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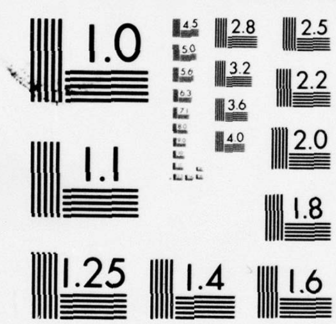

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ANALYTIC APPROACHES TO UNSTABLE RESONATORS:

ANNUAL TECHNICAL REPORT

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ABSTRACT

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A method for obtaining asymptotic solutions of the unstable resonator integral equation which is valid for all values of the magnification was developed. Approximations were made on the Greens functions rather than the eigenmodes, leading to results which are easily generalized to different mirror geometries. "Diffraction dominated eigenmodes" for resonators where each ray escapes after a few transits were differentiated from "waveguide dominated eigenmodes" which are obtained for cavities with a large number of transits per ray. The solutions obtained were seen to agree in the appropriate limits with other asymptotic solutions, numerical results, and geometric optics predictions. To include the effects of gain, the unstable resonator equation was derived from Maxwell's equations in a polarizable medium. The resulting equations have the same structure as the empty resonator equation, and similar approximations can be used. Some features of the effects of saturation on the eigenmodes of an unstable resonator were considered.

Approved for public release; distribution unlimited.

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STATEMENT OF WORK

The contractor shall furnish scientific effort during the period and at the level indicated in paragraph 1a of Section H, together with all related services, facilities, supplies and materials, needed to conduct the following research.

a. Develop and apply analytic techniques for determining the eigenmodes and eigenvalues of cylindriical unstable optical resonators.

b. For systems which display azimuthal symmetry, develop and apply two different techniques: (1) an asymptotic differential equation approach and (2) an approach that is valid for situations in which the resonator magnification is large.

c. Develop a technique that is designed to treat cylindrical resonators in which the system's azimuthal symmetry has been destroyed by optical perturbations.

d. Develop techniques that are specifically designed for resonators with large Fresnel numbers.

e. Develop a technique to incorporate gain in unstable resonator calculations.

ANNUAL REPORT

Recently there has been considerable interest in developing analytic approaches to unstable optical resonators, especially devices with large Fresnel numbers. Two reasons for this interest are:

(1) The fundamental physics of these resonators is not yet fully understood.

(2) There is a need for rapid and accurate techniques for calculating the eigenvalues and eigenmodes of large Fresnel number unstable resonators. Such devices are of greatest practical importance, and are most tractable to asymptotic techniques. We note that computer generated solutions of the Fox-Li resonator equation become more costly and less accurate as the Fresnel number increases.

In the last year we have developed novel approaches towards elucidating the physics of these devices and generating efficient techniques for calculating the eigenmodes and eigenvalues of unstable optical resonators. Below, we have summarized the results and related our work to that of others in the field. We also discuss the relevance of our approach to problems involving unstable optical resonators with gain media.

To clarify the motivation for our research, we first briefly review the work of P. Horwitz[1] and R. Butts and P. Avizonis[2]. These workers have developed asymptotic approaches to unstable resonators which we have been able to complement and extend.

In 1973, Horwitz presented a very interesting asymptotic approach for treating the electrodynamics of large Fresnel number resonators with rectangular geometries. The principle result of this calculation was the discovery of a set of rapidly varying diffraction functions which could be used to obtain the empty resonator eigenmodes. Specifically, the resonator eigenmode was written as the sum of a fundamental spherical wave plus a series of edge diffracted waves. The resultant series converged rapidly for large magnifications (M) with the convergence becoming poorer as M decreased towards unity. Recently, Butts and Avizonis extended Horwitz's analysis to resonators with circular mirrors. The eigenmode was once again written as a spherical wave plus a series of edge diffracted waves appropriate to cylindrical coordinates.

Despite the very interesting results of these authors a number of difficulties still remained. In particular,

(1) Convergence was best for large M. In their analysis Butts and Avizonis found that this approach failed entirely for $M < 1.3$ with $N \sim 10$.

(2) For both cylindrical and rectangular geometries the edge diffracted waves contained unphysical infinities at the shadow boundary of the resonator.

(3) It is difficult to extend this approach to problems involving driven laser resonators, especially in the non-linear regime.

It is clear that an asymptotic approach to solving the resonator equation is relevant in the limit of Fresnel numbers where numerical solutions are difficult to obtain. One would like to eliminate the unphysical singularities that appear in the existing asymptotic solutions, and extend the applicability of asymptotic solutions to devices with small magnifications. Finally, it is necessary to understand the physical consequences of the approximations involved in the asymptotic approach to adequately treat the more complicated case of a resonator with an active gain medium.

Under AFOSR support, we have developed an asymptotic [and analytic] approach to unstable resonators that does not suffer from the above difficulties. In particular,

(1) Our approach demands only that the Fresnel number of the resonator be large, is valid for all magnifications, and is not dependent on the specific symmetry of the systems. Approximations are made on the integral equation itself, rather than the basis functions, leading to results which are generally applicable.

