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STATISTICAL PERT: AN IMPROVED PROJECT SCHEDULING ALGORITHM.(U)

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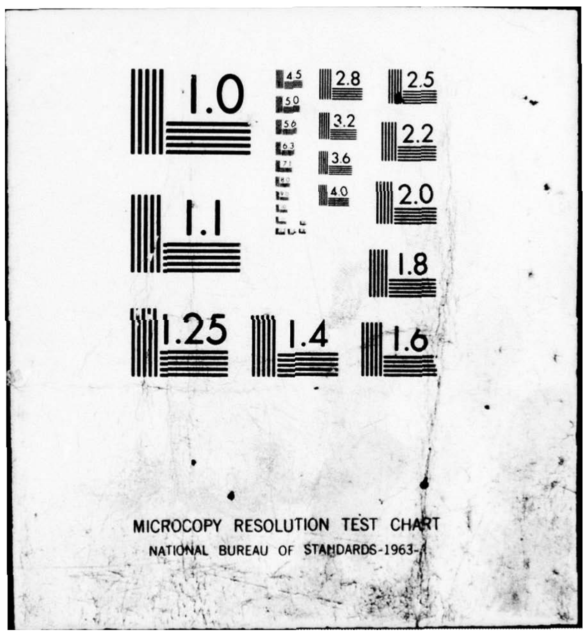
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STATISTICAL PERT: AN IMPROVED PROJECT SCHEDULING ALGORITHM

by

Cynthia S. Dunn and Robert L. Sielken, Jr.

Texas A&M University
Office of Naval Research
Contract N00014-76-C-0038
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(10) Cynthia S. Dunn [redacted] Robert L. Sielken, Jr

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ATTACHMENT II

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Abstract

A project scheduling algorithm is developed and illustrated. For each feasible project deadline time the minimum project cost and corresponding optimal deterministic activity durations are derived. The cost of an activity is assumed to be a convex piecewise linear function of its duration. The algorithm is based upon network-flow techniques including the use of a labeling procedure which preserves complementary slackness. A computer implementation of the algorithm is documented.

TABLE OF CONTENTS

1. INTRODUCTION

This paper describes a scheduling algorithm for a project composed of "jobs" or "activities." These activities are represented by arcs in a directed network. The network nodes represent events in time. The activities at any node can "commence" as soon as <u>all</u> activities "terminating" at that node are completed. Associated with each activity is an interval of possible completion times and an associated piecewise linear cost function. Given that the project must be completed by a specified deadline time, the algorithm determines the individual activity completion times which minimize the total project cost. Repeating the process for all feasible deadline times yields the entire project cost curve and associated optimal activity completion times.

For example, suppose the project consists of activities A, B, C, D, E and the order relations:

A precedes C and D,

B precedes D,

C and D both precede E

and those implied by transitivity. The corresponding network representation is shown in Figure 1 where the arcs represent activities and nodes are events. Notice that arc F does not correspond to any "real" activity but merely represents the order relation that A must precede D. We shall assume that such dummy activities have zero completion times and zero costs.

Using this network representation of the project, the problem of computing the cost curve can be formulated as a network-flow problem. We shall make the following assumptions about the network: there are no directed cycles, and each arc is contained in some directed path from the beginning

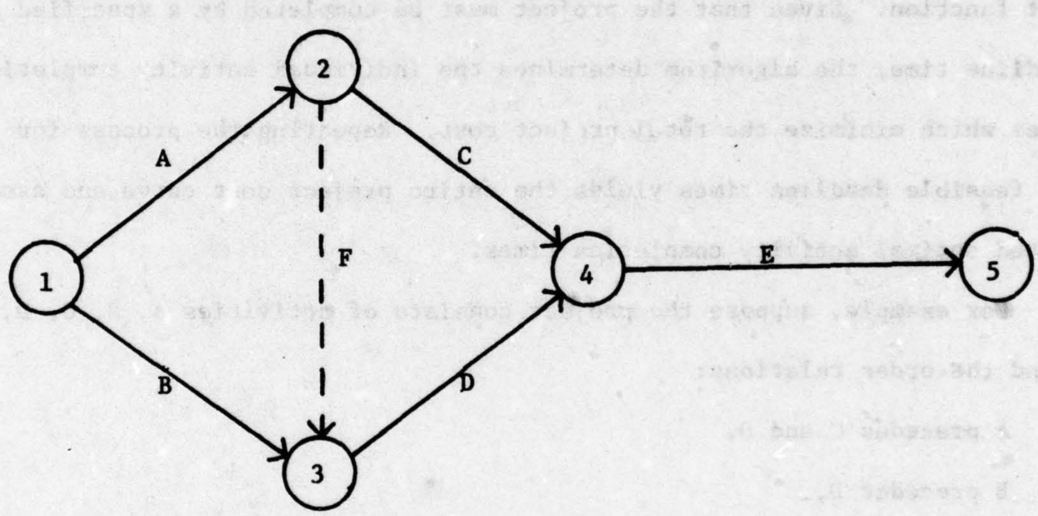


FIGURE 1

node (called the "source") to the terminal node (called the "sink").

This problem can also be formulated as a linear programming problem; however, due to the large number of variables and constraints, it would be impractical storage-wise to solve it using linear programming methods.

D. R. Fulkerson (1961) has formulated a very efficient network-flow algorithm for solving the problem with a linear activity cost function. In this paper, Fulkerson's algorithm has been extended to accept a convex piecewise linear cost function for each individual activity.

2. PROBLEM FORMULATION AND SOLUTION PROCEDURE

2.1. Problem Formulation

The cost of completing an activity is assumed to be a convex piecewise linear function. The cost curve for activity I is depicted in Figure 2. Note that the allowable completion times for activity I have been divided into NK(I)-1 intervals: [TIME(I,1), TIME(I,2)], [TIME(I,2), TIME(I,3)],..., {TIME[I,NK(I)-2], TIME[I,NK(I)-1]}, {TIME[I,NK(I)-1], TIME[I,NK(I)]} with

$$\mathrm{TIME}(I,\ 1) \le \mathrm{TIME}(I,\ 2) \le \ldots \le \mathrm{TIME}[I,\ NK(I)]. \qquad (2.1)$$

Here we interpret TIME(I,1) as the shortest possible completion time and TIME[I,NK(I)] as the cheapest completion time. Even though the duration of activity I could be greater than TIME[I,NK(I)], such durations would be needlessly expensive and hence TIME[I,NK(I)] is the practical upper bound on the duration of activity I. The intermediate times, TIME(I,2), ..., TIME[I,NK(I)-1], will be called "breakpoints."

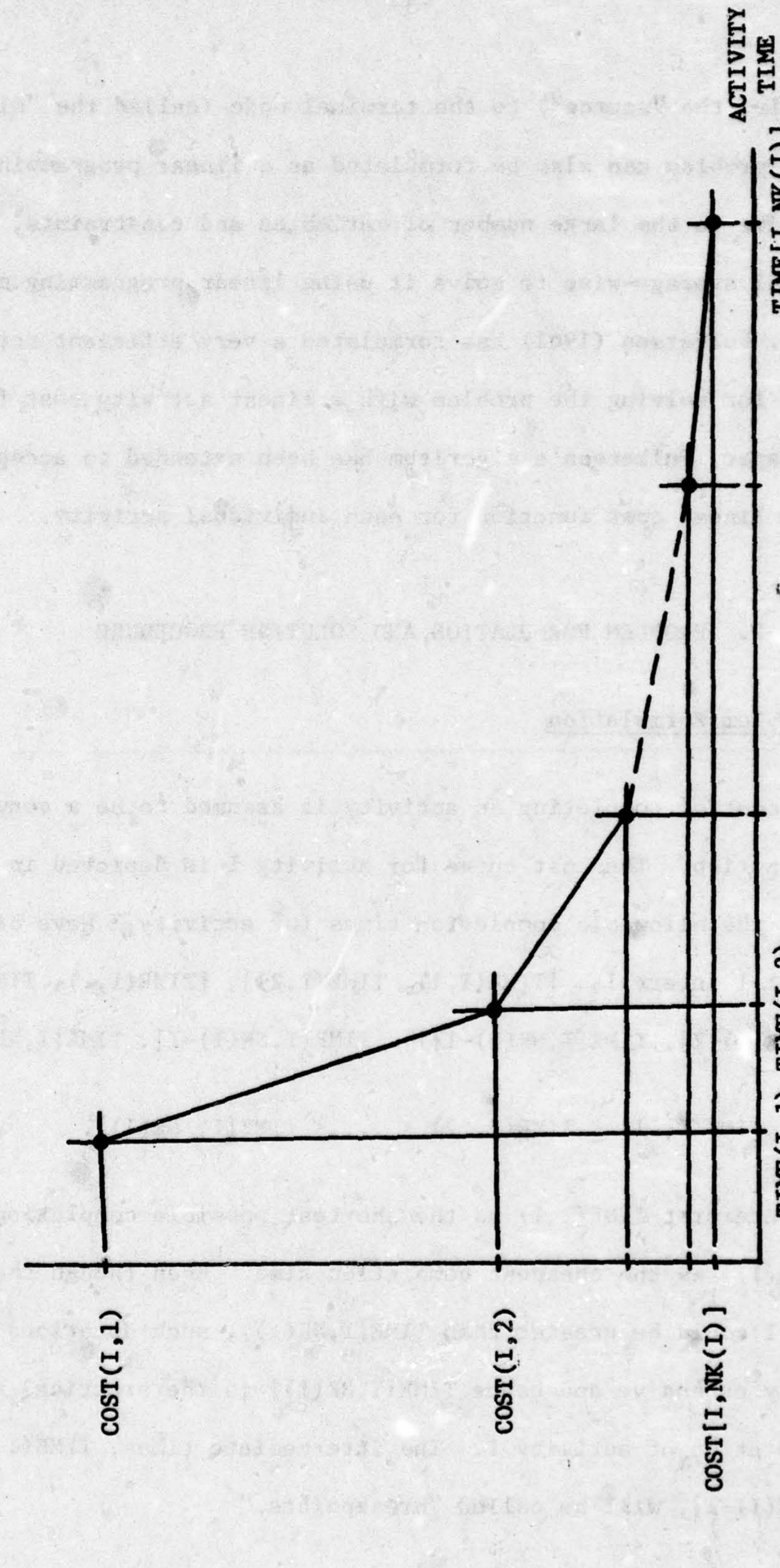

ACTIVITY COST
COST(I,1)
COST(I,2)
COST[I,NK(I)]
ACTIVITY TIME
TIME(I,1)
TIME(I,2)
TIME[I,NK(I)]

FIGURE 2

Breakpoints arise when there are alternative methods of performing an activity. These methods do not differ in the end result, but they do differ in the amount of time they take and their cost. For example, suppose that snow plows rent for a fixed $200/day and cost a varying amount per hour to operate depending upon the speed at which they are operated. A corresponding activity cost curve might be as in Figure 3 where the "breakpoints" correspond to the use of different numbers of plows.

The cost for completing activity I in time TIME(I,M) is COST(I,M) which satisfies

$$\text{COST}(I,1) \geq \text{COST}(I,2) \geq \ldots \geq \text{COST}[I,\text{NK}(I)]. \qquad (2.2)$$

Furthermore, letting C(I,M) represent the rate of decrease in the cost of activity I on the M^{th} interval implies

$$C(I,1) = \frac{\text{COST}(I,1) - \text{COST}(I,2)}{\text{TIME}(I,2) - \text{TIME}(I,1)}, \qquad (2.3)$$

$$\vdots$$

$$C[I,\text{NK}(I)-1] = \frac{\text{COST}[I,\text{NK}(I)-1] - \text{COST}[I,\text{NK}(I)]}{\text{TIME}[I,\text{NK}(I)] - \text{TIME}[I,\text{NK}(I)-1]}.$$

The convexity of the piecewise linear cost function implies that

$$C[I,\text{NK}(I)-1] \leq C[I,\text{NK}(I)-2] \leq \ldots \leq C(I,1). \qquad (2.3a)$$

Let XACT(I) represent the duration time for activity I. This duration time, XACT(I), can be decomposed as

$$\text{XACT}(I) = \sum_{M=1}^{\text{NK}(I)-1} \text{XACT}(I,M) \qquad (2.4)$$

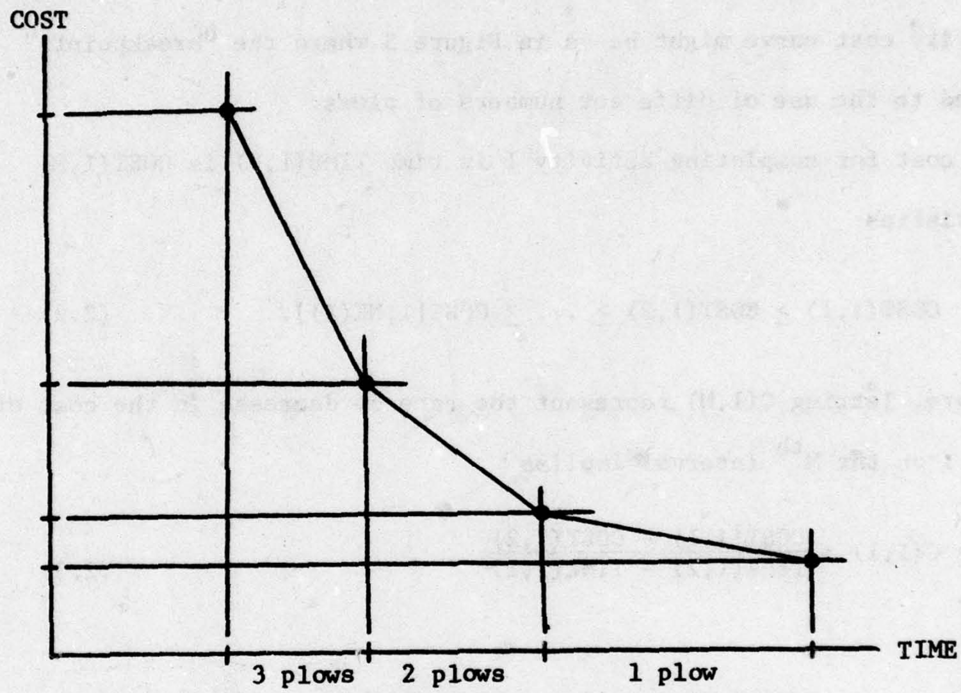


FIGURE 3

where

$$XACT(I,1) = \min[TIME(I,2), XACT(I)] \qquad (2.5)$$

and for M = 2, ..., NK(I) - 1

$$XACT(I,M) = \min\{TIME(I,M+1) - TIME(I,M),$$
$$\max[0, XACT(I) - TIME(I,M)]\}. \qquad (2.6)$$

For example, suppose that in Figure 4 XACT(I) = 25, then

$$XACT(I,1) = \min[TIME(I,2), XACT(I)]$$
$$= \min[10, 25]$$
$$= 10,$$

$$XACT(I,2) = \min\{TIME(I,3) - TIME(I, 2), \max[0, XACT(I) - TIME(I,2)]\}$$
$$= \min\{20 - 10, \max[0, 25 - 10]\}$$
$$= 10,$$

$$XACT(I,3) = \min\{TIME(I,4) - TIME(I, 3), \max[0, XACT(I) - TIME(I,3)]\}$$
$$= \min\{30 - 20, \max[0, 25 - 20]\}$$
$$= 5,$$

and

$$XACT(I) = \sum_{M=1}^{3} XACT(I,M)$$
$$= 10 + 10 + 5$$
$$= 25.$$

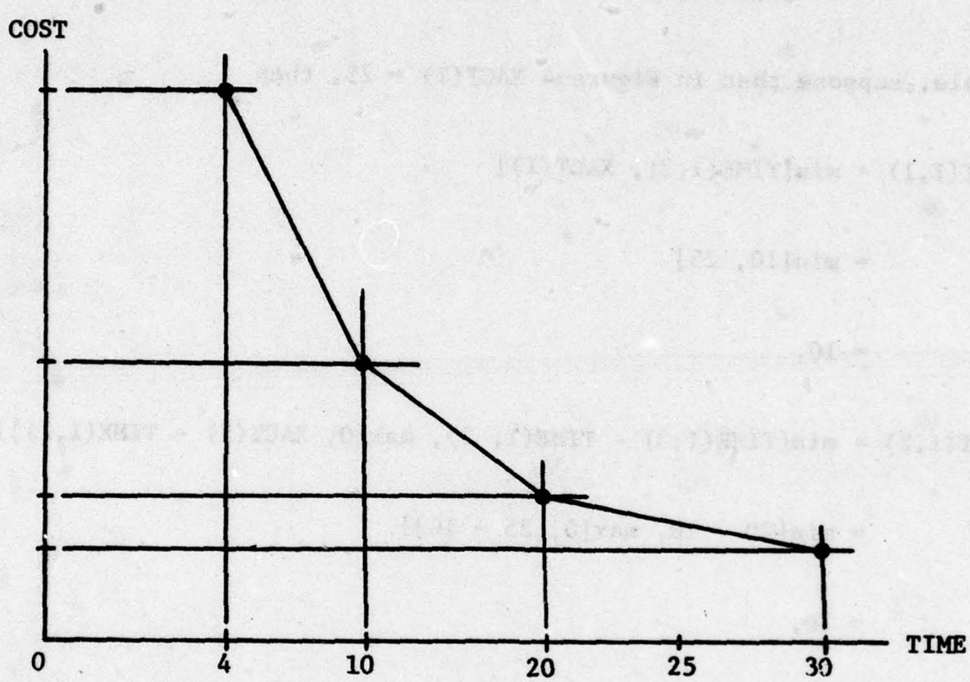


FIGURE 4

The total cost associated with duration time XACT(I) for activity I is

$$KK(I) - \sum_{M=1}^{NK(I)-1} C(I,M)XACT(I,M) \qquad (2.7)$$

where

$$KK(I) = COST(I,1) + C(I,1)TIME(I,1). \qquad (2.8)$$

The total project cost is

$$\sum_{I} [KK(I) - \sum_{M=1}^{NK(I)-1} C(I,M)XACT(I,M)]. \qquad (2.9)$$

Let the node time XNODE(K) be the "length" of the longest path from the source node to node K when the "length" of an arc (activity) is its completion time. Thus, for example, in Figure 1

$$XNODE(1) = 0,$$

$$XNODE(2) = A,$$

$$XNODE(3) = max(B, A + F),$$

$$XNODE(4) = max(A + C, B + D, A + F + D), \text{ and}$$

$$XNODE(5) = max(A + C + E, B + D + E, A + F + D + E).$$

If activity I originates at node 0_I, terminates at node T_I, and takes XACT(I) units of time, feasibility requires that

$$XNODE(0_I) + XACT(I) \leq XNODE(T_I). \qquad (2.10)$$

Note that the time to complete the entire project is

$$XNODE(SINK) - XNODE(SOURCE).$$

In what follows

$$XNODE(SOURCE) \equiv 0 \qquad (2.11)$$

without loss of generality.

The problem is to minimize the total project cost (2.9) subject to the condition that the project is completed by a specified time LAMBDA. This problem can now be formulated as

$$\min\{PCOST(LAMBDA) \equiv \sum_{I}[KK(I) - \sum_{M=1}^{NK(I)-1} C(I,M)XACT(I,M)]\} \qquad (2.12)$$

subject to the constraints

$$XNODE(O_I) + \sum_{M=1}^{NK(I)-1} XACT(I,M) - XNODE(T_I) \le 0, \text{ all } I, \qquad (2.13)$$

$$XNODE(SINK) \le LAMBDA, \qquad (2.14)$$

$$XACT(I,M) \le U(I,M), \qquad \text{all I and M}, \qquad (2.15)$$

$$XACT(I,M) \ge L(I,M), \qquad \text{all I and M}, \qquad (2.16)$$

where

$$U(I,M) = \begin{cases} TIME(I,2) & M = 1, \\ TIME(I,M+1) - TIME(I,M) & M = 2, \ldots, NK(I)-1, \end{cases} \qquad (2.17)$$

$$L(I,M) = \begin{cases} TIME(I,1) & M = 1, \\ 0 & M = 2, \ldots, NK(I)-1, \end{cases} \qquad (2.18)$$

O_I = the origin node of activity I,

T_I = the terminal node of activity I.

Since the addition or subtraction of a constant in the objective function does not change the problem, we can represent the objective function as

$$\max \sum_{I} \sum_{M=1}^{NK(I)-1} C(I,M)XACT(I,M). \qquad (2.19)$$

We shall solve this problem for all feasible values of LAMBDA. The minimum feasible value of LAMBDA, LMIN, is the length of the longest path from the source to the sink when the XACT(I)'s are at their lower bounds, XACT(I) = TIME(I,1) for all I. The maximum value of interest for LAMBDA, LMAX, is the length of the longest path from the source to the sink when the XACT(I)'s represent the cheapest practical times, XACT(I) = TIME[I,NK(I)] for all I. Thus, for a given LAMBDA such that

$$LMIN \leq LAMBDA \leq LMAX,$$

the constraints (2.13) - (2.16) are feasible. The proof for this and all other underlying theorems presented in the problem formulation and algorithm are found in Chapter 3, Section 2. We shall refer to the problem given in (2.13) - (2.19) as the Primal Problem.

