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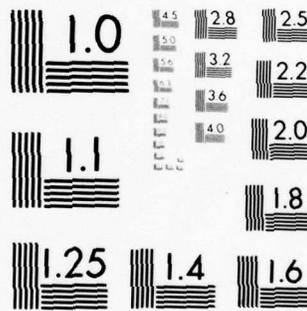
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PIEZOELECTRIC SEMICONDUCTORS

by

M.F. McCarthy and H.F. Tiersten

Office of Naval Research  
Contract N00014-76-C-0368  
Project NR 318-009  
Technical Report No. 28

June 1979

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function of the state of the material immediately ahead of the wavefront. In the special case of plane waves entering a homogeneous steady state, the growth equation for the amplitude of the acceleration wave is determined and, of course, the propagation velocity and coefficients in the growth equation depend on the propagation direction, but otherwise are constant. The relation between acceleration waves and the formation and propagation of acousto-electric domains is indicated. The solutions of the growth equation indicate the formation of a shock in a finite time for conditions conducive to domain formation except in certain unusual cases possibly occurring with purely transverse acceleration waves. In the course of the treatment the condition for the threshold field for domain formation is determined under quite general circumstances. When the electrical conduction equation, which can be quite general in this treatment, is specialized to the simple form usually employed for anisotropic semiconductors, the aforementioned more general condition reduces to the anisotropic generalization of the well-known elementary result. In addition, the behavior of weak waves is discussed.

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# THE GROWTH OF WAVE DISCONTINUITIES IN PIEZOELECTRIC SEMICONDUCTORS

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## ABSTRACT

The reference coordinate description of the general nonlinear differential equations describing the interaction of finitely deformable, polarizable, intrinsic n-type semiconductors with the quasi-static electric field is applied in the study of acceleration waves in piezoelectric semiconductors. As a consequence, the mechanical and dielectric nonlinearities are included in the treatment as well as the semiconduction nonlinearity. The general equation for the propagation velocity of the disturbance is obtained as a function of the state of the material immediately ahead of the wavefront. In the special case of plane waves entering a homogeneous steady state, the growth equation for the amplitude of the acceleration wave is determined and, of course, the propagation velocity and coefficients in the growth equation depend on the propagation direction, but otherwise are constant. The relation between acceleration waves and the formation and propagation of acoustoelectric domains is indicated. The solutions of the growth equation indicate the formation of a shock in a finite time for conditions conducive to domain formation except in certain unusual cases possibly occurring with purely transverse acceleration waves. In the course of the treatment the condition for the threshold field for domain formation is determined under quite general circumstances. When the electrical conduction equation, which can be quite general in this treatment, is specialized to the simple form usually employed for anisotropic semiconductors, the aforementioned more general condition reduces to the anisotropic generalization of the well-known elementary result. In addition, the behavior of weak waves is discussed.

## 1. Introduction

In a previous investigation the theory of one-dimensional acceleration waves was applied<sup>1</sup> to a one-dimensional version of general rotationally invariant nonlinear electroelastic equations derived earlier from a well-defined macroscopic model<sup>2</sup> of deformable semiconductors. In that treatment<sup>1</sup> an analytical description of the formation and propagation of purely longitudinal acoustoelectric domains in piezoelectric semiconductors was obtained. The analysis indicated that for electric fields above a threshold value the amplitude of the acceleration wave would always increase without bound and become a shock. A natural and logical extension of the previous one-dimensional work is the treatment of three-dimensional acceleration waves, in which acoustoelectric domains with transverse mechanical displacement components can be considered. Recently, in the case of the quasi-static electric field the general nonlinear electroelastic equations for deformable intrinsic n-type semiconductors<sup>2</sup> were transformed<sup>3</sup> from the unknown present coordinate description to the known reference coordinate description, which is the form needed here and in general for the treatment of problems.