(2) For a fixed N, the solutions to the resonator equation divide into two classes. For curved mirrors with $M>1+1/\sqrt{N}$ the modes are diffraction dominated, and our work reduces to that of Butts and Avizonis (for circular mirrors) and Horwitz (for rectangular devices).

(3) For nearly flat mirrors with $M<1+1/\sqrt{N}$, the solutions are dominated by the waveguide nature of the cavities, and are given by a smooth core term analogous to that obtained by Weinstein[3], together with a small, rapidly oscillating function representing the effects of edge diffraction. The core term contains the diffraction loss effects, and the oscillating term averages essentially to zero.

(4) In our approach to resonator electrodynamics the fluctuation effects of diffraction appear in the form of edge waves. As discussed in Appendix I as well as references (4) and (5), each edge wave is the Fresnel diffraction pattern of a plane wave around the mirror edge which has propagated k times across the resonator. Furthermore, the edge waves used by Horwitz (for strip systems) and Butts and Avizonis (for cylindrical resonators) are the leading terms in the asymptotic expansions of the Fresnel diffraction functions.

(5) No unphysical infinities appeared in our treatment of the resonator eigenmodes, because the diffraction functions are given

exactly in terms of Lommel functions. The leading terms of the asymptotic expansions of the Fresnel diffraction patterns contain singularities on the shadow boundary.

(6) As a critical test of our approach we have applied it to a marginally stable (M=1) cylindrical resonator with a Fresnel number of ten. The resultant eigenvalues and eigenmodes were in excellent agreement with computer generated solutions of the Fox-Li resonator equation.[6,7] This work was reported in references (4) and (5) which are enclosed with this report.

(7) For the waveguide region $M<1+1/\sqrt{N}$, we have demonstrated that in the limit $N\to\infty$ the core term reduces to the closed resonator or waveguide solution, and the edge diffraction term vanishes. We emphasize that previous asymptotic theories do not apply to this regime, and for large N reduce to geometric optics rather than closed resonator solutions.

(8) For the diffraction dominated region, $M>1+1/\sqrt{N}$, the core term reduces in the limit of large N to the fundamental spherical wave used by Butts and Avizonis (for circular systems) and Horwitz (for rectangular systems).

In Appendix I we have outlined the most salient features of our treatment of empty resonators in a form that is applicable to all values of M. The basic elements of the theory are summarized in Table (I-1) for cylindrical and rectangular geometry.

The essence of our approach is to approximate the Green's function for the differential equation governing resonator dynamics, and then to solve the approximate integral equation exactly. Consequently, the application to gain media is straightforward since the gain is a source term in the Green's function solution. In Appendix II we outline the derivation of the unstable resonator equation for driven resonators. The same type of approximations made for empty resonators may be applied. Approximate solutions both for linear gain and non-linear, saturable gain resonators are also discussed in Appendix II.

Specific points of interest which have arisen from our treatment of driven resonators are:

(1) In the paraxial approximation, Maxwell's equations reduce to a form similar to a diffusion equation in the transverse direction, with the longitudinal coordinate z in place of time. The Green's function spreads transversely and decreases in amplitude as z is increased, in complete analogy with diffusion.

(2) The integral equation for driven resonators implicitly contains diffraction off the edge of a non-uniform gain media in the same manner that the empty resonator equation describes diffraction around the mirror edge.

Also, we expect the equation to describe the phenomena of refraction and internal reflection at the boundary of the gain medium and empty space.

(3) In the waveguide region ($M<1+1/\sqrt{N}$) with a *non-linear* gain medium, the resonator saturates very quickly as single pass unsaturated gain exceeds diffraction losses. The reason for this is that each ray makes a large number of passes through the gain medium before escaping, leading to large amplifications and saturation.

(4) For the diffraction dominated region with $M > 1+1/\sqrt{N}$, each ray makes a relatively few number of transits before leaving the cavity, and saturation is expected to be of less importance. Different methods must be used for this region.

(5) We have shown in Appendix II how a homogeneous linear gain medium leads to the same integral equation as that for an empty resonator, with the eigenvalue λ_{DIFF} modified by the single pass unsaturated gain, $\lambda = e^{\chi L}\lambda_{DIFF}$. The modes have exactly the same form as those of an empty resonator.

(6) For a saturable gain medium in a cavity with flat mirrors the resonator saturates for single pass gain exceeding diffraction losses by as little at .1%. The modes have nearly the same form as for the empty cavity.

In summary, we feel that our research illuminates the fundamental physics that govern the electrodynamics of unstable resonators. Our methods are applicable to different mirror geometries, are valid in the waveguide dominated region where even ray optical solutions become cumbersome, and describe the resonator with a gain-filled cavity. Presently we are obtaining numerical solutions for empty resonators with $1<M<1+1/\sqrt{N}$. These solutions and the methods used to obtain them will be used to predict the amplitude and distribution of radiation in saturable gain unstable resonators.