In the Primal Problem, dummy activities may be assumed to have times and costs equal to zero.

2.2. The Dual Problem

The standard duality theory for linear programming implies that, if the primal problem has the form

$$\max c^T x$$

subject to the constraints

$$Ax \leq b, \qquad (2.20)$$

then the corresponding dual problem is

$$\min b^T w$$

subject to the constraints

$$A^T w = c$$

$$w \geq 0, \qquad (2.21)$$

see for example Hadley (1962). Writing our Primal Problem in the form (2.20) implies that our dual problem can be written as

$$\min[\mathrm{LAMBDA} \cdot V + \sum_I \sum_M U(I,M) \cdot G(I,M) - \sum_I \sum_M L(I,M) \cdot H(I,M)] \qquad (2.22)$$

subject to the constraints

$$F(I) + G(I,M) - H(I,M) = C(I,M) \qquad \text{all } I,M \qquad (2.23)$$

$$\sum_{I \in O_I = K} F(I) - \sum_{I \in T_I = K} F(I) = \begin{cases} 0 & K = \text{node} \neq \text{SOURCE,SINK} \\ -V & K = \text{SINK}, \end{cases} \qquad (2.24)$$

$$F(I),\ V,\ G(I,M),\ H(I,M) \geq 0. \qquad (2.25)$$

Note that the coefficients in (2.21) of the s^{th} dual variable are the coefficients in the s^{th} primal constraint, so that, there is a natural one-to-one correspondence between primal constraints and dual variables.

The dual problem (2.22) - (2.25) can be interpreted as a flow problem for the project network. The dual variable, F(I), associated with constraint (2.13) is the flow for the I^{th} activity. The constraint (2.24) implies that except for the source and the sink the flow going into a node equals the flow coming out of that node. Thus at all nodes other than the source and the sink we have conservation of flow. The total flow of the network is

$$V = \sum_{I \varepsilon T_I = SINK} F(I) - \sum_{I \varepsilon O_I = SINK} F(I) = \sum_{I \varepsilon T_I = SINK} F(I), \qquad (2.26)$$

and V is the dual variable associated with constraint (2.14) for a fixed LAMBDA.

The dual variables G(I,M) and H(I,M) are associated with the upper and lower bounds for XACT(I,M) respectively.

Rearranging (2.23), we have an equation of the form

$$g - h = c - f.$$

For a fixed value of f, we have c - f = r, say, and

$$g = h + r. \qquad (2.27)$$

In (2.22) we want to minimize an expression of the form

$$Ug - Lh,$$

or equivalently using (2.27)

$$Ug - Lh = U(h + r) - Lh = Ur + h(U - L).$$

Since g = h + r and both $g \geq 0$ and $h \geq 0$, making h as small as possible implies

$$h = \max(0, -r).$$

Correspondingly

$$g = h + r = \max(r, 0).$$

Thus

$$g = \max(0, c - f),$$

$$h = \max(0, f - c),$$

and correspondingly

$$G(I,M) = \max[0, C(I,M) - F(I)], \tag{2.28}$$

$$H(I,M) = \max[0, F(I) - C(I,M)]. \tag{2.29}$$

Using (2.28) and (2.29) the dual becomes

$$\min\{LAMBDA \cdot V + \sum_{IM}\sum U(I,M) \cdot \max[0, C(I,M) - F(I)]$$

$$- \sum_{IM}\sum L(I,M) \cdot \max[0, F(I) - C(I,M)]\} \tag{2.30}$$

subject to the constraints

$$\sum_{I\varepsilon O_I=K} F(I) - \sum_{I\varepsilon T_I=K} F(I) = \begin{cases} 0 & K = \text{node} \neq \text{SOURCE, SINK} \\ -V, & K = \text{SINK}, \end{cases}$$

$$F(I),\ V \geq 0.$$

A key observation at this point is that for all (I,M)

$$U(I,M)\max[0, C(I,M) - F(I)] - L(I,M)\max[0, F(I) - C(I,M)] \tag{2.31}$$

is a convex piecewise linear function of F(I) as sketched in Figure 5. The convexity of (2.31) follows from U(I,M) $\geq$ L(I,M). Furthermore, since the sum of convex piecewise linear functions is also a convex piecewise linear function, it follows that

$$\sum_M U(I,M)\max[0, C(I,M) - F(I)] - \sum_M L(I,M)\max[0, F(I) - C(I,M)] \tag{2.32}$$

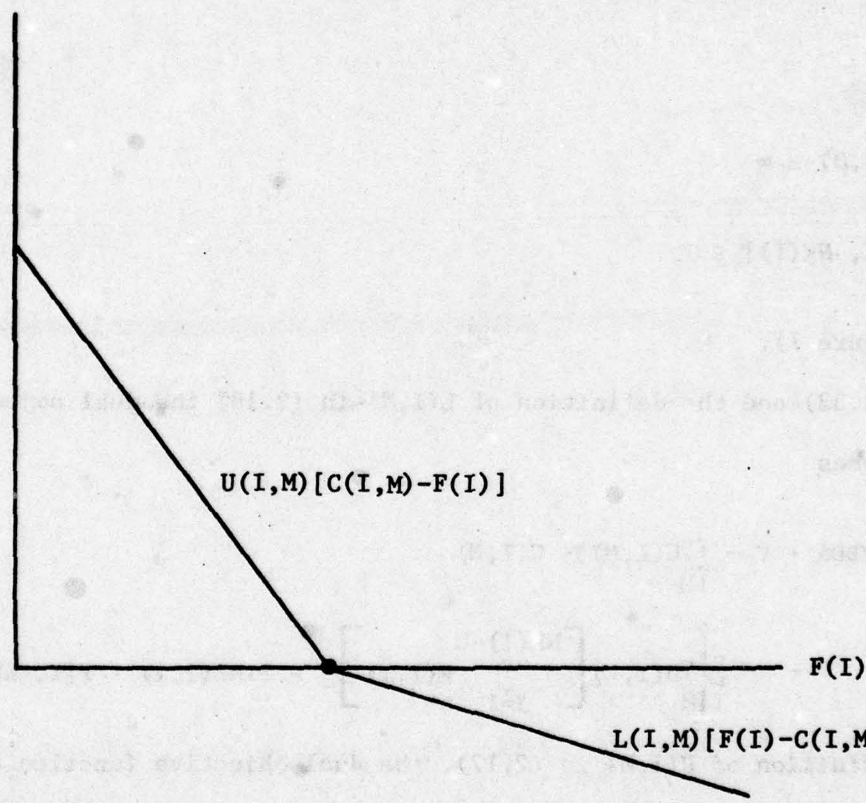


FIGURE 5

is a convex piecewise linear function of F(I) as sketched in Figure 6.

The piecewise linear behavior of (2.32) suggests the following decomposition of F(I):

$$F(I) = \sum_{J=1}^{NK(I)} F(I,J) \qquad (2.33)$$

where

$$0 \leq F(I,J) \leq C[I, NK(I) - J] - C(I, NK(I) - J + 1] \qquad (2.34)$$

and

$$C(I,0) \equiv \infty$$

$$C[I, NK(I)] \equiv 0.$$

(also see Figure 7).

Using (2.33) and the definition of L(I,M) in (2.18) the dual objective function becomes

$$LAMBDA \cdot V + \sum_{IM}\sum U(I,M) \cdot C(I,M)$$

$$- \sum_{I}\left\{\sum_{M} U(I,M)\left[\sum_{J=1}^{NK(I)-M} F(I,J)\right] - TIME(I,1) \cdot F[I, NK(I)]\right\} .$$

Using the definition of U(I,M) in (2.17), the dual objective function can be further simplified to

$$LAMBDA \cdot V + \sum_{IM}\sum U(I,M) \cdot C(I,M) - \sum_{IJ}\sum TIME[I,NK(I)+1-J]F(I,J). \qquad (2.35)$$

Since $\sum_{IM}\sum U(I,M)C(I,M)$ is a constant, (2.35) is equivalent to

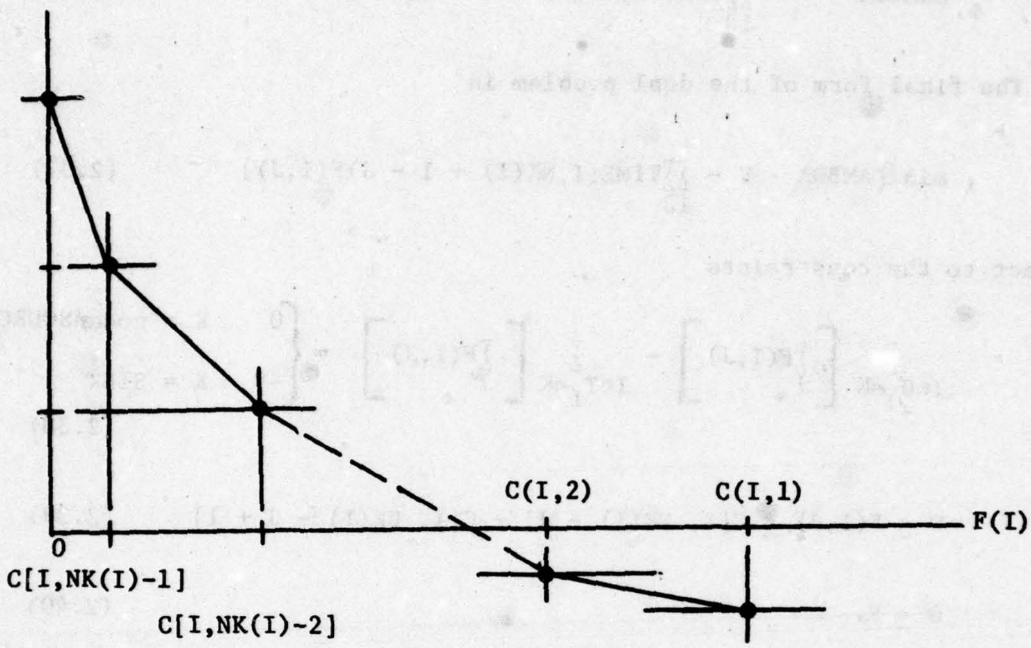


FIGURE 6

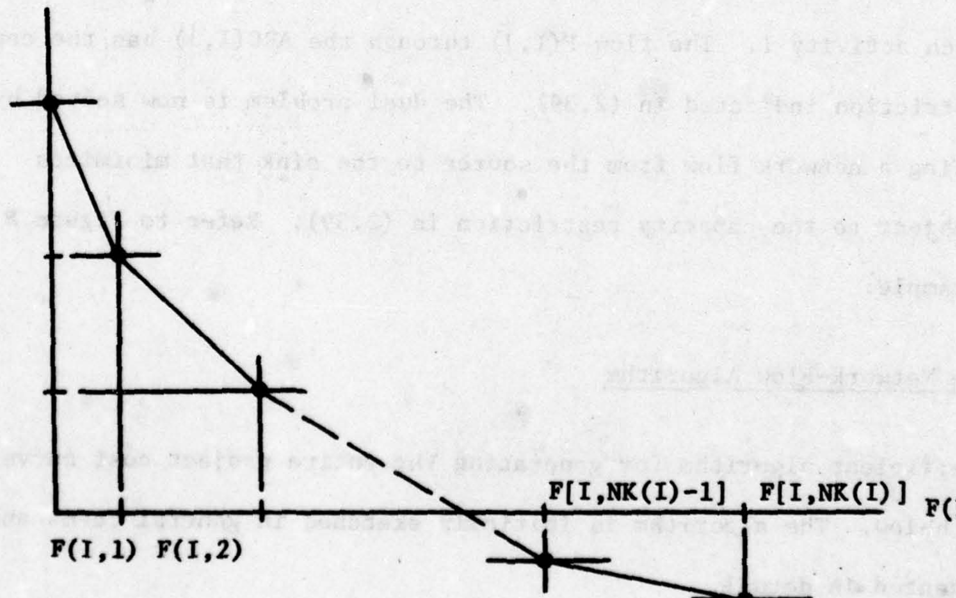


FIGURE 7

$$\text{LAMBDA} \cdot V - \sum_{I}\sum_{J}\text{TIME}(I,\text{NK}(I) + 1 - J)F(I,J). \qquad (2.36)$$

The final form of the dual problem is

$$\min\{\text{LAMBDA} \cdot V - \sum_{I}\sum_{J}\text{TIME}(I,\text{NK}(I) + 1 - J)F(I,J)\} \qquad (2.37)$$

subject to the constraints

$$\sum_{I\varepsilon O_I=K}\left[\sum_J F(I,J)\right] - \sum_{I\varepsilon T_I=K}\left[\sum_J F(I,J)\right] = \begin{cases} 0 & K = \text{node} \neq \text{SOURCE,SINK} \\ -V & K = \text{SINK} \end{cases} \qquad (2.38)$$

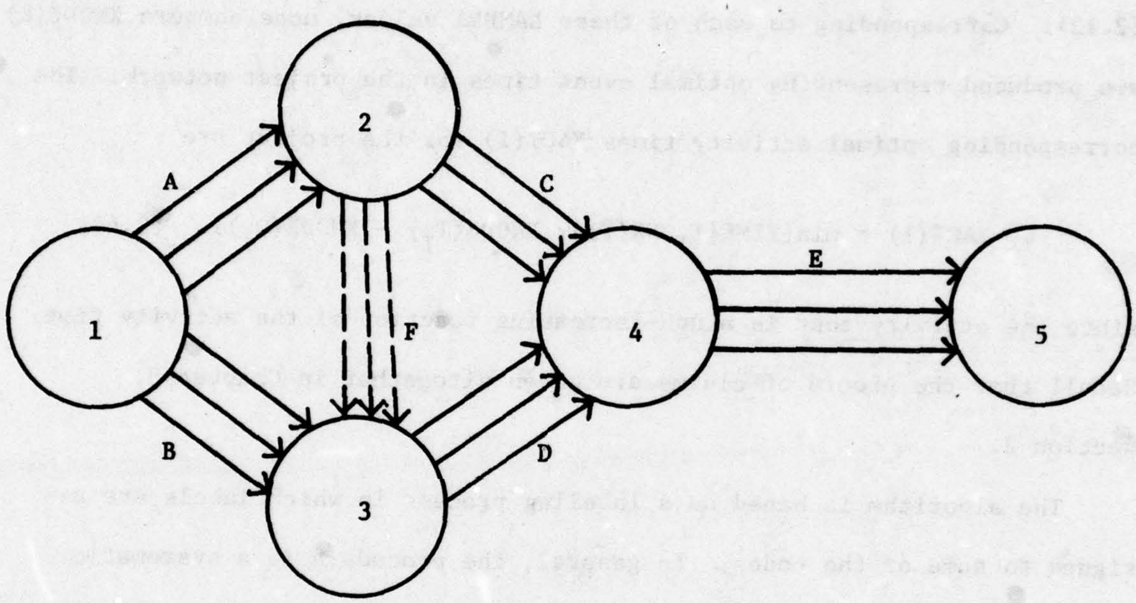
$$0 \leq F(I,J) \leq C[I,\ \text{NK}(I) - J] - C[I,\ \text{NK}(I) - J + 1] \qquad (2.39)$$

$$0 \leq V. \qquad (2.40)$$

This dual problem can be solved as a network flow problem. The original project network is enlarged by adding one arc, say ARC(I,J), for each F(I,J) so that the project network has NK(I) arcs from O_I to T_I corresponding to each activity I. The flow F(I,J) through the ARC(I,J) has the capacity restriction indicated in (2.39). The dual problem is now solved by constructing a network flow from the source to the sink that minimizes (2.36) subject to the capacity restriction in (2.39). Refer to Figure 8 for an example.

2.3. The Network-Flow Algorithm

An efficient algorithm for generating the entire project cost curve is given below. The algorithm is initially sketched in general terms and then presented in detail.



Example: NK(I) = 3, all I.

FIGURE 8

2.3.1. The Sketch

The algorithm starts with the largest LAMBDA of interest, LMAX, and sequentially determines the LAMBDA corresponding to each breakpoint of the convex piecewise linear project cost function PCOST(LAMBDA) defined in (2.12). Corresponding to each of these LAMBDA values, node numbers XNODE(K) are produced representing optimal event times in the project network. The corresponding optimal activity times XACT(I) for the project are

$$\mathrm{XACT(I)} = \min\{\mathrm{TIME[I,\ NK(I)]},\ \mathrm{XNODE}(T_I) - \mathrm{XNODE}(O_I)\} \quad (2.41)$$

since the activity cost is a non-increasing function of the activity time. Recall that the proofs of claims are given altogether in Chapter 3, Section 2.

The algorithm is based on a labeling process in which labels are assigned to some of the nodes. In general, the procedure is a systematic search for a path from the source to the sink having certain desired properties. Flow along this path may travel through arcs either in the same direction as their orientation or in the opposite direction. Such flows will be called forward and reverse flows respectively. Roughly speaking, a reverse flow is really only a reversal or re-routing of earlier flow in the forward direction. No net flow in the reverse direction is allowed.

The labeling process is started with a feasible and optimal solution to the primal and dual problems for LAMBDA = LMAX. The initial node times are found by setting the activity times equal to their upper bounds. These initial XNODE(K)'s and the initial flow - F(I,J) = 0 for all (I,J) - satisfy the following properties:

$$\text{ABAR}(I,J) < 0 \Rightarrow F(I,J) = 0, \text{ and} \tag{2.42}$$

$$\text{ABAR}(I,J) > 0 \Rightarrow F(I,J) = C[I,\ NK(I) - J] - C[I,\ NK(I) - J + 1] \tag{2.43}$$

where

$$\text{ABAR}(I,J) \equiv \text{TIME}[I,\ NK(I) + 1 - J] + \text{XNODE}(O_I) - \text{XNODE}(T_I). \tag{2.44}$$

Note that no restrictions are placed on F(I,J) when ABAR(I,J) = 0. Henceforth, the properties (2.42) and (2.43) will be referred to as the "optimality properties" for LAMBDA = XNODE(SINK). These optimality properties imply that complementary slackness holds and that the flow F(I,J) minimizes (2.36).

The labeling process has been divided into two parts called the first and second labelings, respectively. In both of these procedures we have freedom to label with respect to complementary slackness since we work exclusively with arcs having ABAR(I,J) = 0. The first labeling seeks a path from the source to the sink composed of infinite capacity arcs, i.e. those corresponding to J = NK(I). If such a path is found, the algorithm terminates since the Primal Problem will be infeasible if the current value of LAMBDA is decreased. If no such path is found, we go on to the second labeling in which we search for a path from the source to the sink having the following desired properties: for all forward arcs of the path ABAR(I,J) = 0 and F(I,J) is less than its upper bound in (2.39); for all reverse arcs of the path ABAR(I,J) = 0 and F(I,J) > 0. If at the end of the second labeling the sink has been labeled, we say "breakthrough" has occurred.

If breakthrough occurs, then the minimum arc capacity along the path is determined, say CAP(SINK). The old flow F(I,J) is changed by adding CAP(SINK) to the amount of all forward flows on the path and by subtracting CAP(SINK) from the amount of all reverse flows on the path. This new flow still satisfies the optimality properties and is interpreted as an alternate optimal dual solution for the current LAMBDA=XNODE(SINK). On the other hand, if the sink has not been labeled at the end of the second labeling, we say "nonbreakthrough" has occurred. When this happens, the old dual variables are optimal for the old primal problem and no new alternate dual solution can be found. In this case the node numbers XNODE(K)'s are changed by subtracting a positive quantity DEL from all XNODE(K) corresponding to unlabeled nodes K. This does not change XNODE(SOURCE) = 0 but reduces XNODE(SINK) = LAMBDA by DEL. Through (2.41), these new node times imply a set of optimal activity times for the new LAMBDA where

new LAMBDA = old LAMBDA - DEL.

The definition of DEL guarantees that the new XNODE(K)'s and the old F(I,J)'s still satisfy the optimality properties. Hence, when nonbreakthrough occurs, we have identified the point on the project cost curve corresponding to the new LAMBDA.

The second labeling can terminate only in breakthrough or nonbreakthrough. After either of these, the entire labeling process is repeated.

2.3.2. The Details

Initially, the algorithm sets each activity time to its smallest (most expensive) feasible value and determines the corresponding minimum feasible project completion time (deadline time LMIN). Then, the algorithm sets each activity completion time to its largest (cheapest) feasible value and determines the corresponding minimum project cost and maximum completion time of interest (deadline time LMAX).

