\ 16 \\ -1 \\ 0 \\ -76 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\} , \tag{51}$$

and similarly (Porter and D'Azzo 1977) from equation (45) that

$$\ker \underset{\sim}{T}_1'(-1) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\} \qquad (52)$$

$$\ker \underset{\sim}{T}_1'(-2) = \operatorname{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right\} \qquad (53)$$

$$\ker \underset{\sim}{T}_1'(-3) = \operatorname{span}\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 11 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -3 \end{bmatrix} \right\} \qquad (54)$$

and

$$\ker \underset{\sim}{T}_1'(-4) = \text{span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 19 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -4 \end{bmatrix} \right\} . \quad (55)$$

It is evident from equations (48), (49), (50), and (51) that the closed-loop eigenvectors corresponding to the eigenvalue spectrum (47) must be assigned to the respective subspaces

$$\Sigma_1(-1) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (56)$$

$$\Sigma_1(-2) = \text{span}\left\{ \begin{bmatrix} -2 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (57)$$

$$\Sigma_1(-3) = \text{span}\left\{ \begin{bmatrix} -3 \\ 9 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (58)$$

and

$$\Sigma_1(-4) = \text{span}\left\{\begin{bmatrix}-4\\16\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{59}$$

and it is similarly evident from equations (52), (53), (54), and (55) that the closed-loop reciprocal eigenvectors corresponding to the eigenvalue spectrum (47) must be assigned to the respective subspaces

$$\Gamma_1(-1) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{60}$$

$$\Gamma_1(-2) = \text{span}\left\{\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{61}$$

$$\Gamma_1(-3) = \text{span}\left\{\begin{bmatrix}-4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{62}$$

and

$$\Gamma_1(-4) = \text{span}\left\{\begin{bmatrix}-5\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} . \tag{63}$$

Since the vectors

$$\underset{\sim}{X}_1 = \begin{bmatrix} -5 \\ 5 \\ -5 \\ -7 \end{bmatrix} \in \Sigma_1(-1) \tag{64}$$

$$\underset{\sim}{X}_2 = \begin{bmatrix} -2 \\ 4 \\ -1 \\ -7 \end{bmatrix} \in \Sigma_1(-2) \tag{65}$$

$$\underset{\sim}{X}_3 = \begin{bmatrix} -3 \\ 9 \\ -1 \\ -14 \end{bmatrix} \in \Sigma_1(-3) \tag{66}$$

$$\underset{\sim}{X}_4 = \begin{bmatrix} -4 \\ 16 \\ -1 \\ -23 \end{bmatrix} \in \Sigma_1(-4) \tag{67}$$

$$\underset{\sim}{\phi}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \in \Gamma_1(-1) \tag{68}$$

$$\underset{\sim}{\phi}_2 = \begin{bmatrix} 27/2 \\ -9/2 \\ -11 \\ -5 \end{bmatrix} \in \Gamma_1(-2) \tag{69}$$

$$\underset{\sim}{\phi}_3 = \begin{bmatrix} -16 \\ 4 \\ 13 \\ 5 \end{bmatrix} \in \Gamma_1(-3) \tag{70}$$

and

$$\underset{\sim}{\phi}_4 = \begin{bmatrix} 35/6 \\ -7/6 \\ -14/3 \\ -5/3 \end{bmatrix} \in \Gamma_1(-4) \tag{71}$$

are clearly such that

$$\underset{\sim}{\phi}_j'\underset{\sim}{\chi}_i = \delta_{ij} \qquad (i,j=1,2,3,4) \tag{72}$$

it follows from equations (33) and (34) that equation (47) is satisfied by the output-feedback matrix

$$\underset{\sim}{G}_1 = \begin{bmatrix} -47 \; , & 34 \; , & 10 \\ 49 \; , & -35 \; , & -11 \end{bmatrix} \quad . \tag{73}$$

In view of equations (10), (11), (22), (73), the corresponding dynamic compensator for the open-loop tracking system governed by equations (39) and (40) is governed by the respective state and output equations

$$\dot{w}(t) = -11w(t) - 49e(t) - 35z(t) \tag{74}$$

and

$$u(t) = 10w(t) + 47e(t) + 34z(t) \quad , \tag{75}$$

so that the required error-actuated dynamic controller is characterised by the transfer function

$$T(s) = \bar{u}(s)/\bar{e}(s) = (47s^2+61s+24)/s(s+11) \quad . \tag{76}$$

It can be readily verified that the poles of the closed-loop tracking system governed by equations (37), (38),(74), and (75) are $\{-1,-2,-3,-4\}$ and that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\{r(t)-y(t)\} = 0 \tag{77}$$

for any constant command input $r(t)$ and any constant unmeasurable disturbance input $d(t)$.

5. Conclusion

In this paper, the method of entire eigenstructure assignment has been applied to the design of linear multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. It has been indicated that such tracking systems can be designed for a much larqer class of plants than tracking systems incorporating error-actuated static controllers (Porter and Bradshaw 1976) in view of the fact that eigenstructure assignment by error-actuated static controllers and by output-feedback controllers (Kimura 1975,1977) are essentially equivalent.

Acknowledgement

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005A.

REFERENCES

BRASCH, F. M. and PEARSON, J. B., 1970, I.E.E.E. Trans. autom. Control, 15, 34.

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509; 1977, Ibid., 22, 458.

PORTER, B. and BRADSHAW, A., 1974, Int. J. Systems Sci., 5, 1155; 1976, Ibid., 7, 943; 1978, Ibid. (in the press).

PORTER, B. and D'AZZO, J. J., 1977, Electron. Lett., 13, 230.

APPENDIX 8

DESIGN OF LINEAR MULTIVARIABLE DISCRETE-TIME TRACKING SYSTEMS INCORPORATING ERROR-ACTUATED DYNAMIC CONTROLLERS

DR A BRADSHAW AND PROFESSOR B PORTER

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, VOL 9, PP 1079-1090, 1978)

ABSTRACT

In this paper, the method of entire eigenstructure assignment (Bradshaw and Porter 1978a) is applied to the design of linear multivariable discrete-time tracking systems incorporating error-actuated dynamic controllers. The method is illustrated by designing an error-actuated dynamic controller which causes the output of a second-order discrete-time plant to track a constant command input in the presence of an unmeasurable constant disturbance input.

1. INTRODUCTION

In this paper, the method of entire eigenstructure assignment (Bradshaw and Porter 1978a) is applied to the design of linear multivariable discrete-time tracking systems incorporating error-actuated dynamic controllers. Such tracking systems consist of a controllable and observable nth-order linear multivariable plant governed by state and output equations of the respective forms

$$\underset{\sim}{x}(k+1) = \underset{\sim}{A}\underset{\sim}{x}(k) + \underset{\sim}{B}\underset{\sim}{u}(k) + \underset{\sim}{D}\underset{\sim}{d}(k) \tag{1}$$

and

$$\underset{\sim}{y}(k) = \underset{\sim}{C}\underset{\sim}{x}(k) \tag{2}$$

where $\underset{\sim}{B}$ and $\underset{\sim}{C}$ have full rank, together with an error-actuated dynamic controller which is required to cause the pxl output vector $\underset{\sim}{y}(k)$ of the plant to track a pxl command input vector $\underset{\sim}{r}(k)$ in the sense that

$$\lim_{k\to\infty} \underset{\sim}{e}(k) = \lim_{k\to\infty}\{\underset{\sim}{r}(k)-\underset{\sim}{y}(k)\} = \underset{\sim}{0} \tag{3}$$

for polynomial command and disturbance inputs of the respective forms

$$\underset{\sim}{r}(k) = \sum_{i=1}^{r} \underset{\sim}{\alpha}_{i-1} k^{i-1} \tag{4}$$

and

$$\underset{\sim}{d}(k) = \sum_{i=1}^{s} \underset{\sim}{\beta}_{i-1} k^{i-1} \quad . \tag{5}$$

The theory developed in this paper is analogous to that developed by Porter and Bradshaw (1978) for continuous-time tracking systems but is significantly extended in order to allow the assignment of confluent eigenvalues to the plant matrix of the closed-loop tracking system. It is therefore possible to apply this theory to the design of an error-actuated dynamic controller which causes the output vector of a plant governed by equations (1) and (2) to track a command input vector in the sense that

$$\underset{\sim}{e}(k) = \underset{\sim}{r}(k) - \underset{\sim}{y}(k) = \underset{\sim}{0} \qquad (k=\nu, \nu+1,\ldots) \qquad (6)$$

for command and disturbance inputs defined by equations (4) and (5), where ν is the index of nilpotency of the closed-loop plant matrix of the tracking system. It is important to note that tracking systems incorporating error-actuated <u>dynamic</u> controllers can be designed for a much larger class of plants than tracking systems incorporating error-actuated <u>static</u> controllers (Bradshaw and Porter 1978b) in view of the fact that eigenstructure assignment by error-actuated static controllers and by output-feedback controllers (Kimura 1975,1977) are essentially equivalent.

2. <u>Theory</u>

The first stage in the design of the required error-actuated dynamic controller for the plant governed by equations (1) and (2) involves the introduction (Bradshaw and Porter 1975) of a vector comparator and a series of $q = \max(r,s)$ discrete-time vector integrators in order to generate the q vectors defined by the equations

$$\left.\begin{array}{lcll}
\mathbf{z}_1(k+1) & = & \mathbf{z}_1(k) + \mathbf{e}(k) & , \\
\mathbf{z}_2(k+1) & = & \mathbf{z}_2(k) + \mathbf{z}_1(k) & , \\
\cdots\cdots\cdots & & \cdots\cdots\cdots & , \\
\cdots\cdots\cdots & & \cdots\cdots\cdots & , \\
\mathbf{z}_q(k+1) & = & \mathbf{z}_q(k) + \mathbf{z}_{q-1}(k) & .
\end{array}\right\} \tag{7}$$

It is then evident from equations (1), (2), and (7) that the open-loop tracking system is governed by state and output equations of the respective forms

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}_1(k+1) \\ \cdots\cdots \\ \cdots\cdots \\ \mathbf{z}_{q-1}(k+1) \\ \mathbf{z}_q(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A} , & \mathbf{O} , & \cdots , & \mathbf{O} , & \mathbf{O} \\ -\mathbf{C} , & \mathbf{I}_p , & \cdots , & \mathbf{O} , & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} , & \mathbf{O} , & \cdots , & \mathbf{I}_p , & \mathbf{O} \\ \mathbf{O} , & \mathbf{O} , & \cdots , & \mathbf{I}_p , & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \mathbf{z}_{q-1}(k) \\ \mathbf{z}_q(k) \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{B} \\ \mathbf{O} \\ \cdot \\ \cdot \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} \mathbf{u}(k) + \begin{bmatrix} \mathbf{O} \\ \mathbf{I}_p \\ \cdot\cdot \\ \cdot\cdot \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} \mathbf{r}(k) + \begin{bmatrix} \mathbf{D} \\ \mathbf{O} \\ \cdot \\ \cdot \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} \mathbf{d}(k) \tag{8}$$

and

$$\begin{bmatrix} \mathbf{y}(k) \\ \mathbf{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \mathbf{z}_{q-1}(k) \\ \mathbf{z}_q(k) \end{bmatrix} = \begin{bmatrix} \mathbf{C} , & \mathbf{O} , & \cdots , & \mathbf{O} , & \mathbf{O} \\ \mathbf{O} , & \mathbf{I}_p , & \cdots , & \mathbf{O} , & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} , & \mathbf{O} , & \cdots , & \mathbf{I}_p , & \mathbf{O} \\ \mathbf{O} , & \mathbf{O} , & \cdots , & \mathbf{O} , & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \mathbf{z}_{q-1}(k) \\ \mathbf{z}_q(k) \end{bmatrix} . \tag{9}$$

The open-loop tracking system governed by equations (8) and (9) is controllable if and only if (Bradshaw and Porter 1975)

$$\text{rank} \begin{bmatrix} \underset{\sim}{B} , & \underset{\sim}{A}-\underset{\sim}{I} \\ \underset{\sim}{0} , & -\underset{\sim}{C} \end{bmatrix} = n + p \tag{10}$$

since $(\underset{\sim}{A},\underset{\sim}{B})$ and $(\underset{\sim}{C},\underset{\sim}{A})$ are respectively controllable and observable pairs.

In the case of such controllable and observable open-loop tracking systems, the second stage in the design of the error-actuated dynamic controller involves the introduction of an ℓth-order dynamic compensator (Brasch and Pearson 1970) governed by state and output equations of the respective forms

$$\underset{\sim}{w}(k+1) = \underset{\sim}{F}\underset{\sim}{w}(k) + \underset{\sim}{G}\underset{\sim}{e}(k) + \sum_{i=1}^{q} \underset{\sim}{H}_i \underset{\sim}{z}_i(k) \tag{11}$$

and

$$\underset{\sim}{u}(k) = \underset{\sim}{K}\underset{\sim}{w}(k) + \underset{\sim}{L}\underset{\sim}{e}(k) + \sum_{i=1}^{q} \underset{\sim}{M}_i \underset{\sim}{z}_i(k) \tag{12}$$

where

$$\ell = \min(\nu_c-1, \nu_o-1) \tag{13}$$

and ν_c and ν_o are respectively the controllability and observability indices of the open-loop tracking system governed by equations (8) and (9). It is then evident from equations (8), (9), (11), and (12) that the closed-loop tracking system is governed by state and output equations of the respective forms

$$\begin{bmatrix} \underset{\sim}{x}(k+1) \\ \underset{\sim}{z}_1(k+1) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k+1) \\ \underset{\sim}{w}(k+1) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{L}\underset{\sim}{C} , & \underset{\sim}{B}\underset{\sim}{M}_1 , & \cdots , & \underset{\sim}{B}\underset{\sim}{M}_q , & \underset{\sim}{B}\underset{\sim}{K} \\ -\underset{\sim}{C} , & \underset{\sim}{I}_p , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{I}_p , & \underset{\sim}{O} \\ -\underset{\sim}{G}\underset{\sim}{C} , & \underset{\sim}{H}_1 , & \cdots , & \underset{\sim}{H}_q , & \underset{\sim}{F} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(k) \\ \underset{\sim}{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k) \\ \underset{\sim}{w}(k) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{B}\underset{\sim}{L} \\ \underset{\sim}{I}_p \\ \cdot\cdot \\ \cdot\cdot \\ \underset{\sim}{O} \\ \underset{\sim}{G} \end{bmatrix} \underset{\sim}{r}(k) + \begin{bmatrix} \underset{\sim}{D} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{d}(k) \tag{14}$$

and

$$\begin{bmatrix} \underset{\sim}{y}(k) \\ \underset{\sim}{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k) \\ \underset{\sim}{w}(k) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{C} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{I}_p , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{I}_p , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \cdots , & \underset{\sim}{O} , & \underset{\sim}{I}_\ell \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(k) \\ \underset{\sim}{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k) \\ \underset{\sim}{w}(k) \end{bmatrix} . \tag{15}$$

In view of equations (4) and (5), it is clear by differencing equation (14) (q-1) times that the closed-loop tracking system will behave so that

$$\lim_{k\to\infty} \Delta^{(q)} \underset{\sim}{z}_q(k) = \underset{\sim}{O} \tag{16}$$

and therefore, in view of equations (7), so that equation (3) will be satisfied if the error-actuated dynamic compensator governed by equations (11) and (12) is designed such that all the eigenvalues of the plant matrix of the closed-loop tracking system governed by equations (14) and (15) are assigned to locations within the unit circle of the complex plane.