APPENDIX I

OUTLINE OF ASYMPTOTIC APPROACHES TO EMPTY UNSTABLE RESONATORS

As a specific example, we consider a cylindrically symmetric unstable resonator with magnification M and a Fresnel number N. The Fox-Li resonator equation can be cast in the following form

$$\lambda_{n\ell}\,\phi_{n\ell}(r) = 2\pi i^{\ell+1} N \int_0^1 d\rho\rho\, J_\ell(2\pi N\rho\tfrac{r}{M}) e^{-i\pi N[\rho^2+(r/M)^2]}\,\phi_{n\ell}(\rho), \qquad \text{(I-1)}$$

where the eigenfunction of the (n,ℓ) Fox-Li transverse mode is given by $f_{n\ell} = \phi_{n\ell}\,\exp(-i\pi N_{eq} r^2)$, the eigenvalue is given by $\gamma_{n\ell} = \lambda_{n\ell}/M$, and J_ℓ is the Bessel function of order ℓ. As a first step we rewrite the reduced integral Eq. (I-1) as

$$\lambda_{n\ell}\phi_{n\ell}(r) = 2\pi i^{\ell+1} N\left[\int_0^\infty - \int_1^\infty\right] d\rho\rho\, J_\ell(2\pi N\rho\tfrac{r}{M}) e^{-i\pi N[\rho^2+(r/M^2)]M}\,\phi_{n\ell}(\rho). \qquad \text{(I-2)}$$

We shall refer to the first term on the right hand side of Eq. (I-2) as the core term and the second as the edge term. The physical meaning of these terms will become apparent as we proceed. Next we make the following approximation on the edge term,

$$2\pi i^{\ell+1} N \int_1^\infty d\rho\rho\, J_\ell\left(2\pi N\rho\frac{r}{M}\right)e^{-i\pi N[\rho^2+(\frac{r}{M})]^2} \phi_{n\ell}(\rho) = \phi_{n\ell}(1)\, F_\ell(r,N), \quad \text{(I-3a)}$$

where

$$F_\ell(r,N) = 2\pi i^{\ell+1}N \int_1^\infty d\rho\rho J_\ell\left(2\pi N\rho\frac{r}{M}\right)e^{-i\pi N[\rho^2+(\frac{r}{M})^2]} \quad . \qquad \text{(I-3b)}$$

There are three features that should be noted:

(1) Using asymptotic techniques, one can demonstrate that equation (I-3a) is valid to order $(N)^{-1} << 1$. For $r<1$, the integral Eq. (I-3a) is dominated by the contribution from the lower endpoint of the contour, and $\phi_{n\rho}(\rho)$ may be replaced by $\phi_{n\rho}(1)$ to lowest order in N^{-1}.

(2) $F_\ell(r,N)$ is the ℓ^{th} azimuthal Fresnel diffraction pattern associated with the mirror edge. For $\ell=0$, it is the compliment of the s-wave diffraction pattern of a circular hole, which was discussed extensively by Lommel.

(3) The integral in Eq. (I-3b) can be analytically expressed in terms of a rapidly convergent Neumann series of Bessel functions.

The core term in Eq. (I-2) is dominated by the stationary phase point for $r<M$, and is of order $\phi_{n\rho}(\rho/M)$. The edge term which is of order $1/\sqrt{N}$ arises from diffraction and varies rapidly in space. To obtain a physical understanding of the core term, we neglect diffraction entirely and solve equation (I-2) exactly. Following this, we shall incorporate diffraction and solve equation (I-2) using the approximation (I-3a). The resultant eigenmodes and eigenvalues are exact to order (1/N).

For cylindrically symmetric systems, we find in the absence of diffraction

$$\phi_{s\ell}(r) = \frac{\Gamma(1+\frac{\ell+s}{2})(\pi N)^{\ell-s}}{\ell!\left(i\pi NM\frac{M^2}{M^2-1}\right)^{(\ell-s)/2}} \quad r^{\ell}\ {}_1F_1\left(\frac{\ell-s}{2},\ \ell+1,\ \frac{i\pi Nr^2(M^2-1)}{M^2}\right) \tag{I-4a}$$

$$\lambda_{s\ell} = \left(\frac{1}{M}\right)^s \tag{I-4b}$$

where Γ is the gamma function, ${}_1F_1$ is the confluent hypergeometric function, and s can be complex.

There are several features of Eq. (I-4) that are of interest:

(1) The eigenvalue $\lambda_{s\ell}$ has the geometric optics value of unity for s=0 and the loss per bounce is $1-\frac{1}{M^2}$.

(2) The leading term in the eigenmode is

$$\phi_{s\ell}(r) \to r^{s} + O\left(\frac{1}{N}\right) \tag{I-5}$$

where $O(1/N)$ implies a quantity of order $1/N$. For the geometric optic solution $s=0$ and $\phi_{n\ell}(r) \sim 1$, which gives the fundamental spherical wave used by Horwitz, Butts, and Avizonis.

