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ANALYSES OF PLATES CONSTRUCTED OF
FIBER-REINFORCED BIMODULUS MATERIALS

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ABSTRACT

To implement the structural application of the recently introduced fiber-governed symmetric compliance model for bimodulus composite materials, both classical closed-form and finite-element solutions are being developed. This paper summarizes the results obtained for deflection of single-layer ortho-tropic and two-layer, cross-ply plates of the following configurations and loadings:

1. Thin elliptic plates clamped on the boundary and subjected to uniform pressure.
2. Moderately thick rectangular plate freely supported on the boundary and subjected to sinusoidally distributed pressure.

NOMENCLATURE

A_{ij}	=	stretching stiffness ($i,j=1,2,6$)
a,b	=	plate semi-axes
B_{ij}	=	bending-stretching coupling stiffness ($i,j=1,2,6$)
D_{ij}	=	bending stiffness ($i,j=1,2,6$)
d_x	=	$\alpha(\ )/\alpha x$
F_i	=	finite-element force components ($i=1,2,\ldots,5$)
h	=	total plate thickness
K_{ij}	=	stiffness coefficients in the finite-element formulation
$L_{\alpha\beta}$	=	linear differential operators defined in equations (2)
M_i,N_i	=	stress couples and stress resultants ($i=1,2,6$)
N_i	=	finite-element interpolation functions
$n,\bar{n}$	=	number of nodes per element, number of layers in laminate
p,p_o	=	normal pressure, intensity of normal pressure
Q_x,Q_y	=	thickness shear stress resultants

Q_{ij}	=	plane-stress reduced elastic stiffness ($i,j=1,2,6$)
Q_{ijk}	=	Q_{ij} for tension ($k=1$) or compression ($k=2$)
$Q_{ijk\ell}$	=	Q_{ijk} for layers 1 or 2 ($\ell=1$ or 2)
S_{ij}	=	elastic compliance coefficient ($i,j=1,2,6$)
$S_{ij}^{\xi\eta}$	=	finite-element matrix coefficients
u,v,w	=	midplane displacements in x,y,z directions
$\bar{u},\bar{v}$	=	total displacements in x,y directions
u_0,v_0,w_0	=	coefficients in expressions for u,v,w
x,y	=	in-plane rectangular position coordinates
Z_x,Z_y	=	z_{nx}/h, z_{ny}/h
z	=	thickness-direction position coordinate, measured from midplane
z_{nx},z_{ny}	=	z coordinates of neutral surfaces based on ϵ_x and ϵ_y
ϵ_f	=	fiber-direction strain
ϵ_j	=	strain ($j=1,2,6$)
σ_i	=	stress ($i=1,2,6$)
θ	=	angle between fiber direction of an individual layer and reference direction for the laminate
ψ_x,ψ_y	=	slope functions
Ω	=	middle plane of the plate
Ω_e	=	a typical finite element

INTRODUCTION

An elastic plate is an important structural component in a wide variety of engineering systems. Thus, it is not surprising that the first modern development of the basic constitutive equations of bimodulus[1] materials by Ambartsumyan (1)[2] was followed only one year later by Shapiro's analysis (2) of static deflection of a circular plate constructed of such a material and subjected to a pure radial bending moment. However, in his analysis, he used Love's stress-function formulation rather than plate theory.

Unfortunately, there have been relatively few more recent analyses of bending of bimodulus plates. Notable exceptions are the series of papers by Kamiya (3-5). In (3), he treated large deflections (geometric nonlinearity) of circular plates, using an iterative finite-difference technique, while in (4), he applied the energy method to large deflections of a rectangular plate. The effect of thickness shear deformation on the linear problem was treated in (5).

Ponomarev (6) considered bending of a square plate made of a nonlinear elastic material of a slightly more general nature than bilinear, namely one with a third-degree polynomial stress-strain relation such that the ratio of tension stress to compression stress at the same absolute value of strain remains constant.

All of the analyses mentioned above are limited to isotropic materials with different properties in tension and compression. Apparently, the first

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[1] Here a bimodulus material is understood to be one having different elastic properties in tension (T) and compression (C). Materials with same properties in T and C are referred to herein as ordinary materials.

[2] Underlined numbers in parentheses designate References at end of paper.

analysis considering bimodulus materials other than isotropic is due to Jones and Morgan (7), who presented a closed-form solution for cylindrical bending of a thin, cross-ply laminate of finite width but infinite length. They presented numerical results for examples of both antisymmetric and more generally un-symmetric cross-ply constructions.

Apparently, the first analyses of anisotropic bimodulus plates finite in both directions are those recently carried out by the present authors and their colleagues and summarized here. The planform geometries, boundary conditions, loadings, lamination schemes, degree of nonlinearity, and methods of solution are listed in Table 1.