3. Compensator Design

It is evident from equations (8), (9), (11), and (12) that such a compensator can be designed by the synthesis of an appropriate output-feedback control law of the form

$$\underset{\sim}{u}_\ell(k) = \underset{\sim}{G}_\ell \underset{\sim}{y}_\ell(k) \tag{17}$$

for the augmented open-loop system governed by state and output equations of the respective forms

$$\underset{\sim}{x}_\ell(k+1) = \underset{\sim}{A}_\ell \underset{\sim}{x}_\ell(k) + \underset{\sim}{B}_\ell \underset{\sim}{u}_\ell(k) \tag{18}$$

and

$$\underset{\sim}{y}_\ell(k) = \underset{\sim}{C}_\ell \underset{\sim}{x}_\ell(k) \quad , \tag{19}$$

where

$$\underset{\sim}{u}_\ell(k) = \begin{bmatrix} \underset{\sim}{u}(k) \\ \underset{\sim}{v}(k) \end{bmatrix} \tag{20}$$

$$\underset{\sim}{x}_\ell(k) = \begin{bmatrix} \underset{\sim}{x}(k) \\ \underset{\sim}{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k) \\ \underset{\sim}{w}(k) \end{bmatrix} \tag{21}$$

$$\underset{\sim}{y}_{\ell}(k) = \begin{bmatrix} \underset{\sim}{y}(k) \\ \underset{\sim}{z}_1(k) \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_q(k) \\ \underset{\sim}{w}(k) \end{bmatrix} \tag{22}$$

$$\underset{\sim}{G}_{\ell} = \begin{bmatrix} -\underset{\sim}{L}, & \underset{\sim}{M}_1, & \cdots, & \underset{\sim}{M}_q, & \underset{\sim}{K} \\ -\underset{\sim}{G}, & \underset{\sim}{H}_1, & \cdots, & \underset{\sim}{H}_q, & \underset{\sim}{F} \end{bmatrix} \tag{23}$$

$$\underset{\sim}{A}_{\ell} = \begin{bmatrix} \underset{\sim}{A}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ -\underset{\sim}{C}, & \underset{\sim}{I}_p, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{I}_p, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{O} \end{bmatrix} \tag{24}$$

$$\underset{\sim}{B}_{\ell} = \begin{bmatrix} \underset{\sim}{B}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots \\ \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{I}_{\ell} \end{bmatrix} \tag{25}$$

and

$$\underset{\sim}{C}_{\ell} = \begin{bmatrix} \underset{\sim}{C}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{I}_p, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{I}_p, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \cdots, & \underset{\sim}{O}, & \underset{\sim}{I}_{\ell} \end{bmatrix} . \tag{26}$$

Thus, if the $(m+\ell)\mathrm{x}(p+pq+\ell)$ output-feedback matrix $\mathbf{G}_\ell$ is such that the closed-loop plant matrix $(\mathbf{A}_\ell+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell)$ has a self-conjugate eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$, a corresponding eigenvector and generalised eigenvector set $\{\boldsymbol{\chi}_{\lambda_i}^{(h,j)}: h=1,2,\ldots,m_{ji};j=1,2,\ldots,k_i;i=1,2,\ldots,t\}$, and a corresponding reciprocal eigenvector and reciprocal generalised eigenvector set $\{\boldsymbol{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\ldots,m_{ba};b=1,2,\ldots,k_a;a=1,2,\ldots,t\}$, then

$$(\mathbf{A}_\ell-\lambda_i\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell)\boldsymbol{\chi}_{\lambda_i}^{(1,j)} = \mathbf{0} \tag{27a}$$

$$(\mathbf{A}_\ell-\lambda_i\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell)\boldsymbol{\chi}_{\lambda_i}^{(2,j)} = \boldsymbol{\chi}_{\lambda_i}^{(1,j)} \tag{27b}$$

....

....

$$(\mathbf{A}_\ell-\lambda_i\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell)\boldsymbol{\chi}_{\lambda_i}^{(m_{ji},j)} = \boldsymbol{\chi}_{\lambda_i}^{(m_{ji}-1,j)} \tag{27m_{ji}}$$

$$(j=1,2,\ldots,k_i;i=1,2,\ldots,t) \quad ,$$

and

$$\boldsymbol{\phi}_{\lambda_a}^{(1,b)\prime}(\mathbf{A}_\ell-\lambda_a\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell) = \boldsymbol{\phi}_{\lambda_a}^{(2,b)\prime} \tag{28a}$$

....

....

$$\boldsymbol{\phi}_{\lambda_a}^{(m_{ba}-1,b)\prime}(\mathbf{A}_\ell-\lambda_a\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell) = \boldsymbol{\phi}_{\lambda_a}^{(m_{ba},b)\prime} \tag{28m_{ba}-1}$$

$$\boldsymbol{\phi}_{\lambda_a}^{(m_{ba},b)\prime}(\mathbf{A}_\ell-\lambda_a\mathbf{I}+\mathbf{B}_\ell\mathbf{G}_\ell\mathbf{C}_\ell) = \mathbf{0}' \tag{28m_{ba}}$$

$$(b=1,2,\ldots,k_a;a=1,2,\ldots,t) \quad ,$$

where $\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}$ is the hth vector in the jth string of length m_{ji} associated with the eigenvalue λ_i, and where $\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}$ is the cth vector in the bth string of length m_{ba} associated with the eigenvalue λ_a. The vectors $\underset{\sim}{\chi}_{\lambda_i}^{(1,j)}$ $(j=1,2,\ldots,k_i)$ are the k_i eigenvectors associated with the eigenvalue λ_i, whilst the remaining vectors in each of the k_i strings of vectors satisfying equations (27) are generalised eigenvectors associated with the eigenvalue λ_i. Similarly, the vectors $\underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)}$ $(b=1,2,\ldots,k_a)$ are the k_a reciprocal eigenvectors associated with the eigenvalue λ_a, whilst the remaining vectors in each of the k_a strings of vectors satisfying equations (28) are reciprocal generalised eigenvectors associated with the eigenvalue λ_a. The total number of vectors associated with the eigenvalue λ_f in each set is evidently

$$m_f = \sum_{g=1}^{k_f} m_{gf} \qquad (f=1,2,\ldots,t) \qquad (29)$$

and

$$n + pq + \ell = \sum_{f=1}^{t} m_f \quad . \qquad (30)$$

Equations (27) and (28) can be written in the form

$$[\underset{\sim}{A}_\ell - \lambda_i \underset{\sim}{I} \ , \ \underset{\sim}{B}_\ell] \begin{bmatrix} \underset{\sim}{\chi}_{\lambda_i}^{(1,j)} \\ \\ \underset{\sim}{\omega}_{\lambda_i}^{(1,j)} \end{bmatrix} = \underset{\sim}{0} \qquad (31a)$$

$$[\underset{\sim}{A}_\ell - \lambda_i \underset{\sim}{I} \ , \ \underset{\sim}{B}_\ell] \begin{bmatrix} \underset{\sim}{\chi}_{\lambda_i}^{(2,j)} \\ \underset{\sim}{\omega}_{\lambda_i}^{(2,j)} \end{bmatrix} = \underset{\sim}{\chi}_{\lambda_i}^{(1,j)} \tag{31b}$$

....

....

$$[\underset{\sim}{A}_\ell - \lambda_i \underset{\sim}{I} \ , \ \underset{\sim}{B}_\ell] \begin{bmatrix} \underset{\sim}{\chi}_{\lambda_i}^{(m_{ji},j)} \\ \underset{\sim}{\omega}_{\lambda_i}^{(m_{ji},j)} \end{bmatrix} = \underset{\sim}{\chi}_{\lambda_i}^{(m_{ji}-1,j)} \tag{31m_{ji}}$$

$$(j=1,2,\ldots,k_i ; i=1,2,\ldots,t)$$

and

$$[\underset{\sim}{A}'_\ell - \lambda_a \underset{\sim}{I} \ , \ \underset{\sim}{C}'_\ell] \begin{bmatrix} \underset{\sim}{\phi}_{\lambda_a}^{(1,b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(1,b)} \end{bmatrix} = \underset{\sim}{\phi}_{\lambda_a}^{(2,b)} \tag{32a}$$

....

....

$$[\underset{\sim}{A}'_\ell - \lambda_a \underset{\sim}{I} \ , \ \underset{\sim}{C}'_\ell] \begin{bmatrix} \underset{\sim}{\phi}_{\lambda_a}^{(m_{ba}-1,b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(m_{ba}-1,b)} \end{bmatrix} = \underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)} \tag{32m_{ba}-1}$$

$$[\underset{\sim}{A}'_\ell - \lambda_a \underset{\sim}{I} \ , \ \underset{\sim}{C}'_\ell] \begin{bmatrix} \underset{\sim}{\phi}_{\lambda_a}^{(m_{ba},b)} \\ \underset{\sim}{\zeta}_{\lambda_a}^{(m_{ba},b)} \end{bmatrix} = \underset{\sim}{O} \tag{32m_{ba}}$$

$$(b=1,2,\ldots,k_a ; a=1,2,\ldots,t)$$

where

$$\underset{\sim}{\omega}_{\lambda_i}^{(h,j)} = \underset{\sim}{G}_\ell \underset{\sim}{C}_\ell \underset{\sim}{\chi}_{\lambda_i}^{(h,j)} \tag{33}$$

$$\underset{\sim}{\zeta}_{\lambda_a}^{(c,b)} = \underset{\sim}{G}_\ell' \underset{\sim}{B}_\ell' \underset{\sim}{\phi}_{\lambda_a}^{(c,b)} \tag{34}$$

and

$$\underset{\sim}{\phi}_{\lambda_a}^{\prime(a,b)} \underset{\sim}{\chi}_{\lambda_i}^{(h,j)} = \delta_{ai}\delta_{bj}\delta_{ch} \tag{35}$$

$$(h=1,2,\ldots,m_{ji};j=1,2,\ldots,k_i;i=1,2,\ldots,t)$$

$$(c=1,2,\ldots,m_{ba};b=1,2,\ldots,k_a;a=1,2,\ldots,t) \quad .$$

Conversely, if equations (31), (32), and (35) are satisfied by a self-conjugate set $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$ of complex numbers and corresponding self-conjugate sets $\{\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}: h=1,2,\ldots,m_{ji}; j=1,2,\ldots,k_i;i=1,2,\ldots,t\}$ and $\{\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\ldots,m_{ba};b=1,2,\ldots,k_a; a=1,2,\ldots,t\}$ of linearly independent vectors, then equations (33) and (34) are satisfied by an $(m+\ell)\mathrm{x}(n+pq+\ell)$ matrix $\underset{\sim}{G}_\ell$ such that $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$ is the eigenvalue spectrum of the matrix $(\underset{\sim}{A}_\ell+\underset{\sim}{B}_\ell\underset{\sim}{G}_\ell\underset{\sim}{C}_\ell)$, $\{\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}:k=1,2,\ldots,m_{ji};j=1,2,\ldots,k_i; i=1,2,\ldots,t\}$ constitutes a corresponding eigenvector and generalised eigenvector set, and $\{\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\ldots,m_{ba}; b=1,2,\ldots,k_a;a=1,2,\ldots,t\}$ constitutes a corresponding reciprocal eigenvector and reciprocal generalised eigenvector set. It accordingly follows from equations (33) and (34) respectively that the real output-feedback matrix

$$\underset{\sim}{G}_\ell = [\underset{\sim}{\omega}_1 , \underset{\sim}{\omega}_2 , \ldots , \underset{\sim}{\omega}_{p+pq+\ell}][\underset{\sim}{C}_\ell\underset{\sim}{\chi}_1 , \underset{\sim}{C}_\ell\underset{\sim}{\chi}_2 , \ldots , \underset{\sim}{C}_\ell\underset{\sim}{\chi}_{p+pq+\ell}]^{-1} \tag{36}$$

and the real transposed output-feedback matrix

$$\underset{\sim}{G}'_{\ell} = [\underset{\sim}{\zeta}_1 , \underset{\sim}{\zeta}_2 , \dots , \underset{\sim}{\zeta}_{m+\ell}][\underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_1 , \underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_2 , \dots , \underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_{m+\ell}]^{-1} \tag{37}$$

assign the self-conjugate eigenvalue spectrum $\{\lambda_1,\lambda_2,\dots,\lambda_t\}$, the corresponding eigenvector and generalised eigenvector set $\{\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}:h=1,2,\dots,m_{ji};j=1,2,\dots,k_i;i=1,2,\dots,t\}$, and the corresponding reciprocal eigenvector and reciprocal generalised eigenvector set $\{\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\dots,m_{ba};b=1,2,\dots,k_a;a=1,2,\dots,t\}$ to the closed-loop plant matrix $(\underset{\sim}{A}_{\ell}+\underset{\sim}{B}_{\ell}\underset{\sim}{G}_{\ell}\underset{\sim}{C}_{\ell})$ in case $\{\underset{\sim}{C}_{\ell}\underset{\sim}{\chi}_1,\underset{\sim}{C}_{\ell}\underset{\sim}{\chi}_2,\dots,\underset{\sim}{C}_{\ell}\underset{\sim}{\chi}_{p+pq+\ell}\}$ is a subset of $(p+pq+\ell)$ linearly independent members of the set $\{\underset{\sim}{C}_{\ell}\underset{\sim}{\chi}_{\lambda_i}^{(h,j)};h=1,2,\dots,m_{ji};$ $j=1,2,\dots,k_i;i=1,2,\dots,t\}$ and $\{\underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_1,\underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_2,\dots,\underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_{m+\ell}\}$ is a subset of $(m+\ell)$ linearly independent members of the set $\{\underset{\sim}{B}'_{\ell}\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\dots,m_{ba};b=1,2,\dots,k_a;a=1,2,\dots,t\}$.