The iterative procedure is begun with the node times XNODE(K) corresponding to all activity completion times at their largest (cheapest) values and all flows F(I,J) equal to zero. These node times and flows satisfy the optimality properties.

A. Labeling Process. During this routine, a node is considered to be in one of three states: unlabeled, labeled and unscanned, or labeled and scanned. Initially all nodes are unlabeled.

In general, a node label has four parts [A, B, C, D] when the node is being labeled because it is at "the other end" of an arc associated with some F(I,J). If "the other end" is the terminal node T_I, then the label contains

$A = O_I$, $B = J$, $D =$ maximum allowable flow, and

$C = 0$ [denoting that flow will be in the forward direction ($O_I \rightarrow T_I$)].

If "the other end" is the origin O_I, then the label contains $A = T_I$, $B = J$, $D =$ maximum allowable flow, and $C = 1$ [denoting that flow will be in the reverse direction ($T_I \rightarrow O_I$)].

1. First Labeling. Assign the source node the label [-, -, -, CAP(SOURCE) = ∞]. In general, select any labeled, unscanned node, say node n, and search for all unlabeled nodes T_I such that $n = O_I$ and ARC[I, NK(I)] is an arc with

$$ABAR[I, NK(I)] = 0. \qquad (2.45)$$

Label such nodes T_I with $[0_I, NK(I), 0, CAP(T_I) = \infty]$. Such T_I's are now labeled and unscanned, and node n is labeled and scanned. Repeat this step until either the sink node is labeled and unscanned, or no more nodes can be labeled and the sink node is unlabeled. In the former case, terminate the algorithm. In the latter case, go on to the Second Labeling.

2. Second Labeling. Nodes that were labeled from the First Labeling retain their labels. However, all nodes revert back to an unscanned state. The general step is to select any labeled, unscanned node, say n.

(i) Scan n for all unlabeled nodes T_I such that $n = 0_I$. For each such node T_I find the J (if one exists) such that both

$$ABAR(I,J) = 0 \qquad (2.46)$$

and

$$F(I,J) < C[I, NK(I) - J] - C[I, NK(I) - J + 1]; \qquad (2.47)$$

then assign node T_I the label $[0_I, J, 0, CAP(T_I)]$ where

$$CAP(T_I) = \min\{CAP(0_I), C[I, NK(I) - J] - C[I, NK(I) - J + 1] - F(I,J)\} \qquad (2.48)$$

so that T_I is now labeled and unscanned. If no such J exists, the node T_I is not labeled.

(ii) Scan n for all unlabeled nodes 0_I such that $n = T_I$. For each such node 0_I find the J (if one exists) such that both

$$ABAR(I,J) = 0 \qquad (2.49)$$

and

$$F(I,J) > 0; \tag{2.50}$$

then assign node 0_I the label $[T_I, J, 1, CAP(0_I)]$ where

$$CAP(0_I) = \min[CAP(T_I), F(I,J)] \tag{2.51}$$

so that 0_I is now labeled and unscanned. If no such J exists, the node 0_I is not labeled.

Repeat the general step until either the sink node is labeled and unscanned (breakthrough), or no more nodes can be labeled and the sink node is unlabeled (nonbreakthrough). If breakthrough occurs, go on to routine B; if nonbreakthrough occurs, go to routine C.

B. <u>Flow Change</u>. The labeling process has resulted in breakthrough. The sink node will have a label of the form $[0_I, J, 0, CAP(SINK)]$. The total network flow will now be increased by CAP(SINK). The flows are updated as follows. Add CAP(SINK) to F(I,J); then go on to node $n = 0_I$ and its label. The general step for node n depends on its label and is:

1. Label = $[0_I, J, 0, CAP(T_I)]$. Add CAP(SINK) to F(I,J) since this additional flow along ARC(I,J) will be forward flow from 0_I to $n = T_I$. The next node to consider is $n = 0_I$.

2. Label = $[T_I, J, 1, CAP(0_I)]$. Subtract CAP(SINK) from F(I,J) since this additional flow along ARC(I,J) will be a reversal of previous flow from $n = 0_I$ to T_I. The next node to consider is $n = T_I$.

This iternative procedure is continued until n = SOURCE. At this point a path from the source to the sink has been retraced working backwards from the sink. The arcs on this path that are traversed in the forward direction ($O_I \rightarrow T_I$) as we go from the source to the sink have their flows increased by CAP(SINK) while the arcs on this path that are traversed in the reverse direction ($T_I \rightarrow O_I$) have their flows decreased by CAP(SINK).

All labels are now discarded and the labeling process (A) is started over.

C. Node Number Change. The labeling process has resulted in non-breakthrough. The following subsets of arcs are determined:

$$A_1 = \{(I,J) \mid O_I \text{ labeled, } T_I \text{ unlabeled, } ABAR(I,J) < 0\}, \quad (2.52)$$

$$A_2 = \{(I,J) \mid O_I \text{ unlabeled, } T_I \text{ labeled, } ABAR(I,J) > 0\}. \quad (2.53)$$

We now define

$$DELTA1 = \min_{A_1}[-ABAR(I,J)], \quad (2.54)$$

$$DELTA2 = \min_{A_2}[ABAR(I,J)], \quad (2.55)$$

$$DEL = \min(DELTA1, DELTA2). \quad (2.56)$$

The node numbers XNODE(K) are changed by subtracting DEL from all XNODE(K) corresponding to unlabeled K. All labels are discarded and the labeling process (A) is started over.

2.4. Flowchart of the Algorithm

See Figure 9.

FIGURE 9: Flowchart of the Algorithm

Read the algorithm parameters:

NA = Total number of activities

NN = Total number of nodes

SINK = Number of the sink node

SOURCE = Number of the source node

For each activity I:

O_I, T_I, NK(I)

COST(I,1),...,COST[I, NK(I)]

TIME(I,1),...,TIME[I, NK(I)]

Either the entire project cost curve is determined or optionally just the optimal activity times for one specified project deadline time, LAMBDA.

↓

Set all F(I,J)=0

Starting routines set all activity times at their minimums and determine the corresponding XNODE(K)'s and minimum feasible LAMBDA, LMIN.

Is specified LAMBDA $\leq$ LMIN?

YES → Terminate the algorithm. No feasible activity times.

NO → Set all activity times at their upper bounds and determine the corresponding XNODE(K)'s and maximum practical LAMBDA, LMAX.

Is specified LAMBDA $\geq$ LMAX?

YES → Terminate the algorithm. The optimal activity completion times are their upper bounds.

NO → Assign the SOURCE the label [-,-,-,∞], all other nodes are unlabeled.

↓

Go to (2)

(2)

General step:

Select any labeled node, n, and search for activities I with O_I=n, T_I unlabeled, and ABAR[I,NK(I)]=0.

Label such T_I nodes with $\{O_I, NK(I), 0, CAP[T_I]=\infty\}$

Repeat until no more nodes can be labeled.

Is SINK labeled?

YES: Terminate the algorithm. No feasible activity times.

NO: Labeled nodes retain labels. Current node=SOURCE.

General step:

Select any labeled node, then go to either (1) or (2) according to the node:

(1) If node=O_I, search for unlabeled T_I such that ARC(I,J) exists with ABAR(I,J)=0 and F(I,J)<C[I,NK(I)-J]-C[I,NK(I)-J+1]. Label such nodes $\{O_I$, J,0,CAP$[T_I]\}$ where CAP$[T_I]$=min{CAP$[O_I]$, C[I,NK(I)-J]-C[I,NK(I)-J+1]-F(I,J)}.

(2) If node=T_I, search for unlabeled O_I such that ARC(I,J) exists with ABAR(I,J)=0 and F(I,J)>0. Label such nodes $\{T_I$,J,1,CAP$[O_I]\}$ where CAP$[O_I]$=min{CAP$[T_I]$,F(I,J)}.

Repeat general step until either no more nodes can be labeled or the SINK is labeled.

If SINK is labeled, breakthrough has occurred.

If SINK is not labeled, nonbreakthrough has occurred. → Go to (3)

The path from the SOURCE to the SINK is retraced starting at the SINK. Update the F(I,J)'s by adding CAP(SINK) to all forward flows along the path and subtracting CAP(SINK) from all reverse flows along the path. Continue until the SOURCE is reached. Discard labels and return to (1).

(3)

Find the following subsets:

A_1: {(I,J) such that 0_I is labeled, T_I is unlabeled, and ABAR(I,J)<0}

A_2: {(I,J) such that 0_I is unlabeled, T_I is labeled, and ABAR(I,J)>0}

$$DELTA1 = \min_{A_1}[-ABAR(I,J)]$$

$$DELTA2 = \min_{A_2}[ABAR(I,J)]$$

$$DEL = \min[DELTA1, DELTA2].$$

Subtract DEL from all unlabeled nodes XNODE(K). Then the

$$XACT(I) = \min\{TIME[I,NK(I)], XNODE(T_I)-XNODE(0_I)\}$$

are an alternative optimal solution for the current LAMBDA and also an optimal solution for new LAMBDA = current LAMBDA - DEL.

A new point on the project cost curve has been determined.

Is new LAMBDA $\leq$ specified LAMBDA?

YES: Terminate the algorithm. Desired solution found.

NO: Discard labels and return to (1).

3. VERIFICATION OF CLAIMS

The algorithm described in the previous chapter is based on many claims. The lemmas and theorems given below prove these claims and, at the same time, show that the given algorithm does indeed yield for each project deadline time LAMBDA the individual activity completion times which minimize the total project cost.

3.1. Summary

The initial primal and dual variables XACT(I),XNODE(K) and F(I,J) provide a feasible and optimal solution for the largest LAMBDA of interest, LMAX (Lemmas 1, 5 and 6). The changes applied to these variables arise from either breakthrough or nonbreakthrough and force the variables to remain feasible and satisfy the optimality properties throughout the algorithm (Lemmas 2, 3, 4, 5, 6 and Theorem 1). The optimality properties imply that complementary slackness holds which, combined with feasibility, implies that the solution is optimal for a given LAMBDA (Lemma 7 and Theorem 2).

The algorithm itself terminates after a finite number of applications of the labeling procedure (Theorem 3). At the conclusion of the computations a path from the source to the sink has been identified in the First Labeling step such that along this path

$$TIME(I,1) + XNODE(O_I) = XNODE(T_I).$$

Since (TIMEI,1) is the minimum feasible completion time for activity I, this means that the minimum possible time to complete the sequence of activities along this path is XNODE(SINK) = LAMBDA. Hence any further decrease in LAMBDA would make the Primal Problem infeasible; i.e., the project cannot be completed in any shorter time.

The project cost function PCOST(LAMBDA) is convex and is linear between the successively determined values of LAMBDA generated in the computations (Lemmas 8 and 9). Given two successively determined values of LAMBDA, say L1 and L2 = L1 - DEL, the optimal node times and activity completion times for any project deadline time L between L1 and L2 are

$$XNODE_L(K) = \begin{cases} XNODE_{L1}(K) & \text{if K labeled when LAMBDA=L1,} \\ XNODE_{L1}(K)-(L1-L) & \text{if K unlabeled when LAMBDA=L1,} \end{cases}$$

$$XACT_L(I) = \min\{TIME[I,NK(I)],XNODE_L[T_I]-XNODE_L[O_I]\}$$

where the subscript L1 implies LAMBDA = L1 (Theorem 4).

One additional feature of the algorithm is that, if the problem is specified in terms of integers, then the breakpoints of the project cost curve PCOST(LAMBDA) and the corresponding optimal activity times will all be integers.

3.2. Proofs

Lemma 1: The original set of node integers XNODE(K) and the zero flow F(I,J) satisfy the optimality properties. Furthermore, this F(I,J) minimizes (2.36) implying an optimal solution for LAMBDA = LMAX.

Proof: In a starting routine the activity times XACT(I) are set to their largest feasible (cheapest) values. Then the node times XNODE(K) are set to their corresponding smallest feasible values. This implies that

$$TIME[I, NK(I)] \le XNODE(T_I) - XNODE(O_I)$$

or equivalently

$$TIME[I, NK(I)] + XNODE(O_I) - XNODE(T_I) \le 0.$$

Thus all ABAR(I, J) $\le$ 0. Finally, since all ABAR(I,J) $\le$ 0 and F(I,J) = 0, the optimality properties are satisfied.

The dual objective function is

$$LAMBDA \cdot V - \sum_{I,J} TIME(I,NK(I) + 1 - J) \cdot F(I,J)$$

$$= -[\sum_{I,J} TIME(I,NK(I) + 1 - J) \cdot F(I,J) - LAMBDA \cdot V]$$

$$= -\{\sum_{I,J} TIME(I, NK(I) + 1 - J) \cdot F(I,J) + [XNODE(SOURCE) - XNODE(SINK)] \cdot V\}$$

$$= -\{\sum_{I,J} TIME(I,NK(I) + 1 - J) \cdot F(I,J) + \sum_{I,J} [XNODE(O_I) - XNODE(T_I)] \cdot F(I,J)\}$$

$$= -[\sum_{I,J} ABAR(I,J) \cdot F(I,J)].$$

Thus, since all ABAR(I,J) $\le$ 0, F(I,J) = 0 is optimal. QED.

<u>Lemma 2</u>: If breakthrough occurs, the old node numbers and the new flow satisfy the optimality properties.

<u>Proof</u>: The node numbers XNODE(K) do not change. The new flows are obtained by adding the positive number CAP(SINK) to all F(I,J) corresponding to forward arcs of the path from the source to the sink, and subtracting CAP(SINK) from all F(I,J) corresponding to reverse arcs of the path.

Flow changes occur only in arcs for which ABAR(I,J) = 0. No restriction is imposed on the F(I,J)'s in the optimality properties when ABAR(I,J) = 0. Thus, the old XNODE(K)'s and the new F(I,J)'s still satisfy the optimality properties. QED.

Lemma 3: If nonbreakthrough occurs, the node number change, DEL, is a well-defined positive number.

Proof: For DEL to be well-defined, at least one of the sets of arcs A_1, A_2 (as defined in equations (2.52) and (2.53)) is non-empty.

Suppose A_1 were empty. Since there is a path from the source to the sink in the project network, and since the source is labeled and the sink is unlabeled, there must be a set of arcs {ARC(I,J), J = 1, ..., NK(I)} in the enlarged network with 0_I labeled and T_I unlabeled. The definition of A_1 implies that if A_1 is empty, then ABAR(I,J) $\geq$ 0 for this set of arcs. From labeling rules (2.46) and (2.47), if ABAR(I,J) = 0 then F(I,J) cannot be less than {C[I, NK(I) - J] - C[I, NK(I) - J + 1]}, otherwise T_I would have been labeled from 0_I. From (2.43), if ABAR(I,J) > 0, this implies that F(I,J) = C[I, NK(I) - J] - C[I, NK(I) - J + 1]. Hence we have F[I, NK(I)] = ∞. But this F[I, NK(I)] is part of the actual flow through the network and, if it equals infinity, the first labeling process would have terminated the algorithm. Since this has not happened, there are no infinite flows and A_1 is non-empty.

By definition, DEL is always positive. QED.

Lemma 4: If nonbreakthrough occurs, for any DEL' satisfying 0 $\leq$ DEL' $\leq$ DEL, the new node numbers

$$\text{XNODE}'(K) = \begin{cases} \text{XNODE}(K) & \text{for K labeled,} \\ \text{XNODE}(K) - \text{DEL}' & \text{for K unlabeled,} \end{cases}$$

and the old flow F(I,J) still satisfy the optimality properties.

Proof: The new ABAR'(I,J) = TIME(I,NK(I) + 1 - J) + XNODE'(O_I) - XNODE'(T_I).

(i) Suppose ABAR'(I,J) < 0. Then F(I,J) = 0 because of the following:

(a) If ABAR(I,J) < 0, then F(I,J) = 0 by (2.42).

(b) If ABAR(I,J) = 0, then

$$\text{TIME}(I,\text{NK}(I) + 1 - J) + \text{XNODE}(O_I) - \text{XNODE}(T_I) = 0,$$

or equivalently

$$\text{TIME}(I,\text{NK}(I) + 1 - J) = - \text{XNODE}(O_I) + \text{XNODE}(T_I);$$

so that

$$\text{ABAR}'(I,J) = \text{TIME}(I,\text{NK}(I) + 1 - J) + \text{XNODE}'(O_I) - \text{XNODE}'(T_I) < 0,$$

implies

$$- \text{XNODE}(O_I) + \text{XNODE}(T_I) + \text{XNODE}'(O_I) - \text{XNODE}'(T_I) < 0,$$

and finally

$$\text{XNODE}'(O_I) - \text{XNODE}(O_I) < \text{XNODE}'(T_I) - \text{XNODE}(T_I);$$

but this can happen only when O_I is unlabeled and T_I is labeled. Hence, if ABAR(I,J) = 0, then by labeling rules (2.49) and (2.50), F(I,J) = 0, otherwise O_I would be labeled from T_I.

(c) If ABAR(I,J) > 0, then

$$\text{TIME}(I,\text{NK}(I) + 1 - J) + \text{XNODE}(O_I) - \text{XNODE}(T_I) > 0,$$

or equivalently

$$\text{TIME}(I,\text{NK}(I) + 1 - J) > - \text{XNODE}(O_I) + \text{XNODE}(T_I);$$

so that

$$ABAR'(I,J) = TIME(I,NK(I) + 1 - J) + XNODE'(0_I) - XNODE'(T_I) < 0,$$

implies

$$TIME(I,NK(I) + 1 - J) < XNODE'(T_I) - XNODE'(0_I),$$

and

$$XNODE'(T_I) - XNODE'(0_I) > - XNODE(0_I) + XNODE(T_I),$$

and finally

$$XNODE'(T_I) - XNODE(T_I) > XNODE'(0_I) - XNODE(0_I).$$

Again, this can happen only when 0_I is unlabeled and T_I is labeled. But then the arc ARC(I,J) is in A_2 and DEL $\leq$ ABAR(I,J). This would imply that

$$\begin{aligned} ABAR'(I,J) &= TIME(I,NK(I) + 1 - J) + XNODE(0_I) - DEL - XNODE(T_I) \\ &= ABAR(I,J) - DEL \\ &\geq 0. \end{aligned}$$

which contradicts the assumption ABAR'(I,J) < 0. Hence this case cannot occur.

(ii) Suppose ABAR'(I,J) = 0. There are no restrictions on F(I,J) so the optimality properties still hold.

(iii) Suppose ABAR'(I,J) > 0. Then

$$F(I,J) = C[I, NK(I) - J] - C[I, NK(I) - J + 1]$$

because of the following:

(a) If ABAR(I,J) > 0, F(I,J) = C[I, NK(I) - J] - C[I, NK(I) - J + 1] by (2.43).

(b) If ABAR(I,J) = 0, then

$$ABAR(I,J) < ABAR'(I,J)$$

or equivalently

$$TIME(I,NK(I) + 1 - J) + XNODE(0_I) - XNODE(T_I)$$

$$< TIME(I,NK(I) + 1 - J) + XNODE'(0_I) - XNODE'(T_I);$$

so that

$$XNODE(0_I) - XNODE'(0_I) < XNODE(T_I) - XNODE'(T_I).$$

This can happen only if 0_I is labeled and T_I is unlabeled. Hence, by labeling rules (2.46) and (2.47)

$$F(I,J) = C[I, NK(I) - J] - C[I, NK(I) - J + L],$$

otherwise T_I would be labeled from 0_I.

(c) If ABAR(I,J) < 0, then

$$ABAR(I,J) < ABAR'(I,J),$$

and again

$$XNODE(0_I) - XNODE'(0_I) < XNODE(T_I) - XNODE'(T_I).$$

This can happen only if 0_I is labeled and T_I is unlabeled. But then the arc ARC(I,J) is in A_1 and DEL $\leq$ - ABAR(I,J) which would imply that

$$\begin{aligned} ABAR'(I,J) &= TIME(I,NK(I) + 1 - J) + XNODE'(0_I) - XNODE'(T_I) \\ &= TIME(I,NK(I) + 1 - J) + XNODE(0_I) - XNODE(T_I) + DEL \\ &= ABAR(I,J) + DEL \\ &\leq 0. \end{aligned}$$

This contradicts the assumption ABAR'(I,J) > 0. Hence this case cannot occur.

Cases (i) - (iii) together imply that the new node numbers and the old flow still satisfy the optimality properties. QED.

<u>Theorem 1</u>: The optimality properties (2.42) and (2.43) are maintained throughout the algorithm.

<u>Proof</u>: From Lemma 1, we see that the initial node numbers XNODE(K)'s and the zero flow provide an optimal solution for LAMBDA = LMAX. If breakthrough occurs, we see that the new F(I,J)'s are still optimal (Lemma 2). If nonbreakthrough occurs, we have a well-defined positive number DEL with which to update the XNODE(K)'s (Lemma 3) and, from Lemma 4, these updated values satisfy the optimality properties. QED.