Table 1. Specific Plate Bending Problems Considered
For Anisotropic Bimodulus Materials†

Problem No.	Planform Geometry	Boundary Conditions	Lamination Scheme	Degree of Nonlinearity	Method of Solution
1	Elliptic thin	Clamped	Unidirectional orthotropic	Linear	Closed form (8)
2	Elliptic thin	Clamped	Unidirectional orthotropic & cross-ply	von Karman geometric nonlinearity	Perturbation (9)
3	Elliptic thin	Clamped	Arbitrary cross-ply	Linear	Closed form
4	Rectangular thick	Freely supported	Unidirectional orthotropic & cross-ply	Linear	Closed form and finite element
5	Rectangular thick	Freely supported*	Cross-ply*	Linear	Finite element

GOVERNING DIFFERENTIAL EQUATIONS

Classical small-deflection theory of thin, unsymmetrically laminated, ani-sotropic plates was originated by Reissner and Stavsky (10). Apparently, the first display of such a theory for the completely arbitrary anisotropic lamin-ated plate was due to Whitney and Leissa (11). These same equations govern the small deflections of thin, laminated, bimodulus anisotropic plates and thus they are presented here in concise form for completeness:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p \end{Bmatrix} \tag{1}$$

where the $L_{\alpha\beta}$ are symmetric linear differential operators ($L_{\beta\alpha} = L_{\alpha\beta}$) defined as follows:

$$L_{11} \equiv A_{11}d_x^2 + 2A_{16}d_xd_y + A_{66}d_y^2$$

$$L_{12} \equiv A_{16}d_x^2 + (A_{12} + A_{66})d_xd_y + A_{26}d_y^2$$

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* The program can handle arbitrary lamination arrangement and boundary cond-itions. However, the results for only cross-ply laminates and freely supported edges are included here.
† Uniform loading is considered in all cases, except in Problem 4, in which sinusoidal loading is used.

$$L_{13} \equiv - B_{11}d_x^3 - 3B_{16}d_x^2 d_y - (B_{12} + 2B_{66})d_x d_y^2 - B_{26}d_y^3$$

$$L_{22} \equiv A_{66}d_x^2 + 2A_{26}d_x d_y + A_{22}d_y^2 \qquad\qquad (2)$$

$$L_{23} \equiv - B_{16}d_x^3 - (B_{12} + 2B_{66})d_x^2 d_y - 3B_{26}d_x d_y^2 - B_{22}d_y^3$$

$$L_{33} \equiv D_{11}d_x^4 + 4D_{16}d_x^3 d_y + 2(D_{12} + 2D_{66})d_x^2 d_y^2 + 4D_{26}d_x d_y^3 + D_{22}d_y^4$$

Here u, v, w are the midplane displacements in the x, y (in-plane) and z (normal) directions, $d_x \equiv d(\ )/dx$, p is the normal pressure, and the plate stiffnesses are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2)Q_{ij}(z)dz \qquad (i,j=1,2,6) \qquad (3)$$

Here h is the total thickness of the laminate and z is measured from the mid-plane by convention. The $Q_{ij}(z)$ are the plane-stress reduced elastic stiff-nesses defined as follows:

$$\{\sigma_i\} = [Q_{ij}]\{\epsilon_j\} \qquad\qquad (i,j=1,2,6) \qquad (4)$$

Here the contracted notation of composite-material theory (12) is used. Thus, subscripts 1 and 2 refer to normal stress (or strain) action in the x and y directions and 6 refers to shearing stress (or strain) action with respect to the x,y axes. Due to thermodynamic considerations, the $[Q_{ij}]$ matrix is symmetric.

PLATE STIFFNESSES FOR BIMODULUS-MATERIAL LAMINATES

Even in laminates made of ordinary materials (those having the same elastic properties in tension and compression), the stiffnesses Q_{ij} are piece-wise-constant functions of the thickness-direction coordinate z. The indivi-dual layers generally consist of unidirectional fiber-reinforced composite material which is orthotropic with respect to its material-symmetry axes: the fiber direction and the two directions orthogonal to it. Thus, when the individual layers are oriented parallel to either the x or y axes of the plate, all of the A_{ij}, B_{ij}, and D_{ij} stiffnesses involving shear-normal coupling (all of those with subscripts 16 and 26) identically vanish.

Furthermore, when a laminate is composed of individual layers of ordinary material arranged symmetrically with respect to its midplane, all of the so-called bending-stretching coupling stiffnesses (B_{ij}) vanish. Thus, a single-layer ordinary composite has no B_{ij} terms present. However, as discussed in (7) and (8), even a single-layer bimodulus composite, as well as a so-called symmetrically laminated multi-layer laminate, has certain B_{ij} terms present by virtue of the nature of the bimodulus action. Thus, the terms symmetric, anti-symmetric, and unsymmetric have no significance for bimodulus laminates.

A popular lamination scheme is the so-called balanced angle-ply laminate. This is one consisting of an even number of layers of identical material and thickness and having an equal number of layers oriented at an angle +θ and -θ with respect to an arbitrary reference direction. When such a laminate is made of ordinary material, the shear-normal coupling effects (Q_{16} and Q_{26} terms) for +θ are exactly balanced by the ones at -θ. Hence, $A_{16} = A_{26} = 0$ and the term "balanced". In contrast, if this same laminate is made of bi-modulus material and undergoes sufficient bending action that the fiber-dir-ection strains at the top and bottom surfaces of the laminate are opposite in sign, then the terms at +θ are not balanced by those at -θ. Thus, for this bimodulus laminate, A_{16} and A_{26} do not vanish and the term balanced is inappropriate.