In view of equations (31), (32), (36), and (37), the computation of $\underset{\sim}{G}_{\ell}$ is thus reduced to the determination (Porter and D'Azzo 1978) of the kernels and generalised kernels of each of the t matrices

$$\underset{\sim}{S}_{\ell}(\lambda_i) = [\underset{\sim}{A}_{\ell}-\lambda_i\underset{\sim}{I} , \underset{\sim}{B}_{\ell}] \qquad (i=1,2,\dots,t) \tag{38}$$

together with the kernels and generalised kernels of each of the t matrices

$$\underset{\sim}{T}'_{\ell}(\lambda_a) = [\underset{\sim}{A}'_{\ell}-\lambda_a\underset{\sim}{I} , \underset{\sim}{C}'_{\ell}] \qquad (a=1,2,\dots,t) \tag{39}$$

followed by the selection of sets of linearly independent self-conjugate vectors $\{\underset{\sim}{\chi}_{\lambda_i}^{(h,j)}:h=1,2,\dots,m_{ji};j=1,2,\dots,k_i;i=1,2,\dots,t\}$ and $\{\underset{\sim}{\phi}_{\lambda_a}^{(c,b)}:c=1,2,\dots,m_{ba};b=1,2,\dots,k_a;a=1,2,\dots,t\}$ from sub-spaces determined by the kernels and generalised kernels of

$\underset{\sim}{S}_\ell(\lambda_i)(i=1,2,\ldots,t)$ and $\underset{\sim}{T}'_\ell(\lambda_a)(a=1,2,\ldots,t)$, respectively, such that the orthogonality conditions (35) are satisfied. It is finally evident from equations (11), (12), and (23) that the matrices in the respective state and output equations of the required ℓth-order dynamic compensator are determined by the sub-matrices of the output-feedback matrix $\underset{\sim}{G}_\ell$.

4. Illustrative Example

The results presented in Sections 2 and 3 can be conveniently illustrated by designing an error-actuated dynamic controller which will cause the output of the controllable and observable linear plant governed by the respective state and output equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 , & 1 \\ 1 , & -1/2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(k) \tag{40}$$

and

$$y(k) = [1 , 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{41}$$

to track any constant command input $r(k)$ in the presence of any unmeasurable constant disturbance input $d(k)$ in such a way that $e(k) = 0$ after a finite number of discrete-time intervals. In this case it is clear that $r = s = q = 1$, so that the open-loop tracking system is governed by the respective state and output equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 , & 0 \\ 1 , & -1/2 , & 0 \\ -1 , & 0 , & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(k) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} d(k) \tag{42}$$

and

$$\begin{bmatrix} y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} 1 , 0 , 0 \\ 0 , 0 , 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ z(k) \end{bmatrix} \quad . \tag{43}$$

This system is clearly controllable and observable with $\nu_c = 3$ and $\nu_o = 2$ so that (Brasch and Pearson 1970) equation (13) indicates that $\ell = 1$. Furthermore, in the notation of equations (24), (25), and (26), it follows from equations (42) and (43) that

$$\underset{\sim}{A}_1 = \begin{bmatrix} 0 , & 1 & , 0 , 0 \\ 1 , & -1/2 & , 0 , 0 \\ -1 , & 0 & , 1 , 0 \\ 0 , & 0 & , 0 , 0 \end{bmatrix} \tag{44}$$

$$\underset{\sim}{B}_1 = \begin{bmatrix} 0 , 0 \\ 1 , 0 \\ 0 , 0 \\ 0 , 1 \end{bmatrix} \tag{45}$$

and

$$\underset{\sim}{C}_1 = \begin{bmatrix} 1 , 0 , 0 , 0 \\ 0 , 0 , 1 , 0 \\ 0 , 0 , 0 , 1 \end{bmatrix} \quad . \tag{46}$$

It is thus evident from equations (38), (39), (44), (45), and (46) that

$$\mathbf{S}_1(\lambda) = \begin{bmatrix} -\lambda, & 0 & , 0 & , 0 & , 0 & , 0 \\ 1 & , -\lambda-1/2 & , 0 & , 0 & , 1 & , 0 \\ -1 & , 0 & , 1-\lambda & , 0 & , 0 & , 0 \\ 0 & , 0 & , 0 & , -\lambda & , 0 & , 1 \end{bmatrix} \tag{47}$$

and

$$\mathbf{T}_1'(\lambda) = \begin{bmatrix} -\lambda, & 1 & , -1 & , 0 & , 1 & , 0 & , 0 \\ 1 & , -\lambda-1/2 & , 0 & , 0 & , 0 & , 0 & , 0 \\ 0 & , 0 & , 1-\lambda & , 0 & , 0 & , 1 & , 0 \\ 0 & , 0 & , 0 & , -\lambda & , 0 & , 0 & , 1 \end{bmatrix} . \tag{48}$$

In order to design an error-actuated dynamic compensator for the open-loop tracking system governed by equations (42) and (43) such that the error is eliminated after a finite number of discrete-time intervals, it is necessary to assign the value zero to all the eigenvalues of the closed-loop plant matrix. Such an assignment clearly requires that $t = 1$, $m_1 = 4$, and $\lambda_1 = 0$ and therefore, in consonance with the results of Rosenbrock and Hayton (1977), that $k_1 = 1$ and $m_{11} = 4$. It follows (Porter and D'Azzo 1978) from equation (47) that

$$\ker \mathbf{S}_1(0) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{49}$$

and similarly (Porter and D'Azzo 1978) from equation (48) that

$$\ker \underset{\sim}{T}_1'(0) = \text{span}\left\{\begin{bmatrix}1/2\\1\\0\\0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\end{bmatrix}\right\} \quad . \tag{50}$$

It is evident from equation (49) that the closed-loop eigenvector $\underset{\sim}{\chi}_o^{(1,1)}$ corresponding to the eigenvalue $\lambda_1 = 0$ must be assigned to the subspace

$$\Sigma_1(0) = \text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{51}$$

whilst the associated string of generalised eigenvectors $\underset{\sim}{\chi}_o^{(2,1)}$, $\underset{\sim}{\chi}_o^{(3,1)}$, and $\underset{\sim}{\chi}_o^{(4,1)}$ must be generated in accordance with equations (31), and it is similarly evident from equation (50) that the closed-loop reciprocal eigenvector $\underset{\sim}{\phi}_o^{(4,1)}$ corresponding to the eigenvalue $\lambda_1 = 0$ must be assigned to the subspace

$$\Gamma_1(0) = \text{span}\left\{\begin{bmatrix}1/2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \tag{52}$$

whilst the associated string of generalised reciprocal eigenvectors $\underset{\sim}{\phi}_o^{(3,1)}$, $\underset{\sim}{\phi}_o^{(2,1)}$, and $\underset{\sim}{\phi}_o^{(1,1)}$ must be generated in accordance with equations (32). Since the vectors

$$\begin{bmatrix} \underset{\sim}{\chi}_o^{(1,1)} \\ \underset{\sim}{\omega}_o^{(1,1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \tag{53}$$

$$\begin{bmatrix} \underset{\sim}{\chi}_o^{(2,1)} \\ \underset{\sim}{\omega}_o^{(2,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1/2 \\ 0 \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} \underset{\sim}{\chi}_o^{(3,1)} \\ \underset{\sim}{\omega}_o^{(3,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} \underset{\sim}{\chi}_o^{(4,1)} \\ \underset{\sim}{\omega}_o^{(4,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \tag{56}$$

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(4,1)} \\ \underset{\sim}{\xi}_o^{(4,1)} \end{bmatrix} = \begin{bmatrix} -1/4 \\ -1/2 \\ 1/4 \\ -1/4 \\ 3/4 \\ -1/4 \\ -1/4 \end{bmatrix} \tag{57}$$

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(3,1)} \\ \underset{\sim}{\xi}_o^{(3,1)} \end{bmatrix} = \begin{bmatrix} -3/4 \\ -1/2 \\ 3/4 \\ 1/4 \\ 1 \\ -1/2 \\ -1/4 \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(2,1)} \\ \underset{\sim}{\xi}_o^{(2,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -7/4 \\ 3/4 \\ 1/4 \end{bmatrix} \tag{59}$$

and

$$\begin{bmatrix} \underset{\sim}{\phi}_o^{(1,1)} \\ \underset{\sim}{\xi}_o^{(1,1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{60}$$

are clearly such that

$$\underset{\sim}{\phi}_o^{(c,1)\prime}\underset{\sim}{\chi}_o^{(h,1)} = \delta_{ch} \qquad (c,h=1,2,3,4) \tag{61}$$

it follows from equations (36) and (37) that the required eigenstructure assignment is achieved by the output feedback matrix

$$\underset{\sim}{G}_1 = \begin{bmatrix} -7/4 \;, & 3/4 \;, & 1/4 \\ 1/2 \;, & -1/2 \;, & -1/2 \end{bmatrix} . \tag{62}$$

In view of equations (11), (12), (23), and (62), the corresponding dynamic compensator for the open-loop tracking system governed by equations (42) and (43) is governed by the respective state and output equations

$$w(k+1) = -1/2\ w(k) - 1/2\ e(k) - 1/2\ z(k) \tag{63}$$

and

$$u(k) = 1/3\ w(k) + 7/4\ e(k) + 3/4\ z(k) \tag{64}$$

so that the required error-actuated dynamic controller is characterised by the transfer function

$$T(z) = \bar{u}(z)/\bar{e}(z) = (7z^2-z-2)/(4z+2)(z-1) . \tag{65}$$

It can be readily verified that the closed-loop tracking system governed by equations (42), (43), (63), and (64) tracks any constant command input r(k) in the presence of any constant unmeasurable disturbance input d(k) in such a way that

$$e(k) = r(k) - y(k) = 0 \qquad (k=4,5,\ldots) \quad . \quad (66)$$

4. Conclusion

In this paper, the method of entire eigenstructure assignment has been applied to the design of linear multivariable discrete-time tracking systems incorporating error-actuated dynamic controllers. The theory developed in this paper is analogous to that developed by Porter and Bradshaw (1978) for continuous-time tracking systems. However, in this paper the theory has been extended in order to allow the assignment of confluent eigenvalues to the plant matrix of the closed-loop tracking system. It is therefore possible to apply the theory to the design of error-actuated dynamic controllers which eliminate completely the error between the command input vector and the output vector after a finite number of discrete-time intervals.

Acknowledgement

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005A.

References

BRADSHAW, A. and PORTER, B., 1975, Int. J. Systems Sci., 6, 117; 1978a, Ibid. (in the press); 1978b, Ibid. (in the press).

BRASCH, F. M. and PEARSON, J. B., 1970, I.E.E.E. Trans. autom Control, 15, 34.

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509; 1977, Ibid., 22, 458.

PORTER, B. and BRADSHAW, A., 1978, Int. J. Systems Sci., (in the press).

PORTER, B. and D'AZZO, J. J., 1978, Int. J. Control, (in the press).

ROSENBROCK, H. H. and HAYTON, G. E., 1977, Control Systems Centre Report No. 288, University of Manchester, Institute of Science and Technology.

APPENDIX 9

DESIGN OF LINEAR MULTIVARIABLE DISCRETE-TIME TRACKING SYSTEMS INCORPORATING ERROR-ACTUATED CONTROLLERS

DR A BRADSHAW AND PROFESSOR B PORTER

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, VOL 9, PP185-191, 1978)

ABSTRACT

In this paper, the controllability results of Bradshaw and Porter (1975) are applied to the design of linear mutlivariable discrete-time tracking systems incorporating plants with inaccessible states for which appropriate error-actuated controllers exist. The design method is illustrated by the presentation of the results of simulation studies.

1. INTRODUCTION

It is the purpose of this paper to develop a basis for the design of linear multivariable discrete-time tracking systems incorporating error-actuated controllers which is directly analogous to that developed for continuous-time tracking systems by Porter and Bradshaw (1976). The controllability results of Bradshaw and Porter (1975) are accordingly applied to the design of linear multivariable discrete-time tracking systems incorporating plants with inaccessible states for which appropriate error-actuated controllers exist and for which it is therefore possible to achieve satisfactory tracking behaviour without the need to incorporate observers in the manner of Bradshaw and Porter (1976). Such discrete-time tracking systems consist of a controllable nth-order linear multivariable plant governed by state and output equations of the respective forms

$$\underset{\sim}{x}(k+1) = \underset{\sim}{A}\underset{\sim}{x}(k) + \underset{\sim}{B}\underset{\sim}{u}(k) \tag{1}$$

and

$$\underset{\sim}{y}(k) = \underset{\sim}{C}\underset{\sim}{x}(k) \tag{2}$$

together with a controller which is required to cause the pxl output vector $\underset{\sim}{y}(k)$ of the plant to track a pxl command input vector $\underset{\sim}{v}(k)$ in the sense that

$$\lim_{k\to\infty} \underset{\sim}{e}(k) = \lim_{k\to\infty}\{\underset{\sim}{v}(k) - \underset{\sim}{y}(k)\} = \underset{\sim}{0} \tag{3}$$

for polynomial command inputs, i.e., for command inputs with the property that

$$\Delta^{(r)}\underset{\sim}{v}(k) = \underset{\sim}{0} \tag{4}$$

where

$$\left.\begin{array}{lcll} \Delta^{(1)}\underset{\sim}{v}(k) & = & \underset{\sim}{v}(k+1) - \underset{\sim}{v}(k) & , \\ \Delta^{(2)}\underset{\sim}{v}(k) & = & \Delta^{(1)}\underset{\sim}{v}(k+1) - \Delta^{(1)}\underset{\sim}{v}(k) & , \\ \cdots\cdots & & \cdots\cdots\cdots\cdots & , \\ \cdots\cdots & & \cdots\cdots\cdots\cdots & , \\ \cdots\cdots & & \cdots\cdots\cdots\cdots & , \\ \Delta^{(m)}\underset{\sim}{v}(k) & = & \Delta^{(m-1)}\underset{\sim}{v}(k+1) - \Delta^{(m-1)}\underset{\sim}{v}(k) & . \end{array}\right\} \tag{5}$$

It is important to note that, although these discrete-time tracking systems reduce to the error-actuated sampled-data servomechanisms of classical control theory (Bergen and Ragazzini 1954) in the special case $p = 1$, the design of error-actuated multivariable servomechanisms in the general case $p > 1$ is always non-trivial - and sometimes impossible - in view of the fact that the assignment of prescribed eigenvalue spectra by error-actuated controllers and by output-feedback controllers (Kimura 1975) are essentially equivalent.