(3) When s is cast into the form

$$s = ia^{2}M^{2}/2\pi N(M^{2}-1), \tag{I-6}$$

where a is a complex number, then the solution given by Eq. (I-4a) can be simply represented in terms of a rapidly convergent Neumann series of Bessel functions,

$$\phi_{a\ell}(r) = \sum_{k=0} c_{k} r^{k} J_{\ell+k}(ar), \tag{I-7}$$

where

$$c_{0} = 1,\ c_{1} = \ell b/a,\quad b = i\pi N(M^{2}-1)/M^{2}, \text{ and} \tag{I-8}$$

$$c_{k+1} - b(\ell+2k)c_{k} + abc_{k-1} = 0.$$

For M=1, b=0 and Eq. (I-7) reduces to a single Bessel function of complex argument,

$$\phi_{a\ell}(r) = J_\ell(ar) \ . \tag{I-9}$$

(4) For a strip resonator one obtains parabolic cylindrical functions. For $M \neq 1$, the asymptotic expansion of these modes are the geometric optics solutions which correspond to the fundamental spherical wave used by Horwitz in reference (1). For M=1, they reduce to the modes of a closed strip resonator, sin(ax).

If diffraction is included, then to order (1/N) the empty resonator electrodynamics are specified by

$$\lambda_{n\ell}\phi_{n\ell}(r) = 2\pi i^{\ell+1} N \int_0^\infty d\rho\rho \; J_\ell(2\pi N\rho \frac{r}{M}) e^{-i\pi N \left[\rho^2 + (\frac{r}{M})^2\right]} \phi_{n\ell}(\rho) \quad - \quad \phi_{n\ell}(1)F(r,N). \tag{I-10}$$

The exact solution of Eq. (I-10) is

$$\phi_{n\ell}(r) = \phi_{n\ell}(1) \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_{n\ell}}\right)^k F_\ell \left(\frac{r}{M^k} \; ; \frac{N}{M_{k-1}}\right) \tag{I-11}$$

where

$$M_{k-1} \equiv \sum_{j=0}^{k-1} \frac{1}{M^{2J}} \tag{I-12}$$

and the $\lambda_{n\ell}$ are obtained from continuity of the eigenmode at the mirror edge. Eq. (I-11) corresponds to a coherent superposition of diffracted waves which propagate back and forth across the resonator and are modified by the magnification. The usefulness of this solution is set by the convergence of the series which is turn is determined by the resonator's magnification. There are in fact several different regions of interest. These are: (1) M >> 1, (2) M=1 and (3) $1<M\leq 1+1/\sqrt{N}$.

Case (I) M>>1

For this case convergence is excellent. To use equation (I-11) we note that for sufficiently large k, $F_\ell(\frac{r}{M^k}, \frac{N}{M_{k-1}})$ is independent of both r and as k, i.e.,

$$F_\ell(\frac{r}{M^k}, \frac{N}{M_{k-1}}) \simeq F(0,\frac{N}{M_{k_o-1}}) \quad \text{if } k \geq k_o \tag{I-13}$$

and one can then rewrite Eq. (I-11) as

$$\phi_{n\ell}(r) = \phi_{n\ell}(1) \sum_{k=1}^{k_o} \left(\frac{1}{\lambda_{n\ell}}\right)^k F_\ell\left(\frac{r}{M^k}; \frac{N}{M^{k-1}}\right) +$$

$$+ \phi_{n\ell}(1)\, F_\ell\left(0; \frac{N}{M^{k_o+1}}\right) \sum_{k_o+1}^{\infty} \left(\frac{1}{\lambda_{n\ell}}\right)^k$$

$$= \phi_{n\ell}(1) \sum_{k=1}^{k} \left(\frac{1}{\lambda_{n\ell}}\right)^k F_\ell\left(\frac{r}{M^k}; \frac{N}{M^{k-1}}\right) + \phi_{n\ell}(1) F_\ell\left(0; \frac{N}{M^{k_o-1}}\right) / \lambda_{n\ell}^{ko} (\lambda_{n\ell}-1) \qquad \text{(I-14)}$$

The first term in Eq. (I-14) corresponds to a sequence of edge diffracted waves and the second to a spherical wave. We obtain the Butts and Avizonis description if the edge waves are given by their asymptotic expansion.

Case (2) M=1

For this case, one requires all of the terms in the series (I-11) and the techniques of reference (2) can not be used. Eq. (I-11) becomes

$$\phi_{n\ell}(r) = \phi_{n\ell}(1) \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_{n\ell}}\right)^k F_\ell(r;N/k) \qquad \text{(I-15)}$$

which consists of a coherent superposition of Fresnel diffraction patterns that propagate 1,2,...,k,... times across the resonator. To obtain solutions from Eq. (I-15) we note that diffraction will perturb the low lying transverse modes only weakly from the closed resonator form [e.g., the loss per bounce is only a few percent for the $TEM_{0,0}$ mode]. Thus, we assume that all but the first term in Eq. (I-15) coherently sum to form a core term which is conveniently represented in terms of a Fourier-Bessel series,

$$\phi_{n\ell}(r) = \sum_{j=1}^{n} C_{j\ell}^{(n)} J_{\ell}(a_{j\ell}\, r) + \frac{1}{\lambda_{n\ell}} F_{\ell}(r;N)\phi_{n\ell}(1), \tag{I-16}$$

where the $a_{j\ell}$ are determined by requiring that the Bessel functions be orthogonal within the resonator and the $C_{j\ell}^{(n)}$ are obtained by setting Eq. (I-16) equal to Eq. (I-15). Convergence is rapid and only a few Bessel functions are required. This particular case is discussed in references (4) and (5). Note that the eigenmodes consist of a slowly varying core term with small oscillations from the Fresnel diffraction pattern of the mirror edge. The core term can be shown in the limit of large N to approach a single Bessel function $J_{\ell}(b_{n\ell}r)$, where $b_{n\ell}$ is the n^{th} zero of the ℓth Bessel function.