<u>Lemma 5</u>: The starting values of the XNODE(K)'s and XACT(I)'s are feasible and remain feasible throughout the algorithm.

<u>Proof</u>: The starting values are found by an algorithm that sets the XACT(I)'s to their largest feasible times, TIME[I, NK(I)]. Correspondingly the XACT(I,M)'s are set equal to their upper bounds and hence (2.15) and (2.16) are satisfied. Then the algorithm sets XNODE(K) equal to the length of the longest path from the source to node K, which implies that (2.13) is satisfied. We also define XNODE(SOURCE) ≡ 0 and LAMBDA = XNODE(SINK); hence (2.14) is satisfied and the initial values are feasible.

If breakthrough occurs, the XNODE(K)'s and XACT(I)'s are not changed and hence remain feasible.

If nonbreakthrough occurs, the labeled XNODE(K)'s are unchanged, and the unlabeled XNODE(K)'s are updated by subtracting DEL, determined by (2.56). Then

$$\text{new LAMBDA} \equiv \text{XNODE(SINK)} - \text{DEL}$$

so (2.14) is satisfied.

(i) Suppose both 0_I and T_I are labeled for activity I. Then neither these nodes nor XACT(I) are updated and hence XACT(I) remains feasible.

(ii) Suppose both 0_I and T_I are unlabeled for activity I. Then

$$\text{new XNODE}(0_I) = \text{old XNODE}(0_I) - \text{DEL},$$

$$\text{new XNODE}(T_I) = \text{old XNODE}(T_I) - \text{DEL, and}$$

$$\begin{aligned}\text{new XACT(I)} &= \min\{\text{TIME[I, NK(I)], old XNODE}(T_I) - \text{DEL} \\ &\quad - \text{old XNODE}(0_I) + \text{DEL}\} \\ &= \text{old XACT(I)} \\ &\leq \text{old XNODE}(T_I) - \text{old XNODE}(0_I) \\ &= \text{new XNODE}(T_I) - \text{new XNODE}(0_I),\end{aligned}$$

or equivalently

$$\text{new XACT(I)} + \text{new XNODE}(0_I) - \text{new XNODE}(T_I) \leq 0;$$

so that (2.13) is satisfied. Since XACT(I) has not changed, (2.15) and (2.16) are still satisfied. Therefore, in this case, feasibility is maintained.

(iii) Suppose 0_I is labeled and T_I is unlabeled for activity I. Then

$$\text{new XNODE}(T_I) = \text{old XNODE}(T_I) - \text{DEL},$$

$$\text{new XNODE}(0_I) = \text{old XNODE}(0_I), \text{ and}$$

$$\begin{aligned}\text{new XACT(I)} &= \min\{\text{TIME[I, NK(I)], old XNODE}(T_I) - \text{DEL} \\ &\quad - \text{old XNODE}(0_I)\};\end{aligned}$$

so that (2.13) and (2.15) are satisfied. The lower bound constraint, (2.16), is also satisfied because of the following:

(a) Suppose ABAR[I,NK(I)] < 0. Then since 0_I is labeled and T_I is unlabeled, the definition of DEL implies that

$$\text{XNODE}(T_I) - \text{XNODE}(0_I) - \text{TIME}(I,1) \geq \text{DEL}$$

and hence

$$\text{XNODE}(T_I) - \text{XNODE}(0_I) - \text{DEL} \geq \text{TIME}(I,1)$$

which implies that XACT(I) $\geq$ TIME(I,1).

(b) Now ABAR[I, NK(I)] = 0 cannot occur, since T_I would have been labeled from 0_I.

(c) Also ABAR[I, NK(I)] > 0 cannot happen since this would imply that

$$\text{old XNODE}(0_I) + \text{TIME}(I,1) \geq \text{old XNODE}(T_I)$$

which contradicts the feasibility of the previous node times.

(iv) Suppose 0_I is unlabeled and T_I is labeled for activity I. Then

$$\text{new XNODE}(0_I) = \text{old XNODE}(0_I) - \text{DEL},$$

$$\text{new XNODE}(T_I) = \text{old XNODE}(T_I), \text{ and}$$

$$\text{new XACT(I)} = \min\{\text{TIME[I, NK(I)], old XNODE}(T_I) - \text{old XNODE}(0_I) + \text{DEL}\};$$

so that (2.13) and (2.15) are satisfied. Since

$$\text{TIME}(I,1) \leq \text{old XACT(I)} \leq \text{new XACT(I)},$$

the lower bound constraint, (2.16), is trivially satisfied. QED.

Lemma 6: The starting values of the F(I,J)'s and V are feasible and remain feasible throughout the algorithm.

Proof: Initially, the values of the F(I,J)'s and V are set to zero. Conservation of flow, (2.24), is trivially satisfied since the flow going into each node is equal to zero which is also equal to the flow going out of each node, i.e.

$$\sum_{I\varepsilon O_I = K} [\sum_J F(I,J)] = 0 = \sum_{I\varepsilon T_I = K} [\sum_J F(I,J)].$$

Since all F(I,J) are set equal to zero, they satisfy their upper and lower bounds. Hence, the starting values are feasible.

If nonbreakthrough occurs, the values for F(I,J) do not change, hence remain feasible.

If breakthrough occurs, the F(I,J) along the path from the source to the sink are updated by a positive number CAP(SINK) determined by (2.48) or (2.51); all other flows remain unchanged. Suppose activity I is an arc along the path from the source to the sink. Then either T_I is labeled from O_I or O_I is labeled from T_I.

(i) In the former case, F(I,J) < C[I, NK(I) - J] - C[I,NK(I) -J + 1] by labeling rules (2.46) and (2.47), and CAP(I) is given by (2.48). This CAP(I) is the minimum of the previous CAP and C[I, NK(I) - J] - C[I, NK(I) - J + 1] - F(I, J) > 0. Now CAP(SINK) $\leq$ CAP(I) and new F(I,J) = old F(I,J) + CAP(SINK). Conservation of flow is satisfied since the same value is added to or subtracted from all activities along this path and V, the total flow, is increased by CAP(SINK). Also, (2.34) is satisfied since

$$0 \leq \text{old } F(I,J) + CAP(SINK) \leq \text{old } F(I,J) + CAP(I)$$
$$< C[I, NK(I) - J] - C[I, NK(I) - J + 1].$$

Hence the new F(I,J)'s are feasible.

(ii) In the latter case, F(I,J) > 0 by labeling rules (2.49) and (2.50), and CAP(I) is given by (2.51); i.e., the minimum of the previous CAP(K) and F(I,J). Conservation of flow is again satisfied. The following also shows that (2.34) is satisfied: Now

$$\text{old } F(I,J) \leq C[I, NK(I) - J] - C[I, NK(I) - J + 1], \text{ and}$$

$$\text{new } F(I,J) = \text{old } F(I,J) - CAP(SINK).$$

Since CAP(SINK) $\leq$ CAP(I) $\leq$ old F(I,J), this implies that

$$0 \leq \text{new } F(I,J) \leq C[I, NK(I) - J] - C[I, NK(I) - J + 1].$$

Hence, F(I,J) remains feasible for this case as well and, therefore, remains feasible throughout the algorithm. QED.

Lemma 7: The optimality properties (2.42) and (2.43) imply that complementary slackness holds between the primal and the dual problems.

Proof: We will use the original pair of primal and dual problems ((2.13) - (2.19) and (2.22) - (2.25) respectively) along with the definitions of G(I,M), H(I,M) and F(I,J) to show that the complementary slackness conditions are satisfied; i.e.,

(i) $XACT(I) + XNODE(O_I) - XNODE(T_I) < 0$ implies that F(I) = 0;

(ii) XACT(I,M) < U(I,M) implies that G(I,M) = 0; and

(iii) XACT(I,M) > L(I,M) implies that H(I,M) = 0.

(i) If $XACT(I) + XNODE(O_I) - XNODE(T_I) < 0$, then

$$XACT(I) < XNODE(T_I) - XNODE(O_I).$$

Since

$$XACT(I) = \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(O_I)\},$$

this implies that

$$XACT(I) = TIME[I, NK(I)].$$

Hence,

$$TIME[I, NK(I)] + XNODE(O_I) - XNODE(T_I) < 0.$$

Since

$$TIME(I, 1) \leq TIME(I, 2) \leq \ldots \leq TIME[I, NK(I)],$$

it follows that

$$TIME(I, NK(I) + 1 - J) + XNODE(O_I) - XNODE(T_I) < 0$$

for $J = 1, 2, \ldots, NK(I)$. From optimality property (2.42), $F(I,J) = 0$ for $J = 1, 2, \ldots, NK(I)$, and finally $F(I) = \sum_J F(I,J) = 0$.

(Remark: Since

$$ABAR(I,J) = TIME[I, NK(I) + 1 - J] + XNODE(O_I) - XNODE(T_I)$$

and

$$TIME(I,1) \leq TIME(I,2) \leq \ldots \leq TIME[I, NK(I)],$$

it follows that

$$ABAR(I,1) \geq ABAR(I,2) \geq \ldots \geq ABAR[I, NK(I)].$$

Now the TIME(I,J)'s will be <u>strictly</u> increasing and the ABAR(I,J)'s <u>strictly</u> decreasing unless there is only one possible value for XACT(I) in which case the upper and lower bounds for F(I) and the F(I,J)'s are 0.

Therefore in the Second Labeling part (i), page 24, there can only be one J such that

$$ABAR(I,J) = 0$$

and

$$F(I,J) < C[I, NK(I) - J] - C[I, NK(I) - J + 1].$$

For this J

$$0 > ABAR(I,J + 1) > \ldots > ABAR[I, NK(I)],$$

so that by optimality property (2.42)

$$F(I, J + 1) = \ldots = F[I, NK(I)] = 0.$$

Also, for this J

$$ABAR(I, 1) > \ldots > ABAR(I,J - 1) > 0,$$

so that by optimality property (2.43) F(I, 1), ..., F(I, J - 1) are all at their upper bounds. Thus, when F(I) is increased, it is the F(I,J) with the smallest index J such that F(I,J) is less than its upper bound which is increased.

Similarly the Second Labeling part (ii) and the optimality properties imply that when F(I) is decreased it is the F(I,J) with the largest index J such that F(I,J) > 0 which is decreased. Therefore, if F(I,J) is positive, then F(I,1),...F(I,J - 1) are all at their upper bounds; and, if F(I,J) = 0, then F(I, J + 1), ..., F[I, NK(I)] also equal 0. These natural properties of the F(I,J)'s are used in parts (ii) and (iii) below.)

(ii) Show that XACT(I,M) < U(I,M) implies G(I,M) = 0 where, as in (2.15) and (2.28),

$$U(I,M) = \begin{cases} TIME(I,2) & M = 1 \\ TIME(I,M+1) - TIME(I,M) & M = 2, \ldots, NK(I) - 1, \end{cases}$$

$$G(I,M) = \max\{0, C(I,M) - F(I)\},$$

and

$$XACT(I,M) = \begin{cases} \min[U(I,M), XACT(I)] & M = 1 \\ \min\{U(I,M), \max[0, XACT(I) - TIME(I,M)]\} & M = 2, \ldots, NK(I) - 1. \end{cases}$$

If XACT(I,M) < U(I,M), then

$$XACT(I,M) = \begin{cases} XACT(I) & M = 1 \\ \max[0, XACT(I) - TIME(I,M)] & M = 2, \ldots, NK(I)-1. \end{cases}$$

<u>Case I</u>: M = 1.

Since

$$TIME(I, 2) = U(I, 1) > XACT(I, 1) = XACT(I),$$

it follows that

$$TIME(I, 2) > XACT(I) = \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(O_I)\}.$$

Since

$$TIME(I, 2) \le TIME[I, NK(I)],$$

this implies that

$$TIME(I, 2) > XNODE(T_I) - XNODE(O_I)$$

and

$$ABAR[I, NK(I) - 1] = TIME(I,2) + XNODE(O_I) - XNODE(T_I) > 0.$$

By (2.43),

$$F[I, NK(I) - 1] = C(I, 1) - C(I, 2).$$

Therefore $F(I,J) = C[NK(I) - J] - C[NK(I) - J + 1] \qquad J = 1, ..., NK(I) - 1.$

Hence

$$\begin{aligned} F(I) = \sum_J F(I,J) &= F[I, NK(I)] + \sum_{J=1}^{NK(I)-1} F(I,J) \\ &= F[I, NK(I)] + C[I, NK(I) - 1] - C[I, NK(I)] \\ &\qquad + C[I, NK(I) - 2] - C[I, NK(I) - 1] \\ &\qquad + \ldots \\ &\qquad + C(I, 1) - C(I, 2) \\ &= F[I, NK(I)] + C(I, 1) - C[I, NK(I)]. \end{aligned}$$

Since $C[I, NK(I)] \equiv 0$, $F(I) \ge C(I, 1)$. Therefore,

$$\begin{aligned} G(I, 1) &= \max[0, C(I, 1) - F(I)] \\ &= 0. \end{aligned}$$

<u>Case II</u>: M = 2, ..., NK(I) - 1.

Now U(I,M) > XACT(I,M) = max[0, XACT(I) - TIME(I,M)] implies

$$U(I,M) > XACT(I) - TIME(I,M)$$

and

$$U(I,M) > \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(0_I)\} - TIME(I,M);$$

so that

$$U(I,M) + TIME(I,M) > \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(0_I)\}.$$

Since U(I,M) = TIME(I,M + 1) - TIME(I,M),

$$U(I,M) + TIME(I,M) = TIME(I, M + 1)$$

and

$$TIME(I, M + 1) > \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(0_I)\}.$$

Since TIME(I,M + 1) $\leq$ TIME[I, NK(I)], this implies

$$TIME(I, M + 1) > XNODE(T_I) - XNODE(0_I)$$

or

$$TIME(I,M + 1) + XNODE(0_I) - XNODE(T_I) > 0.$$

By (2.43),

$$F[I, NK(I) - M] = C(I,M) - C(I,M + 1).$$

Therefore F(I,1), ..., F[I, NK(I) - M - 1] are also at their upper bounds.

Of course $\sum_{J=NK(I)-M+1}^{NK(I)} F(I,J) \geq 0.$

Thus

$$F(I) = \sum_{J} F(I,J) \geq \sum_{J}^{NK(I)-M} F(I,J)$$

$$= C[I, NK(I) - 1] - C[I, NK(I)]$$

$$+ C[I, NK(I) -2] - C[I, NK(I) - 1]$$

$$+ \ldots$$

$$+ C(I,M) - C(I, M + 1)$$

$$= C(I, M) - C[I,NK(I)].$$

Since $C[I,NK(I)] \equiv 0$, $F(I) \geq C(I,M)$.

Therefore,

$$G(I,M) = \max[0, C(I,M) - F(I)]$$

$$= 0$$

for $M = 2, \ldots, NK(I) - 1$.

(iii) Show that $XACT(I,M) > L(I,M)$ implies that $H(I,M) = 0$ where, as in (2.16) and (2.29),

$$L(I,M) = \begin{cases} TIME(I, 1) & M = 1 \\ 0 & M = 2, \ldots, NK(I) - 1, \end{cases}$$

$$H(I,M) = \max[0, F(I) - C(I,M)],$$

and

$$XACT(I,M) = \begin{cases} \min[U(I,M),\ XACT(I)] & M = 1 \\ \min\{U(I,M),\ \max[0, XACT(I) - TIME(I,M)]\} & M = 2, \ldots, NK(I)-1. \end{cases}$$

<u>Case I</u>: M = 1.

If XACT(I,1) > L(I,1), then

$$TIME(I,1) = L(I,1) < XACT(I,1) = \min[U(I,1),\ XACT(I)].$$

Since U(I,1) = TIME(I,2),

$$TIME(I,1) < \min[TIME(I,2),\ XACT(I)]$$

and

$$TIME(I,1) < XACT(I).$$

Thus

$$TIME(I,1) < XACT(I) = \min\{TIME[I,\ NK(I)],\ XNODE(T_I) - XNODE(0_I)\}$$

and

$$TIME(I,1) < XNODE(T_I) - XNODE(0_I).$$

Therefore

$$ABAR[I,\ NK(I)] = TIME(I,1) + XNODE(0_I) - XNODE(T_I) < 0.$$

Then by (2.42)

$$F[I,\ NK(I)] = 0,$$

and

$$F(I) = \sum_{J=1}^{NK(I)} F(I,J) = \sum_{J=1}^{NK(I)-1} F(I,J)$$

$$\leq \sum_{J=1}^{NK(I)-1} \{C[I, NK(I) - J] - C[I, NK(I) - J + 1]\}$$

$$= C[I, NK(I) - 1] - C[I, NK(I)]$$

$$+ C[I, NK(I) - 2] - C[I, NK(I) - 1]$$

$$+ \ldots$$

$$+ C(I, 1) - C(I, 2)$$

$$= C(I, 1) - C[I, NK(I)].$$

Since $C[I, NK(I)] \equiv 0$, $F(I) \leq C(I,1)$. Therefore

$$H(I,1) = \max[0, F(I) - C(I,1)]$$

$$= 0.$$

Case II: $M = 2, \ldots, NK(I) - 1$.

If

$$0 = L(I,M) < XACT(I,M) = \min\{U(I,M), \max[0, XACT(I) - TIME(I,M)]\},$$

then

$$0 < \min\{U(I,M), \max[0, XACT(I) - TIME(I,M)]\},$$

$$0 < \max[0, XACT(I) - TIME(I,M)], \text{ and}$$

$$0 < XACT(I) - TIME(I,M).$$

This implies that

$$TIME(I,M) < XACT(I) = \min\{TIME[I, NK(I)], XNODE(T_I) - XNODE(0_I)\};$$

so that

$$TIME(I,M) < XNODE(T_I) - XNODE(O_I),$$

and

$$ABAR[I, NK(I) - M + 1] = TIME(I,M) + XNODE(O_I) - XNODE(T_I) < 0.$$

By (2.42),

$$F[I, NK(I) - M + 1] = 0.$$

Therefore F[I, NK(I) - M + 2], ..., F[I, NK(I)] are all equal to 0. Hence,

$$F(I) = \sum_J F(I,J) \leq \sum_{J=1}^{NK(I)-M} \{C[I, NK(I) - J] - C[I, NK(I) - J + 1]\}$$

$$= C[I, NK(I) - 1] - C[I, NK(I)]$$

$$+ C[I, NK(I) -2] - C[I, NK(I) - 1]$$

$$+ \ldots$$

$$+ C(I,M) - C(I,M + 1)$$

$$= C(I,M) - C[I, NK(I)].$$

Since $C[I,NK(I)] \equiv 0$, $F(I) \leq C(I,M)$. Therefore,

$$H(I,M) = \max[0, F(I) - C(I,M)]$$

$$= 0$$

for M = 2, ..., NK(I) - 1. QED.

<u>Theorem 2</u>: Since the XNODE(K)'s, XACT(I,M)'s, V, AND F(I)'s are feasible and complementary slackness holds, they are optimal.

Proof: The primal problem (2.13) - (2.19) is in the form

$$\max c^T x$$

subject to

$$Ax \leq b,$$

where the x vector contains the XNODE(K)'s and XACT(I,M)'s. The dual problem (2.22) - (2.25) is in the form

$$\min b^T w$$

subject to

$$A^T w = c$$

$$w \geq 0$$

where the w vector contains V and the F(I)'s.

For any feasible x

$$Ax \leq b;$$

so that for any feasible w

$$w^T Ax \leq w^T b.$$

Since $w^T A = c^T$ for any feasible w,

$$c^T x \leq b^T w$$

holds for any feasible x and w. When $Ax \leq b$ is rewritten in the form

$$Ax + x_s = b$$

where x_s is a vector of slack variables, complementary slackness means

$$w^T x_s = 0.$$

Therefore, since for any feasible x and w

$$w^T Ax + w^T x_s = w^T b$$

or

$$c^T x + w^T x_s = b^T w,$$

complementary slackness implies

$$c^T x = b^T w$$

and hence that both x and w are optimal. QED.

Theorem 3: The algorithm terminates after finitely many applications of the labeling procedure.

Proof: In order that the algorithm fail to terminate, an infinite sequence of breakthroughs and nonbreakthroughs would have to occur.

Since the flow change following a breakthrough has a positive minimum, an infinite number of breakthroughs would produce flows having arbitrarily large values V. However, when a sufficiently large value V is reached, there will be a path from the source to the sink with F[I, NK(I)] > 0 all along this path. Since ABAR[I, NK(I)] $\leq$ 0 throughout the computations, we would have ABAR[I,NK(I)] = 0 for arcs on this path. But then the first labeling procedure would terminate. Therefore, there can only be a finite number of breakthroughs.