2. THEORY

The first stage in the design of the required error-actuated controller for the plant governed by equations (1) and (2) involves the introduction (Bradshaw and Porter 1975) of a vector comparator and a series of r discrete-time vector integrators in order to generate the r vectors defined by the equations

$$\left.\begin{array}{llll} \underset{\sim}{z}_1(k+1) & = & \underset{\sim}{z}_1(k) + \underset{\sim}{e}(k) & , \\ \underset{\sim}{z}_2(k+1) & = & \underset{\sim}{z}_2(k) + \underset{\sim}{z}_1(k) & , \\ \underset{\sim}{z}_3(k+1) & = & \underset{\sim}{z}_3(k) + \underset{\sim}{z}_2(k) & , \\ \cdots\cdots & & \cdots\cdots & , \\ \cdots\cdots & & \cdots\cdots & , \\ \cdots\cdots & & \cdots\cdots & , \\ \underset{\sim}{z}_r(k+1) & = & \underset{\sim}{z}_r(k) + \underset{\sim}{z}_{r-1}(k) & . \end{array}\right\} \qquad (6)$$

It is then evident from equations (1), (2), and (6) that the open-loop tracking system is governed by a state equation of the form

$$\begin{bmatrix} \underset{\sim}{x}(k+1) \\ \underset{\sim}{z}_1(k+1) \\ \underset{\sim}{z}_2(k+1) \\ \underset{\sim}{z}_3(k+1) \\ \cdots \\ \cdots \\ \cdots \\ \underset{\sim}{z}_{r-1}(k+1) \\ \underset{\sim}{z}_r(k+1) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}, & \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ -\underset{\sim}{C}, & \underset{\sim}{I}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{I}, & \underset{\sim}{I}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \underset{\sim}{I}, & \ldots, & \underset{\sim}{O}, & \underset{\sim}{O} \\ \cdots & & & & & \\ \cdots & & & & & \\ \cdots & & & & & \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{I}, & \underset{\sim}{O} \\ \underset{\sim}{O}, & \underset{\sim}{O}, & \underset{\sim}{O}, & \ldots, & \underset{\sim}{I}, & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(k) \\ \underset{\sim}{z}_1(k) \\ \underset{\sim}{z}_2(k) \\ \underset{\sim}{z}_3(k) \\ \cdots \\ \cdots \\ \cdots \\ \underset{\sim}{z}_{r-1}(k) \\ \underset{\sim}{z}_r(k) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{B} \\ \underset{\sim}{O} \\ \underset{\sim}{O} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{u}(k) + \begin{bmatrix} \underset{\sim}{O} \\ \underset{\sim}{I} \\ \underset{\sim}{O} \\ \underset{\sim}{O} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{v}(k) \quad . \quad (7)$$

The second stage in the design of the error-actuated controller involves the introduction of (r+1) vector feedback loops in order to generate the mxl input vector $\underset{\sim}{u}(k)$ according to the error control-law equation

$$\underset{\sim}{u}(k) = \underset{\sim}{K}_o \underset{\sim}{e}(k) + \sum_{i=1}^{r} \underset{\sim}{K}_i \underset{\sim}{z}_i(k) \quad , \tag{8}$$

where the $\underset{\sim}{K}_i$ (i=0,1,2,...,r) are mxp feedback matrices. It is then evident from equations (7) and (8) that the closed-loop tracking system is governed by a state equation of the form

$$\begin{bmatrix} \underset{\sim}{x}(k+1) \\ \underset{\sim}{z}_1(k+1) \\ \underset{\sim}{z}_2(k+1) \\ \underset{\sim}{z}_3(k+1) \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_{r-1}(k+1) \\ \underset{\sim}{z}_r(k+1) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{K}_o\underset{\sim}{C} , & \underset{\sim}{B}\underset{\sim}{K}_1 , & \underset{\sim}{B}\underset{\sim}{K}_2 , & \ldots , & \underset{\sim}{B}\underset{\sim}{K}_{r-1} , & \underset{\sim}{B}\underset{\sim}{K}_r \\ -\underset{\sim}{C} , & \underset{\sim}{I} , & \underset{\sim}{O} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{I} , & \underset{\sim}{I} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \underset{\sim}{I} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \underset{\sim}{O} , & \ldots , & \underset{\sim}{O} , & \underset{\sim}{O} \\ \underset{\sim}{O} , & \underset{\sim}{O} , & \underset{\sim}{O} , & \ldots , & \underset{\sim}{I} , & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}(k) \\ \underset{\sim}{z}_1(k) \\ \underset{\sim}{z}_2(k) \\ \underset{\sim}{z}_3(k) \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \underset{\sim}{z}_{r-1}(k) \\ \underset{\sim}{z}_r(k) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{B}\underset{\sim}{K}_o \\ \underset{\sim}{I} \\ \underset{\sim}{O} \\ \underset{\sim}{O} \\ \cdots \\ \cdots \\ \cdots \\ \underset{\sim}{O} \\ \underset{\sim}{O} \end{bmatrix} \underset{\sim}{v}(k) \quad . \tag{9}$$

In the case of px1 vector polynomial command inputs of the form

$$\underset{\sim}{v}(k) = \sum_{i=1}^{r} \underset{\sim}{\alpha}_{i-1} k^{i-1} \quad , \tag{10}$$

it is clear by differencing equation (8) (r-1) times that the closed-loop tracking system will behave so that

$$\lim_{k\to\infty} \Delta^{(r)} \underset{\sim}{z}_r(k) = \underset{\sim}{O} \tag{11}$$

and therefore, in view of equations (6), so that equation (3) will be satisfied if the error control law (8) can be synthesised in such a way that all the eigenvalues of the plant matrix of the closed-loop tracking system governed by equation (9) are assigned to any desired locations within the unit circle.

However, in view of the presence of the sub-matrix $(\underset{\sim}{A}-\underset{\sim}{B}\underset{\sim}{K}_o\underset{\sim}{C})$ in the plant matrix of the closed-loop tracking system, an error control law of this class will not always exist (Kimura 1975) even if the open-loop tracking system governed by equation (7) is controllable in the sense that (Bradshaw and Porter 1975)

$$\text{rank} \begin{bmatrix} \underset{\sim}{B} , & \underset{\sim}{A}-\underset{\sim}{I} \\ \underset{\sim}{O} , & -\underset{\sim}{C} \end{bmatrix} = n + p \quad . \tag{12}$$

It is nevertheless evident that such a control law will certainly exist if, for example, a stabilising state-feedback control law of the form

$$\underset{\sim}{u}(k) = \underset{\sim}{K}\underset{\sim}{x}(k) + \sum_{i=1}^{r} \underset{\sim}{K}_i \underset{\sim}{z}_i(k) \tag{13}$$

can be synthesised such that there exists a matrix $\underset{\sim}{K}_o$ with the special property that

$$-\underset{\sim}{K}_o\underset{\sim}{C} = \underset{\sim}{K} \quad . \tag{14}$$

However, the existence of a stabilising error control law of the form (8) can in general only be investigated systematically by using decision methods in the manner of Anderson, Bose, and Jury (1975).

3. Illustrative Example

The theory presented in Section 2 can be conveniently illustrated by designing an error-actuated controller which will cause the output of the controllable second-order linear plant governed by the respective state and output equations (Bradshaw and Porter 1975)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & , & 1 \\ -6 & , & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & , & 1 \\ 0 & , & 2 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & , & 0 \\ -1 & , & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{16}$$

to track the command input vector

$$\begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} \qquad (0 \le k < \infty) \quad . \tag{17}$$

In this case it is clear that the command input is of the form (10) with $r = 2$, so that the state equation (7) of the open-loop tracking system assumes the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ z_{11}(k+1) \\ z_{21}(k+1) \\ z_{12}(k+1) \\ z_{22}(k+1) \end{bmatrix} = \begin{bmatrix} 0, & 1, & 0, & 0, & 0, & 0 \\ -6, & 5, & 0, & 0, & 0, & 0 \\ -1, & 0, & 1, & 0, & 0, & 0 \\ 1, & -1, & 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1, & 0, & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ z_{11}(k) \\ z_{21}(k) \\ z_{12}(k) \\ z_{22}(k) \end{bmatrix}$$

$$+ \begin{bmatrix} 1, & 1 \\ 0, & 2 \\ 0, & 0 \\ 0, & 0 \\ 0, & 0 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} 0, & 0 \\ 0, & 0 \\ 1, & 0 \\ 0, & 1 \\ 0, & 0 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} 2k \\ k \end{bmatrix} \qquad (18)$$

where $z_{ij}(k)$ is the ith element of the vector $\underset{\sim}{z}_j(k)$. Since (Bradshaw and Porter 1975)

$$\text{rank} \begin{bmatrix} 1, & 1, & -1, & 1 \\ 0, & 2, & -6, & 4 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 1, & -1 \end{bmatrix} = 4 \quad , \qquad (19)$$

the controllability condition (12) is satisfied in this case: it is therefore certainly possible to synthesise a state-feedback control law of the form (13) and a corresponding error control law of the form (8) such that the eigenvalues of the plant matrix of the resulting closed-loop tracking system assume arbitrary values since also the output matrix in equation (14) is invertible. In the particular case when these eigenvalues are all assigned the value zero by the implementation of the error control law

$$\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = \begin{bmatrix} 2.5 , & -2.5 , & 1.5 , & -1.5 , & 0.5 , & -0.5 \\ 0.5 , & 3.5 , & 1.5 , & 1.5 , & 0.5 , & 0.5 \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \\ z_{11}(k) \\ z_{21}(k) \\ z_{12}(k) \\ z_{22}(k) \end{bmatrix} \tag{20}$$

the behaviour of the initially quiescent tracking system is as shown by the full lines in Figs 1 and 2 : it is evident from Fig 1 that

$$\lim_{k\to\infty} e_1(k) = \lim_{k\to\infty}\{v_1(k) - y_1(k)\} = \lim_{k\to\infty}\{v_1(k) - x_1(k)\} = 0 \tag{21a}$$

and from Fig 2 that

$$\lim_{k\to\infty} e_2(k) = \lim_{k\to\infty}\{v_2(k) - y_2(k)\} = \lim_{k\to\infty}\{v_2(k) + x_1(k) - x_2(k)\} = 0 \tag{21b}$$

as required.

The corresponding behaviour of the initially quiescent tracking system in case a state-feedback control law is implemented (Bradshaw and Porter 1975) is as shown by the dotted lines in Figs 1 and 2: it is again evident from Figs 1 and 2 that equations (21a) and (21b) are satisfied, but that the transient behaviour of the tracking system incorporating a state-feedback controller (Bradshaw and Porter 1975) is slower and less oscillatory than the corresponding behaviour of the tracking system incorporating an error-actuated controller.

4. CONCLUSIONS

In this paper, the simple matricial methods developed by Bradshaw and Porter (1975) for the design of linear multivariable discrete-time tracking systems for plants with accessible states have been applied to the design of linear multivariable discrete-time tracking systems incorporating plants with inaccessible states for which appropriate error-actuated controllers exist. The results of simulation studies have been presented which indicate that the transient behaviour of tracking systems of the latter class is faster but more oscillatory than the corresponding behaviour of tracking systems incorporating state-feedback controllors.

ACKNOWLEDGEMENT

This research was supported in part by the Air Force Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005A.

REFERENCES

ANDERSON, B. D. O, BOSE, N. K., and JURY, E. I., 1975, I.E.E.E. Trans. Autom. Control, 20, 53.

BERGEN, A. R. and RAGAZZINI, J. R., 1954, Trans. Am. Inst. elect. Engrs, 73, 236.

BRADSHAW, A. and PORTER, B., 1975, Int. J. Systems Sci., 6, 117.

BRADSHAW, A. and PORTER, B., 1976, Int. J. Control, 24, 275.

KIMURA, A., 1975, I.E.E.E. Trans. Autom. Control, 20, 509.

PORTER, B. and BRADSHAW, A., 1976, Int. J. Systems Sci., 7, 943.

APPENDIX 10

SYNTHESIS OF OUTPUT-FEEDBACK CONTROL LAWS FOR LINEAR MULTIVARIABLE CONTINUOUS-TIME SYSTEMS

DR A BRADSHAW*, DR L R FLETCHER+, AND PROFESSOR B PORTER*

*DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING

+DEPARTMENT OF MATHEMATICS

UNIVERSITY OF SALFORD

SALFORD M5 4WT

ENGLAND

(INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, VOL 9, PP 1331-1340, 1978)

ABSTRACT

It is known (Porter and Bradshaw 1978a) that, in the case of self-conjugate distinct eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the eigenvectors and reciprocal eigenvectors lie in well-defined subspaces. In this paper, a technique is presented which can be used to select the eigenvectors and reciprocal eigenvectors from these subspaces in the case of appropriately augmented (Kimura 1975) controllable and observable continuous-time systems. This technique is ideally suited to digital-computer implementation and therefore greatly facilitates the synthesis of both static (Porter and Bradshaw 1978a) and dynamic (Porter and Bradshaw 1978b) output-feedback controllers.

1. INTRODUCTION

It has been shown (Porter and Bradshaw 1978a,b) that the method of entire eigenstructure assignment can be applied to the design of output-feedback controllers for multivariable linear continuous-time systems governed by state and output equations of the respective forms

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}\underset{\sim}{x}(t) + \underset{\sim}{B}\underset{\sim}{u}(t) \tag{1}$$

and

$$\underset{\sim}{y}(t) = \underset{\sim}{C}\underset{\sim}{x}(t) \quad , \tag{2}$$

where $\underset{\sim}{A} \in R^{nxn}$, $\underset{\sim}{B} \in R^{nxm}$, $\underset{\sim}{C} \in R^{pxn}$, rank $\underset{\sim}{B}$ = m, and rank $\underset{\sim}{C}$ = p. Thus, if output feedback is applied to the system governed by the state equation (1) in accordance with the control-law equation

$$\underset{\sim}{u}(t) = \underset{\sim}{G}\underset{\sim}{y}(t) \tag{3}$$

and the output-feedback matrix $\underset{\sim}{G} \in R^{mxp}$ is such that the closed-loop plant matrix $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C})$ has the self-conjugate distinct eigenvalue spectrum $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$, then the corresponding eigenvector and reciprocal eigenvector sets $\{\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_n\}$ and $\{\underset{\sim}{v}_1,\underset{\sim}{v}_2,\ldots,\underset{\sim}{v}_n\}$ must be such that

$$\begin{bmatrix} \underset{\sim}{u}_j \\ \underset{\sim}{w}_j \end{bmatrix} \in \ker[\underset{\sim}{A}-\lambda_j\underset{\sim}{I} \ , \ \underset{\sim}{B}] \qquad (j=1,2,\ldots,n) \quad , \tag{4}$$

$$\begin{bmatrix} \underset{\sim}{v}_i \\ \underset{\sim}{z}_i \end{bmatrix} \in \ker[\underset{\sim}{A}'-\lambda_i\underset{\sim}{I} \ , \ \underset{\sim}{C}'] \qquad (i=1,2,\ldots,n) \quad , \tag{5}$$

and

$$\underset{\sim}{v}_i' \underset{\sim}{u}_j = \delta_{ij} \qquad (i,j=1,2,\ldots,n) \quad . \tag{6}$$

The output-feedback matrix is then given by the formulae

$$\underset{\sim}{G} = [\underset{\sim}{w}_1 \, , \, \underset{\sim}{w}_2 \, , \, \ldots \, , \, \underset{\sim}{w}_p][\underset{\sim}{C}\underset{\sim}{u}_1 \, , \, \underset{\sim}{C}\underset{\sim}{u}_2 \, , \, \ldots \, , \, \underset{\sim}{C}\underset{\sim}{u}_p]^{-1} \tag{7}$$

and

$$\underset{\sim}{G}' = [\underset{\sim}{z}_1 \, , \, \underset{\sim}{z}_2 \, , \, \ldots \, , \, \underset{\sim}{z}_m][\underset{\sim}{B}'\underset{\sim}{v}_1 \, , \, \underset{\sim}{B}'\underset{\sim}{v}_2 \, , \, \ldots \, , \, \underset{\sim}{B}'\underset{\sim}{v}_m]^{-1} \quad , \tag{8}$$

where, in this paper, the state and output equations (1) and (2) represent appropriately augmented (Kimura 1975) controllable and observable continuous-time systems.