Even in the case of a so-called symmetric balanced angle-ply (SBAP)

laminate, such as one having a lamination scheme $0/-0/-0/0$, of ordinary composite material, even though A_{16} and A_{26} vanish, D_{16} and D_{26} do not. A detailed derivation of general expressions for D_{16} and D_{26} for such a laminate with an arbitrary even number of layers is given in (13). In view of these considerations, it is clear that for an angle-ply laminate of bimodulus material, there are no vanishing plate stiffnesses.

A listing of all of the plate stiffnesses that vanish for a variety of lamination arrangements of both ordinary and bimodulus composite materials is presented in Table 2.

Table 2. Plate Stiffnesses for Some Popular Lamination Arrangements of Ordinary and Bimodulus Composite Materials

Case	Name of Arrangement	Example Laminate	List of Vanishing Plate Stiffnesses	
			Ordinary Material	Bimodulus Material
1	Aligned single ply or aligned parallel ply	0 or 90/90	$A_{16}, A_{26}, D_{16}, D_{26}$, all B_{ij}	$A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$
	Cross-ply:			
2	$\bar{n}$ = odd*	0/90/0	$A_{16}, A_{26}, D_{16}, D_{26}$, all B_{ij}	$A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$
3	$\bar{n}$ = even	0/90	$A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$	$A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$
	Alternating angle-ply:			
4	$\bar{n}$ = odd	30/-30/30	A_{16}, A_{26}, all B_{ij}	None
5	$\bar{n}$ = even	10/-10	$A_{16}, A_{26}, B_{11}, B_{12}, B_{22}, D_{16}, D_{26}$	None
6	SBAP	30/-30/-30/30	A_{16}, A_{26}, all B_{ij}	None
7	Quasi-isotropic	0/-45/+45/90	$A_{16}, A_{26}, D_{16}, D_{26}$	None
8	$0/\pm\theta$	-30/0/30	$A_{16}, A_{26}, B_{11}, B_{12}, B_{22}, D_{16}, D_{26}$	None
9	$(0/\pm\theta)_s$	-30/30/0/30/-30	A_{16}, A_{26}, all B_{ij}	None

The remainder of this discussion is limited to the particular bimodulus-material model originated by Bert (14,15) and known as the symmetric fiber-governed-compliance model. This model can be expressed mathematically as follows:

$$\{\epsilon_j\} = [S_{ji}]\{\sigma_i\} \tag{5}$$

where the compliance takes on different sets of values as follows:

$$[S_{ji}] = \begin{cases} [S_{ji}]^{(t)} & \text{if } \sigma_f \geq 0 \\ [S_{ji}]^{(c)} & \text{if } \sigma_f < 0 \end{cases} \tag{6}$$

Here σ_f denotes the fiber stress along its axis.

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* The symbol $\bar{n}$ denotes the total number of layers in the laminate.

To implement the material model in plate theory, we introduce a third subscript (k) to the Q_{ij} plane-stress stiffness matrix in which k=1,2 denote tension and compression properties, respectively. Then by invoking the well-known Voigt hypothesis of uniformity of fiber-direction strain in the fibers and matrix of each respective individual layer, one obtains the following criteria for use of the appropriate Q_{ijk}:

$$[Q_{ijk}] = \begin{cases} [Q_{ij1}] & \text{if } \varepsilon_f \geq 0 \\ [Q_{ij2}] & \text{if } \varepsilon_f < 0 \end{cases} \tag{7}$$

where ε_f denotes the fiber-direction strain at any arbitrary point. The locus of points at which $\varepsilon_f = 0$ is traditionally called the neutral surface.

Depending upon the plate geometry, boundary conditions, and loading, even in the case of a single-layer plate, the neutral surface in general is not a horizontal plane surface. Since the properties (Q_{ijk}) depend upon the sign of ε_f and the plate stiffnesses (A_{ij}, B_{ij}, D_{ij}) upon the Q_{ijk} as in equations (3), it is clear that in general a bimodulus plate is nonhomogeneous in its plane, i.e., the plate stiffnesses depend upon position (x,y). It is noted that, unlike a plate with linearly tapering thickness, here the plate stiffness in general is not only not a simple function of position, its functional form is not even known à priori. In this regard, there is a qualitative analogy between a bimodulus problem and an elastoplastic problem, for which the elastoplastic boundary is unknown à priori.

From the above discussion, it is clear that in order for a finite element to be applicable to any arbitrary combination of planform geometry, boundary conditions, and loading for a bimodulus laminated plate, it must have a full array of plate stiffnesses (none zero).

SMALL DEFLECTIONS OF THIN, CLAMPED ELLIPTIC PLATES

In studying structural mechanics problems involving new classes of materials, it is often quite instructive to obtain closed-form solutions for certain special cases. Then there are no questions regarding numerical approximations, convergence, etc. In the case of plates laminated of ordinary materials, the closed-form solutions due to Kicher (16) for elliptic plates and to Whitney and Leissa (11) for rectangular plates are most outstanding. Thus, it was decided to investigate the applicability of these forms of solutions to plates laminated of bimodulus materials.

Kicher's solution (16) is the closed-form solution for a uniformly-loaded, clamped-edge elliptic plate of cross-ply construction with an even number of layers (Case 3 in Table 2). The form of this solution is

$$u = u_0[1 - (x/a)^2 - (y/b)^2](x/a) \; ; \; v = v_0[1 - (x/a)^2 - (y/b)^2](y/b)$$

$$w = w_0[1 - (x/a)^2 - (y/b)^2]^2 \tag{8}$$

Here the displacement coefficients are readily determined by direct substitution into governing equations (1).