Following a nonbreakthrough, all nodes previously labeled can again be labeled. (This follows from the fact that for labeled 0_I and T_I, the new ABAR(I,J) is equal to the old ABAR(I,J)). In addition, at least one more node can be labeled (the node(s) corresponding to the arc(s) in A_1' and A_2 that determine DEL). Eventually, the number of nodes that can be labeled will reach the total number of nodes implying that the sink can be labeled and the occurrence of a breakthrough. Therefore, infinitely many successive nonbreakthroughs cannot occur.

Hence, there can only be a finite number of applications of the labeling procedure. QED.

<u>Definition</u>: A function P(X) is said to be <u>convex</u> over some interval in X, if for any two points X1, X2 in the interval and for all α, $0 \leq \alpha \leq 1$,

$$P[\alpha \cdot X2 + (1 - \alpha)X1] \leq \alpha \cdot P(X2) + (1 - \alpha) \cdot P(X1).$$

<u>Lemma 8</u>: PCOST(LAMBDA) is convex for LMAX $\geq$ LAMBDA $\geq$ LMIN, where

LMAX $\equiv$ the longest (cheapest) time to complete the project

and

LMIN $\equiv$ the shortest time to complete the project.

<u>Proof</u>: Let L1 > L2 both be in the interval [LMIN, LMAX]. Let

$$L = \alpha L2 + (1 - \alpha)L1$$

for some α in [0, 1]. Also let $XACT_1(I)$, $XNODE_1(0_I)$ $XNODE_1(T_I)$, $XACT_2(I)$, $XNODE_2(0_I)$, and $XNODE_2(T_I)$ represent optimal solutions to the problems corresponding to LAMBDA = L1 and LAMBDA = L2 respectively. We first want to show that $[\alpha XNODE_2(K) + (1 - \alpha) XNODE_1(K)]$ and $[\alpha \cdot XACT_2(I,M) + (1 - \alpha) XACT_1(I,M)]$ are feasible when LAMBDA = L. This result follows easily since the constraints (2.13), (2.14), (2.15), and (2.16) are linear:

(i) Since $XNODE_1(0_I) + \sum_M XACT_1(I,M) - XNODE_1(T_I) \leq 0$

and

$$XNODE_2(0_I) + \sum_M XACT_2(I,M) - XNODE_2(T_I) \leq 0,$$

it follows that

$$[\alpha XNODE_2(0_I) + (1-\alpha)\ XNODE_1(0_I)] + \sum_M [\alpha XACT_2(I,M) + (1-\alpha)\ XACT_1(I,M)]$$

$$- [\alpha XNODE_2(T_I) + (1-\alpha)\ XNODE_1(T_I)]$$

$$\leq 0$$

and the constraints (2.13) are satisfied.

(ii) Now $XNODE_1(SINK) \leq L1$ and $XNODE_2(SINK) \leq L2$; so that

$$(1-\alpha)XNODE_1(SINK) + \alpha XNODE_2(SINK) \leq (1-\alpha)L1 + \alpha L2 = L$$

and constraint (2.14) is satisfied.

(iii) Also, $L(I,M) \leq XACT_1(I,M) \leq U(I,M)$ and $L(I,M) \leq XACT_2(I,M) \leq U(I,M)$ implies

$$L(I,M) \leq (1-\alpha)XACT_1(I,M) + \alpha XACT_2(I,M)$$

$$\leq U(I,M)$$

and hence constraints (2.15) and (2.16) are satisfied.

Recall that

$$PCOST(LAMBDA) = KK - \sum_{I,M} [C(I,M)XACT(I,M)]$$

where

$$KK = \sum_{I}[COST(I,1) + C(I,1)TIME(I,1)].$$

Hence,

$$\alpha PCOST(L2) + (1-\alpha)PCOST(L1)$$

$$= \alpha\{KK - \sum_{I,M}[C(I,M)XACT_2(I,M)]\} + (1-\alpha)\{KK - \sum_{I,M}[C(I,M)XACT_1(I,M)]\}$$

$$= \alpha KK + (1-\alpha)KK - \alpha\sum_{I,M} C(I,M)XACT_2(I,M) - (1-\alpha)\sum_{I,M} C(I,M)XACT_1(I,M)$$

$$= KK - \sum_{I,M} C(I,M)[\alpha XACT_2(I,M)] - \sum_{I,M} C(I,M)[(1-\alpha)XACT_1(I,M)]$$

$$= KK - \sum_{I,M} C(I,M)[\alpha XACT_2(I,M) + (1-\alpha)XACT_1(I,M)].$$

Furthermore $\alpha PCOST(L2) + (1-\alpha)PCOST(L1)$ is the objective function value corresponding to $[\alpha XNODE_2(K) + (1-\alpha)XNODE_1(K)]$ and $[\alpha XACT_2(I,M) + (1-\alpha)XACT_1(I,M)]$ which we have just shown are feasible. Therefore, since we are minimizing PCOST(LAMBDA),

$$PCOST(L) = PCOST[\alpha L2 + (1-\alpha)L1]$$

$$\leq \alpha PCOST(L2) + (1-\alpha)PCOST(L1),$$

and PCOST(LAMBDA) is convex.

<u>Lemma 9</u>: The project cost function, PCOST(LAMBDA), is piecewise linear.

<u>Proof</u>: Let L1 > L2 = L1 - DEL be two successively determined LAMBDA's where DEL is determined by (2.56). (Of course, L1 could be the initial value of LAMBDA.) Suppose L1 $\geq$ LAMBDA $\geq$ L2 and that F(I,J)'s and V were the flows when LAMBDA was changed from L2 to L1. Recall that

$$PCOST(LAMBDA) = PCOST[XNODE(SINK)] = \sum_{I}[KK(I) - \sum_{M}C(I,M)XACT(I,M)]$$

which is the primal objective function. Since the primal and dual objective functions are equal under optimality, we have, for all LAMBDA with $L1 \geq LAMBDA \geq L2$,

$$PCOST(LAMBDA) = Z - LAMBDA \cdot V + \sum_{IJ} F(I,J) TIME[I, NK(I) - J + 1]$$

where Z is a constant. Therefore

$$\begin{aligned} PCOST(LAMBDA) - PCOST(L1) \\ &= Z - LAMBDA \cdot V + \sum_{IJ} F(I,J) TIME[I, NK(I) - J + 1] \\ &\quad - Z + L1 \cdot V - \sum_{IJ} F(I,J) TIME[I, NK(I) - J + 1] \\ &= - LAMBDA \cdot V + L1 \cdot V \\ &= (L1 - LAMBDA) \cdot V \end{aligned}$$

for $L1 \geq LAMBDA \geq L2$, so that PCOST(LAMBDA) is linear on the given interval. QED.

Theorem 4: If $L1 > L2 = L1 - DEL$ are two successively determined values of LAMBDA where DEL is determined by (2.56), then for any value of L such that $L1 > L \geq L2$ the optimal values of the XNODE(K)'s and XACT(I)'s for that L are given by

$$XNODE_L(K) = \begin{cases} XNODE_{L1}(K) & \text{if K is labeled when LAMBDA = L1,} \\ XNODE_{L1}(K) - (L1 - L) & \text{if K is unlabeled when LAMBDA = L1,} \end{cases}$$

$$XACT_L(I) = \min\{TIME[I, NK(I)], XNODE_L(T_I) - XNODE_L(O_I)\}$$

where the subscripts L and L1 imply LAMBDA = L and L1 respectively.

Proof: Since Lemma 1 states that we begin with an optimal solution when LAMBDA = LMAX, we can without loss of generality assume that we have found optimal solutions for all LAMBDA values produced by the nonbreakthrough procedure up to LAMBDA = L1. We will now show that the above XNODE(K)'s and XACT(I)'s are optimal for all LAMBDA between L1 and L2 including L2. The terms "labeled" and "unlabeled" below refer to "labeled when LAMBDA = L1" and "unlabeled when LAMBDA = L1" respectively.

We first want to show that for L1 > L ≥ L2 the $XNODE_L(K)$'s and $XACT_L(I)$'s are feasible. Since the definition of $XACT_L(I)$ implies that

$$XACT_L(I) + XNODE_L(0_I) - XNODE_L(T_I) \leq 0$$

and

$$XACT_L(I) \leq TIME[I, NK(I)],$$

(2.13) and (2.15) are satisfied. Therefore, the only aspect of feasibility left to show is (2.16), i.e.

$$TIME(I,1) \leq XACT_L(I)$$

or equivalently

$$TIME(I,1) \leq XNODE_L(T_I) - XNODE_L(0_I).$$

(i) Suppose 0_I and T_I are both labeled for a specific activity I. Then

$$XNODE_L(T_I) - XNODE_L(0_I) = XNODE_{L1}(T_I) - XNODE_{L1}(0_I) \geq TIME(I,1)$$

since the solution at L1 is feasible.

(ii) Suppose 0_I is labeled and T_I is unlabeled. Then

$$XNODE_L(T_I) - XNODE_L(0_I) = XNODE_{L1}(T_I) - (L1 - L) - XNODE_{L1}(0_I).$$

The definition of DEL implies that if ABAR[I, NK(I)] < 0, then

$$XNODE_{L1}(T_I) - XNODE_{L1}(0_I) - TIME(I,1) \geq DEL \geq L1 - L.$$

Hence,

$$XNODE_{L1}(T_I) - XNODE_{L1}(0_I) - (L1 - L) \geq TIME(I, 1)$$

and

$$XNODE_L(T_I) - XNODE_L(0_I) \geq TIME(I, 1).$$

If ABAR[I, NK(I)] = 0, then T_I would have been labeled from 0_I. Since feasibility is satisfied at LAMBDA = L1, it follows that

$$TIME(I, 1) + XNODE_{L1}(0_I) - XNODE_{L1}(T_I) \leq 0,$$

and consequently, since

$$ABAR[I, NK(I)] = TIME(I,1) + XNODE_{L1}(0_I) - XNODE_{L1}(T_I),$$

ABAR[I, NK(I)] cannot be positive.

(iii) Suppose 0_I and T_I are both unlabeled. Then

$$\begin{aligned} XNODE_L(T_I) - XNODE_L(0_I) &= XNODE_{L1}(T_I) - (L1 - L) - [XNODE_{L1}(0_I) - (L1 - L)] \\ &= XNODE_{L1}(T_I) - XNODE_{L1}(0_I) \\ &\geq TIME(I, 1) \end{aligned}$$

since the solution at L1 is feasible.

(iv) Suppose 0_I is unlabeled and T_I is labeled. Then

$$XNODE_L(T_I) - XNODE_L(0_I) = XNODE_{L1}(T_I) - [XNODE_{L1}(0_I) - (L1 - L)]$$

$$= XNODE_{L1}(T_I) - XNODE_{L1}(0_I) + (L1 - L).$$

Since feasibility is satisfied at LAMBDA = L1,

$$XNODE_{L1}(T_I) - XNODE_{L1}(0_I) \geq TIME(I,1)$$

and trivially

$$XNODE_{L1}(T_I) - XNODE_{L1}(0_I) + (L1 - L) \geq TIME(I,1).$$

Hence,

$$XNODE_L(T_I) - XNODE_L(0_I) \geq TIME(I, 1).$$

Now, we have just shown that the $XNODE_L(K)$'s and $XACT_L(I)$'s are feasible. Lemma 6 implies that the F(I,J)'s are always kept feasible. Lemma 4 implies that the optimality properties (2.42) and (2.43) are satisfied for these $XNODE_L(K)$'s and F(I,J)'s; so that by Lemma 7 these $XNODE_L(K)$'s and F(I,J)'s also satisfy complementary slackness. Since we have shown that feasibility and complementary slackness are satisfied, Theorem 3 implies that the $XNODE_L(K)$'s and $XACT_L(I)$'s for $L1 > L \geq L2$ are optimal. QED.

4. A COMPUTER IMPLEMENTATION

A computer program implementing the improved project scheduling algorithm described in Chapter 2 is available. The basic imput to the program is

(a) an acyclic project network with one source and one sink, and

(b) a collection of activity completion times and their associated costs.

The program's output for each feasible project deadline time consists mainly of

(a) the optimal activity completion times and costs, and

(b) the total project cost.

Optional output may include node labels, optimal node times for each project deadline time, and dual variables (flows).

Incorporated in this program is the option to have the minimum project cost and corresponding optimal activity completion times determined for only one specific project deadline time.

A listing of the computer program is given in the appendix. The flowchart for this program is given in Chapter 2, Section 4, pages 27-29.

4.1. Specific Input Instructions

Card 1. Col. 1 - 4: The number of nodes in the network, Format (I4).

Col. 6 - 9: The number of activities in the network, Format (I4).

Col. 11: TEST1 = 0 print the input data,
= 1 do not print the input data.

Col. 13: TEST2 = 0 print the intermediate output,
= 1 do not print the intermediate output.

Col. 15-18: The number of the source node, Format (I4).

Col. 20-23: The number of the sink node, Format (I4).

Col. 25: TEST3 = 0 do not wish to specify a single value for LAMBDA,
= 1 do wish to specify a single value for LAMBDA and print the intermediate output,

= 2 do wish to specify a single value for LAMBDA but do not print the intermediate output.

For each activity I one set of 3 - 5 cards:

Card 1. Col. 1 - 4: O_I = the number of the origin node, Format (I4).

Col. 6 - 9: T_I = the number of the terminal node, Format (I4).

Col. 11-12: NK(I) = the number of activity completion times and costs that are read in (≤ 11), Format (I2).

Card(s) 2 - 3. Format (8I10): TIME(I,1),...,TIME[I, NK(I)] = the activity completion times in increasing order (8 on Card 2, 3 on Card 3 if needed).

Card(s) 4 - 5. Format (8I10): COST(I,1),...,COST[I,NK(I)] = the cost associated with each activity completion time (8 on Card 4, 3 on Card 5 if needed).

The next card is present only if TEST3 = 1 or 2.

Last Card. Col. 1 -10:Specified project deadline time, LAMBDA, Format (I10).

The nodes and activities may be numbered in any order. The current dimensions will allow 3000 nodes, 3000 activities, and at most 11 different completion times and costs.

4.2. An Example

The program's input and output are illustrated in terms of the example network in Figure 10. The input data are found in Table 1. As an example, the activity cost curve for activity 7 is illustrated in Figure 11.

A listing of the computer input is given in Figure 12. The optimal project cost curve determined by the algorithm is plotted in Figure 13. The optimal activity durations for two values of the project deadline time, LAMBDA, are given in Table 2. The actual computer output is given in Figure 14.

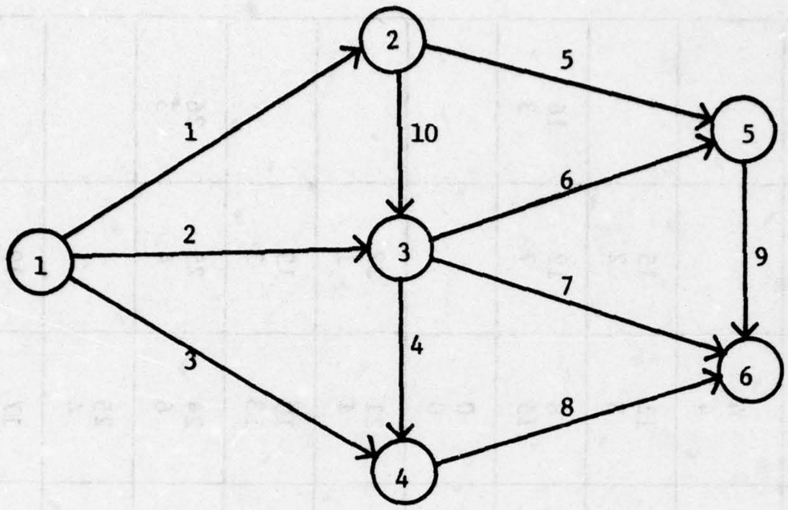


FIGURE 10

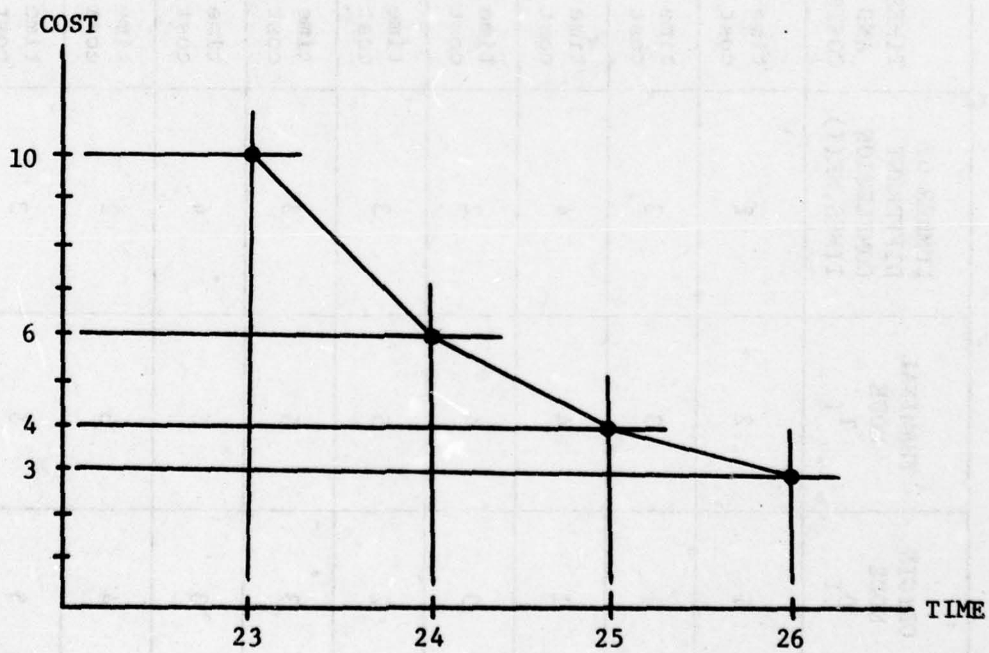


FIGURE 11

TABLE 1: EXAMPLE DATA

ACTIVITY NUMBER I	ORIGIN NODE O_I	TERMINAL NODE T_I	NUMBER OF DIFFERENT COMPLETION TIMES, NK(I)	TIMES AND COSTS	J = 1	J = 2	J = 3	J = 4
1	1	2	2	time cost	2 8	4 4		
2	1	3	3	time cost	7 23	12 8	15 2	
3	1	4	4	time cost	4 27	8 15	12 7	16 3
4	3	4	2	time cost	0 0	0 0		
5	2	5	3	time cost	20 8	21 4	22 1	
6	3	5	3	time cost	5 28	10 13	15 3	
7	3	6	4	time cost	23 10	24 6	25 4	26 3
8	4	6	2	time cost	23 8	25 4		
9	5	6	3	time cost	16 12	17 7	19 3	
10	2	3	2	time cost	6 4	6 4		

<u>Figure 11</u>: Computer Input for Example

```
6    10  0  0     1        6  0
1     2  2
      2           4
      8           4
1     3  3
      7          12          15
     23           8           2
1     4  4
      4           8          12          16
     27          15           7           3
3     4  2
      0           0
      0           0
2     5  3
     20          21          22
      8           4           1
3     5  3
      5          10          15
     28          13           3
3     6  4
     23          24          25          26
     10           6           4           3
4     6  2
     23          25
      8           4
5     6  3
     16          17          19
     12           7           3
2     3  2
      6           6
      4           4
```

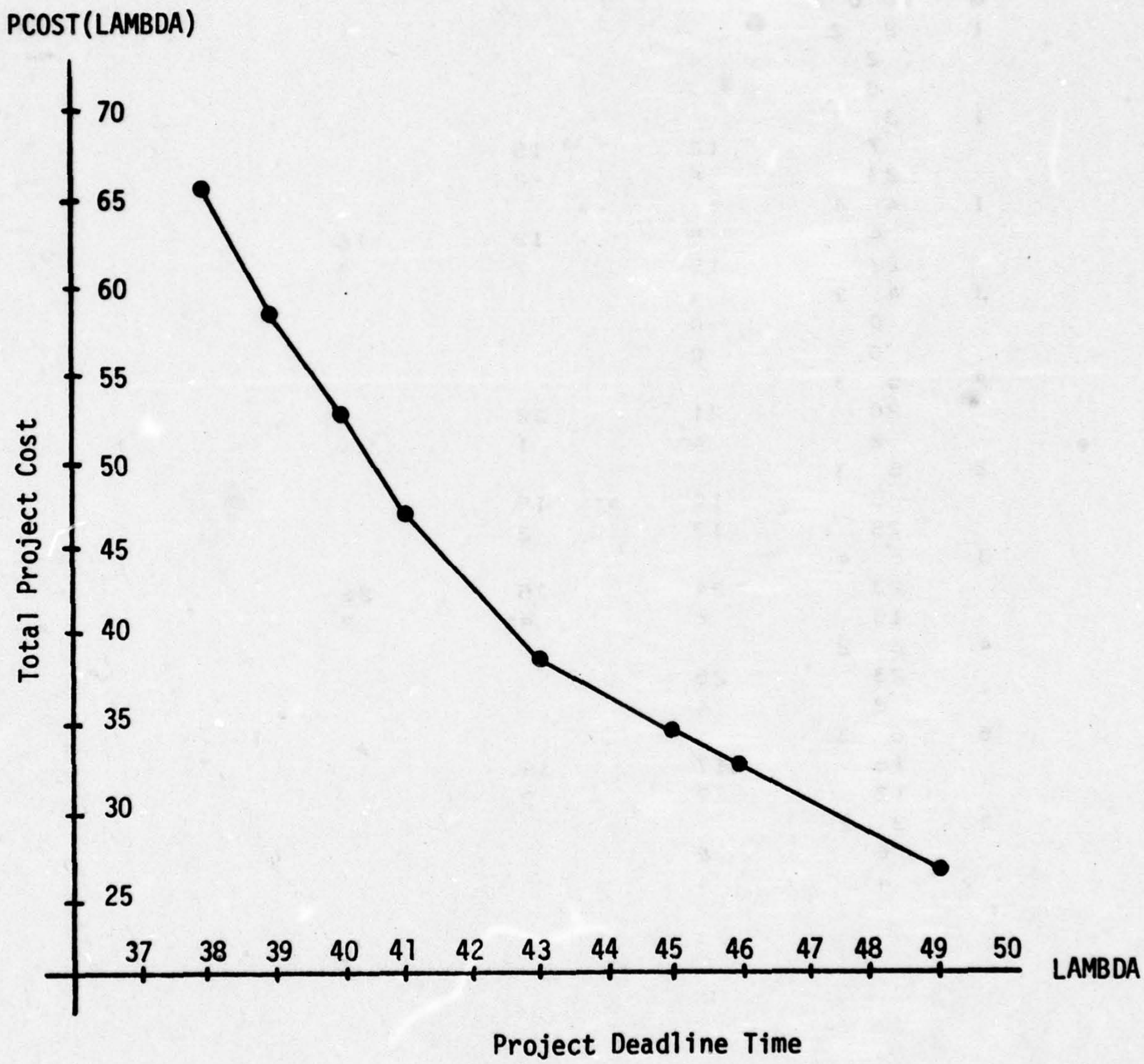

PCOST(LAMBDA)
Total Project Cost
70
65
60
55
50
45
40
35
30
25
37
38
39
40
41
42
43
44
45
46
47
48
49
50
LAMBDA
Project Deadline Time

Figure 13.