Case (3) $1<M\leq 1 + \frac{1}{\sqrt{N}}$

A spatially varying core term which is not described by geometric optics remains and a number of diffraction functions are required. Recalling that the solutions Eq. (I.4) for $M\neq 1$ consist of confluent hypergeometric functions one is tempted to use a series of such functions for the core term. For computational purposes, it is easier to work with Bessel functions and one can express the hypergeometric functions in terms of Bessel functions via a Neumann series as in (I-7). One then sets

$$\phi_{n\ell}(r) = \sum_{m=1}^{P} \sum_{j=0}^{\infty} c_{j\ell}^{m} J_{\ell+j}(ar) + \sum_{k=1}^{k_0} \left(\frac{1}{\lambda_{n\ell}}\right)^{k} F_{\ell}\left(\frac{r}{M^k}, \frac{N}{M_{k-1}}\right) \phi_{n\ell}(1)$$

and proceeds from there. Numerical calculations indicate that as M increases, the core term declines in magnitude and moves outwards towards the shadow boundary. This case is presently under study.

One can carry out an entirely analogous procedure for strip resonators. For this case, the Fresnel diffraction patterns can be expressed in terms of the well known sin and cosine integrals. We find the analysis to be entirely analogous to the cylindrical case. The results are summarized below in Table (I-1).

	STRIP MIRROR	CIRCULAR MIRROR	ANY MIRROR
closed resonator solution	$\sin(n\pi x)$	$J_\ell(b_{n\ell}x)$	Solution to $\lambda\phi = \oiint_{S^\infty} K(r,r')\phi(r')ds'$ which vanishes on the edge.
core term (M=1)	$\sin(ax)$	$J_\ell(ax)$	any solution to above equation
core term ($1<M<1+1\sqrt{N}$)	parabolic cylinder function	confluent hypergeometric function	solution to above equation, with the K for the appropriate curved coordinate system
Fresnel Diffraction Pattern	$F = \sqrt{\frac{i}{2}} - S(\sqrt{2N}(1+x))$ $-iC(\sqrt{2N}(1+x))$ S,C are Fresnel integrals	$F = \frac{e^{-i\pi N(1+x^2)}}{ix^\ell}(V_\ell(2\pi N, 2\pi Nx)$ $- iV_{\ell+1}(2\pi N, 2\pi Nx))$ V_ℓ = Lommel function for $\ell=0$	$F = [\oiint_{S^\infty} - \oiint_{S^{MIRROR}}] K(r,r')ds'$
asymptotic limit	$F = e^{-i\pi N(1+x^2)}/(1+x)$	$F = \frac{e^{-i\pi N(1+x^2)}}{\sqrt{x(1-x^2)}}(\cos\chi + ix\sin\chi)$ $\chi = 2\pi Nx - \ell\pi/2 - \pi/4$	———

TABLE I-1

APPENDIX II

UNSTABLE RESONATOR EQUATION WITH GAIN

The Unstable Resonator Equation with or without gain is derived from the Maxwell equations with the boundary condition that the transverse field vanish on the mirror surface. The time dependence is separated as follows:

$$\vec{E}\,(\vec{r},t) = \vec{E}(\vec{r})\; e^{-iwt} \; . \tag{II-1}$$

Then the Maxwell equations become, assuming $\mu=\mu_0$,

$$(\bar{\nabla}^2 + k^2)\vec{E}(\vec{r}) = -\frac{k^2}{\varepsilon_0}\,\vec{P}(\vec{r}) \;, \tag{II-2}$$

where k=w/c and $\vec{P}$ is the electric polarization. For a resonator, the transverse field will consist of a left moving and a right moving wave:

$$E_T(\vec{r}) = \sum_{\ell=0} [\varepsilon_L^\ell(r,z)e^{i\ell\theta}\;\; e^{-ikz} + C\; \varepsilon_R^\ell(r,L-z)e^{i\ell\theta}\; e^{ikz}] \; . \tag{II-3}$$

where cylindrical coordinates appropriate to circular mirrors have been used, the amplitudes ε_L^ℓ and ε_R^ℓ are slowly varying, and C is a constant to be determined. Assuming that $P_T(\vec{r})$ satisfies a similar relation, and making the paraxial approximation[8], we find that each amplitude in Eq. (II-3) satisfies an equation of the form

$$\left(\frac{\partial^2}{\delta r^2} + \frac{1}{r}\frac{\delta}{\delta r} - \frac{\ell^2}{r^2} - 2ik\frac{d}{dz'}\right) \Sigma^{\ell}_{L,R}(r,z') = -\frac{k^2}{\varepsilon_o} P^{\ell}_{L,R}(r,z') \qquad \text{(II-4)}$$

where $z' = z$ for Σ_L and $z' = L-z$ for Σ_R.

