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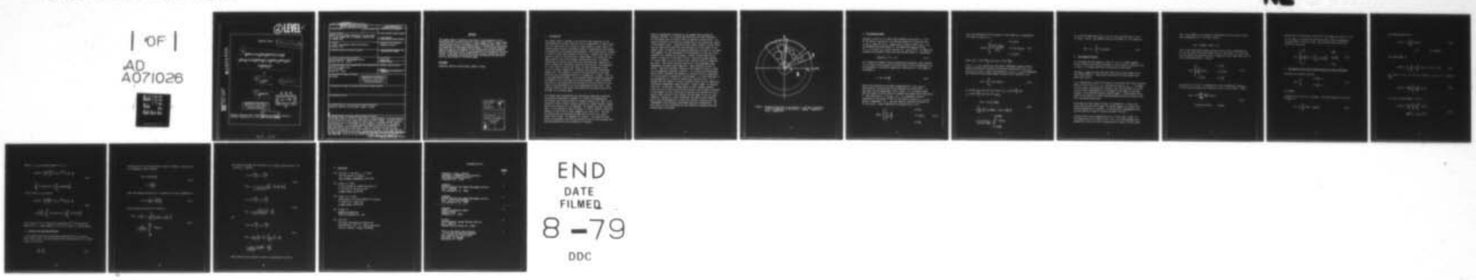

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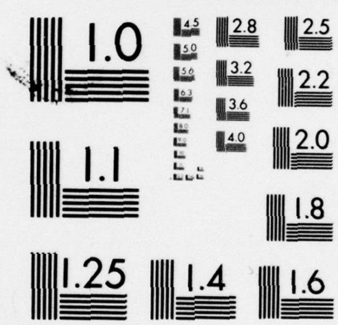

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6 MOMENTS OF THE RANDOM PROPORTION OF AN ARC

WHICH IS IN THE SHADOWS CAST BY RANDOMLY LOCATED DISKS

OF RANDOM SIZE

BY

10 S./ZACKS

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The present paper is devoted to the study of some properties of the distribution of the total portion of an arc length which is under shadow. The total portion of an arc which is under shadow (or under light) is a random variable having a distribution which is determined by the stochastic properties of the shadow casting objects. In the present Technical Report we show how the moments of this distribution can be determined. Explicit formulas are given for the expection and variance of the total portion in the light, of any specified arc on the circumference of the circle. Preliminary results on the actual distribution are available, but will be provided in a different paper.

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ABSTRACT

The present paper is devoted to the study of some properties of the distribution of the total portion of an arc length which is under shadow. The total portion of an arc which is under shadow (or under light) is a random variable having a distribution which is determined by the stochastic properties of the shadow casting objects. In the present Technical Report we show how the moments of this distribution can be determined. Explicit formulas are given for the expectation and variance of the total portion in the light, of any specified arc on the circumference of the circle. Preliminary results on the actual distribution are available, but will be provided in a different paper.

KEY WORDS

Shadowing, Masking, Surveillance, Combat in woods.

1. Introduction

The present paper is devoted to the study of some properties of the distribution of the total portion of an arc length which is under shadow. The total portion of an arc which is under shadow (or under light) is a random variable having a distribution which is determined by the stochastic properties of the shadow casting objects. More specifically, consider a circle of radius T[m] centered at the origin. Suppose that there is a source of light in the origin and a random number, N, of disks are randomly placed in the circle without covering the origin. We further assume that the radius of the disks are relatively small compared to T and of random size, following a specified distribution. The projection of a disk, which is within the circle, on the circumference of the circle is called a shadow arc. Different disks may have overlapping shadow arcs. The problem is to derive the distribution of the total portion of the circumference which is in shadow (or in light). This is a very complicated problem. In the present Technical Report we show how the moments of this distribution can be determined. Explicit formulas are given for the expectation and variance of the total portion in the light, of any specified arc on the circumference of the circle. Preliminary results on the actual distribution are available, but will be provided in a different paper.