Table 2: OPTIMAL PROJECT SCHEDULES FOR TWO SPECIFIED DEADLINE TIMES

Activity # (I)	Project Deadline Time LAMBDA = 40		Project Deadline Time LAMBDA = 44	
	Activity Duration Time XACT(I)	Activity Cost	Activity Duration Time XACT(I)	Activity Cost
1	2	8	4	4
2	12	8	12	8
3	15	4	16	3
4	0	0	0	0
5	21	4	22	1
6	11	11	14	5
7	26	3	26	3
8	25	4	25	4
9	17	7	18	5
10	6	4	6	4

Figure 14: Computer Output for Example

THE NUMBER OF NODES IS 6.
THE NUMBER OF ACTIVITIES IS 10.
THE SOURCE NODE IS NUMBERED 1 AND THE SINK NODE IS NUMBERED 6.

** NODES: **

K	1	2	3	4	5	6
INITIAL XNODE(K)	0	4	15	16	30	49

** ACTIVITIES: **

I	XACT	ORIG	TERM	J	TIME	COST	C	ABAR
1	4	1	2	1	2	8	0.20000E 01	0
				2	4	4		-2
2	15	1	3	1	7	23	0.30000E 01	0
				2	12	8	0.20000E 01	-3
				3	15	2		-8
3	16	1	4	1	4	27	0.30000E 01	0
				2	8	15	0.20000E 01	-4
				3	12	7	0.10000E 01	-8
				4	16	3		-12
4	0	3	4	1	0	0	0.0	-1
				2	0	0		-1
5	22	2	5	1	20	8	0.40000E 01	-4
				2	21	4	0.30000E 01	-5
				3	22	1		-6
6	15	3	5	1	5	28	0.30000E 01	0
				2	10	13	0.20000E 01	-5
				3	15	3		-10
7	26	3	6	1	23	10	0.40000E 01	-8
				2	24	6	0.20000E 01	-9
				3	25	4	0.10000E 01	-10
				4	26	3		-11
8	25	4	6	1	23	8	0.20000E 01	-8
				2	25	4		-10
9	19	5	6	1	16	12	0.50000E 01	0
				2	17	7	0.20000E 01	-2
				3	19	3		-3
10	6	2	3	1	6	4	0.0	-5
				2	6	4		-5

THE ENTIRE PROJECT COST CURVE IS GOING TO BE DETERMINED.

LAMBDA = PROJECT COMPLETION TIME

THE STARTING VALUE OF LAMBDA IS 49.

THE CORRESPONDING TOTAL PROJECT COST IS 0.27000E 02.

THE SOURCE HAS A VALUE OF ZERO AND IS ASSIGNED THE LABEL (-,-,-,INF).

*** ITERATICN NUMBER 1 ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 2 HAS THE LABEL (1, 1, 0, 0.20000E 01).

THE NODE 3 HAS THE LABEL (1, 1, 0, 0.20000E 01).

THE NODE 4 HAS THE LABEL (1, 1, 0, 0.10000E 01).

THE NODE 5 HAS THE LABEL (3, 1, 0, 0.20000E 01).

THE NODE 6 HAS THE LABEL (5, 1, 0, 0.20000E 01).

BREAKTHROUGH: UPDATE THE DUAL VARIABLES.

ACTIVITY #: I	J	NEW FLOW: F(I,J)
1	1	0.0
1	2	0.0
2	1	0.20000E 01
2	2	0.0
2	3	0.0
3	1	0.0
3	2	0.0
3	3	0.0
3	4	0.0
4	1	0.0
4	2	0.0
5	1	0.0
5	2	0.0
5	3	0.0
6	1	0.20000E 01
6	2	0.0
6	3	0.0
7	1	0.0
7	2	0.0
7	3	0.0
7	4	0.0
8	1	0.0
8	2	0.0
9	1	0.20000E 01
9	2	0.0
9	3	0.0
10	1	0.0
10	2	0.0

*** ITERATICN NUMBER 2 ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 2 HAS THE LABEL (1, 1, 0, 0.20000E 01).

THE NODE 4 HAS THE LABEL (1, 1, 0, 0.10000E 01).

NONBREAKTHROUGH: UPDATE THE PRIMAL VARIABLES;
I.E. ; DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA = 46.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO 3.
LAMBDA RANGES FROM 49 TO 46.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE = 49-D:

NODE #: K	NEW VALUE: XNODE(K)
1	0
2	4
3	15-D
4	16
5	30-D
6	49-D

PROJECT COMPLETION TIME = 49-D.

ACTIVITY #: I	NEW VALUE: XACT(I)	ACTIVITY COST
1	4	0.40000E 01
2	15-D	0.20000E 01 + (0.20000E 01*D)
3	16	0.30000E 01
4	0	0.0
5	22	0.10000E 01
6	15	0.30000E 01
7	26	0.30000E 01
8	25	0.40000E 01
9	19	0.30000E 01
10	6	0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS 0.27000E 02 + (0.20000E 01*D).

NEW VALUES OF ABAR FOR J=1,2,...,NK(I)

I J:	1	2	3	4	5	6	7	8	9
1	0	-2							
2	3	0	-5						
3	0	-4	-8	-12					
4	-4	-4							
5	-1	-2	-3						
6	0	-5	-10						
7	-8	-9	-10	-11					
8	-5	-7							
9	0	-2	-3						
10	-2	-2							

*** ITERATION NUMBER 3 ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 2 HAS THE LABEL (1, 1, 0, 0.20000E 01).

THE NODE 3 HAS THE LABEL (1, 2, 0, 0.10000E 01).

THE NODE 4 HAS THE LABEL (1, 1, 0, 0.10000E 01).

```
NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES;
I.E.: DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA =          45.


DELTA (REPRESENTED BY "D") RANGES FROM 0 TO    1.
LAMBDA RANGES FROM           46 TO           45.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE =          46-D:
  NODE #: K      NEW VALUE: XNODE(K)
          1                    0
          2                    4
          3                   12
          4                   16
          5                   27-D
          6                   46-D


PROJECT COMPLETION TIME =            46-D.

  ACTIVITY #: I    NEW VALUE: XACT(I)          ACTIVITY COST
        1                    4                 0.40000E 01
        2                   12                 0.80000E 01
        3                   16                 0.30000E 01
        4                    0                 0.0
        5                   22                 0.10000E 01
        6                   15-D               0.30000E 01 + (   0.20000E 01*D)
        7                   26                 0.30000E 01
        8                   25                 0.40000E 01
        9                   19                 0.30000E 01
       10                    6                 0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS       0.33000E 02 + (   0.20000E 01*D).


NEW VALUES OF ABAR FOR J=1,2,...,NK(I)

   I    J:    1       2       3       4       5       6       7       8       9
   1          0      -2
   2          3       0      -5
   3          0      -4      -8     -12
   4         -4      -4
   5          0      -1      -2
   6          1      -4      -9
   7         -7      -8      -9     -10
   8         -4      -6
   9          0      -2      -3
  10         -2      -2

***      ITERATION NUMBER       4      ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE     2 HAS THE LABEL (    1.     1.    0.     0.20000E 01).

THE NODE     3 HAS THE LABEL (    1.     2.    0.     0.10000E 01).

THE NODE     4 HAS THE LABEL (    1.     1.    0.     0.10000E 01).

THE NODE     5 HAS THE LABEL (    2.     1.    0.     0.20000E 01).
```

NONBREAKTHROUGH: UPDATE THE PRIMAL VARIABLES;
I.E. :DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA = 43.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO 2.
LAMBDA RANGES FROM 45 TO 43.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE = 45-D:

NODE #: K	NEW VALUE: XNODE(K)
1	0
2	4
3	12
4	16
5	26
6	45-D

PROJECT COMPLETION TIME = 45-D.

ACTIVITY #: I	NEW VALUE: XACT(I)	ACTIVITY COST
1	4	0.40000E 01
2	12	0.80000E 01
3	16	0.30000E 01
4	0	0.0
5	22	0.10000E 01
6	14	0.50000E 01
7	26	0.30000E 01
8	25	0.40000E 01
9	19-D	0.30000E 01 + (0.20000E 01*D)
10	6	0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS 0.35000E 02 + (0.20000E 01*D).

NEW VALUES OF ABAR FOR J=1,2,...,NK(I)

I J:	1	2	3	4	5	6	7	8	9
1	0	-2							
2	3	0	-5						
3	0	-4	-8	-12					
4	-4	-4							
5	0	-1	-2						
6	1	-4	-9						
7	-5	-6	-7	-8					
8	-2	-4							
9	2	0	-1						
10	-2	-2							

*** ITERATION NUMBER 5 ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 2 HAS THE LABEL (1, 1, 0, 0.20000E 01).

THE NODE 3 HAS THE LABEL (1, 2, 0, 0.10000E 01).

THE NODE 4 HAS THE LABEL (1, 1, 0, 0.10000E 01).

THE NODE 5 HAS THE LABEL (2, 1, 0, 0.20000E 01).

```
THE NODE    6 HAS THE LABEL (   5.   2.   0.     0.20000E 01).

BREAKTHROUGH:  UPDATE THE DUAL VARIABLES.

  ACTIVITY #: I    J           NEW FLOW: F(I,J)
              1    1                0.20000E 01
              1    2                0.0
              2    1                0.20000E 01
              2    2                0.0
              2    3                0.0
              3    1                0.0
              3    2                0.0
              3    3                0.0
              3    4                0.0
              4    1                0.0
              4    2                0.0
              5    1                0.20000E 01
              5    2                0.0
              5    3                0.0
              6    1                0.20000E 01
              6    2                0.0
              6    3                0.0
              7    1                0.0
              7    2                0.0
              7    3                0.0
              7    4                0.0
              8    1                0.0
              8    2                0.0
              9    1                0.20000E 01
              9    2                0.20000E 01
              9    3                0.0
             10    1                0.0
             10    2                0.0

***      ITERATION NUMBER      6      ***

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE    3 HAS THE LABEL (   1.   2.   0.     0.10000E 01).

THE NODE    4 HAS THE LABEL (   1.   1.   0.     0.10000E 01).

NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES;
I.E.  DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA =          41.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO   2.
LAMBDA RANGES FROM          43 TO          41.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE =          43-D:
  NODE #: K       NEW VALUE: XNODE(K)
          1                  0
          2                  4-D
          3                 12
          4                 16
          5                 26-D
```

6 43-D

PROJECT COMPLETION TIME = 43-D.

ACTIVITY #: I	NEW VALUE: XACT(I)	ACTIVITY COST
1	4-D	0.40000E 01 + (0.20000E 01*D)
2	12	0.80000E 01
3	16	0.30000E 01
4	0	0.0
5	22	0.10000E 01
6	14-D	0.50000E 01 + (0.20000E 01*D)
7	26	0.30000E 01
8	25	0.40000E 01
9	17	0.70000E 01
10	6	0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS 0.39000E 02 + (0.40000E 01*D).

NEW VALUES OF ABAR FOR J=1,2,....,NK(I)

I J:	1	2	3	4	5	6	7	8	9
1	2	0							
2	3	0	-5						
3	0	-4	-8	-12					
4	-4	-4							
5	0	-1	-2						
6	3	-2	-7						
7	-3	-4	-5	-6					
8	0	-2							
9	2	0	-1						
10	-4	-4							

*** ITERATION NUMBER 7 ***

THE NODE 2 HAS THE LABEL (1, 2,0,INF).

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 3 HAS THE LABEL (1, 2, 0, 0.10000E 01).

THE NODE 4 HAS THE LABEL (1, 1, 0, 0.10000E 01).

THE NODE 5 HAS THE LABEL (2, 1, 0, 0.10000E 01).

THE NODE 6 HAS THE LABEL (4, 1, 0, 0.10000E 01).

BREAKTHROUGH: UPDATE THE DUAL VARIABLES.

ACTIVITY #: I	J	NEW FLOW: F(I,J)
1	1	0.20000E 01
1	2	0.0
2	1	0.20000E 01
2	2	0.0
2	3	0.0

3	1	0.10000E 01
3	2	0.0
3	3	0.0
3	4	0.0
4	1	0.0
4	2	0.0
5	1	0.20000E 01
5	2	0.0
5	3	0.0
6	1	0.20000E 01
6	2	0.0
6	3	0.0
7	1	0.0
7	2	0.0
7	3	0.0
7	4	0.0
8	1	0.10000E 01
8	2	0.0
9	1	0.20000E 01
9	2	0.20000E 01
9	3	0.0
10	1	0.0
10	2	0.0

*** ITERATION NUMBER 8 ***

THE NODE 2 HAS THE LABEL (1, 2.0,INF).

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 3 HAS THE LABEL (1, 2, 0, 0.10000E 01).

THE NODE 5 HAS THE LABEL (2, 1, 0, 0.10000E 01).

THE NODE 6 HAS THE LABEL (5, 2, 0, 0.10000E 01).

BREAKTHROUGH: UPDATE THE DUAL VARIABLES.

ACTIVITY #: I	J	NEW FLOW: F(I,J)
1	1	0.20000E 01
1	2	0.0
2	1	0.20000E 01
2	2	0.0
2	3	0.0
3	1	0.10000E 01
3	2	0.0
3	3	0.0
3	4	0.0
4	1	0.0
4	2	0.0
5	1	0.30000E 01
5	2	0.0
5	3	0.0
6	1	0.20000E 01
6	2	0.0
6	3	0.0

```
     7   1          0.0
     7   2          0.0
     7   3          0.0
     7   4          0.0
     8   1          0.10000E 01
     8   2          0.0
     9   1          0.20000E 01
     9   2          0.30000E 01
     9   3          0.0
    10   1          0.0
    10   2          0.0

***     ITERATICN NUMBER      9      ***

THE NODE    2 HAS THE LABEL (    1,    2.0,INF).

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE    3 HAS THE LABEL (    1,   2,   0,      0.10000E 01).

NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES;
I.E.  DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA =            40.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO    1.
LAMBDA RANGES FROM           41 TO          40.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE =          41-D:
  NODE #: K       NEW VALUE: XNODE(K)
        1                     0
        2                     2
        3                    12
        4                    16-D
        5                    24-D
        6                    41-D

PROJECT COMPLETICN TIME =           41-D.

  ACTIVITY #: I    NEW VALUE: XACT(I)           ACTIVITY COST
       1                      2                 0.80000E 01
       2                     12                 0.80000E 01
       3                     16-D               0.30000E 01 + (  0.10000E 01*D)
       4                      0                 0.0
       5                     22-D               0.10000E 01 + (  0.30000E 01*D)
       6                     12-D               0.90000E 01 + (  0.20000E 01*D)
       7                     26                 0.30000E 01
       8                     25                 0.40000E 01
       9                     17                 0.70000E 01
      10                      6                 0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS       0.47000E 02 + (  0.60000E 01*D).

NEW VALUES OF AEAR FOR J=1,2,...,NK(I)

    I   J:      1         2         3         4         5         6         7         8         9
    1                2         0
```

2	3	0	-5	
3	1	-3	-7	-11
4	-3	-3		
5	1	0	-1	
6	4	-1	-6	
7	-2	-3	-4	-5
8	0	-2		
9	2	0	-1	
10	-4	-4		

```
***     ITERATION NUMBER     10     ***

THE NODE     2 HAS THE LABEL (    1.    2,0,INF).

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE     3 HAS THE LABEL (    1.    2.    0.       0.10000E 01).

THE NODE     5 HAS THE LABEL (    2.    2.    0.       0.10000E 01).

NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES;
I.E.  DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA =          39.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO    1.
LAMBDA RANGES FROM          40 TO         39.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE =          40-D:
```

NODE #: K	NEW VALUE: XNODE(K)
1	0
2	2
3	12
4	15-D
5	23
6	40-D

PROJECT COMPLETION TIME =   40-D.

ACTIVITY #: I	NEW VALUE: XACT(I)	ACTIVITY COST
1	2	0.80000E 01
2	12	0.80000E 01
3	15-D	0.40000E 01 + (0.10000E 01*D)
4	0	0.0
5	21	0.40000E 01
6	11	0.11000E 02
7	26	0.30000E 01
8	25	0.40000E 01
9	17-D	0.70000E 01 + (0.50000E 01*D)
10	6	0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS   0.53000E 02 + (0.60000E 01*D).

NEW VALUES OF ABAR FOR J=1,2,....,NK(I)

I J:	1	2	3	4	5	6	7	8	9
1	2	0							

2	3	0	-5		
3	2	-2	-6	-10	
4	-2	-2			
5	1	0	-1		
6	4	-1	-6		
7	-1	-2	-3	-4	
8	0	-2			
9	3	1	0		
10	-4	-4			

*** ITERATICN NUMBER 11 ***

THE NODE 2 HAS THE LABEL (1. 2.0.INF).

THE SINK HAS NCT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE 3 HAS THE LABEL (1. 2. 0. 0.10000E 01).

THE NODE 5 HAS THE LABEL (2. 2. 0. 0.10000E 01).

THE NODE 6 HAS THE LABEL (5. 3. 0. 0.10000E 01).

BREAKTHROUGH: UPDATE THE DUAL VARIABLES.