Thus, the synthesis of the output-feedback control law (3) requires the selection of linearly independent sets of vectors $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_n\}$ and $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$ from the respective subspaces defined by relations (4) and (5) which satisfy the orthogonality condition (6). It is shown that this selection can be effected by performing restricted elementary row and column operations on matrices formed from the spanning vectors of these subspaces, and that the resulting synthesis procedure is therefore ideally suited to digital computer implementation.

2. THEORY

The first stage in the synthesis of the control law is clearly the determination (Porter and D'Azzo 1977) of the n kernels

$$S(\lambda_j) = \ker\left[\underset{\sim}{A} - \lambda_j \underset{\sim}{I} \ , \ \underset{\sim}{B}\right] = \text{span}\left\{\begin{bmatrix} \underset{\sim}{\chi}_k(\lambda_j) \\ \underset{\sim}{\omega}_k(\lambda_j) \end{bmatrix} : k=1,\ldots,m\right\}$$

$$(j=1,2,\ldots,n) \qquad (9)$$

and the n kernels

$$T'(\lambda_i) = \ker\left[\underset{\sim}{A}' - \lambda \underset{\sim}{I} \ , \ \underset{\sim}{C}'\right] = \text{span}\left\{\begin{bmatrix} \underset{\sim}{\phi}_k(\lambda_i) \\ \underset{\sim}{\zeta}_k(\lambda_i) \end{bmatrix} : k=1,\ldots,p\right\} .$$

$$(i=1,2,\ldots,n) \qquad (10)$$

It follows from relation (4) and equation (9) that

$$\underset{\sim}{u}_j \in U(\lambda_j) = \text{span}\ \{\underset{\sim}{\chi}_k(\lambda_j) : k=1,\ldots,m\}$$

$$(j=1,2,\ldots,n) \qquad (11)$$

and from relation (5) and equation (10) that

$$\underset{\sim}{v}_i \in V(\lambda_i) = \text{span}\ \{\underset{\sim}{\phi}_k(\lambda_i) : k=1,\ldots,p\}$$

$$(i=1,2,\ldots,n) \quad , \qquad (12)$$

that is

$$\underset{\sim}{u}_j = \underset{\sim}{X}(\lambda_j)\underset{\sim}{n}_j \qquad (j=1,2,\ldots,n) \qquad (13)$$

and

$$\underset{\sim}{v}_i' = \underset{\sim}{\ell}_i' \underset{\sim}{\Phi}'(\lambda_i) \qquad (i=1,2,\ldots,n) \quad , \qquad (14)$$

where

$$\underset{\sim}{X}(\lambda_j) = \left[\underset{\sim}{\chi}_1(\lambda_j) \ , \ \underset{\sim}{\chi}_2(\lambda_j) \ , \ \ldots \ , \ \underset{\sim}{\chi}_m(\lambda_j)\right]$$

$$(j=1,2,\ldots,n) \qquad (15)$$

and

$$\underset{\sim}{\phi}(\lambda_i) = [\underset{\sim}{\phi}_1(\lambda_i) \; , \; \underset{\sim}{\phi}_2(\lambda_i) \; , \; \ldots \; , \; \underset{\sim}{\phi}_p(\lambda_i)] \quad ,$$

$$(i=1,2,\ldots,n) \qquad (16)$$

where $\underset{\sim}{n}_j$ is an mxl vector and $\underset{\sim}{\ell}_i'$ is an lxp vector. Equations (13) and (14) can be more conveniently expressed in the form

$$\underset{\sim}{U}_j = \underset{\sim}{X}(\lambda_j)\underset{\sim}{N}_j \qquad (j=1,2,\ldots,n) \qquad (17)$$

and

$$\underset{\sim}{V}_i' = \underset{\sim}{L}_i\underset{\sim}{\phi}'(\lambda_i) \qquad (i=1,2,\ldots,n) \quad , \; (18)$$

where $\underset{\sim}{u}_j$ and $\underset{\sim}{n}_j$ are the first columns of the nxm matrix $\underset{\sim}{U}_j$ and the mxm matrix $\underset{\sim}{N}_j$ respectively, and $\underset{\sim}{v}_i'$ and $\underset{\sim}{\ell}_i'$ are the first rows of the pxn matrix $\underset{\sim}{V}_i'$ and the pxp matrix $\underset{\sim}{L}_i$ respectively. The orthogonality condition (7) then requires that the element m_{ij} in the first row and the first column of each of the pxm matrices

$$\underset{\sim}{M}_{ij} = \underset{\sim}{V}_i'\underset{\sim}{U}_j = \underset{\sim}{L}_i\underset{\sim}{\phi}'(\lambda_i)\underset{\sim}{X}(\lambda_j)\underset{\sim}{N}_j = \underset{\sim}{L}_i\underset{\sim}{M}_{ij}^{(o)}\underset{\sim}{N}_j$$

$$(i,j=1,2,\ldots,n) \qquad (19)$$

be such that

$$m_{ij} = \delta_{ij} \qquad (i,j=1,2,\ldots,n) \; , \; (20)$$

where

$$\underset{\sim}{M}_{ij}^{(o)} = \underset{\sim}{\phi}'(\lambda_i)\underset{\sim}{X}(\lambda_j) \qquad (i,j=1,2,\ldots,n) \; . \; (21)$$

It is evident that the condition (20) can be satisfied by performing restricted elementary row and column operations on the npxnm matrix

$$\underset{\sim}{M}^{(o)} = \left[\underset{\sim}{M}_{ij}^{(o)}\right] \qquad (i,j=1,2,\ldots,n) \qquad (22)$$

according to the equation

$$\underset{\sim}{M} = \underset{\sim}{L}\underset{\sim}{M}^{(o)}\underset{\sim}{N} \qquad , \qquad (23)$$

where the npxnm matrix

$$\underset{\sim}{M} = \left[\underset{\sim}{M}_{ij}\right] \qquad (i,j=1,2,\ldots,n), \qquad (24)$$

the npxnp matrix

$$\underset{\sim}{L} = \mathrm{diag}\left[\underset{\sim}{L}_i\right] \qquad (i=1,2,\ldots,n) \quad , \qquad (25)$$

and the nmxnm matrix

$$\underset{\sim}{N} = \mathrm{diag}\left[\underset{\sim}{N}_j\right] \qquad (j=1,2,\ldots,n) \quad . \qquad (26)$$

These computations can be conveniently organized in the following steps in view of the results of Kimura (1975):

(1) Set $\underset{\sim}{M}^{(o)} = \left[\underset{\sim}{M}_{ij}^{(o)}\right]$, $\underset{\sim}{L}^{(o)} = \underset{\sim}{I}_{np}$, and $\underset{\sim}{N}^{(o)} = \underset{\sim}{I}_{nm}$;

(2) By restricted elementary column operations on $\underset{\sim}{M}_{ij}^{(o)}$ and $\underset{\sim}{N}_j^{(o)}$ $(i=1,2,\ldots,n; j=1,2,\ldots,n-m)$ determine $\underset{\sim}{u}_j \in \mathcal{U}(\lambda_j)$ $(j=1,2,\ldots,n-m)$ such that $\{\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_{n-m}\}$ is a linearly independent set, $\underset{\sim}{M}^{(o)} \rightarrow \underset{\sim}{M}^{(1)}$, $\underset{\sim}{N}^{(o)} \rightarrow \underset{\sim}{N}^{(1)}$, and

$$\underset{\sim}{M}^{(1)} = \underset{\sim}{M}^{(o)}\underset{\sim}{N}^{(1)} \qquad ; \qquad (27)$$

(3) By restricted elementary row operations on $\underset{\sim}{M}_{ij}^{(1)}$ and $\underset{\sim}{L}_i^{(o)}$ $(i=n-m+1,n-m+2,\ldots,n; j=1,2,\ldots,n)$ determine $\underset{\sim}{v}_i' \in \mathcal{V}'(\lambda_i)$ $(i=n-m+1,n-m+2,\ldots,n)$ such that $\underset{\sim}{v}_i'\underset{\sim}{u}_j = m_{ij}^{(2)} = 0$ $(i=n-m+1,n-m+2,\ldots,n; j=1,2,\ldots,n-m)$, $\underset{\sim}{M}^{(1)} \rightarrow \underset{\sim}{M}^{(2)}$,

$\underset{\sim}{L}^{(o)} \rightarrow \underset{\sim}{L}^{(1)}$, and

$$\underset{\sim}{M}^{(2)} = \underset{\sim}{L}^{(1)}\underset{\sim}{M}^{(o)}\underset{\sim}{N}^{(1)} \quad ; \tag{28}$$

(4) By restricted elementary column operations on $\underset{\sim}{M}_{ij}^{(2)}$ and $\underset{\sim}{N}_{j}^{(1)}$ $(i=1,2,\ldots,n; j=n-m+1,n-m+2,\ldots,n)$ determine $\underset{\sim}{u}_j \in U(\lambda_j)$ $(j=n-m+1,n-m+2,\ldots,n)$ such that $\underset{\sim}{v}_i'\underset{\sim}{u}_j = m_{ij}^{(3)} = 0$ $(i=n-m+1,n-m+2,\ldots,n; j=n-m+1,n-m+2,\ldots,n; i\neq j)$, $\underset{\sim}{M}^{(2)} \rightarrow \underset{\sim}{M}^{(3)}$, $\underset{\sim}{N}^{(1)} \rightarrow \underset{\sim}{N}^{(2)}$, and

$$\underset{\sim}{M}^{(3)} = \underset{\sim}{L}^{(1)}\underset{\sim}{M}^{(o)}\underset{\sim}{N}^{(2)} \quad ; \tag{29}$$

(5) By restricted elementary row operations on $\underset{\sim}{M}_{ij}^{(3)}$ and $\underset{\sim}{L}_{i}^{(1)}$ $(i=1,2,\ldots,n-m; j=1,2,\ldots,n)$ determine $\underset{\sim}{v}_i' \in V'(\lambda_i)$ $(i=1,2,\ldots,n-m)$ such that $\underset{\sim}{v}_i'\underset{\sim}{u}_j = m_{ij}^{(4)} = 0$ $(i=1,2,\ldots,n-m; j=n-m+1,n-m+2,\ldots,n)$, $\underset{\sim}{M}^{(3)} \rightarrow \underset{\sim}{M}^{(4)}$, $\underset{\sim}{L}^{(1)} \rightarrow \underset{\sim}{L}^{(2)}$, and

$$\underset{\sim}{M}^{(4)} = \underset{\sim}{L}^{(2)}\underset{\sim}{M}^{(o)}\underset{\sim}{N}^{(2)} \quad ; \tag{30}$$

(6) By restricted elementary row or column operations on $\underset{\sim}{M}_{ij}^{(4)}$ $(i=1,2,\ldots,n; j=1,2,\ldots,n)$ normalize $\underset{\sim}{v}_i'$ or $\underset{\sim}{u}_i$ such that $\underset{\sim}{v}_i'\underset{\sim}{u}_i = m_{ii}^{(5)} = 1$ $(i=1,2,\ldots,n)$, $\underset{\sim}{M}^{(4)} \rightarrow \underset{\sim}{M}^{(5)}$, $\underset{\sim}{L}^{(2)} \rightarrow \underset{\sim}{L}^{(3)}$, $\underset{\sim}{N}^{(2)} \rightarrow \underset{\sim}{N}^{(3)}$, and

$$\underset{\sim}{M}^{(5)} = \underset{\sim}{L}^{(3)}\underset{\sim}{M}^{(o)}\underset{\sim}{N}^{(3)} = \underset{\sim}{M} = \underset{\sim}{L}\underset{\sim}{M}^{(o)}\underset{\sim}{N} \quad ; \tag{31}$$

(7) Compute the $\underset{\sim}{u}_j$ $(j=1,2,\ldots,n)$ using equation (17) and compute the $\underset{\sim}{v}_i'$ $(i=1,2,\ldots,n)$ using equation (18).

In certain pathological cases (Kimura 1975), special spectra exist for which no corresponding output feedback matrix exists and for which this computational procedure therefore fails: in such cases, however, it is only necessary slightly to perturb the spectra in order to obtain solutions. It is also possible for this computational procedure to fail for certain pathological choices of $\underset{\sim}{u}_j \in \mathcal{U}(\lambda_j)$ $(j=1,2,\ldots,n-m)$ in step (2): in such cases, however, it is only necessary slightly to perturb the $\underset{\sim}{u}_j \in \mathcal{U}(\lambda_j)$ $(j=1,2,\ldots,n-m)$.