The problem under consideration was motivated by questions concerning the possible concealment of targets in a forest with homogeneously distributed trees, having trunks of randomly varying size. What is the probability that at least a certain portion of a target located T meters away from an observer will be observed? The targets could be relatively small in size compared to their distance, T, from an observer (tanks for example) or they could be large (segments of a road, etc.). We treat the problem here by considering targets which are arcs on a circle rather than straight line segments (or chords). This simplifies the solution considerably and from the practical point of view provides very good approximations to many situations of interest.

There is considerable literature on the probabilistic aspects of coverage problems. For an excellent exposition and many references see the recent book of Solomon (1978) on geometric probabilities. The results provided in this book are related but do not overlap the results of the present report. Chernoff and Daly (1957) studied a similar problem, but they derive implicitly the distribution of the length of one shadow, which is formed by overlapping projections of disks on a straight line segment. We are interested in the distribution of the sum of the lengths of all shadows on the target. Another related study can be found in Volume III of TRW report (1971). They have studied problems of line of sight in a hilly terrain with different kinds of vegetation and trees. That report provides formulas for non-concealment under homogeneous and under mixed vegetation. These formulas are easy to derive. They also provided a formula of the probability that a target will be completely concealed (covered). In a recently published paper, Siegel (1978) considered the moments of a portion of the circumference of a circle which is covered by a fixed number, n, of arcs (obtained as the intersection of n disks of random size) with the circle. It is assumed that the lengths of these n randomly placed arcs are identically distributed independent random variables. Siegel's model is different from ours. We consider a random number of disks randomly centered within the circle (and not covering the origin), while Siegel considered a fixed number of disks, which may be centered also outside the circle. Kaplan (1978) derived a limit thereon for the number of arcs of fixed length, α, required to cover a proportion not smaller than $1-K\alpha/2\pi R$, $K\geq 1$ of the circumference of a circle with radius R, when the arcs are independently and uniformly distributed along the circumference. The main result is very interesting but the problem is different from the problem which we consider.