In view of the Kirchhoff hypothesis, upon which the present theory is based, the x-direction normal strain at any arbitrary location (x,y,z) is

$$\varepsilon_1 = u,_x - zw,_{xx} \tag{9}$$

Thus, if one sets $\varepsilon_1 = 0$ and solves for z, one obtains the neutral surface position z_{nx} associated with the normal strain in the x direction. It can readily be shown that when this procedure is applied to the Kicher displacements, equations (8), the resulting expression for z_{nx} is independent of x and y.

This means that for this particular problem, the plate stiffnesses are uniform throughout the plate surface of a single-layer bimodulus plate provided that the fibers are aligned in the x direction. (A similar conclusion can be reached for z_{ny} and thus, a single-layer bimodulus plate with the fibers in the y direction.)

For the single-layer plate, it can be shown (8) that the plate stiffnesses depend upon $Z(\pm z_n/h)$ in the following manner:

$$A_{ij}/h = (1/2)(Q_{ij1} + Q_{ij2}) + (Q_{ij2} - Q_{ij1})Z$$
$$B_{ij}/h^2 = (1/8)(Q_{ij1} - Q_{ij2}) + (Q_{ij2} - Q_{ij1})(Z^2/2) \qquad (10)$$
$$D_{ij}/h^3 = (1/24)(Q_{ij1} + Q_{ij2}) + (Q_{ij2} - Q_{ij1})(Z^3/3)$$

The only thing remaining is the determination of the neutral-surface location Z. In principle, an equation in Z can be obtained by combining equations (10) with the expression for u_0/w_0 as required by solution of equations (1). Although the resulting expression is only cubic in Z, the algebraic structure of the expressions for the coefficients of the cubic is so lengthy that it was found to be more expedient computationally to obtain Z by direct iteration. From physical considerations, only one unique solution for Z is meaningful (i.e. real, with $|Z| < 1/2$). In (8), numerical results were presented for a composite material with considerable bimodulus action, namely aramid-cord/rubber which has the properties listed in Table 3, taken from experimental data of Patel et al. (17). The exact bimodulus results were compared with those obtained by ordinary theory using average properties. The necessity of using bimodulus theory is clearly demonstrated as shown in Figures 1 and 2.

Table 3. Bimodulus Properties of Aramid Tire-Cord/Rubber Composite, Reduced from Experimental Data in (17)

Property	Units	Tension	Compression
Major Young's modulus	MPa	3,580	12.0
Transverse Young's modulus	MPa	9.09	12.0
Inplane shear modulus	MPa	3.70	3.70
		(3.83)	(3.83)
Major in-plane Poisson's ratio	-	0.416	0.205
Minor in-plane Poisson's ratio	-	0.01105	0.205

In view of the nature of Cases 2 and 3 in Table 2, one would expect the Kicher displacement functions to be applicable to an arbitrary cross-ply plate of bimodulus material. In (18), this is shown to be true and it is implemented for the "most unsymmetric" layer, namely the case of n=2. In this case, equations (10) are replaced by

$$A_j/h = (1/2)(Q_{ij11} + Q_{ij22}) + (Q_{ij21} - Q_{ij11})Z_x + (Q_{ij22} - Q_{ij12})Z_y$$
$$B_{ij}/h^2 = (1/8)(Q_{ij11} - Q_{ij22}) + (Q_{ij21} - Q_{ij11})(Z_x^2/2) + (Q_{ij22} - Q_{ij12})(Z_y^2/2)$$
$$D_{ij}/h^3 = (1/24)(Q_{ij11} + Q_{ij22}) + (Q_{ij21} - Q_{ij11})(Z_x^3/3) + (Q_{ij22} - Q_{ij12})(Z_y^3/3)$$

$$(11)$$

Here the last subscript ℓ in stiffnesses $Q_{ijk\ell}$ denotes the layer number (either layer 1 or 2) and the ijk subscripts have the same meaning as before.

In the case of cross-ply laminates, the elastic properties used for the layers having the fibers oriented in the x direction are determined by the x-direction neutral surface, while those for the other layers are determined by the y-direction neutral surface. Thus, to a certain extent a two-layer cross-
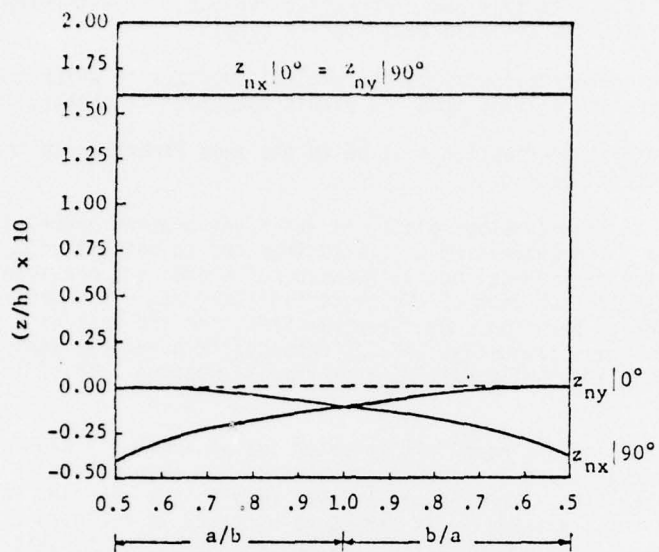
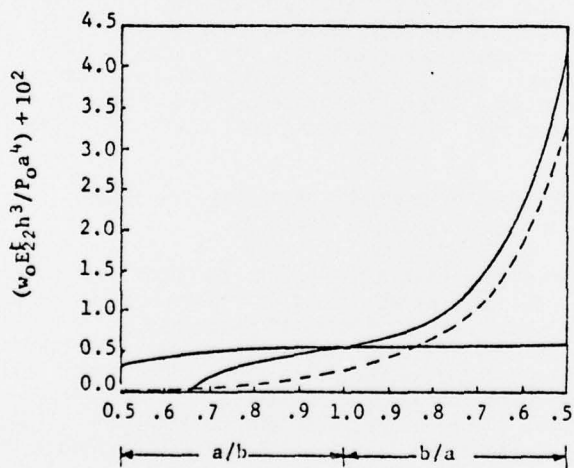Fig. 1 Neutral-surface location vs. plate aspect ratio for single-layer aramid-rubber plate.