3. Illustrative Example

The procedure can be conveniently illustrated by the synthesis of an output-feedback control law for the continuous-time system governed by the respective state and output equations (Porter and Bradshaw 1978b)

$$\dot{\underset{\sim}{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{u}(t) \qquad (32)$$

and

$$\underset{\sim}{y}(t) = \begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{bmatrix} \underset{\sim}{x}(t) \tag{33}$$

such that the eigenvalue spectrum of the closed-loop plant matrix is

$$\sigma(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}\underset{\sim}{C}) = \{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} = \{-1,-2,-3,-4\} \quad . \tag{34}$$

Indeed, it is evident from equations (9), (10), (32), (33), and (34) that

$$S(-1) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \quad , \tag{35}$$

$$S(-2) = \text{span}\left\{ \begin{bmatrix} -2 \\ 4 \\ -1 \\ 0 \\ -10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\} \quad , \tag{36}$$

$$S(-3) = \text{span}\left\{\begin{bmatrix}-3\\9\\-1\\0\\-33\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\-3\end{bmatrix}\right\} , \tag{37}$$

and

$$S(-4) = \text{span}\left\{\begin{bmatrix}-4\\16\\-1\\0\\-76\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\-4\end{bmatrix}\right\} , \tag{38}$$

and that

$$T'(-1) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\-1\end{bmatrix}\right\} , \tag{39}$$

$$T'(-2) = \text{span}\left\{\begin{bmatrix}-3\\1\\0\\0\\5\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\-2\end{bmatrix}\right\}, \qquad (40)$$

$$T'(-3) = \text{span}\left\{\begin{bmatrix}-4\\1\\0\\0\\11\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\-3\end{bmatrix}\right\}, \qquad (41)$$

and

$$T'(-4) = \text{span}\left\{\begin{bmatrix}-5\\1\\0\\0\\19\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\-4\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\-4\end{bmatrix}\right\}. \qquad (42)$$

It is evident from equations (11), (35), (36), (37), and (38) that the closed-loop eigenvectors corresponding to the eigenvalue spectrum (34) must be assigned to the respective subspaces

$$\mathcal{U}(-1) = \text{span}\left\{\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{43}$$

$$\mathcal{U}(-2) = \text{span}\left\{\begin{bmatrix}-2\\4\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{44}$$

$$\mathcal{U}(-3) = \text{span}\left\{\begin{bmatrix}-3\\9\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{45}$$

and

$$\mathcal{U}(-4) = \text{span}\left\{\begin{bmatrix}-4\\16\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{46}$$

and it is similarly evident from equations (12), (39), (40), (41), and (42) that the closed-loop reciprocal eigenvectors corresponding to the eigenvalue spectrum (34) must be assigned to the respective subspaces

$$\mathcal{V}(-1) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{47}$$

$$V(-2) = \text{span}\left\{\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{48}$$

$$V(-3) = \text{span}\left\{\begin{bmatrix}-4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} , \tag{49}$$

and

$$V(-4) = \text{span}\left\{\begin{bmatrix}-5\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} . \tag{50}$$

It therefore follows from equations (11), (12), (15), (16), (21), and (22) that

$$\underset{\sim}{M}^{(o)} = \left[\begin{array}{cc|cc|cc|cc}
3 , & 0 & 8 , & 0 & 15 , & 0 & 24 , & 0 \\
-1 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & 0 , & 1 & 0 , & 1 & 0 , & 1 \\
\hline
4 , & 0 & 10 , & 0 & 18 , & 0 & 28 , & 0 \\
-1 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & 0 , & 1 & 0 , & 1 & 0 , & 1 \\
\hline
5 , & 0 & 12 , & 0 & 21 , & 0 & 32 , & 0 \\
-1 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & 0 , & 1 & 0 , & 1 & 0 , & 1 \\
\hline
6 , & 0 & 14 , & 0 & 24 , & 0 & 36 , & 0 \\
-1 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & 0 , & 1 & 0 , & 1 & 0 , & 1
\end{array}\right] . \tag{51}$$

Then, by performing the restricted elementary column operations corresponding to setting

$$\underset{\sim}{n}_1 = \begin{bmatrix} 5 \\ -7 \end{bmatrix} \tag{52}$$

and

$$\underset{\sim}{n}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix} , \tag{53}$$

it follows from equation (27) that

$$\underset{\sim}{M}^{(1)} = \left[\begin{array}{c:c:c:c} 15\ ,\ 0 & 8\ ,\ 0 & 15\ ,\ 0 & 24\ ,\ 0 \\ -5\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 \\ -7\ ,\ 1 & -7\ ,\ 1 & 0\ ,\ 1 & 0\ ,\ 1 \\ \hdashline 20\ ,\ 0 & 10\ ,\ 0 & 18\ ,\ 0 & 28\ ,\ 0 \\ -5\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 \\ -7\ ,\ 1 & -7\ ,\ 1 & 0\ ,\ 1 & 0\ ,\ 1 \\ \hdashline 25\ ,\ 0 & 12\ ,\ 0 & 21\ ,\ 0 & 32\ ,\ 0 \\ -5\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 \\ -7\ ,\ 1 & -7\ ,\ 1 & 0\ ,\ 1 & 0\ ,\ 1 \\ \hdashline 30\ ,\ 0 & 14\ ,\ 0 & 24\ ,\ 0 & 36\ ,\ 0 \\ -5\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 & -1\ ,\ 0 \\ -7\ ,\ 1 & -7\ ,\ 1 & 0\ ,\ 1 & 0\ ,\ 1 \end{array}\right] , \tag{54}$$

from equation (28) that

$$
\underset{\sim}{M}^{(2)} = \left[\begin{array}{cc|cc|cc|cc}
15 , & 0 & 8 , & 0 & 15 , & 0 & 24 , & 0 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
-7 , & 1 & -7 , & 1 & 0 , & 1 & 0 , & 1 \\
\hline
20 , & 0 & 10 , & 0 & 18 , & 0 & 28 , & 0 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
-7 , & 1 & -7 , & 1 & 0 , & 1 & 0 , & 1 \\
\hline
0 , & 5/4 & 0 , & 5/4 & 71/4 , & 5/4 & 115/4 , & 5/4 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & 7/5 , & 1 & 7/5 , & 1 \\
\hline
0 , & 10/7 & 0 , & 10/7 & 20 , & 10/7 & 32 , & 10/7 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & 7/5 , & 1 & 7/5 , & 1
\end{array}\right] , \tag{55}
$$

from equation (29) that

$$
\underset{\sim}{M}^{(3)} = \left[\begin{array}{cc|cc|cc|cc}
15 , & 0 & 8 , & 0 & 15 , & 0 & 24 , & 0 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
-7 , & 1 & -7 , & 1 & -14 , & 1 & -23 , & 1 \\
\hline
20 , & 0 & 10 , & 0 & 18 , & 0 & 28 , & 0 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
-7 , & 1 & -7 ,\ , & 1 & -14 , & 1 & -23 , & 1 \\
\hline
0 , & 5/4 & .0 , & 5/4 & .\,1/4 , & 5/4 & 0 , & 5/4 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1 \\
\hline
0 , & 10/7 & 0 , & 10/4 & 0 , & 10/7 & -6/7 , & 10/7 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1
\end{array}\right] , \tag{56}
$$

from equation (30) that

$$
\underset{\sim}{M}^{(4)} = \left[\begin{array}{cc|cc|cc|cc}
3 , & 1 & 0 , & 1 & 0 , & 1 & 0 , & 1 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
63 , & 1 & 7 , & 1 & 0 , & 1 & -9 , & 1 \\
\hline
0 , & 10/9 & -2/9 , & 10/9 & 0 , & 10/9 & 0 , & 10/9 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
63 , & 1 & 7 , & 1 & 0 , & 1 & -9 , & 1 \\
\hline
0 , & 5/4 & 0 , & 5/4 & 1/4 , & 5/4 & 0 , & 5/4 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1 \\
\hline
0 , & 10/7 & 0 , & 10/7 & 0 , & 10/7 & -6/7 , & 10/7 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1
\end{array}\right] , \qquad (57)
$$

and from equation (31) that

$$
\underset{\sim}{M} = \left[\begin{array}{cc|cc|cc|cc}
1 , & 1/3 & 0 , & 1/3 & 0 , & 1/3 & 0 , & 1/3 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
63 , & 1 & 7 , & 1 & 0 , & 1 & -9 , & 1 \\
\hline
0 , & -5 & 1 , & -5 & 0 , & -5 & 0 , & -5 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
63 , & 1 & 7 , & 1 & 0 , & 1 & -9 , & 1 \\
\hline
0 , & 5 & 0 , & 5 & 1 , & 5 & 0 , & 5 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1 \\
\hline
0 , & -5/3 & 0 , & -5/3 & 0 , & -5/3 & 1 , & -5/3 \\
-5 , & 0 & -1 , & 0 & -1 , & 0 & -1 , & 0 \\
0 , & 1 & -28/5 , & 1 & -63/5 , & 1 & -108/5 , & 1
\end{array}\right] , \qquad (58)
$$

$$\underset{\sim}{L} = \mathrm{diag}\left[\begin{array}{c|c|c|c} 1/3,1/3,1/3 & -9/2,-11,-5 & 4,\ 13\ ,5 & -7/6,-14/3,-5/3 \\ 0\ ,\ 1\ ,\ 0 & 0\ ,\ 1\ ,0 & 0,\ 1\ ,0 & 0\ ,\ 1\ ,\ 0 \\ 0\ ,-14,\ 1 & 0\ \ -14,\ 1 & 0,-7/5,1 & 0\ ,-7/5\ ,\ 1 \end{array}\right] , \tag{59}$$

and

$$\underset{\sim}{N} = \mathrm{diag}\left[\begin{array}{c|c|c|c} 5\ ,\ 0 & 1\ ,\ 0 & 1\ \ \ 0 & 1\ ,\ 0 \\ -7\ ,\ 1 & -7\ ,\ 1 & -14\ ,\ 1 & -23\ ,\ 1 \end{array}\right] . \tag{60}$$

It therefore follows from equation (17) that

$$\{\underset{\sim}{u}_1,\underset{\sim}{u}_2,\underset{\sim}{u}_3,\underset{\sim}{u}_4\} = \left\{\begin{bmatrix}-5\\5\\-5\\-7\end{bmatrix},\begin{bmatrix}-2\\4\\-1\\-7\end{bmatrix},\begin{bmatrix}-3\\9\\-1\\-14\end{bmatrix},\begin{bmatrix}-4\\16\\-1\\-23\end{bmatrix}\right\} \tag{61}$$

and from equation (18) that

$$\{\underset{\sim}{v}_1,\underset{\sim}{v}_2,\underset{\sim}{v}_3,\underset{\sim}{v}_4\} = \left\{\begin{bmatrix}-2/3\\1/3\\1/3\\1/3\end{bmatrix},\begin{bmatrix}27/2\\-9/2\\-11\\-5\end{bmatrix},\begin{bmatrix}-16\\4\\13\\5\end{bmatrix},\begin{bmatrix}35/6\\-7/6\\-14/3\\-5/3\end{bmatrix}\right\} . \tag{62}$$

In view of equations (4) and (9) the results (61) imply that

$$\{\underset{\sim}{w}_1,\underset{\sim}{w}_2,\underset{\sim}{w}_3,\underset{\sim}{w}_4\} = \left\{\begin{bmatrix}-5\\7\end{bmatrix},\begin{bmatrix}-10\\14\end{bmatrix},\begin{bmatrix}-33\\42\end{bmatrix},\begin{bmatrix}-76\\92\end{bmatrix}\right\}, \tag{63}$$

and in view of equations (5) and (10) the results (62) imply that

$$\{\underset{\sim}{z}_1, \underset{\sim}{z}_2, \underset{\sim}{z}_3, \underset{\sim}{z}_4\} = \left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} -67/2 \\ 22 \\ 10 \end{bmatrix}, \begin{bmatrix} 57 \\ -39 \\ -15 \end{bmatrix}, \begin{bmatrix} -161/6 \\ 56/3 \\ 20/3 \end{bmatrix} \right\} . \tag{64}$$

It finally follows from either equation (7) or equation (8) that the output-feedback control law (Porter and Bradshaw 1978b)

$$\underset{\sim}{u}(t) = \begin{bmatrix} -47 , & 34 , & 10 \\ 49 , & -35 , & -11 \end{bmatrix} \underset{\sim}{x}(t) \tag{65}$$

assigns the spectrum (34), the eigenvectors (61), and the reciprocal eigenvectors (62) to the closed-loop plant matrix of the system governed by equations (32) and (33).

4. Conclusion

It is known (Porter and Bradshaw 1978a) that, in the case of self-conjugate distinct eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the eigenvectors and reciprocal eigenvectors lie in well-defined subspaces. In this paper, a technique has been presented which can be used to select the eigenvectors and reciprocal eigenvectors from these subspaces in the case of appropriately augmented (Kimura 1975) controllable and observable continuous-time systems by performing restricted elementary row and column operations on matrices formed from the spanning vectors of these subspaces. This technique is ideally suited to digital-computer implementation and therefore greatly facilitates the synthesis of both static (Porter and Bradshaw 1978a) and dynamic (Porter and Bradshaw 1978b) output-feedback controllers.

ACKNOWLEDGEMENT

This research was supported in part by the Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005B.

REFERENCES

KIMURA, H., 1975, I.E.E.E. Trans. autom. Control, 20, 509.

PORTER, B. and BRADSHAW, A., 1978a, Int. J. Systems Sci. (in the press); 1978b, Ibid. (in the press).

PORTER, B. and D'AZZO, J. J., 1977, Electron. Lett., 13, 230.

APPENDIX 11

COMPUTER AIDED DESIGN OF DYNAMIC COMPENSATORS FOR LINEAR MULTIVARIABLE CONTINUOUS-TIME SYSTEMS

PROFESSOR B PORTER AND DR A BRADSHAW

DEPARTMENT OF AERONAUTICAL AND MECHANICAL ENGINEERING
UNIVERSITY OF SALFORD
SALFORD M5 4WT
ENGLAND

(PROC IFAC SYMPOSIUM ON
COMPUTER-AIDED DESIGN OF CONTROL SYSTEMS
ZURICH, SWITZERLAND, AUGUST 1979)

ABSTRACT

In view of the fundamental new insights into the structure of linear multivariable continuous-time systems provided by the method of entire eigenstructure assignment, the design of dynamic compensators is equivalent to the selection of pairwise-orthogonal eigenvectors and reciprocal eigenvectors from two families of well-defined subspaces which are parametrised by associated self-conjugate eigenvalue spectra. This selection is effected by the use of a powerful new algorithm which requires the performance of restricted elementary row and column operations on matrices formed from the spanning vectors of these subspaces. The digital computer implementation of the resulting procedure incorporating this algorithm is described and is illustrated by the design of an error-actuated dynamic compensator for a linear multivariable plant.