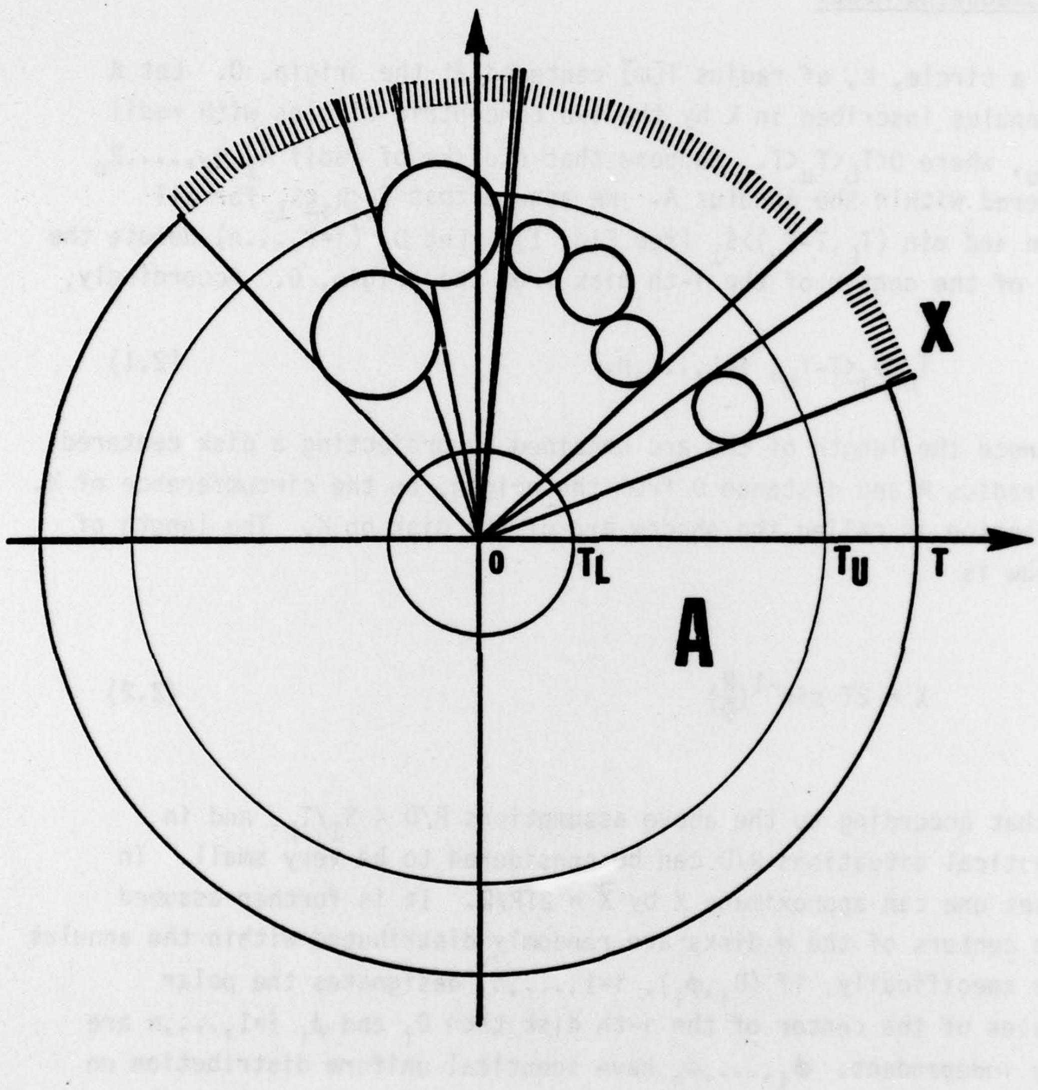

Figure 1 Randomly Placed Disks in the Annulus, A, and Their Projections on the Circumference of the Circle K. (Dashed - Shadow Arc, Plain - Lighted Arc)

2. The Shadowing Model

Consider a circle, K, of radius T[m] centered at the origin, 0. Let A be the annulus inscribed in K by the two concentric circles with radii T_L and T_u, where $0<T_L<T_u<T$. Suppose that n disks of radii $R_1,R_2,\ldots,R_n$ are centered within the annulus A. We assume that $S_0\leq R_i\leq S_1$ for all $i=1,\ldots,n$ and $\min(T_L,T-T_u)>S_1$ (see Fig. 1). Let D_i ($i=1,\ldots n$) denote the distance of the center of the i-th disk from the origin, 0. Accordingly,

$$T_L\leq D_i\leq T-T_u,\ i=1,\ldots,n. \tag{2.1}$$

Let X denote the length of the arc obtained by projecting a disk centered in A of radius R and distance D from the origin, on the circumference of K. The projection is called the shadow arc of the disk on K. The length of the shadow is

$$X = 2T\sin^{-1}\left(\frac{R}{D}\right) \tag{2.2}$$

Notice that according to the above assumptions $R/D < S_1/T_L$, and in many practical situations R/D can be considered to be very small. In such cases one can approximate X by $\widetilde{X} = 2TR/D$. It is further assumed that the centers of the N disks are randomly distributed within the annulus A. More specifically, if (D_i,ϕ_i), $i=1,\ldots,n$, designates the polar coordinates of the center of the i-th disk then D_i and ϕ_i $i=1,\ldots,n$ are mutually independent. $\phi_i,\ldots,\phi_n$ have identical uniform distribution on $(0, 2\pi)$ and $D_i,\ldots D_n$ are identically distributed with a c.d.f.

$$F_D(d) = \begin{cases} 0 & ,\text{if } d<T_L \\ \dfrac{d^2 - T_L^2}{T_u^2 - T_L^2} & ,\text{if } T_L\leq d\leq T_u \\ 1 & ,\text{if } d\geq T_u \end{cases} \tag{2.3}$$

Thus, the distribution of the length X of the shadow arc, corresponding to a disk of radius R, is

$$F_X(x|R) = \begin{cases} 0 & ,\text{if } x \le \xi_L(R) \\ \dfrac{T_u^2 - R^2/\sin^2(\frac{x}{2T})}{T_u^2 - T_L^2} & ,\text{if } \xi_L(R) < x \le \xi_u(R) \\ 1 & ,\text{if } x > \xi_u(R), \end{cases} \tag{2.4}$$

where $\xi_L(R) = 2T\sin^{-1}(\frac{R}{T_u})$ and $\xi_u(R) = 2T\sin^{-1}(\frac{R}{T_L})$.