Fig. 2 Maximum deflection vs. plate aspect ratio for single-layer aramid-rubber plate. Bi-modulus solutions for fibers parallel (0°) and perpendicular (90°) to the x axis. Dashed line represents 0° case based on average of tension and compression properties.

ply bimodulus laminate is like a four-layer, cross-ply ordinary laminate (with undetermined individual layer thicknesses). Thus, there arises a question as to the proper order of the tensile and compressive regions. To determine the correct order, the following criteria were used in ([18]):

1. For bending-producing loadings, the neutral surfaces to which the iteration procedure leads must lie within the plate thickness, i.e. $|Z_v| < 1/2$.
2. The maximum plate deflection must be in the same direction as the normal-pressure loading.

In the case of an aramid/rubber plate, it was found that an order of stress-state tension T or compression C (listed from top to bottom) of C/T/T/C resulted in a negative maximum deflection (upward for a downward pressure), while an order of C/T/C/T met both of the criteria listed above and thus was judged to be the correct solution. On the other hand, for the case of a polyester/rubber plate, it was found that C/T/C/T resulted in a neutral surface lying outside of the plate, while C/T/T/C gave results meeting both of the above criteria.

In attempting to find a closed-form solution for an angle-ply bimodulus plate, the second author found a closed-form solution for an arbitrarily laminated ordinary laminate ([19]). However, it did not result in a constant value of neutral surface and thus, apparently it cannot be extended to a closed-form solution of the angle-ply bimodulus plate. Alternate approaches to this problem are currently being investigated.

NONLINEAR DEFLECTIONS OF THIN, CLAMPED ELLIPTIC PLATES

The geometrically nonlinear midplane strain-displacement relations originated by von Karman were incorporated into thin, laminated, anisotropic plate theory by Whitney and Leissa ([11]), although they did not solve the resulting nonlinear partial differential equations. Although the Ritz-Galerkin and Rayleigh-Ritz (energy) methods are probably equally popular, in ([9]) it was elected to use a perturbation procedure first used for plates by Nash, Cooley ([20]) for clamped, elliptic isotropic plates under uniform pressure. The only change in the procedure is the additional iteration procedure (necessary to take into account the bimodulus stiffnesses) and the additional elastic constants due the bimodulus action.

SMALL DEFLECTIONS OF RECTANGULAR PLATES INCLUDING THICKNESS-SHEAR DEFORMATION

It has been demonstrated ([21]) that fiber-reinforced composite materials exhibit much larger thickness shear effects than do plates having the same geometry but constructed of homogeneous, isotropic materials. The explanation for this is due to the relatively low thickness shear moduli relative to the in-plane moduli. The numerous theories including these effects for laminated anisotropic plates were reviewed in ([12]). It suffices here to mention that the two most widely used of these theories are those due to Yang et al. ([22]) and Whitney and Pagano ([23]). It was shown in ([24]) that the latter theory is more accurate.

In ([25]), the modal shapes used by ([26]) for free vibration and buckling analyses of shear deformable cross-ply rectangular plates are used to analyze the static deflection of similar plates constructed of bimodulus composite materials and subjected to a sinusoidally distributed normal pressure. Again, as in the case of the elliptic plate problem discussed in the preceding section, the two criteria for constancy of the neutral-surface position are satisfied, and thus the solutions are exact, closed-form solutions.