ACTIVITY #: I	J	NEW FLOW: F(I,J)
1	1	0.20000E 01
1	2	0.0
2	1	0.20000E 01
2	2	0.0
2	3	0.0
3	1	0.10000E 01
3	2	0.0
3	3	0.0
3	4	0.0
4	1	0.0
4	2	0.0
5	1	0.30000E 01
5	2	0.10000E 01
5	3	0.0
6	1	0.20000E 01
6	2	0.0
6	3	0.0
7	1	0.0
7	2	0.0
7	3	0.0
7	4	0.0
8	1	0.10000E 01
8	2	0.0
9	1	0.20000E 01
9	2	0.30000E 01
9	3	0.10000E 01
10	1	0.0
10	2	0.0

*** ITERATICN NUMBER 12 ***

```
THE NODE     2 HAS THE LABEL (    1.    2.0.INF).

THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY - CONTINUE WITH THE LABELING PROCESS.
THE NODES THAT HAVE BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERATION.

THE NODE     3 HAS THE LABEL (    1.    2.    0.      0.10000E 01).

NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES;
I.E.  DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA =          38.

DELTA (REPRESENTED BY "D") RANGES FROM 0 TO    1.
LAMBDA RANGES FROM       39 TO      38.
THE MINIMUM COST PROJECT SCHEDULE FOR PROJECT DEADLINE =          39-D:
  NODE #: K      NEW VALUE: XNODE(K)
          1                     0
          2                     2
          3                    12
          4                    14-D
          5                    23-D
          6                    39-D

PROJECT COMPLETION TIME =           39-D.

  ACTIVITY #: I   NEW VALUE: XACT(I)         ACTIVITY COST
       1                     2                 0.80000E 01
       2                    12                 0.80000E 01
       3                    14-D               0.50000E 01 + (  0.10000E 01*D)
       4                     0                 0.0
       5                    21-D               0.40000E 01 + (  0.40000E 01*D)
       6                    11-D               0.11000E 02 + (  0.20000E 01*D)
       7                    26                 0.30000E 01
       8                    25                 0.40000E 01
       9                    16                 0.12000E 02
      10                     6                 0.40000E 01

THE CURRENT VALUE OF THE PROJECT COST IS       0.59000E 02 + (  0.70000E 01*D).

NEW VALUES OF ABAR FOR J=1,2,...,NK(I)

      I    J:      1         2         3         4         5         6         7         8         9
      1            2         0
      2            3         0        -5
      3            3        -1        -5        -9
      4           -1        -1
      5            2         1         0
      6            5         0        -5
      7            0        -1        -2        -3
      8            0        -2
      9            3         1         0
     10           -4        -4

***      ITERATION NUMBER     13      ***

THE NODE     2 HAS THE LABEL (    1.    2.0.INF).
```

THE NODE 5 HAS THE LABEL (2, 3.0,INF).

THE NODE 6 HAS THE LABEL (5, 3.0,INF).

* * * * *

THE SINK WAS REACHED WITH INFINITE CAPACITY IMPLYING AN INFEASIBLE SOLUTION TO THE PRIMAL PROBLEM
IF LAMBDA DROPS BELOW ITS CURRENT VALUE. 38.