Let $R_1, \ldots, R_n$ be identically distributed independent random variables, having an absolutely continuous distribution concentrated on $[S_0, S_1]$, with a c.d.f. $F_R(r)$. Hence, the shadow arcs $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, with a marginal c.d.f.

$$F_X(x) = \int_{S_0}^{S_1} F_X(x|r)dF_R(r) \tag{2.5}$$

It follows from (2.4) and (2.5) that if $\xi_L = 2T\sin^{-1}(\frac{S_0}{T_u})$ and $\xi_u = 2T\sin^{-1}(\frac{S_1}{T_L})$ then for all $\xi_L \le x \le \xi_u$

$$\begin{aligned} F_X(x) &= F_R(T_L\sin(\tfrac{x}{2T})) \\ &+ \frac{T_u^2}{T_u^2 - T_L^2}\left[F_R(T_u\sin(\tfrac{x}{2T})) - F_R(T_L\sin(\tfrac{x}{2T}))\right] \\ &- \frac{1}{(T_u^2 - T_L^2)\sin^2(\frac{x}{2T})}\int_{T_L\sin(\frac{x}{2T})}^{T_u\sin(\frac{x}{2T})} r^2 dF_R(r) \end{aligned} \tag{2.6}$$

It is easy to check that $F_X(x) = 0$ for all $x \le \xi_L$ and that $F_X(x) = 1$ for all $x \ge \xi_u$. Finally, the expected value of the length, X, of a shadow arc

$$E\{X\} = \xi_L + \int_{\xi_L}^{\xi_u} [1-F_X(y)]dy. \tag{2.7}$$

3. <u>The Shadowing Process</u>

It is assumed that the number of disks, N, in A is a random variable having a Poisson distribution with mean $\nu = 2\pi T\lambda$, where $\lambda = \mu(T_u^2 - T_L^2)/2T$ and μ, $0<\mu<\infty$, is a proper coefficient (the density of the disks).

Let $0 \le t_1 \le \ldots \le t_N < 2\pi T$ be the right hand limit point of the shadow arcs of the N disks within A. These points constitute a Poisson process with mean λ(per radian).

It is interesting to observe the anology of this model and an M/G/∞ queueing model, with arrivals at t_1, t_2,...and service times X_1, X_2,... The number of disks, $N(\phi)$, intersected by a straight line from the origin to a point (T, ϕ) on the circumference of K corresponds to the number of occupied channels in the queueing system at time $\phi T, 0 \le \phi < 2\pi$. A busy period corresponds to the length of a single shadow generated possibly by several disks with overlapping projections.

Given that the total number of disks, N, centered at A, is N=n, the conditional distribution of $N(\phi)$, for each ϕ, $0 \le \phi \le 2\pi$, is binomial with success probability $\theta = E\{X\}/2\pi T$. It follows that, independently of ϕ, the distribution of $N(\phi)$ is Poisson with mean $\rho = \lambda \cdot E\{X\}$.

A point (T,ϕ) on the circumference of K is in the light if $N(\phi) = 0$. The probability of this event is $e^{-\rho}$. This is also the expected total proportion of any arc on the circumference of K, which is in the light.