FINITE-ELEMENT FORMULATION FOR PLATES INCLUDING
TRANSVERSE SHEAR DEFORMATION

While considerable effort has been expended in the finite-element analysis of isotropic plates, only limited investigations of laminated anisotropic plates can be found in the literature. Pryor and Barker (27), and Barker, Lin and Dara (28) used the conventional displacement finite-element method to analyze thick laminated plates. The element has seven degrees of freedom (three displacements, two rotations, and two shear rotations) per node. Exploiting the symmetries exhibited by anisotropic plates, Noor and Mathers (29-31) studied the effects of shear deformation and anisotropy on the accuracy and convergence of several shear-flexible displacement finite-element models based on a form of Reissner's plate theory. The analysis was limited to symmetrically laminated cross-ply plates and the element used involved 80 degrees of freedom per element. The conventional finite element, when applied to relatively thick laminated plates, either has failed to predict accurately the local deformations and stresses of a plate under bending or is too expensive to use due to large degrees of freedom involved for even relatively simple problems. Mau and Witmer (32), and Mau, Tong, and Pian (33) have employed the so-called hybrid-stress finite-element method to analyze composite plates including shear deformation. The hybrid elements have proven (see Gallagher (34)) to have some convergence problems, and in some cases they give erroneous results. Most recently, Panda and Natarajan (35) used, following Mawenya and Davies (36), the quadratic shell element of Ahmad, Irons and Zienkiewicz (37) with the same normal rotation through the thickness to claim improved accuracy over Mawenya and Davies (36). The 'thickness concept' mentioned in there is essentially the same as that used in the YNS theory (21). The authors were primarily concerned with the accuracy of the element, and no attempt was made to solve new problems for which there do not exist any closed-form, or exact solutions. In an effort to relax the continuity requirements on the shape functions, Reddy (38) recently devised a simple finite element and successfully tested the closed-form solutions of Bert and Chen (39). Only investigation that concerns with the finite element formulations of bimodulus materials is due to Crose, et al. (40). Here, following (38), we present the finite-element formulation.

Consider a plate of constant thickness h composed of thin anisotropic layers oriented at angles $\theta_1, \theta_2, \ldots$. The origin of the coordinate system is located within the middle plane (x-y) with the z-axis being normal to the midplane. The material of each layer is assumed to possess a plane of elastic symmetry parallel to the xy-plane. We shall denote the middle plane with Ω.

The YNS theory is based on the following assumed displacement field:

$$\bar{u} = u(x,y) + z\psi_x(x,y)$$
$$\bar{v} = v(x,y) + z\psi_y(x,y) \tag{12}$$
$$w = w(x,y)$$

where $\bar{u}$, $\bar{v}$, and w are the displacement components in the x,y and z-directions, respectively, u and v are the in-plane (stretching) displacements of the middle plane, and ψ_x and ψ_y are the slope functions.

The equations of motion associated with the YNS theory are,

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = 0$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = p$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_x = 0 \qquad\qquad (13)$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_y = 0$$

where $p = p(x,y)$ is the transversely distributed load, and N_i, M_i, Q_x, and Q_y are the stress resultants per unit length,

$$(N_1,N_2,N_6) = \int_{-h/2}^{h/2} (\sigma_x,\sigma_y,\tau_{xy})dz \ , \quad (Q_x,Q_y) = \int_{-h/2}^{h/2} (\tau_{xz},\tau_{yz})dz \qquad (14)$$

$$(M_1,M_2,M_6) = \int_{-h/2}^{h/2} (\sigma_x,\sigma_y,\tau_{xy})zdz \qquad (15)$$

A thin orthotropic material with provision for a shift of the neutral surface (due to different properties in tension and compression) has the following plate constitutive relations:

$$\begin{Bmatrix} N_1 \\ N_2 \\ Q_y \\ Q_x \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & 0 & A_{26} & B_{12} & B_{22} & B_{21} \\ 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 & 0 \\ A_{16} & A_{26} & 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & 0 & 0 & B_{16} & D_{11} & D_{12} & D_{11} \\ B_{12} & B_{22} & 0 & 0 & B_{26} & D_{12} & D_{22} & D_{16} \\ B_{16} & B_{26} & 0 & 0 & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \cdot \begin{Bmatrix} u,_x \\ v,_y \\ w,_y + \psi_y \\ w,_x + \psi_x \\ u,_y + v,_x \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{Bmatrix} , \ k=1 \text{ or } 2 \quad (16)$$

where the material coefficients A_{ij}, B_{ij} and D_{ij} are given by

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} Q_{ij} (1,z,z^2)dz \ , \quad (i,j=1,2,6)$$

$$= \sum_{\ell=1} \int_{z_\ell}^{z_n} Q_{ij1\ell}dz + \sum_{\ell=1} \int_{z_n}^{z_{\ell+1}} Q_{ij2\ell}dz \qquad (17)$$

Here $Q_{ijk\ell}$ denote the stiffness coefficients in the plate coordinates of the ℓ-th layer in tension (k=1) or compression (k=2), and z_n is the distance from the midsurface to the neutral plane (which is unknown à priori).

We assume, over each element Ω_e, the same kind of interpolation for all of the variables,

$$u^e = \sum_1^n u_i^e N_i^e \ , \quad v^e = \sum_1^n v_i^e N_i^e \ , \ \text{ etc. (n=nodes per element)} \qquad (18)$$

where N_i^e are the element interpolation (or shape) functions, and u_i^e, and v_i^e are the nodal values of u_o^e and v_o^e, respectively. Substituting (18) into the first variation of the total potential energy associated with (13) and collecting the coefficients of the variations (δu_i, δv_i, etc.) we obtain

$$[K^e]\{\Delta^e\} = \{F^e\} \tag{19}$$

where $\{\Delta^e\} = \{\{u_e\}, \{v_e\}, \{w_e\}, \{\psi_x^e\}, \{\psi_y^e\}\}$, and $K_{ij}^{\alpha\beta}$ are given by

$$K_{ij}^{11} = A_{11}S_{ij}^x + A_{16}(S_{ij}^{xy} + S_{ji}^{xy}) + A_{66}S_{ij}^y$$