1. INTRODUCTION

In most practical cases, it is of course impossible to implement state-feedback control laws since the state of the plant is inaccessible and only the plant output is available for control purposes. The method of entire eigenstructure assignment (Porter and D'Azzo, 1977) has accordingly been applied to the design of linear multivariable continuous-time output-feedback regulators by Porter and Bradshaw (1978a). It has been shown that, in the case of self-conjugate distinct eigenvalue spectra, the closed-loop eigenstructure assignable by output feedback is constrained by the requirement that the eigenvectors and reciprocal eigenvectors of the closed-loop plant matrix lie in two families of well-defined subspaces and satisfy appropriate orthogonality conditions. In contrast, the closed-loop eigenstructure assignable by state feedback (Moore, 1976) is constrained only by the requirement that the eigenvectors of the closed-loop plant matrix lie in just one family of well-defined subspaces. It is because of the severe constraints on the closed-loop eigenstructure assignable by output feedback that it is frequently impossible to achieve satisfactory closed-loop behaviour by means of static continuous-time output-feedback regulators, and that it is consequently necessary to introduce dynamic compensators (Brash and Pearson, 1970; Kimura, 1975). However, it has been shown by Porter and Bradshaw (1978b) that the design of such dynamic compensators can be effected by applying the method of entire eigenstructure assignment to appropriately augmented (Kimura, 1975) continuous-time systems.

In view of these fundamental new insights into the structure of linear multivariable systems, the design of dynamic compensators is equivalent to the selection of pairwise-orthogonal eigenvectors and reciprocal eigenvectors from two families of well-defined subspaces which are parametrised by associated self-conjugate eigenvalue spectra. This selection can be effected by the use of a powerful new algorithm (Bradshaw, Fletcher, and Porter, 1978) which requires the performance of restricted elementary row and column operations on matrices formed from the spanning vectors of these subspaces. The digital computer implementation of a procedure incorporating this algorithm is described and is illustrated by the design of an error-actuated dynamic compensator for a linear multivariable plant. The principal computational attraction of the procedure is that no operations with polynomial matrices are involved, so that error-actuated dynamic compensators for large-scale systems can be readily designed.

2. Compensator Structure

The linear multivariable continuous-time tracking systems considered by Porter and Bradshaw (1978b) consist of a controllable and observable nth-order plant governed by state and output equations of the respective forms

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \tag{1}$$

and

$$y(t) = Cx(t) \qquad , \tag{2}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$, $d(t) \in R^h$, $A \in R^{nxn}$, $B \in R^{nxm}$, $C \in R^{pxn}$, $D \in R^{nxh}$, rank $B = m$, and rank $C = p$, together with an error-actuated dynamic compensator which is required to cause the output vector, $y(t)$, to track a command input vector, $r(t)$, in the sense that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\{r(t)-y(t)\} = 0 \tag{3}$$

for unmeasurable command and disturbance inputs of the respective forms

$$r(t) = \sum_{i=1}^{r} \alpha_{i-1} t^{i-1} \tag{4}$$

and

$$d(t) = \sum_{i=1}^{s} \beta_{i-1} t^{i-1} \quad . \tag{5}$$

Such an error-actuated dynamic compensator is governed by state and output equations of the respective forms

$$\dot{w}(t) = Fw(t) + Ge(t) + \sum_{i=1}^{q} H_i z_i(t) \tag{6}$$

and

$$u(t) = Kw(t) + Le(t) + \sum_{i=1}^{q} M_i z_i(t) \quad , \tag{7}$$

where

$$\left.\begin{array}{l}\dot{z}_1(t) = e(t) \quad , \\ \dot{z}_2(t) = z_1(t) \quad , \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{z}_q(t) = z_{q-1}(t) \quad ,\end{array}\right\} \tag{8}$$

$$q = \max(r,s) \quad , \tag{9}$$

$w(t) \in R^{\ell}$, $e(t) \in R^{p}$, $z_i(t) \in R^{p}$ $(i=1,2,\ldots,q)$, $u(t) \in R^{m}$, $F \in R^{\ell x \ell}$, $G \in R^{\ell xp}$, $H_i \in R^{\ell xp}$ $(i=1,2,\ldots,q)$, $K \in R^{mx\ell}$, $L \in R^{mxp}$, $M_i \in R^{mxp}$ $(i=1,2,\ldots,q)$, and (Kimura, 1975)

$$\ell = \max(0,n-m-p+1) \quad . \tag{10}$$

It is then evident from equations (1), (2), (6), (7), and (8) that the closed-loop system is governed by state and output equations of the respective forms

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}_1(t)\\ \cdots\\ \cdots\\ \dot{z}_q(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}A-BLC & BM_1 & \ldots & BM_q & BK\\ -C & 0 & \ldots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \ldots & 0 & 0\\ -GC & H_1 & \ldots & H_q & F\end{bmatrix}\begin{bmatrix}x(t)\\ z_1(t)\\ \cdots\\ \cdots\\ z_q(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}BL\\ I_p\\ \cdot\cdot\\ \cdot\cdot\\ 0\\ G\end{bmatrix} r(t) + \begin{bmatrix}D\\ 0\\ \cdot\\ \cdot\\ 0\\ 0\end{bmatrix} d(t) \tag{11}$$

and

$$\begin{bmatrix} y(t) \\ z_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ z_q(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} C, & 0, & \ldots, & 0, & 0 \\ 0, & I_p, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \ldots, & I_p, & 0 \\ 0, & 0, & \ldots, & 0, & I_\ell \end{bmatrix} \begin{bmatrix} x(t) \\ z_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ z_q(t) \\ w(t) \end{bmatrix} . \qquad (12)$$

In view of equations (4) and (5), it is clear by differentiating equation (11) $(q-1)$ times that the closed-loop system will behave so that

$$\lim_{t\to\infty} \frac{d^q z_q}{dt^q} = 0 \qquad (13)$$

and therefore, in view of equations (8), so that equation (3) will be satisfied if the error-actuated dynamic compensator governed by equations (6) and (7) is designed such that all the eigenvalues of the plant matrix of the closed-loop system governed by equations (11) and (12) are assigned to the open left-half of the complex plane.

It is evident from equations (11) and (12) that such a compensator can be designed by the synthesis of an appropriate output-feedback control law of the form

$$u_\ell(t) = G_\ell y_\ell(t) \qquad (14)$$

for the augmented open-loop system governed by state and output equations of the respective forms

$$\dot{x}_\ell(t) = A_\ell x_\ell(t) + B_\ell u_\ell(t) \qquad (15)$$

and

$$y_\ell(t) = C_\ell x_\ell(t) \tag{16}$$

where

$$u_\ell(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} , \tag{17}$$

$$x_\ell(t) = \begin{bmatrix} x(t) \\ z_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ z_q(t) \\ w(t) \end{bmatrix} , \tag{18}$$

$$y_\ell(t) = \begin{bmatrix} y(t) \\ z_1(t) \\ \cdots\cdots \\ \cdots\cdots \\ z_q(t) \\ w(t) \end{bmatrix} , \tag{19}$$

$$G_\ell = \begin{bmatrix} -L , & M_1 , & \ldots , & M_q , & K \\ -G , & H_1 , & \ldots , & H_q , & F \end{bmatrix} , \tag{20}$$

$$A_\ell = \begin{bmatrix} A , & 0 , & \ldots , & 0 , & 0 \\ -C , & 0 , & \ldots , & 0 , & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 , & 0 , & \ldots , & 0 , & 0 \\ 0 , & 0 , & \ldots , & 0 , & 0 \end{bmatrix} , \tag{21}$$

$$B_\ell = \begin{bmatrix} B & , & O \\ O & , & O \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ O & , & O \\ O & , & I_\ell \end{bmatrix} , \tag{22}$$

and

$$C_\ell = \begin{bmatrix} C & , & O & , & \ldots & , & O & , & O \\ O & , & I_p & , & \ldots & , & O & , & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O & , & O & , & \ldots & , & I_p & , & O \\ O & , & O & , & \ldots & , & O & , & I_\ell \end{bmatrix} . \tag{23}$$

Thus, if the $(m+\ell)\times(p+pq+\ell)$ output-feedback matrix G_ℓ is such that the closed-loop plant matrix $(A_\ell+B_\ell G_\ell C_\ell)$ has the self-conjugate distinct eigenvalue spectrum $\Lambda = \{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$, then the corresponding eigenvector and reciprocal eigenvector sets $\{u_1,u_2,\ldots,u_{n+pq+\ell}\}$ and $\{v_1,v_2,\ldots,v_{n+pq+\ell}\}$ must be such that

$$\begin{bmatrix} u_j \\ w_j \end{bmatrix} \in \ker[A_\ell-\lambda_j I_{n+pq+\ell} \;,\; B_\ell] \quad (j=1,2,\ldots,n+pq+\ell) \tag{24}$$

$$\begin{bmatrix} v_i \\ z_i \end{bmatrix} \in \ker[A'_\ell-\lambda_i I_{n+pq+\ell} \;,\; C'_\ell] \quad (i=1,2,\ldots,n+pq+\ell) \tag{25}$$

and

$$v'_i u_j = \delta_{ij} \qquad (i,j=1,2,\ldots,n+pq+\ell). \tag{26}$$

The output-feedback matrix is then given by the equivalent formulas

$$G_\ell = [w_1 , w_2 , \ldots , w_{p+pq+\ell}][C_\ell u_1 , C_\ell u_2 , \ldots , C_\ell u_{p+pq+\ell}]^{-1} \tag{27}$$

and

$$G'_\ell = [z_1 , z_2 , \ldots , z_{m+\ell}][B'_\ell v_1 , B'_\ell v_2 , \ldots , B'_\ell v_{m+\ell}]^{-1} . \tag{28}$$

In view of equations (24), (25), (27), and (28), the computation of G_ℓ is reduced to the determination of the kernels of each of the n+pq+ℓ matrices

$$S_\ell(\lambda_j) = [A_\ell - \lambda_j I , B_\ell] \qquad (j=1,2,\ldots,n+pq+\ell) \tag{29}$$

together with the kernels of each of the n+pq+ℓ matrices

$$T'_\ell(\lambda_i) = [A'_\ell - \lambda_i I , C'_\ell] \qquad (i=1,2,\ldots,n+pq+\ell) \tag{30}$$

followed by the selection of sets of linearly independent self-conjugate vectors $\{u_1, u_2, \ldots, u_{n+pq+\ell}\}$ and $\{v_1, v_2, \ldots, v_{n+pq+\ell}\}$ from subspaces determined by the kernels of $S_\ell(\lambda_j)$ $(j=1,2,\ldots,n+pq+\ell)$ and $T_\ell(\lambda_i)$ $(i=1,2,\ldots,n+pq+\ell)$, respectively, such that the orthogonality conditions (26) are satisfied. It is finally evident from equations (6), (7), and (20) that the matrices in the respective state and output equations of the required ℓth-order error-actuated dynamic compensator are determined by the sub-matrices of the output-feedback matrix G_ℓ.

3. COMPENSATOR DESIGN PROCEDURE

3.1 System Augmentation Procedure

The first stage in the compensator design procedure involves the formation of the augmented plant, input, and output matrices by the following steps which constitute the routine AUGMENT:

(i) Set $q = \max(r,s)$;

(ii) Set $\ell = \max(0,n-m-p+1)$

(iii) Form the augmented open-loop plant, input, and output matrices A_ℓ, B_ℓ, and C_ℓ.

3.2 Kernel Computation Procedure

The second stage in the compensator design procedure involves the computation of the closed-loop eigenvector and reciprocal eigenvector subspaces by the following steps which constitute the routine KERNELS:

(i) Select the closed-loop eigenvalue spectrum $\Lambda = \{\lambda_1,\lambda_2,\ldots,\lambda_{n+pq+\ell}\}$;

(ii) Form $S_\ell(\lambda_j) = [A_\ell-\lambda_j I \ , \ B_\ell]$ $(j=1,2,\ldots,n+pq+\ell)$;

(iii) Form $T'_\ell(\lambda_i) = [A'_\ell-\lambda_i I \ , \ C'_\ell]$ $(i=1,2,\ldots,n+pq+\ell)$;

(iv) Compute

$$\ker S_\ell(\lambda_j) = \text{span}\left\{\begin{bmatrix}\chi_k(\lambda_j)\\ \omega_k(\lambda_j)\end{bmatrix}:k=1,2,\ldots,m+\ell\right\}$$

$$(j=1,2,\ldots,n+pq+\ell);$$

(v) Compute

$$\ker T_\ell'(\lambda_i) = \text{span}\left\{\begin{bmatrix}\phi_k(\lambda_i)\\ \zeta_k(\lambda_i)\end{bmatrix} : k=1,2,\ldots,p+pq+\ell\right\}$$

$(i=1,2,\ldots,n+pq+\ell)$;

(vi) Form $X(\lambda_j) = [\chi_1(\lambda_j)\ ,\ \chi_2(\lambda_j)\ ,\ \ldots\ ,\ \chi_{m+\ell}(\lambda_j)]$

$(j=1,2,\ldots,n+pq+\ell)$;

(vii) Form $\Omega(\lambda_j) = [\omega_1(\lambda_j)\ ,\ \omega_2(\lambda_j)\ ,\ \ldots\ ,\ \omega_{m+\ell}(\lambda_j)]$

$(j=1,2,\ldots,n+pq+\ell)$;

(viii) Form $\Phi(\lambda_i) = [\phi_1(\lambda_i)\ ,\ \phi_2(\lambda_i)\ ,\ \ldots\ ,\ \phi_{p+pq+\ell}(\lambda_i)]$

$(i=1,2,\ldots,n+pq+\ell)$;

(ix) Form $Z(\lambda_i) = [\zeta_1(\lambda_i)\ ,\ \zeta_2(\lambda_i)\ ,\ \ldots\ ,\ \zeta_{p+pq+\ell}(\lambda_i)]$

$(i=1,2,\ldots,n+pq+\ell)$.

3.3 Eigenvector Selection Procedure

The third stage in the compensator design procedure involves the selection of the pairwise-orthogonal closed-loop eigenvectors and reciprocal eigenvectors from the respective subspaces im $X(\lambda_j)$ $(j=1,2,\ldots,n+pq+\ell)$ and im $\Phi(\lambda_i)$ $(i=1,2,\ldots,n+pq+\ell)$ by the following steps which constitute the routine SELECT:

(i) Select $u_j \in \text{im}\ X(\lambda_j)$ $(j=1,2,\ldots,n+pq-m)$ such that $\{u_1,u_2,\ldots,u_{n+pq-m}\}$ is a linearly independent set;

(ii) Compute $v_i \in \text{im}\ \Phi(\lambda_i)$ $(i=n+pq-m+1,n+pq-m+2,\ldots,n+pq+\ell)$ such that $v_i'u_j = 0$ $(i=n+pq-m+1,n+pq-m+2,\ldots,n+pq+\ell;$ $j=1,2,\ldots,n+pq-m)$;

(iii) Compute $u_j \in \text{im}\ X(\lambda_j)$ $(j=n+pq-m+1,n+pq-m+2,\ldots,n+pq+\ell)$ such that $v_i'u_j = 0$ $(i=1,2,\ldots,n+pq-m;j=n+pq-m+1,$ $n+pq-m+2,\ldots,n+pq+\ell)$;

(iv) Compute $v_i \in \text{im}\ \Phi(\lambda_i)$ $(i=1,2,\ldots,n+pq-m)$ such that $v_i'u_j = 0$ $(i=1,2,\ldots,n+pq-m;j=n+pq-m+1,n+pq-m+2,\ldots,$ $n+pq+\ell)$;

(v) Normalise v_i or u_i such that $v_i'u_i = 1$ $(i=1,2,\ldots,n+pq+\ell)$.