Let $t = \phi T$, $0 \le t \le 2\pi T$, be a parametric representation of the points (T,ϕ) on the circumference of K. For each t define

$$p(t) = P[N(\tfrac{t}{T}) = 0 | N(0) = 0].$$

This is the conditional probability that a point on the circumference of K is in the light, given that another point, t units away, is in the light. Let $\bar{F}(t)$ denote the probability that a shadow arc with a right hand limit in $(0,t)$ terminates before t. We have

$$\bar{F}(t) = \begin{cases} 0 & ,\text{if } t \le \xi_L \\ \frac{1}{t}\int_{\xi_L}^{t} F_X(x)dx & ,\text{if } \xi_L < t \le \xi_u \\ 1 - \frac{1}{t} E\{X\} & ,\text{if } t \ge \xi_u \end{cases} \tag{3.2}$$

The points of "arrivals" of shadow arcs on the circumference, $0 \le t_1 \le t_2 \le \ldots$, follow a Poisson process with intensity λ per arc unit (radian). Hence,

$$\begin{aligned} p(t) &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} (\bar{F}(t))^n \\ &= \exp\left\{-\lambda t(1-\bar{F}(t))\right\} \quad , 0 \le t \le 2\pi. \end{aligned} \tag{3.3}$$

Notice from (3.2) that $p(t) = \exp\{-\lambda t\}$ for all $0 \le t \le \xi_L$ and that $p(t) = e^{-\rho}$ for all $t \ge \xi_u$. This means that the events $\{n(\frac{t}{T}) = 0\}$ are independent of the event $\{N(0) = 0\}$ for all $t \ge \xi_u$. We write the function $p(t)$ more explicitly in the form

$$p(t) = \begin{cases} e^{-\lambda t} & ,t \le \xi_L \\ \exp\left\{-\lambda t\left(1 - \frac{1}{t}\int_{\xi_L}^{t} F_X(x)dx\right)\right\} & ,\xi_L \le t \le \xi_u \\ e^{-\rho} & ,t \ge \xi_u \end{cases} \tag{3.4}$$

4. The Moments of the Total Lighted Proportion of the Circumference

Introduce the indicator function

$$I(t) = \begin{cases} 1 & ,\text{if } N(\frac{t}{T}) = 0 \\ 0 & ,\text{if } N(\frac{t}{T}) > 0 \end{cases} \tag{4.1}$$

for $0 \le t \le 2\pi T$.

Consider the arc from 0 to L, $0 < L \le 2\pi T$. The total proportion of this arc in light is

$$Y(L) = \frac{1}{L}\int_0^L I(t)dt \quad ,0 < L \le 2\pi T. \tag{4.2}$$

The expected proportion is

$$E\{Y(L)\} = \frac{1}{L}\int_0^L E\{I(t)\}dt \tag{4.3}$$

$$= e^{-\rho} \qquad ,\text{all } 0 \leq L \leq 2\pi T.$$

The second moment is

$$E\{Y^2(L)\} = \frac{2}{L^2}\int_0^L ds \int_s^L P\{I(s) = 1, I(t) = 1\}dt. \tag{4.4}$$

But, $P\{I(s) = 1, I(t) = 1\} = P\{I(s) = 1\}P\{I(t) = 1|I(s) = 1\} = e^{-\rho}p(t-s)$.
Thus,

$$E\{Y^2(L)\} = \frac{2e^{-\rho}}{L^2}\int_0^L ds \int_s^L p(t-s)dt \tag{4.5}$$

If $L \leq \xi_L$ the second moment of $Y(L)$ is

$$E\{Y^2(L)\} = \frac{2e^{-\rho}}{L^2}\int_0^L ds \int_0^{L-s} e^{-\lambda u}\, du$$

$$= \frac{2e^{-\rho}}{\lambda L}\left[1 - \frac{1}{L}(1-e^{-\lambda L})\right]. \tag{4.6}$$

When $\xi_L \leq L \leq \xi_u$ the second moment of $Y(L)$ is

$$E\{Y^2(L)\} = \frac{2e^{-\rho}}{L^2}\left\{\frac{\xi_L}{\lambda} + \frac{1}{\lambda}(1-e^{-\lambda\xi_L})(L-\xi_L-\frac{1}{\lambda})\right. \tag{4.7}$$

$$\left. + \int_{\xi_L}^{L}(L-u)\exp\{-\lambda u(1-\frac{1}{u}\int_{\xi_L}^{u}F_X(x)dx)\}du\right\}$$