$$K_{ij}^{12} = A_{11}S_{ij}^x + A_{16}S_{ij}^x + A_{26}S_{ij}^y + A_{66}S_{ji}^{xy}$$

$$K_{ij}^{14} = B_{11}S_{ij}^x + B_{16}(S_{ij}^{xy} + S_{ji}^{xy}) + B_{66}S_{ij}^y$$

$$K_{ij}^{15} = B_{12}S_{ij}^{xy} + B_{16}S_{ij}^x + B_{26}S_{ij}^y + B_{66}S_{ji}^{xy}$$

$$K_{ij}^{22} = A_{26}(S_{ij}^{xy} + S_{ji}^{xy}) + A_{22}S_{ij}^y + A_{66}S_{ij}^x$$

$$K_{ij}^{24} = B_{16}S_{ij}^x + B_{66}S_{ij}^{xy} + B_{12}S_{ji}^{xy} + B_{26}S_{ij}^y$$

$$K_{ij}^{25} = B_{26}(S_{ij}^{xy} + S_{ij}^{xy}) + B_{66}S_{ij}^x + B_{22}S_{ij}^y$$

$$K_{ij}^{33} = A_{44}S_{ij}^x + \epsilon_2^2 S_{ij}^y + A_{45}(S_{ij}^{xy} + S_{ji}^{xy})$$

$$K_{ij}^{34} = A_{44}S_{ij}^{xo} + A_{45}S_{ij}^{yo} , \quad K_{ij}^{35} = A_{45}S_{ij}^{xo} + A_{55}S_{ij}^{yo}$$

$$K_{ij}^{44} = D_{11}S_{ij}^x + D_{16}(S_{ij}^{xy} + S_{ji}^{xy}) + D_{66}S_{ij}^y + A_{45}S_{ij}^o$$

$$K_{ij}^{45} = D_{12}S_{ij}^{xy} + D_{66}S_{ji}^{xy} + D_{16}S_{ij}^x + D_{26}S_{ij}^y + A_{45}S_{ij}^o$$

$$K_{ij}^{55} = D_{26}(S_{ij}^{xy} + S_{ji}^{xy}) + D_{66}S_{ij}^x + D_{22}S_{ij}^y + A_{55}S_{ij}^o$$

$$K_{ij}^{13} = K_{ij}^{23} = 0 , \quad S_{ij}^{\xi\eta} = \int_{\Omega^e} N_{i,\xi}N_{j,\eta} \, dxdy , \quad (\xi,\eta=0,x,y)$$

$$F_i^3 = \int_{\Omega^e} p N_i \, dxdy , \quad F_i^1 = F_i^2 = F_i^4 = F_i^5 = 0 \tag{21}$$

The element stiffness matrices are assembled in the usual manner, and boundary conditions of the problem are imposed before solving for $\{\Delta\}$. In the present study, the four-node quadrilateral element of the serendepity family is used. The element stiffness matrices for this element is of order 20x20.

Figure 3 shows the influence of the aspect ratio (b/a) and side-to-thickness ratio (a/h) on the location of neutral surfaces for a single-layer, isotropic, bimodulus, simply supported rectangular plate subjected to sinusoidal loading,

$$p = P_o \sin(\pi x/a) \sin(\pi y/b)$$

The following elastic properties are used:

$$E_{11}^t = 3.584 \text{ GPa} , \quad E_{11}^c = 1.792 \text{ GPa} , \quad E_{22}^t = E_{11}^t , \quad E_{22}^c = E_{11}^c$$

$$G_{12}^t = G_{12}^c = 1.27 \text{ GPa} , \quad \nu_{12}^t = \nu_{21}^t = 0.4 , \quad \nu_{12}^c = \nu_{21}^c = 0.2$$
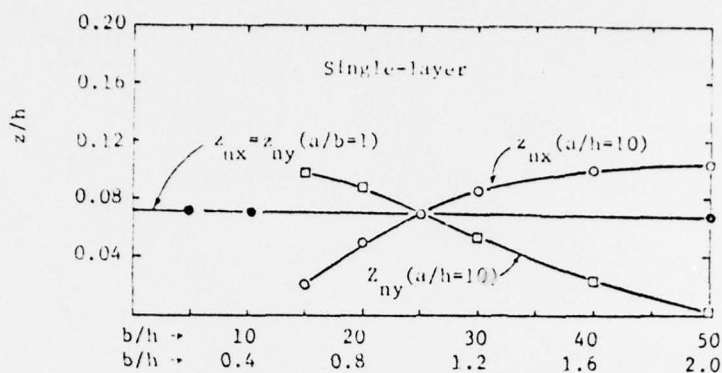
Fig. 3 Neutral-surface location vs. plate aspect ratio,
and side-to-thickness ratio for single-layered
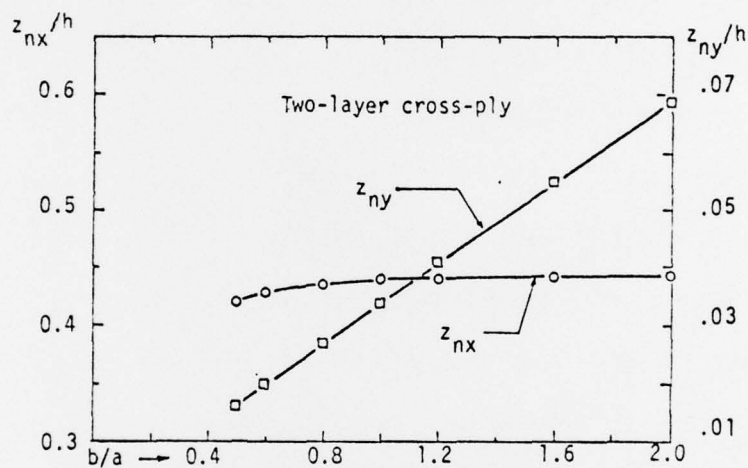rectangular plate under sinusoidal loading.