In this paper the theory of three-dimensional acceleration waves<sup>4-12</sup> is applied to the above-mentioned reference coordinate description<sup>3</sup> of the general rotationally invariant nonlinear electroelastic equations for deformable intrinsic n-type semiconductors in order to analytically describe the formation and propagation of acoustoelectric domains, with both transverse and longitudinal components of mechanical displacement, in piezoelectric semiconductors subject to high electric fields. The analysis results in an expression for the amplitude of the acceleration wave (or domain) which exhibits the competition between dissipation due to electrical conduction

and the semiconduction and mechanical nonlinearities in producing decay or growth of the acceleration wave (or domain). As in the case of the purely longitudinal acceleration wave treated earlier<sup>1</sup>, the possibility of the amplitude of the more general three-dimensional plane acceleration wave increasing without bound and becoming a shock is clearly indicated. However, in the special case of purely transverse acceleration waves, circumstances can exist under which it is not possible for the amplitude to grow. Nevertheless, if any longitudinal motion is present in the acceleration wave, the possibility of the amplitude increasing without bound always exists. During the course of the analysis the expression for the velocity of the wave (or domain) as a function of the state of the material immediately ahead of the wavefront naturally is obtained.

## 2. Basic Formulae and Equations

The macroscopic model of an elastic intrinsic n-type semiconductor employed in Ref.2 consists of three interacting, interpenetrating continua, which consist of (i) a lattice continuum which has a positive charge density; (ii) a bound electronic continuum which has a negative charge density and which can displace slightly from the lattice continuum and thus produce electric polarization, and (iii) a free electronic continuum which has a negative charge density, negligible inertia and is a conducting compressible fluid which experiences a force of resistance from its motion with respect to the lattice continuum.

Initially, the lattice continuum and the bound electronic continuum all occupy the same region of space and, hence, have the same reference coordinates  $X_L$ . The motion of a point of the lattice continuum is described by the mapping

$$y_i = y_i(x_L, t), \quad \underline{y} = \underline{y}(\underline{x}, t), \quad (2.1)$$

which is one-to-one and differentiable as often as required. Here the  $y_i$  denote the present coordinates of material (lattice continuum) points and  $x_L$ , the reference coordinates, and  $t$  denotes the time. We consistently use the convention that capital indices denote the Cartesian components of  $\underline{x}$  and lower case indices, the Cartesian components of  $\underline{y}$ . A comma followed by an index denotes partial differentiation with respect to a coordinate

$$g_{,i} = \frac{\partial g}{\partial y_i}(y_r, t), \quad G_{,L} = \frac{\partial G}{\partial x_L}(x_K, t), \quad (2.2)$$

and the summation convention for repeated tensor indices is employed.

Since reference coordinates are employed in our study of the propagation of acceleration waves in elastic semiconductors, the integral forms of the equations required in this work consist of Eqs. (2.41) - (2.44) of Ref.3, which we reproduce here in the form

$$\int_{S_0} N_L (K_{Lj} + M_{Lj} - \phi_{Lj}^e) dS_0 = \frac{d}{dt} \int_{V_0} \rho_0 v_j dV_0, \quad (2.3)$$

$$\int_{S_0} N_L \theta_L dS_0 = \int_{V_0} \bar{\mu} dV_0, \quad (2.4)$$

$$\int_{V_0} w_L^e dV_0 = \int_{S_0} N_L (\varphi + \varphi^e) dS_0, \quad (2.5)$$

$$\int_{S_0} N_L \theta_L dS_0 = - \frac{d}{dt} \int_{V_0} \bar{\mu} dV_0, \quad (2.6)$$

where  $N_L$  denotes the outwardly directed unit normal to a reference element of area and  $S_0$  denotes the surface enclosing the reference volume  $V_0$ . Equations (2.3) - (2.6) are the reference integral forms of the conservation of linear momentum of the combined continuum, the charge equation of