This equation may be solved using Green's functions. The polarization on the right is regarded as a source, and the solution is determined by boundary conditions on the mirror. The field on the mirror is naturally assumed to vanish away from the edge:

$$\Sigma(r,z') = \begin{cases} \Sigma(r,o) & r \leq a \\ o & r > a \end{cases} \qquad \text{(II-5)}$$

This condition, together with the paraxial approximation, are the only approximations which are made in the derivation of the resonator equation. It is assumed that both the field and its derivations vanish at $r=\infty$. Then using Hankel transforms, or results for the diffusion equation, the Green's function for Eq. (II-4) is found to be

$$G(r,z;r'z') = \frac{2\pi i^{\ell}}{z-z'} e^{\frac{-ik(r^2+r'^2)}{2(z-z')}} J\,(krr'/z-z')u(z-z') \qquad \text{(II-6)}$$

where

$$u(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{cases} \tag{II-7}$$

Treating the polarization as a source, and using the boundary conditions given by Eq. (II-5), the solution to Eq. (II-4) is given by

$$\Sigma^{\ell}_{L,R}(r,z') = \frac{k}{z'} i^{\ell+1} e^{-ikr^2/2z'} \int_0^a \rho d\rho e^{-ik\rho^2/2z'} J_\ell(kr\rho/z') \Sigma^{\ell}_{L,r}(\rho,0)$$

$$+ \frac{k^2}{2\epsilon_0} i^{\ell} \int_0^{z'} \frac{dz''}{z'-z''} \int_0^{\infty} \rho d\rho e^{-ik(r^2+\rho^2)/2(z'-z'')} J_\ell(kr\rho/(z'-z'')) P^{\ell}_{L,R}(\rho,z'')$$

(II-8)

The plane $z'=0$ coincides with the mirror surface only for flat mirrors. Assuming a curvature R for the mirror, and R>>a, $\Sigma^{\ell}_{L,R}(r,o)$ is related to an amplitude ϕ on the mirror surface by

$$\Sigma^{\ell}_{L,R}(r,o) = e^{ikr^2/2R} \phi^{\ell}_{L,R}(r) \tag{II-9}$$

This assumption may be seen to yield the correct results for curved mirrors by obtaining the Green's function in spheroidal coordinates. The Green's function in spheroidal coordinates has the same form as Eq. (II-6). In the limit of slightly curved mirror, $R/L >> 1$, the following results are obtained. Substituting Eq. (II-9) into Eq. (II-8), we arrive at the amplitude at the point r, z' given the amplitude on the mirror surface and the polarization:

$$\Sigma^{\ell}_{L,R}(r,z') = \frac{k}{z'} i^{\ell+1} \int_0^a \rho d\rho e^{-ik(r^2+\rho^2)/2z'} e^{ik\rho^2/2R} J_\ell(kr\rho/z') \phi^{\ell}_{L,R}(\rho)$$

$$+ \frac{k^2}{2\varepsilon_0} i^{\ell} \int_0^{z'} \frac{dz''}{z'-z''} \int_0^{\infty} \rho d\rho \, e^{-ik(r^2+\rho^2)/2(z'-z'')} J_\ell(kr\rho/(z'-z'')) P^{\ell}_{L,R}(\rho,z'')$$

(II-10)

To determine ϕ, we develop an eigenvalue equation. Letting $z'=L$, we first use Eq. (II-10) to determine $\Sigma^{\ell}(r,L)$, which is then related to the amplitude ϕ on the mirror surface with Eq. (II-9). Using the boundary condition that the total field given by Eq. (II-3) vanishes on both mirrors, we obtain

$$\Sigma_L(r,L)e^{-ikL} + C\Sigma_R(r,o)e^{+ikL} = 0$$

$$\Sigma_L(r,o) + C\Sigma_R(r,L) = 0 \qquad \text{(II-11)}$$

Define the following integral transforms:

$$K^a_{L,M} f = \frac{k}{L} i^{\ell+1} \int_0^a \rho d\rho e^{-ik(M^2+1)(r^2+\rho^2)/4ML} J_\ell(kr\rho/L) f(\rho) \qquad \text{(II-12)}$$

where the magnification is related to mirror curvature by

$$\frac{M^2+1}{2M} = 1-L/R \tag{II-13}$$

Using Eqs. (II-9)-(II-13), we arrive at two coupled equations for the resonator with equal mirrors:

$$\lambda_R \ \phi_R(r) = K^a_{L,M}\phi_L(r) - \frac{ik}{2\epsilon} \underset{o}{e^{-ikr\ /2R^2}} \int_0^L dz\ K^{\infty}_{L-z,1}P_L(r,z)$$

$$\lambda_L \ \phi_L(r) = K^a_{L,M}\ \phi_R(r) - \frac{ik}{2\epsilon_o}\ e^{-ikr^2/2R} \int_0^L dz\ K^{\infty}_{L-z,1}P_R(r,z) \tag{II-14}$$