Fig. 4 Neutral-surface location vs. plate aspect ratio
for two-layer, cross-ply (0°/90°) square plate
under sinusoidal loading.

Note that for b/a = 1, the neutral surfaces associated with x- and y-directions
coincide (i.e., $z_{nx} = z_{ny}$).

Similar results are presented in Figures 4 and 5 for a two-layer, cross-ply
(0°/90°), rectangular plate under sinusoidal loading. The bimodulus properties
used are the same as those listed in Table 3. Note from Fig. 4 that the
neutral-surface location, z_{nx} is virtually unchanged for aspect ratio greater
than 1, while the neutral-surface location, z_{ny} increases proportional to the
aspect raio. It should also be noted that the neutral surfaces do not coincide
in the cross-ply case for b/a = 1.

Figure 6 shows the influence of the aspect ratio, and side-to-thickness
ratio on the transverse deflection for single-layer, and two-layer cross-ply
problems discussed above. The effect of thickness on the deflection is more
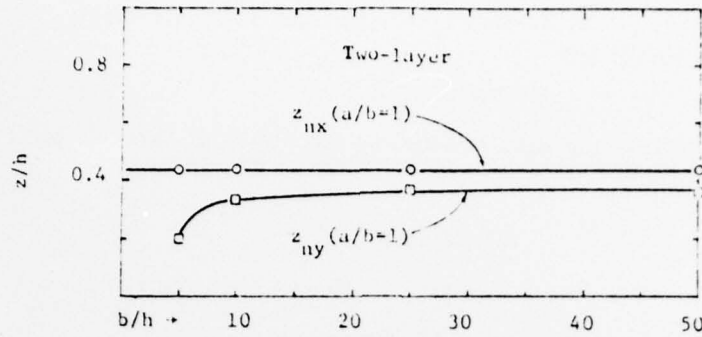pronounced than the effect of the aspect ratio.

Fig. 5 Neutral-surface location vs. side-to-thickness
ratio for two-layer, cross-ply (0°/90°) rect-
angular plate under sinusoidal loading.

CONCLUDING REMARKS

A theory of plate bending for laminated, anistropic bimodulus materials
were presented. This theory is based on the fiber-governed symmetric compliance
model for bimodulus materials.

Closed-form and finite-element solutions have been presented for selected
problems involving different plate geometrics, lamination arrangements,
boundary conditions, both without and with thickness-shear deformation.

It was shown that even for single-layer bimodulus plates, the neutral-sur-
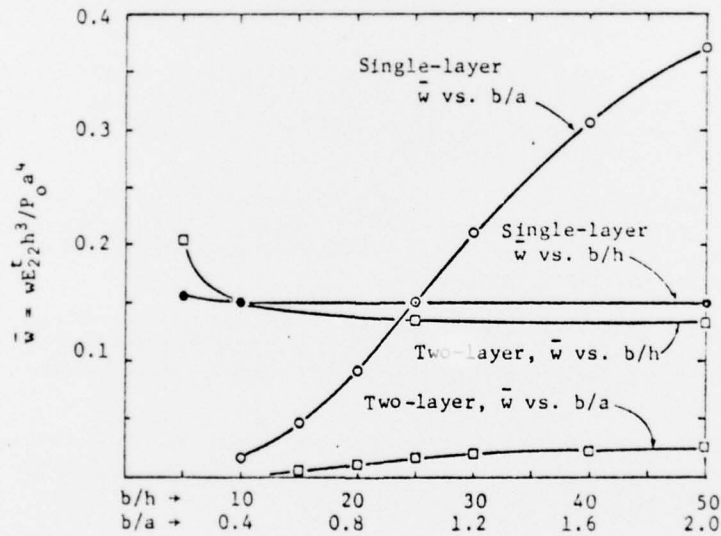face location may vary considerably from the geometric midplane, depending upon

Fig. 6 Transverse deflection vs. plate aspect ratio,
and side-to-thickness ratio for single-layer
and two-layer cross-ply plates under sinusoidal
loading.

the degree of bimodularity. Also, the plate deflection is significantly affected by the bimodulus action.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

To implement the structural application of the recently introduced fiber-governed symmetric compliance model for bimodulus composite materials, both classical closed-form and finite-element solutions are being developed. This paper summarizes the results obtained for deflection of single-layer ortho-tropic and two-layer, cross-ply plates of the following configurations and loadings: ⟶

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1. Thin elliptic plates clamped on the boundary and subjected to uniform pressure.

2. Moderately thick rectangular plate freely supported on the boundary and subjected to sinusoidally distributed pressure.