Finally, when $L \geq \xi_u$ we obtain

$$E\{Y^2(L)\} = \frac{2e^{-\rho}}{L^2}\left\{\frac{\xi_L}{\lambda} + \frac{1}{\lambda}(1-e^{-\lambda\xi_L})(\xi_u-\xi_L-\frac{1}{\lambda})\right. \tag{4.8}$$

$$\left. + e^{-\rho}\frac{L^2-\xi_u^2}{2} + \int_{\xi_L}^{\xi_u}(\xi_u-u)\exp\{-\lambda u(1-\frac{1}{u}\int_{\xi_L}^{u}F_X(x)dx)\}du\right\}.$$

The variance of $Y(L)$ is obtained by subtracting $e^{-2\rho}$ from the second moment of $Y(L)$. Higher moments of $Y(L)$ can be found in a similar manner.

5. A Special Case and Approximations

In the present section we provide some approximations for the case of small values of R/D. We also assume that the distribution of R is uniform on $[S_0, S_1]$ and that

$$\frac{S_0}{T_L} < \frac{S_1}{T_u}. \tag{5.1}$$

We have seen in (2.2) that when R/D is small $X \cong 2TR/D$. Since R and D are independent random variables

$$E\{X\} \cong 2TE\{R\}E\{\frac{1}{D}\}$$

$$= 2T\,\frac{S_0+S_1}{T_u+T_L}. \tag{5.2}$$

Hence, the expected proportion of a lighted arc on the circumference is

$$e^{-\rho} \cong \exp\left\{-2\lambda T\,\frac{S_0+S_1}{T_L+T_u}\right\}. \tag{5.3}$$

The distribution function of $\widetilde{X} = 2TR/D$ is

$$\widetilde{F}(x) = F_R(\frac{T_L}{2T}x) + \frac{T_u^2}{T_u^2-T_L^2}\left[F_R(\frac{T_u}{2T}x)-F_R(\frac{T_L}{2T}x)\right]$$

$$-\frac{4T^2}{(T_u^2-T_L^2)x^2}\int_{\frac{T_L}{2T}x}^{\frac{T_u x}{2T}} r^2 dF_R(r). \tag{5.4}$$

This function assumes the value zero for $x \leq 2TS_0/T_u$ and the value 1 for $x \geq 2TS_1/T_L$. Moveover,

(i) for $\dfrac{2TS_0}{T_u} \leq x \leq \dfrac{2TS_0}{T_L}$

$$\tilde{F}(x) = \frac{x}{(S_1-S_0)(T_u^2-T_L^2)}\left[\frac{T_u^3}{3T} + \frac{S_0}{x}\left(\frac{4}{3}T^2-T_u^2\right)\right], \tag{5.5}$$

(ii) for $\dfrac{2TS_0}{T_L} \leq x \leq \dfrac{2TS_1}{T_u}$

$$\tilde{F}(x) = \frac{x}{S_1-S_0}\left[\frac{T_u^2+T_uT_L+T_L^2}{6T(T_u+T_L)} - \frac{S_0}{x}\right], \tag{5.6}$$

and

(iii) for $\dfrac{2TS_1}{T_u} \leq x \leq \dfrac{2TS_0}{T_L}$

$$\tilde{F}(x) = \frac{x}{S_1-S_0}\left[\frac{T_L}{2T} - \frac{S_0}{x} + \frac{T_u^2}{T_u^2-T_L^2}\left(\frac{S_1}{x} - \frac{T_L}{2T}\right) - \frac{1}{6T(T_u^2-T_L^2)}\left(\left(S_1\frac{2T}{x}\right)^3 - T_L^3\right)\right]. \tag{5.7}$$

These formulas can be applied to obtain an approximation for $F(t)$.

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