where $\lambda_R = -Ce^{2ikL}$

$$\lambda_L = -1/C \tag{II-14a}$$

For equal mirrors, symmetry implies $C=(-1)^{q+1}\ e^{-ikL}$, $\lambda_R=\lambda_L=\lambda$, $\phi_R=\phi_L=\phi$, and $P_L=P_R=P$. In this case, the eigenvalue equation for the amplitudes on the mirror edge is:

$$\lambda\phi = K^a_{L,M}\phi - \frac{ik}{2\epsilon_o}\ e^{ikr^2/2R} \int_0^L dz\ K^{\infty}_{L-z,1}P \tag{II-15}$$

where

$$\lambda = e^{-i(kL-\pi q)} \quad . \tag{II-16}$$

Eq. (II-15) is first solved to yield the amplitude ϕ on the mirror, and the eigenvalue λ which determines the single transit gain or loss. The amplitude at point z' may then be found by using Eq. (II-10). Finally, the total field at any point is found by adding the left and right moving waves as in Eq. (II-3). Since P is in general a function of the total field, Eq. (II-15) is usually difficult to solve.

Before discussing specific examples, we note three points of interest:

(1) For P=0, Eq. (II-15) reduces to the usual Fresnel-Kirchoff equations for empty resonators.

(2) Diffraction off the edge of the gain media is included if we assume that P=0 for r>a. Then K^a rather than K^∞ is used, analogous to the case of mirror diffraction for empty resonators.

(3) The transform of ϕ in Eq. (II-15) depends on M as appropriate for a mirror term, while the transform of P is the usual Green's function which is independent of mirror curvature.

We first consider the case of linear gain, described by $P = \chi E$ where χ is a constant. We use the following properties of the integral transforms for unit magnification:

$$K_{L-z}^{\infty} K_z^a = K_L^a \; . \tag{II-17}$$

We find that the eigenvalue equation reduces on interation to:

$$\lambda\phi = \exp(-ik\chi L/2\varepsilon_0) K_L^a \phi \tag{II-18}$$

which is identical with the empty resonator equation except for the additional phase factor. Consequently the eigenvalues will differ by the phase factor, and the modes will be identical. This is the usual case where single pass gain is multiplied by the eigenvalue for empty resonators to yield the total gain, for which no stable solution exists.

To qualitatively discuss solutions for saturable gain media, we choose a polarization of the form

$$P = \chi E e^{-|E|^2/E_s^2} \; , \tag{II-19}$$

which leads to analytic results if various approximations are made.

(1) K^{∞}_{L-z} in the gain integral is evaluated in the geometric optics limit.

(2) The gain is assumed to be large enough so that saturation occurs, and the amplitude for flat mirrors is nearly that of an empty resonator,

$$\phi \sim E_0 J_\ell(ax) \ , \qquad \text{(II-20}$$

where a is determined in refs 4 and 5 and the equilibrium amplitude E_0 is to be determined.

(3) The phase changes induced by polarization in the modes and eigenvalues are neglected.

For equilibrium to occur, the radiation in the cavity neither grows or decays and $|\lambda|=1$. The eigenvalue equation is then easily solved for E_0. Leaving out the details, the solutions are presented below in figure (II-1) where (E_0/E_s) is graphed as a function of gain.

From the figure, we see that the cavity saturates ($E_0 \sim E_s$) for single pass unsaturated gain exceeding diffraction losses by as little as .1%. The eigenmodes were found to be essentially identical with the empty resonator modes, with the greatest differences near the mirror edge.