3.4 Compensator Matrix Computation Procedure

The final stage in the compensator design procedure involves the computation of the compensator matrices by the following steps which constitute the routine COMPENSATE:

(i) Select a set $\{C_\ell u_1\ ,\ C_\ell u_2\ ,\ \ldots\ ,\ C_\ell u_{p+pq+\ell}\}$ of linearly independent vectors and a set $\{B_\ell' v_1\ ,\ B_\ell' v_2\ ,\ \ldots\ ,$ $B_\ell' v_{m+\ell}\}$ of linearly independent vectors;

(ii) Compute the output-feedback matrix

$$G_\ell = [w_1,w_2,\ldots,w_{p+pq+\ell}]\,[C_\ell u_1,C_\ell u_2,\ldots,C_\ell u_{p+pq+\ell}]^{-1}$$

and the transposed output-feedback matrix

$$G_\ell' = [z_1,z_2,\ldots,z_{m+\ell}]\,[B_\ell' v_1,B_\ell' v_2,\ldots,B_\ell' v_{m+\ell}]^{-1} \quad ;$$

(iii) Form the compensator matrices $K,L,M_1,M_2,\ldots,M_q$ and $F,G,H_1,H_2,\ldots,H_q$.

4. Illustrative Example

This procedure can be conveniently illustrated by designing an error-actuated dynamic compensator which will cause the output of the controllable and observable linear plant governed by the respective state and output equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0, & 1, & 0, & 0 \\ 0, & -1, & -1, & 0 \\ 1, & 0, & 0, & 0 \\ -1, & 0, & 1, & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1, & 0 \\ 0, & 0 \\ 0, & 0 \\ 0, & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} d(t) \tag{31}$$

and

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1, & -1, & 2, & 0 \\ 0, & 0, & 1, & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{32}$$

to track any constant command input $r(t) = [r_1(t), r_2(t)]' \in R^2$ in the presence of any unmeasurable constant disturbance input $d(t) \in R$.

In this case, the outputs of the routines AUGMENT, KERNELS, SELECT, and COMPENSATE when $\Lambda = \{-1.0,-1.5,-2.0,-2.5,-3.0,-3.5,-4.0\}$ are listed in the Appendix. These listings indicate that the required error-actuated dynamic compensator is governed by the respective state and output equations

$$\dot{w}(t) = -1.989\, w(t) + [-0.5431\ ,\ -2.258]\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

$$+ [-1.260\ ,\ 5.227]\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \qquad (33)$$

and

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -19.40 \\ 1.823 \end{bmatrix} w(t) + \begin{bmatrix} 10.26 & , & -51.98 \\ 0.6437 & , & 4.961 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -9.521 & , & 0.2316 \\ 2.645 & , & -3.509 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \qquad (34)$$

where $[e_1(t)\ ,\ e_2(t)]' = [r_1(t)-y_1(t)\ ,\ r_2(t)-y_2(t)]' \in R^2$ and $[\dot{z}_1(t)\ ,\ \dot{z}_2(t)]' = [e_1(t)\ ,\ e_2(t)]' \in R^2$.

5. CONCLUSION

The method of entire eigenstructure assignment has yielded fundamental new insights into the structure of linear multivariable systems and, in particular, into the closed-loop eigenstructure assignable by output feedback (Porter and Bradshaw, 1978a,b). The design of dynamic compensators has accordingly been reduced to the selection (Bradshaw, Fletcher, and Porter, 1978) of pairwise-orthogonal eigenvectors and reciprocal eigenvectors from two families of well-defined subspaces which are parametrised by associated self-conjugate eigenvalue spectra. The resulting procedure for the design of dynamic compensators is computationally attractive since its constituent routines AUGMENT, KERNELS, SELECT, and COMPENSATE involve only numerically stable operations. Indeed, the entire procedure has been coded in FORTRAN for the routine computer-aided design of error-actuated dynamic compensators, and forms part of a comprehensive suite of design procedures for various classes of controllers for both continuous-time and discrete-time linear multivariable systems.

ACKNOWLEDGMENT

This research was supported in part by the Wright Aeronautical Laboratories, United States Air Force under Grant AFOSR-76-3005B.

REFERENCES

Bradshaw, A, L R Fletcher, and B Porter (1978). Synthesis of output-feedback control laws for linear multivariable continuous-time systems. Int J Syst Sci, 9, 1331-1340.

Brasch, F M and J B Pearson (1970). Pole-placement using dynamic compensators. IEEE Trans Autom Control, AC-15, 34-43.

Kimura, H (1975). Pole assignment by gain output feedback. IEEE Trans Autom Control, AC-20, 509-516.

Moore, B C (1976). On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment. IEEE Trans Autom Control, AC-21, 689-692.

Porter, B and A Bradshaw (1978). Design of linear multivariable continuous-time output-feedback regulators. Int J Syst Sci, 9, 445-450.

Porter, B and A Bradshaw (1978). Design of linear multivariable continuous-time tracking systems incorporating error-actuated dynamic controllers. Int J Syst Sci, 9, 627-637.

Porter, B and J J D'Azzo (1977). Algorithm for the synthesis of state-feedback regulators by entire eigenstructure assignment. Electron Lett, 13, 230-231.

APPENDIX

AUGMENT

AUGMENTED MATRIX A

0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	-1.0000E 00	-1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
-1.0000E 00	0.0000E 00	1.0000E 00	2.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
-1.0000E 00	1.0000E 00	-2.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	-1.0000E 00	-1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00

AUGMENTED MATRIX B

1.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	1.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	1.0000E 00

AUGMENTED MATRIX C

1.0000E 00	-1.0000E 00	2.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	1.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00

KERNELS

UPPER SPANNING VECTORS
CHI(LAMBDA)

1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00
0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	0.0000E 00	5.0000E-01	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	0.0000E 00	2.8570E-01	-1.4290E-01	0.0000E 00	2.5000E-01
1.0000E 00	-1.0000E 00	0.0000E 00	6.6670E-01	-3.6380E-12	0.0000E 00	5.0000E-01
0.0000E 00	0.0000E 00	0.0000E 00	1.9050E-01	2.3810E-01	0.0000E 00	1.2500E-01
0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00

0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00
1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00
1.0000E 00	0.0000E 00	-3.6380E-12	1.0000E 00	0.0000E 00	0.0000E 00	2.0000E 00
-2.5000E-01	0.0000E 00	2.2720E-01	0.0000E 00	0.0000E 00	2.0000E-01	-4.0000E-01
5.0000E-01	0.0000E 00	4.0000E-01	0.0000E 00	0.0000E 00	3.3330E-01	1.0000E 00
3.7500E-01	0.0000E 00	8.8890E-02	0.0000E 00	0.0000E 00	6.6670E-02	5.3330E-01
0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00

0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00
0.0000E 00	0.0000E 00	2.5000E 00	0.0000E 00	3.6380E-12	3.0000E 00	0.0000E 00
0.0000E 00	1.8180E-01	-4.5450E-01	0.0000E 00	1.6670E-01	-5.0000E-01	0.0000E 00
0.0000E 00	2.8570E-01	1.1430E 00	0.0000E 00	2.5000E-01	1.2500E 00	0.0000E 00
0.0000E 00	5.1950E-02	5.8440E-01	0.0000E 00	4.1670E-02	6.2500E-01	0.0000E 00
1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00

LOWER SPANNING VECTORS
OMEGA(LAMBDA)

-1.0000E 00	-1.0000E 00	0.0000E 00	-1.5000E 00	-1.0000E 00	0.0000E 00	-2.0000E 00
-1.0000E 00	0.0000E 00	0.0000E 00	-1.0000E 00	-7.5000E-01	0.0000E 00	-1.0000E 00
0.0000E 00	0.0000E 00	-1.0000E 00	0.0000E 00	0.0000E 00	-1.5000E 00	0.0000E 00

-1.0000E 00	0.0000E 00	-2.5000E 00	-1.0000E 00	0.0000E 00	-3.0000E 00	-1.0000E 00
-2.0000E 00	0.0000E 00	-1.0000E 00	-3.7500E 00	0.0000E 00	-1.0000E 00	-6.0000E 00
0.0000E 00	-2.0000E 00	0.0000E 00	0.0000E 00	-2.5000E 00	0.0000E 00	0.0000E 00

0.0000E 00	-3.5000E 00	-1.0000E 00	0.0000E 00	-4.0000E 00	-1.0000E 00	0.0000E 00
0.0000E 00	-1.0000E 00	-8.7000E 00	0.0000E 00	-1.0000E 00	-1.2000E 01	0.0000E 00
-3.0000E 00	0.0000E 00	0.0000E 00	-3.5000E 00	0.0000E 00	0.0000E 00	0.0000E 00

UPPER RECIPROCAL SPANNING VECTORS
PHI(LAMBDA)

```
 1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00
 0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
-2.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -4.2500E 00 -1.2500E 00
 0.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.7500E 00 -7.5000E-01
 0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00

 0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00 -7.0000E 00 -2.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00 -4.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00
 1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00
 0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
 0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00

 0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
 0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00 -1.0250E 01 -3.2500E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.4000E 01
 0.0000E 00 -6.7500E 00 -1.7500E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.0000E 01
 0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00
 1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00

 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00
 1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00
-5.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.8250E 01 -7.2500E 00  0.0000E 00
-3.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.3750E 01 -4.7500E 00  0.0000E 00
 0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
 0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00

 0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00 -2.3000E 01 -1.0000E 01  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00 -1.8000E 01 -7.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00
 1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00
 0.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
```

LOWER RECIPROCAL SPANNING VECTORS
ZETA(LAMBDA)

```
 1.0000E 00 -5.4570E-12  1.0000E 00 -3.6380E-12  0.0000E 00  1.0000E 00  5.0000E-01
 0.0000E 00  3.0000E 00  0.0000E 00  1.0000E 00  0.0000E 00  6.1250E 00  2.6250E 00
 0.0000E 00  0.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00 -3.3420E-11 -8.8680E-12
 0.0000E 00  0.0000E 00  0.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -1.0000E 00  0.0000E 00  0.0000E 00

 1.0000E 00 -1.0910E-11  0.0000E 00  1.0000E 00  1.0000E 00  1.0000E 00 -1.0910E-11
 0.0000E 00  1.0000E 00  0.0000E 00  1.6000E 01  4.0000E 00  0.0000E 00  1.0000E 00
-1.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -2.0000E 00  0.0000E 00
 0.0000E 00 -1.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -2.0000E 00
 0.0000E 00  0.0000E 00 -1.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00

 0.0000E 00  1.0000E 00  1.5000E 00  1.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00
 0.0000E 00  3.0370E 01  7.8750E 00  0.0000E 00  1.0000E 00  0.0000E 00  5.0000E 01
 0.0000E 00  0.0000E 00  0.0000E 00 -2.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -2.5000E 00  0.0000E 00  0.0000E 00
-2.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -2.5000E 00  0.0000E 00

 2.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00  1.0000E 00  2.5000E 00  1.0000E 00
 1.5000E 01  0.0000E 00  1.0000E 00  0.0000E 00  7.5630E 01  2.6130E 01  0.0000E 00
 0.0000E 00 -3.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -3.5000E 00
 0.0000E 00  0.0000E 00 -3.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00 -3.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00

-7.9580E-13  0.0000E 00  1.0000E 00  3.0000E 00  1.0000E 00  0.0000E 00  0.0000E 00
 1.0000E 00  0.0000E 00  1.0800E 02  4.2000E 01  0.0000E 00  1.0000E 00  0.0000E 00
 0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -4.0000E 00  0.0000E 00  0.0000E 00
-3.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -4.0000E 00  0.0000E 00
 0.0000E 00 -3.5000E 00  0.0000E 00  0.0000E 00  0.0000E 00  0.0000E 00 -4.0000E 00
```

SELECT

CLOSED-LOOP EIGENVECTORS

U(J)

1.0000E 00	1.0000E 00	1.0000E 00	1.0000E 00	-1.0940E 02	-2.9070E 01	1.0000E 00
1.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	1.0000E 00	3.3880E-01	-1.2730E-02
0.0000E 00	5.0000E-01	0.0000E 00	-3.6380E-12	2.0000E 00	8.4700E-01	-3.6680E-02
3.3330E-01	1.4290E-01	2.5000E-01	2.2720E-01	-2.2270E 01	-5.4390E 00	1.7280E-01
0.0000E 00	6.6670E-01	5.0000E-01	4.0000E-01	-3.5450E 01	-7.9180E 00	2.3470E-01
3.3330E-01	4.8260E-01	1.2500E-01	8.8890E-02	-6.7570E 00	-1.3120E 00	3.4020E-02
1.0000E 00	1.0000E 00	0.0000E 00	0.0000E 00	2.4100E 00	1.0000E 00	-4.1240E-02

CLOSED-LOOP RECIPROCAL EIGENVECTORS

V(J)

1.7860E-01	-4.6450E-01	-7.4350E 00	6.1630E 01	1.2540E 00	-3.7090E 00	-2.3610E 01
-2.1270E 00	5.2940E 00	6.5570E 01	-4.2720E 02	-7.4040E 00	2.0860E 01	1.5010E 02
1.7650E 00	-4.6440E 00	-7.9110E 01	7.4040E 02	1.9460E 01	-8.3530E 01	-9.5780E 02
2.1720E 00	-3.1580E 00	-3.5840E 01	3.2780E 02	9.6730E 00	-4.8070E 01	-6.2560E 02
-3.2830E 00	6.1770E 00	6.8160E 01	-7.0090E 02	-1.7820E 01	7.2180E 01	7.7910E 02
7.4590E 00	-1.4680E 01	-1.3350E 02	1.6650E 03	4.1910E 01	-1.6290E 02	-1.6760E 03
-2.5020E-01	1.1150E 00	-1.6960E 00	-3.0200E 02	-1.1050E 01	5.3180E 01	6.4080E 02

COMPENSATE

OUTPUT-FEEDBACK MATRIX G

-1.0260E 01	5.1980E 01	-9.5210E 00	2.3160E-01	-1.9400E 01
-6.4370E-01	-4.9610E 00	2.6450E 00	-3.5090E 00	1.8230E 00
5.4130E-01	-2.2580E 00	-1.2600E 00	5.2270E 00	-1.9890E 00