REFERENCES

1. Fulkerson, D. R. (1961). "A Network Flow Computation for Project Cost Curves," Management Science, 7, 167-178.

2. Hadley, G., Linear Programming. Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1963, 221-272.

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Appendix: Program Listing

```
C
C***********************************************************************
C
C     THIS PROGRAM IS DESIGNED TO FIND THE MINIMUM PROJECT COST AS A
C     FUNCTION OF PROJECT DEADLINE TIME.  CURRENT DIMENSIONS WILL
C     ALLOW A PROJECT NETWORK WITH UP TO 3000 NODES, 3000 ACTIVITIES,
C     AND 11 LEVELS OF COSTS AND TIMES.  ALL VARIABLES ARE INTEGER*2.
C     (IF ANY VARIABLE IS NOT ALREADY IN INTEGER FORM, THE VALUES MUST
C     BE RESCALED - THAT IS, MULTIPLIED BY AN APPROPRIATE POWER OF 10 -
C     UNTIL THE VALUES ARE INTEGER.)
C
C***********************************************************************
C
C     THE INPUT IS AS FOLLOWS (ALL RIGHT-JUSTIFIED):
C
C                   COLUMN       DESCRIPTION
C       CARD1:        1-4     NUMBER OF NODES
C                     6-9     NUMBER OF ACTIVITIES
C                      11     OPTION TO SUPPRESS PRINTING OF INPUT - TEST1
C                                (0=PRINT, 1=NO PRINT)
C                      13     OPTION TO SUPPRESS INTERMEDIATE OUTPUT-TEST3
C                                (0=PRINT, 1=NO PRINT)
C                   15-18     SOURCE NODE
C                   20-23     SINK NODE
C                      25     OPTION TO SPECIFY VALUE FOR LAMBDA - TEST3
C                                (0=NO, 1=YES AND SEE INTERMEDIATE
C                                OUTPUT, 2=YES BUT NO INTERMEDIATE OUTPUT)
C
C     THE FOLLOWING CARDS ARE IN SETS OF 3-5 CARDS
C        (ONE SET FOR EACH ACTIVITY).
C
C                   COLUMN       DESCRIPTION
C       CARD1:        1-4     ORIGIN NODE
C                     6-9     TERMINAL NODE
C                   11-12     NUMBER OF ACTIVITY COMPLETION TIMES
C                                AND COSTS THAT ARE READ IN (<=11)
C       CARD(S)2-3:    FORMAT 8I10    COMPLETION TIMES    (8 ON CARD 2,
C                                         3 ON CARD 3 IF NEEDED)
C       CARD(S)4-5:    FORMAT 8I10    COST ASSOCIATED W/EACH COMPLETION
C                                         TIME (8 ON CARD 4, 3 ON CARD 5)
C
C     LAST CARD (USE ONLY IF TEST3 = 1 OR 2):
C                   COLUMN       DESCRIPTION
C                    1-10     SPECIFIC VALUE OF LAMBDA
C
C***********************************************************************
C
C     DEFINITION OF VARIABLES:
C
C        ABAR(I,J) = TIME(I,NK(I)+1-J) + XNODE(ORIG(I))-XNODE(TERM(I))
C           C(I,J) = DECREASE IN I TH ACT'S COST PER UNIT FOR J TH TIME
C              CAP = MIN(FLOW REACHING ORIGIN NODE, EXCESS CAPACITY TO
C                       TERMINAL NODE)
C        COST(I,J) = COST OF COMPLETING ACTIVITY I AT TIME(I,J)
C              DEL = MIN(DELTA1,DELTA2)
C           DELTA1 = MIN(-ABAR(I,J) WITH I LABELED AND J UNLABELED,
C                       ABAR(I,J)<0)
C           DELTA2 = MIN(ABAR(I,J) WITH I UNLABELED AND J LABELED,
C                       ABAR(I,J)>0)
C          DIREC(J) = DIRECTION OF FLOW REACHING NODE J
C                       (0=FORWARD, 1=REVERSE)
```

```
C         FLOW(I,J) = FLOW IN J TH PIECE OF ACTIVITY I
C               INF = ANY NUMBER GREATER THAN MAX(CAP)
C                     (CURRENTLY SET AT (2*MAX +1))
C             K1(I) = THE NUMBER OF THE TIME-COST PIECE USED IN
C                     LABELING TERM(I) FROM ORIG(I)
C             KOUNT = KEEPS TRACK OF ORDER IN WHICH NODES WERE LABELED
C          LABEL(I) = 0 IF NODE I UNLABELED
C                     1 IF NODE I LABELED
C            LINPUT = SPECIFIC VALUE OF LAMBDA IF TEST3=1 OR 2
C                NA = TOTAL NUMBER OF ACTIVITIES
C             NK(I) = NUMBER OF DIFFERENT TIMES AND COSTS FOR ACTIVITY I
C                NN = TOTAL NUMBER OF NODES
C           ORIG(I) = ORIGIN NODE FOR ACTIVITY I
C          ORIG2(I) = WHERE THE FLOW IS FROM - USED IN LABELING ONLY
C             PCOST = PROJECT COST FUNCTION
C              SINK = NUMBER OF THE SINK NODE
C            SOURCE = NUMBER OF THE SOURCE NODE
C           TERM(I) = TERMINAL NODE FOR ACTIVITY I
C             TEST1 = OPTION TO SUPPRESS PRINTING OF INPUT
C                     (0=PRINT, 1=NO PRINT)
C             TEST2 = OPTION TO SUPPRESS INTERMEDIATE OUTPUT
C                     (0=PRINT, 1=NO PRINT)
C             TEST3 = OPTION TO SPECIFY VALUE FOR LAMBDA
C                     (0=NO, 1=YES AND SEE INTERMEDIATE OUTPUT,
C                     2=YES BUT NO INTERMEDIATE OUTPUT)
C         TIME(I,J) = J TH BREAKPOINT (DURATION TIME) FOR ACTIVITY I
C           XACT(I) = ACTIVITY DURATION TIME
C          XNODE(I) = NODE TIME
C          XDIFF(I) = XNODE(ORIG(I))-XNODE(TERM(I)),  AN UPPER BOUND ON
C                     THE ACTIVITY DURATION TIME
C     I,J,K,M,N,P = INDICES
C   INODE,ITERM,IACT,IORIG,IDIFF,ETC.
C                 = NON-INDEXED VERSIONS OF XNODE(I),TERM(I),XACT(I),
C                     ORIG(I),XDIFF(I),ETC.
C
C*******************************************************************
C
C   DIMENSIONS:
C                NN = TOTAL NUMBER OF NODES
C                NA = TOTAL NUMBER OF ACTIVITIES
C               MAX = MAX(NK(I))
C     CAP(NN),FLOW( NA,MAX),C(NA,MAX),ORIG(NA),TERM(NA),TIME(NA,MAX),
C     COST(NA,MAX),NK(MAX),ABAR(NA,MAX),XDIFF(NN),XNODE(NN),XACT(NA),
C     DIREC(NN),LABEL(NN),K1(NN),ORIG2(NN),KOUNT(NN),AORD(NA),
C     ND(NN),NDD(NN),IP(NA),CTIME(NA)
C
C*******************************************************************
C
      IMPLICIT INTEGER*2(A-Z)
      REAL*4 CAP(3000),FLOW(3000,11),C(3000,11),PCOST,INF,PCOST1,
     1KCOST,ACOST ,PNEW
      COMMON TIME,CTIME,XNODE,ORIG,TERM,AORD,NK,NN,NA,LMIN,LMAX,TEST1
      DIMENSION ORIG(3000),TERM(3000),TIME(3000,11),COST(3000,11),
     1NK(3000),          ABAR(3000,11),XDIFF(3000),XNODE(3000),
     2XACT(3000),              DIREC(3000),              LABEL(3000),
     3K1(3000),ORIG2(3000),KOUNT(3000),AORD(3000),CTIME(3000),
     4ND(3000),NDD(3000),IP(3000)
C
C   INPUT DATA
C
      READ(5,100) NN,NA,TEST1,TEST2,SOURCE,SINK,TEST3
```

```
      INF=0.
      PCOST=0.
      WRITE(6,228)
      IF(TEST1.EQ.1) GO TO 401
      WRITE(6,150) NN,NA,SOURCE,SINK
  401 DO 12 I=1,NA
      READ(5,230) ORIG(I),TERM(I),NK(I)
      KN=NK(I)
      READ(5,231) (TIME(I,J),J=1,KN)
      READ(5,231) (COST(I,J),J=1,KN)
   12 CONTINUE
      CALL ORDER
C
C     SET UP INITIAL VALUES
C
      IF(TEST1.EQ.1) GO TO 193
      K3=1
  192 K2=K3+8
      IF(K2.GT.NN) K2=NN
      WRITE(6,151) (K,K=K3,K2)
      WRITE(6,157) (XNODE(K),K=K3,K2)
      IF(K2.GE.NN) GO TO 191
      K3=K2+1
      GO TO 192
  191 WRITE(6,152)
  193 DO 10 I=1,NA
      LABEL(I)=0
      XDIFF(I)=XNODE(ORIG(I))-XNODE(TERM(I))
      NKM1=NK(I)-1
      KN=NK(I)
      DO 9 J=1,NKM1
      IF(TIME(I,J+1)-TIME(I,J)) 7,8,7
    7 C(I,J)=(COST(I,J)-COST(I,J+1))/(TIME(I,J+1)-TIME(I,J))
      GO TO 6
    8 C(I,J)=0.
    6 IF(INF.LT.C(I,J))  INF=C(I,J)
      XACT(I)=XDIFF(I)
      IF(XACT(I).LT.TIME(I,J+1)) XACT(I)=TIME(I,J+1)
      JJ=NK(I)-J+1
      ABAR(I,J)=TIME(I,JJ)+XDIFF(I)
      FLOW(I,J)=0
    9 CONTINUE
      ABAR(I,KN)=TIME(I,1)+XDIFF(I)
      FLOW(I,KN)=0
      IF(TEST1.EQ.1) GO TO 10
      WRITE(6,153) I,XACT(I),ORIG(I),TERM(I),(J,TIME(I,J),COST(I,J),
     1C(I,J),ABAR(I,J),J=1,NKM1)
      WRITE(6,156) KN,TIME(I,KN),COST(I,KN),ABAR(I,KN)
   10 CONTINUE
      INF=2.*INF+1.
      DO 417 I=1,NA
      C(I,NK(I))=0.
      NKM1=NK(I)-1
      PCOST1=0.
      IKK=0
      DO 418 K=1,NKM1
      IF(K.NE.1) GO TO 40
      XIJ=XACT(I)
      IF(XIJ.GT.TIME(I,2)) XIJ=TIME(I,2)
      GO TO 41
   40 XIJ=XACT(I)-TIME(I,K)
```

```
      IF(XIJ.LT.0) XIJ=0
      IF(XIJ.GT.(TIME(I,K+1)-TIME(I,K))) XIJ=TIME(I,K+1)-TIME(I,K)
      IF(IKK.EQ.1) GO TO 41
      IF(C(I,K).GT.C(I,K-1)) GC TC 50
   41 PCOST1=PCOST1+C(I,K)*XIJ
      GO TO 418
   50 IKK=1
      WRITE(6,237) I,I
      PCOST1=PCOST1+C(I,K)*XIJ
  418 CONTINUE
      PCOST=PCOST+COST(I,1)+C(I,1)*TIME(I,1)-PCOST1
      PNEW=PCOST
  417 CONTINUE
      LAMBDA=LMAX
      IF (TEST3.GE.1) GO TC 700
      WRITE(6,154)
      LINPUT=0
      GO TO 96
  700 READ(5,232) LINPUT
      IF(LINPUT.LT.LMIN) GC TC 705
      IF(LINPUT.GE.LMAX) GO TO 704
      IF (TEST3.EQ.2) GC TO 724
      WRITE(6,155) LINPUT
   96 WRITE(6,200) LAMBDA, PCOST
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 724
      WRITE(6,235)
  724 CAP(SOURCE)=INF
      ITER=0
   99 LABEL(SOURCE)=1
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 97
      ITER=ITER+1
      WRITE(6,225) ITER
C
C     INITIAL LABELING ITERATION
C
   97 I=1
      J=SOURCE
      M=0
C     IF ACTIVITY STARTS AT DESIGNATED ORIGIN, TRY TO LABEL,
C     OTHERWISE, CHANGE ORIGINS.
   14 IF (ORIG(I).NE.J) GC TO 13
      ITERM=TERM(I)
C     CHECK IF NCDE ALREADY LABELED AND
C     CHECK IF ABAR(I,NK(I))=0.
      IF (LABEL(ITERM).NE.0.OR.AEAR(I,NK(I)).NE.C) GO TO 13
C     IF NODE NOT ALREADY LABELED AND ABAR(I,NK(I))=0,
C     PROCEDE WITH LABELING.
      LABEL(ITERM)=1
      ORIG2(ITERM)=J
      K1(ITERM)=NK(I)
      DIREC(ITERM)=0
      CAP(ITERM)=INF
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 403
      WRITE(6,201) ITERM,ORIG2(ITERM),K1(ITERM)
C     IF CAN REACH SINK, TERMINATE (IMPLIES INFEASIBLE)
  403 IF (ITERM.EQ.SINK) GC TO 15
      M=M+1
      KOUNT(M)=ITERM
C     IF EVERY PATH TESTED AND INFINITE FLOW NOT PCSSIBLE,
C     GO ON TO LABELING PART(II).
   13 I=I+1
```

```
      IF (I.GT.NA) GO TO 11
      GO TO 14
C     CHANGE DESIGNATED ORIGINS.
   11 IF (J.EQ.SOURCE) P=1
      IF(P.GT.M) GO TC 16
C     IF ALL LABELED NODES HAVE BEEN SCANNED AND NC NEW NODES
C     HAVE BEEN LABELED, GO ON TO LABELING PART (II).
      J=KOUNT(P)
      P=P+1
      I=1
      GO TO 14
   15 IF(TEST3.GE.1) GO TO 404
      WRITE(6,202) LAMBDA
  404 GO TO 999
   16 IF(TEST2.EG.1.OR.TEST3.GE.1) GO TO 405
      WRITE(6,203)
C
C     NEXT LABELING PROCEDURE
C
  405 I=1
      J=SOURCE
C     AGAIN, CHECK ALL CONDITIONS FOR LABELING
C     IE. CHECK IF NODE IS ALREADY LABELED, IF ABAR(I,J)=0, AND
C     IF THE FLOW(I,J) IS LESS THAN ITS UPPER BCUND.
   20 IF (ORIG(I).NE.J) GO TO 24
      ITERM=TERM(I)
      KN=NK(I)
      DO 25 K=1,KN
      IF (K.EQ.KN) GO TO 27
      IF(LABEL(ITERM).NE.0.OR.ABAR(I,K).NE.0.OR.FLOW(I,K).GE.
     1(C(I,NK(I)-K)-C(I,NK(I)-K+1)))GC TO 25
      DIREC(ITERM)=0
C     CAPACITY IS MIN OF PREVICUS FLOW AND THE EXCESS CAPACITY
      CAP(ITERM)=C(I,NK(I)-K)-C(I,NK(I)-K+1) - FLOW(I,K)
      GO TO 23
   27 IF(LABEL(ITERM).NE.0.OR.ABAR(I,K).NE.0.OR.FLCW(I,K).GE.INF)
     1  GO TO 25
C     IF THE NODE HAS NCT ALREACY BEEN LABELED, ABAR(I,J)=0, AND
C     THE FLOW IS LESS THAN ITS UPPER BCUND, PROCEDE WITH THE LABELING
C     OF THE NODE.
      DIREC(ITERM)=0
      CAP(ITERM)=INF
   23 LABEL(ITERM)=1
      ORIG2(ITERM)=J
      K1(ITERM)=K
      IF (CAP(ITERM).GT.CAP(CRIG(I)))  CAP(ITERM)=CAP(ORIG(I))
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 406
      WRITE(6,204) ITERM,ORIG2(ITERM),K1(ITERM),DIREC(ITERM),CAP(ITERM)
C     IF SINK LABELED, GO TO UPDATE PROCEDURE
  406 IF (ITERM.EQ.SINK) GO TO 21
      M=M+1
      KOUNT(M)=ITERM
C     CHECK IF ALL FATHS TRIED
   25 CONTINUE
      GO TO 19
   24 IF(TERM(I).NE.J) GO TO 19
      IORIG=ORIG(I)
      KN=NK(I)
      DO 26 K=1,KN
      IF(LABEL(IORIG).NE.0.OR.ABAR(I,K).NE.C.OR.FLOW(I,K).LE.0)
     2 GO TO 26
```

```
      DIREC(IORIG)=1
      CAP(IORIG)=FLCW(I,K)
      LABEL(IORIG)=1
      ORIG2(IORIG)=J
      K1(IORIG)=K
      IF(CAP(IORIG).GT.CAP(TERM(I))) CAP(IORIG)=CAP(TERM(I))
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 402
      WRITE(6,204) IORIG,ORIG2(ICRIG),K1(IORIG),DIREC(IORIG),CAP(IORIG)
  402 M=M+1
      KOUNT(M)=ICRIG
   26 CONTINUE
   19 I=I+1
      IF (I.GT.NA) GO TO 18
      GO TO 20
   18 IF (J.EQ.SOURCE) P=1
      IF (P.GT.M) GO TO 22
      J=KOUNT(P)
      P=P+1
      I=1
      GO TO 20
C
C     NONBREAKTHROUGH HAS OCCURED. DELTAS ARE FOUND AND UPDATING
C     MADE IN THE XNODES AND XACTS.
C
   22 DELTA1=INF+1
      DELTA2=INF+1
      DO 4 I=1,NA
      KN=NK(I)
      IF (LABEL(ORIG(I)).EQ.1.AND.LABEL(TERM(I)).EQ.0) GO TO 1
C     A1 IS SET CF I LAEELEC AND J UNLABELED,
C     A2 IS SET OF I UNLABELED AND J LABELEC.
      IF(LABEL(CRIG(I)).EQ.0.AND.LABEL(TERM(I)).EQ.1) GO TO 2
      GO TO 4
C     FINDING DELTA1'S.
    1 DO 3 J=1,KN
      IF (ABAR(I,J).GE.0) GO TC 3
      IF (-ABAR(I,J).LT.DELTA1) DELTA1=-ABAR(I,J)
    3 CONTINUE
      GO TO 4
C     FINDING DELTA2'S
    2 DO 5 J=1,KN
      IF(ABAR(I,J).LE.0) GO TO 4
      IF (ABAR(I,J).LT.DELTA2) CELTA2= ABAR(I,J)
    5 CONTINUE
    4 CONTINUE
C     DEL=MIN(DELTA1,DELTA2)
      DEL=DELTA1
      IF (DELTA2.LT.DEL) DEL=DELTA2
      LAMBDA=LAMBDA-DEL
C     UPDATING THE XNODES.
      IF(TEST2.EC.1.OR.TEST3.GE.1) GO TO 407
      WRITE(6,206) LAMBDA
  407 IF (TEST3.EQ.2) GC TO 721
      DELTA= LAMBDA + DEL
      WRITE(6,209)          DEL,DELTA,LAMBDA,DELTA
  721 IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 408
      WRITE(6,207)
  408 DO 80 I=1,NN
      INODE=XNODE(I)
      IF(LABEL(I).EQ.0) GO TC 81
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 409
```

```
      WRITE(6,210) I,INODE
  409 XNODE(I)=INODE
      GO TO 80
   81 IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 410
      WRITE(6,211) I,INODE
  410 XNODE(I)=INODE-DEL
   80 CONTINUE
      IF (TEST3.EQ.2) GO TO 722
      WRITE(6,212) DELTA
  722 PCOST=0.
      DO 82 I=1,NA
      IP(I)=0
      PCOST1=0.
      NKM1=NK(I)-1
      IACT=TIME(I,NK(I))
      IORIG=ORIG(I)
      ITERM=TERM(I)
      IDIFF=XNODE(ITERM)-XNODE(ICRIG)
      XDIFF(I)=-IDIFF
      IF (IDIFF.GE.IACT) GO TO 86
      XACT(I)=IDIFF
      DO 550 K=1,NKM1
      IF(K.NE.1) GO TC 43
      XIJ=XACT(I)
      IF(XIJ.GT.TIME(I,2)) XIJ=TIME(I,2)
      FLAG1=0
      GO TO 42
   43 XIJ=XACT(I)-TIME(I,K)
      IF(XIJ.LT.0) GO TO 552
      IF(XIJ.GT.(TIME(I,K+1)-TIME(I,K))) XIJ=TIME(I,K+1)-TIME(I,K)
      FLAG1=0
      GO TO 42
  552 FLAG1=1
      FLAG2=K-1
      GO TO 553
   42 PCOST1=PCOST1+C(I,K)*XIJ
  550 CONTINUE
  553 KCOST=COST(I,1)+C(I,1)*TIME(I,1)
      ACOST=KCOST-PCOST1
      PCOST=PCOST+ACOST
      IF (TEST3.EQ.2) GO TC 82
      IF (LABEL(IORIG)-LABEL(ITERM)) 83,84,85
   83 IDIFF=IDIFF-DEL
      IF(FLAG1.EQ.1) GO TO 59
      ACOST=ACOST + C(I,NKM1)*DEL
      IP(I)=1
      WRITE(6,214) I,IDIFF,ACOST,C(I,NKM1)
      GO TO 82
   59 ACOST=ACOST+C(I,FLAG2)*DEL
      IP(I)=1
      WRITE(6,214) I,IDIFF,ACOST,C(I,FLAG2)
      GO TO 82
   84 WRITE(6,216) I,XACT(I),ACOST
      GO TO 82
   85 IDIFF=IDIFF+DEL
      IF(FLAG1.EQ.1) GO TC 58
      ACOST=ACOST - C(I,NKM1)*DEL
      IP(I)=2
      WRITE(6,213) I,IDIFF,ACOST,C(I,NKM1)
      GO TO 82
   58 ACOST=ACOST-C(I,FLAG2)*DEL
```

```
      IP(I)=2
      WRITE(6,213) I,IDIFF,ACOST,C(I,FLAG2)
      GO TO 82
   86 XACT(I)=IACT
      DO 551 K=1,NKM1
      IF(K.NE.1) GO TO 45
      XIJ=XACT(I)
      IF(XIJ.GT.TIME(I,2)) XIJ=TIME(I,2)
      GO TO 46
   45 XIJ=XACT(I)-TIME(I,K)
      IF(XIJ.LT.0) XIJ=0
      IF(XIJ.GT.(TIME(I,K+1)-TIME(I,K))) XIJ=TIME(I,K+1)-TIME(I,K)
   46 PCOST1=PCOST1+C(I,K)*XIJ
  551 CONTINUE
      KCOST=COST(I,1)+C(I,1)*TIME(I,1)
      ACOST=KCOST-PCOST1
      PCOST=PCOST+ACOST
      IF (TEST3.EQ.2) GO TO 82
      WRITE(6,216) I,XACT(I),ACOST
   82 CONTINUE
      IF (TEST3.EQ.2) GO TO 723
      PCOST1=PNEW
      PNEW=(PCOST-PNEW)/DEL
      WRITE(6,224) PCOST1,PNEW
  723 PNEW=PCOST
      IF (TEST3.NE.0.AND.LAMEDA.LE.LINPUT) GO TO 703
C     RESET LABELS TO 0 AND REFIGURE ABARS.
C     THEN START OVER.
      DO 87 I=1,NN
      LABEL(I)=0
   87 CONTINUE
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 420
      WRITE(6,226) (J,J=1,11)
  420 DO 88 I=1,NA
      NKM1=NK(I)-1
      DO 500 K=1,NKM1
      J=NKM1+2-K
  500 ABAR(I,K)=TIME(I,J)+XDIFF(I)
      ABAR(I,NK(I))=TIME(I,     1)+XDIFF(I)
      IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO   88
      NK1=NK(I)
      WRITE(6,227)I,(ABAR(I,J),J=1,NK1)
   88 CONTINUE
      IF (LAMBDA.LT.LMIN) GO TO 998
      GO TO 99
C
C     UPDATE THE FLOW AFTER BREAKTHROUGH.
C
   21 IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO   34
      WRITE(6,205)
   34 FLOW(I,K1(ITERM))=FLOW(I,K1(ITERM))+CAP(ITERM)
C     IF DIREC =0 THEN CAP ADDED TO FLOW.
C     IF DIREC =1 THEN CAP IS SUBTRACTED.
   30 ITERM=ORIG2(ITERM)
C     CHECK IF BACK AT SOURCE.
      IF(ITERM.EQ.SOURCE) GO TO 33
C     FIND WHERE FLOW CAME FROM.
      I=NA
   32 I=I-1
      IF (ORIG(I).EQ.ORIG2(ITERM).AND.TERM(I).EQ.ITERM) GO TO 31
      GO TO 32
```

```
C       CHECK IF DIRECTION OF FLOW IS POSITIVE OR NEGATIVE,
C       ALSO CHECK IF CAPACITY IS INFINITE.
    31 IF(CAP(ITERM).EQ.INF) GC TC 33
       IF (DIREC(ITERM).EQ.0) GC TO 34
       FLOW(I,K1(ITERM))=FLOW(I,K1(ITERM))-CAP(ITERM)
       GO TO 30
C       RELABEL AND START OVER.
    33 IF(TEST2.EQ.1.OR.TEST3.GE.1) GO TO 415
       DO 560 I=1,NA
       NK1=NK(I)
       DO 560 K=1,NK1
   560 WRITE(6,220) I,K,FLOW(I,K)
   415 DO 98 I=1,NN
       LABEL(I)=0
    98 CONTINUE
       GO TO 99
C       PROGRAM TERMINATES WHEN EVENTUALLY AN INFINITE FLOW IS ACHIEVED
C       FROM THE SCURCE TO THE SINK,  OR WHEN THE VALUE OF LAMBDA DROPS
C       BELOW THE MINIMUM LENGTH CF THE NETWORK.
   998 IF(TEST3.NE.0) GO TO 999
       WRITE(6,202) LAMBDA
       GO TO 999
   705 WRITE(6,233) LINPUT,LMIN
       GO TO 999
   704 WRITE(6,236) LINPUT,LMAX
       WRITE(6,238) LINPUT
       D=0
       DO 60 I=1,NA
    60 IP(I)=C
       GO TO 707
   703 WRITE(6,234) LINPUT
   706 WRITE(6,238) LINPUT
       D=LINPUT-LAMBCA
   707 PCOST=0.
       DO 57 I=1,NA
       IF(IP(I).EC.1.AND.D.GT.0) XACT(I)=XACT(I)-D
       IF(IP(I).EQ.2.AND.D.GT.0) XACT(I)=XACT(I)+D
       PCOST1=0.
       NKM1=NK(I)-1
       DO 51  K=1,NKM1
       IF(K.NE.1) GO TC 52
       XIJ=XACT(I)
       IF(XIJ.GT.TIME(I,2)) XIJ=TIME(I,2)
       GO TO 53
    52 XIJ=XACT(I)-TIME(I,K)
       IF(XIJ.LT.0) XIJ=0
       IF(XIJ.GT.(TIME(I,K+1)-TIME(I,K))) XIJ=TIME(I,K+1)-TIME(I,K)
    53 PCOST1=PCOST1+C(I,K)*XIJ
    51 CONTINUE
       KCOST=COST(I,1)+C(I,1)*TIME(I,1)
       ACOST=KCCST-PCOST1
       WRITE(6,216) I,XACT(I),ACOST
    57 PCOST=PCOST+ACOST
       WRITE(6,239) PCCST
   999 WRITE(6,228)
       STOP
   100 FORMAT(I4,1X,I4,1X,I1,1X,I1,1X,I4,1X,I4,1X,I1)
   150 FORMAT('-','THE NUMBER OF NODES IS ',I4,'.',/,1X,'THE NUMBER OF AC
      1TIVITIES IS ',I4,'.',/,1X,'THE SCURCE NODE IS NUMBERED ',I4,' AND
      2THE SINK NODE IS NUMBERED ',I4,'.',/,'-',' ** NCDES: **')
   151 FORMAT('0',16X,'K',7X,9(3X,I4,5X))
```

```
152 FORMAT('-',' ** ACTIVITIES: **',//,6X,'I',7X,'XACT',6X,'ORIG',3X,
   1'TERM',4X,'J',6X,'TIME',5X,'COST',14X,'C',13X,'ABAR')
153 FORMAT(' ',3X,I4,3X,I10,3X,I4,3X,I4,(T39,I2,3X,I10,3X,I10,3X,
   1E16.5,3X,I10))
154 FORMAT('-','THE ENTIRE PROJECT COST CURVE IS GOING TO BE DETERMINE
   1D.')
155 FORMAT('-','THE OPTIMAL ACTIVITY COMPLETION TIMES FOR A SPECIFIED
   1PROJECT DEADLINE TIME = ',I10,' ARE GOING TO BE DETERMINED.')
156 FORMAT(' ',T39,I2,3X,I10,3X,I10,22X,I10)
157 FORMAT('0',                               4X,'INITIAL XNODE(K)',3X,
   19(I10,2X))
200 FORMAT('0','LAMBDA = PROJECT COMPLETION TIME',//,
   1         1X,'THE STARTING VALUE OF LAMBDA IS ',I10,'.',//
   2           ,1X,'THE CORRESPONDING TOTAL PROJECT COST IS ',E16.5,'.')
201 FORMAT('0','THE NODE ',I4,' HAS THE LABEL (',I4,',',I4,',0,INF).')
202 FORMAT('0',////,30X,'* * * * * *',/////,1X,
   1         'THE SINK WAS REACHED WITH INFINITE CAPACITY IMPLYING A
   1N INFEASIBLE SOLUTION TO THE PRIMAL PROBLEM ',/,20X,'IF LAMBDA DRO
   2PS BELOW ITS CURRENT VALUE, ',I10,'.')
203 FORMAT('-','THE SINK HAS NOT BEEN REACHED WITH INFINITE CAPACITY -
   1 CONTINUE WITH THE LABELING PROCESS.',/  ,1X,'THE NODES THAT HAVE
   2BEEN LABELED WILL RETAIN THAT LABEL FOR THE REMAINDER OF THE ITERA
   3TION.')
204 FORMAT('0','THE NODE ',I4,' HAS THE LABEL (',I4,',',I4,',',I4,',',
   1E16.5,').')
205 FORMAT('-','BREAKTHROUGH:  UPDATE THE DUAL VARIABLES.'
   1,///,1X,'   ACTIVITY #: I',3X,'J',9X,'NEW FLOW: F(I,J)')
206 FORMAT('-','NONBREAKTHROUGH:  UPDATE THE PRIMAL VARIABLES:',/,1X,
   1'I.E.  DETERMINE OPTIMAL ACTIVITY TIMES FOR LAMBDA = ',I10,'.')
207 FORMAT(' ','   NODE #: K',5X,'NEW VALUE: XNODE(K)')
209 FORMAT('-','DELTA (REPRESENTED BY "D") RANGES FROM 0 TO'
   1 ,I4,'.',/,1X,'LAMBDA RANGES FROM',I10,' TO',I10,'.',
   2                    /,1X,'THE MINIMUM COST PROJECT SCHEDULE FOR PRO
   3JECT DEADLINE = ',I10,'-D:')
210 FORMAT(' ',7X,I4,12X,I10)
211 FORMAT(' ',7X,I4,12X,I10,'-D')
212 FORMAT('-','PROJECT COMPLETION TIME = ',I10,'-D.',//,1X,
   1      '   ACTIVITY #: I',3X,'NEW VALUE: XACT(I)',9X,'ACTIVITY CO
   2ST')
213 FORMAT(' ',5X,I4,12X,I10,'-D',9X,E16.5,' + (',E13.5,'*D)')
214 FORMAT(' ',5X,I4,12X,I10,'+D',9X,E16.5,' + (',E13.5,'*D)')
216 FORMAT(' ',5X,I4,12X,I10,11X,E16.5)
220 FORMAT(' ',12X,I4,2X,I2,7X,E16.5)
224 FORMAT('0','THE CURRENT VALUE OF THE PROJECT COST IS ',E16.5,
   1' + (',E13.5,'*D).')
225 FORMAT('-','***       ITERATION NUMBER',I6,'       ***')
226 FORMAT('-','NEW VALUES OF ABAR FOR J=1,2,...,NK(I)',//,6X,'I',3X,
   1'J:',11(5X,I2,3X))
227 FORMAT(' ',2X,I4,7X,11(I8,2X))
228 FORMAT(1H1)
230 FORMAT(I4,1X,I4,1X,I2)
231 FORMAT(8I10)
232 FORMAT(I10)
233 FORMAT('-','THE SPECIFIED VALUE OF LAMBDA,',I10,',IS LESS THAN THE
   1 MINIMUM VALUE,',I10,',IMPLYING AN INFEASIBLE SOLUTION.',//,1X,
   2'THE PROBLEM WILL NOT BE WORKED.')
234 FORMAT('1','THE SPECIFIED VALUE OF LAMBDA,',I10,',HAS BEEN REACHED
   1.')
235 FORMAT('0',   'THE SOURCE HAS A VALUE OF ZERO AND IS ASSIGNED THE
   3LABEL (-,-,-,INF).',// )
236 FORMAT('-','THE SPECIFIED VALUE OF LAMBDA, ',I10,',IS GREATER THAN
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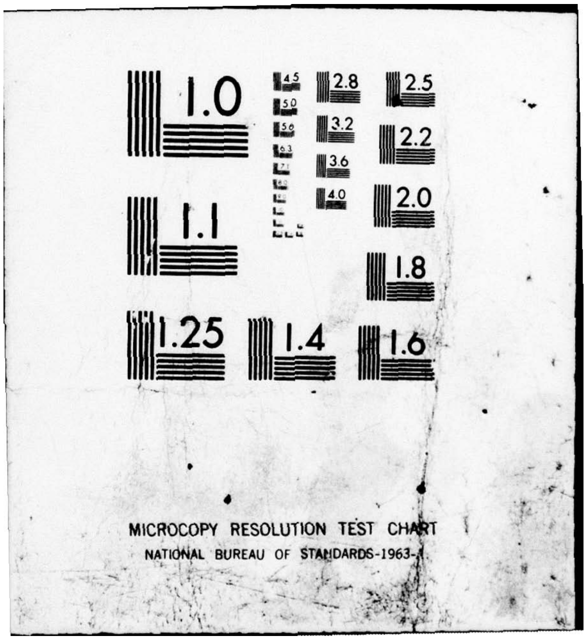
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MICROCOPY RESOLUTION TEST CHART
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```
     1 OR EQUAL TO THE MAXIMUM VALUE. ',I10,';',//,
     1                                          1X,'THEREFORE, THE ORIGINAL
     2XNODE(K)''S AND XACT(I)''S ARE OPTIMAL.')
 237 FORMAT(' ','** WARNING:  ACTIVITY NUMBER ',I4,' HAS A NON-CONVEX C
    1OST FUNCTION;',/,12X,'IE. THE C(',I4,',M)''S ARE NOT NON-INCREASIN
    2G.')
 238 FORMAT('-','FOR PROJECT COMPLETION TIME = ',I10,', THE OPTIMAL SOL
    1UTION IS:',//,1X,
     1         '  ACTIVITY #: I',3X,'NEW VALUE: XACT(I)',9X,'ACTIVITY CO
     2ST')
 239 FORMAT('-','THE CORRESPONDING PROJECT COST IS ',E16.5,'.')
     END
     SUBROUTINE ORDER
C
C       THIS SUBROUTINE DETERMINES THE ORDER IN WHICH TO CONSIDER
C       THE ACTIVITIES FOR THE CALCULATION OF THE CRITICAL PATH TIME
C    DIMENSIONS:
C    NA=M= THE NUMBER OF ACTIVITIES IN THE NETWORK
C    NN=N= THE NUMBER OF NODES IN THE NETWORK
C    ORIG(NA),TERM(NA),AORD(NA),CTIME(NA),XNODE(NN),ND(NN),NDD(NN),
C    TIME(NA,MAX),NK(MAX)
C
     IMPLICIT INTEGER*2(A-Z)
     COMMON TIME,CTIME,XNODE,ORIG,TERM,AORD,NK,NN,NA,LMIN,LMAX,TEST1
     DIMENSION ORIG(3000),TERM(3000),AORD(3000),CTIME(3000),
    1XNODE(3000),ND(3000),NDD(3000),TIME(3000,11),NK(3000)
     N=NN
     M=NA
     NDD(1)=1
     DO 5 I=2,N
 5   NDD(I)=0
     DO 6 I=1,M
 6   AORD(I)=0
     K=0
     MP=M+1
     DO 1 II=1,MP
     DO 20 I=1,N
 20  ND(I)=NDD(I)
     III=0
     IP=II+1
     DO 2 J=1,M
     IF(ND(ORIG(J)).NE.II) GO TO 2
     NDD(TERM(J))=IP
     III=1
     IF(K.EQ.0) GO TO 14
     DO 10 L=1,K
     IF(AORD(L).EQ.J) GO TO 11
 10  CONTINUE
 14  K=K+1
     GO TO 13
 11  IF(L.EQ.K) GO TO 2
     KM=K-1
     DO 12 LL=L,KM
 12  AORD(LL)=AORD(LL+1)
 13  AORD(K)=J
 2   CONTINUE
     IF(III.EQ.0) GO TO 3
 1   CONTINUE
 3   CONTINUE
     DO 30 I=1,NA
 30  CTIME(I)=TIME(I,1)
```

```
      LMIN=CPTIME(CPATHT)
      DO 31 I=1,NA
      NK1=NK(I)
31    CTIME(I)=TIME(I,NK1)
      LMAX=CPTIME(CPATHT)
      RETURN
      END
      FUNCTION CPTIME(CPATHT)
C
C         DETERMINE THE CRITICAL PATH TIME: CPTIME
C         XNODE(I) = EARLIEST TIME THAT AN ACTIVITY BEGINNING AT NODE I
C                    CAN COMMENCE
C     DIMENSIONS:
C     NA=M= THE NUMBER OF ACTIVITIES IN THE NETWORK
C     NN=N= THE NUMBER OF NODES IN THE NETWORK
C     ORIG(NA),TERM(NA),AORD(NA),CTIME(NA),XNODE(NN),ND(NN),NDD(NN),
C     TIME(NA,MAX),NK(MAX)
C
      IMPLICIT INTEGER*2(A-Z)
      COMMON TIME,CTIME,XNODE,ORIG,TERM,AORD,NK,NN,NA,LMIN,LMAX,TEST1
      DIMENSION ORIG(3000),TERM(3000),AORD(3000),CTIME(3000),
     1XNODE(3000),ND(3000),NDD(3000),TIME(3000,11),NK(3000)
      DO 1 I=1,NN
1     XNODE(I)=0
      DO 2 II=1,NA
      I=AORD(II)
2     IF(XNODE(ORIG(I))+CTIME(I).GT.XNODE(TERM(I)))
     1   XNODE(TERM(I))=XNODE(ORIG(I))+CTIME(I)
      CPTIME=XNODE(NN)
      RETURN
      END
```

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13. ABSTRACT

See page following Attachment II

ATTACHMENT III

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Project Scheduling						
Network-flow Algorithm						
Minimum Cost Schedule						
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