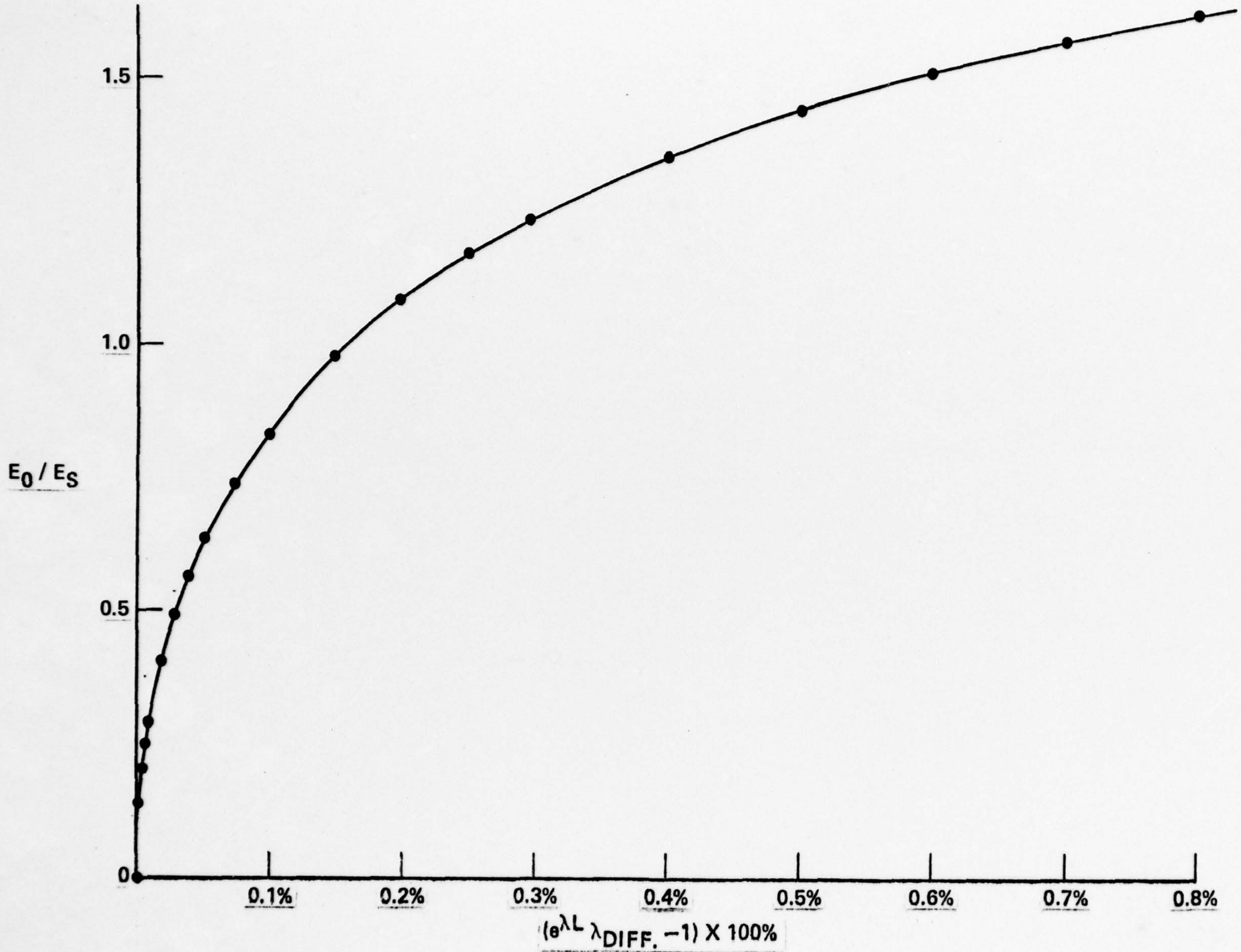

FIGURE II–1. GRAPH OF E_0 / E_S VS. $(e^{\lambda L} \lambda_{DIFF.} - 1) \times 100\%$. $e^{\lambda L}$ IS THE SINGLE PASS LINEAR GAIN, $\lambda_{DIFF.}$ IS THE DIFFRACTION EIGENVALUE FOR EMPTY RESORATORS, AND E_S IS THE E – FIELD SATURATION PARAMETER.

REFERENCES

1. P. Horowitz, "Asymptotic Theory of Unstable Resonator Modes," J. Opt. Soc. Am. 63, 1528-1542 (1973).

2. R.R. Butts and P.V. Avizonis, "Asymptotic Analysis of Unstable Resonators with Circular Mirrors," J. Opt. Soc. Am. 68, 1071-1076 (1978).

3. L. Weinstein, Open Resonators and Open Waveguides, Golem Press, Boulder (1969).

4. J. Nagel, D. Rogovin, P. Avizonis, and R. Butts, "Analytic Approaches to Unstable Optical Resonators", Proceedings Laser '78, 1979.

5. J. Nagel, D. Rogovin, P. Azivonis and R. Butts, "Asymptotic Approaches to Marginally Stable Resonators", submitted to Optics Letters, 1979.

6. H. Kogelnik and T. Li, "Laser Beams and Resonators," Proc. IEEE 54, 1312-1329 (1966).

7. A. Fox and T. Li, Resonant Modes in a Maser Interferometer," Bell Syst. Tech. J. 40, 453-488 (1961).

8. W.H. Louisell, "From Maxwell to Optical Resonators," Laser Induced Fusion and X-Ray Laser Studies, Vol. 3, Ed. S.F. Jacobs, M.O. Scully, M. Sargent, and C.D. Cantrell, Addison Wesley, 1974.

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"Asymptotic Methods for Unstable Resonators" - Talk given at Quantum Electronics Conference, Snowbird, Utah, 1978.

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J. Nagel, D. Rogovin, R. Butts, and P. Avizonis, "Asymptotic Approaches to Marginally Unstable Optical Resonators", submitted to Optics Letters.

J. Nagel and D. Rogovin, "Analytic Iterative Solutions for Marginally Stable Resonators", to be submitted to Optics Letters.

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A method for obtaining asymptotic solutions of the unstable resonator integral equation which is valid for all values of the magnification was developed. Approximations were made on the Greens functions rather than the eigenmodes, leading to results which are easily generalized to different mirror geometries. "Diffraction dominated eigenmodes" for resonators where each ray escapes after a few transits were differentiated from "waveguide dominated eigenmodes" which are obtained for cavities with a large number of transits per ray. The solutions

obtained were seen to agree in the appropriate limits with other asymptotic solutions, numerical results, and geometric optics predictions. To include the effects of gain, the unstable resonator equation was derived from Maxwell's equations in a polarizable medium. The resulting equations have the same structure as the empty resonator equations, and similar approximations can be used. Some features of the effects of saturation on the eigenmodes of an unstable resonator were considered.