electrostatics, the conservation of linear momentum of the free electronic continuum and the conservation of total electric charge. In Eqs. (2.3) - (2.6)  $K_{Lj}$ ,  $M_{Lj}$  and  $\mathcal{P}_{Lj}^e$  denote the reduced mechanical Piola-Kirchhoff stress tensor, the reference free-space Maxwell electrostatic stress tensor and the reference free electronic pressure tensor, respectively;  $v_j$ ,  $\mathcal{D}_L$ ,  $w_L^e$  and  $\mathcal{J}_L$  denote the velocity of the solid, the reference electric displacement vector, the reference local electric field exerted on the free electronic fluid and the reference electric current vector, respectively;  $\rho_0$ ,  $\bar{\mu}$ ,  $\varphi$  and  $\varphi^e$  denote the reference mass density, net reference charge density, electric potential and free electronic chemical potential, respectively; and  $d/dt$  is the material time derivative. The associated constitutive equations and additional required relations take the form

$$\begin{aligned} K_{Lj} &= \rho_0 y_{j,K} \frac{\partial \chi}{\partial E_{LK}}, \quad M_{Lj} = J X_{L,i} T_{ij}^{ES}, \\ \mathcal{P}_{Lj}^e &= J X_{L,j} p^e, \quad \mathcal{D}_L = \epsilon_0 J X_{L,i} E_i - \rho_0 \frac{\partial \chi}{\partial \mathcal{D}_L}, \\ p^e &= (\mu^e)^2 \frac{\partial \epsilon^e}{\partial \mu^e}, \quad \mathcal{J}_L = J \mu^e \Omega_L^e, \end{aligned} \quad (2.7)$$

$$\begin{aligned} J &= \det y_{i,K}, \quad T_{ij}^{ES} = \epsilon_0 E_i E_j - \frac{1}{2} \epsilon_0 E_k E_k \delta_{ij}, \quad v_i = \frac{\partial y_i}{\partial t} \\ E_i &= -\varphi_{,i}, \quad E_{KL} = \frac{1}{2} (C_{KL} - \delta_{KL}), \quad C_{KL} = y_{i,K} y_{i,L}, \quad \mathcal{D}_L = -\varphi_{,L}, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \chi &= \chi(E_{KL}, \mathcal{D}_L), \quad \epsilon^e = \epsilon^e(\mu^e), \quad \varphi^e = \partial(\mu^e \epsilon^e) / \partial \mu^e, \\ \Omega_L^e &= \Omega_L^e(\mu^e, w_L^e, E_{LM}, \mathcal{D}_L), \end{aligned} \quad (2.9)$$

where  $T_{ij}^{ES}$ ,  $E_{KL}$  and  $C_{KL}$  denote the free-space Maxwell electrostatic stress tensor, the material (reference or Lagrangian) strain tensor and Green's deformation tensor, respectively;  $E_i$ ,  $\mathcal{D}_L$  and  $\Omega_L^e$  denote the Maxwell electric field, the reference (or rotationally invariant) measure of the electric field and the rotationally invariant constitutive vector that accounts for

the relative flow velocity of the free electronic fluid, respectively;  $p^e$ ,  $\chi$ ,  $\mu^e$  and  $\epsilon^e$  denote the free electronic pressure, a particularly convenient thermodynamic state function related to the stored internal energy per unit mass of the deformable solid, the free electronic charge density and the stored internal energy per unit charge of the free electronic fluid, respectively; and  $\epsilon_0$  is the permittivity of free space.

When the variables are appropriately differentiable, from (2.3) - (2.6), we obtain the differential equations

$$S_{Lj, L} = \rho_0 \dot{v}_j, \quad (2.10)$$

$$\mathcal{J}_{L, L} = \bar{\mu}, \quad (2.11)$$

$$w_L^e = \varphi_{, L} + \varphi_{, L}^e, \quad (2.12)$$

$$\mathcal{J}_{L, L} + \dot{\bar{\mu}} = 0 \quad (2.13)$$

where

$$S_{Lj} = K_{Lj} + M_{Lj} - \mathcal{E}_{Lj}^e, \quad (2.14)$$

and we have employed the dot notation for partial differentiation with respect to time. We now note that we have an additional relation between the net reference charge density  $\bar{\mu}$  and the free electronic charge density  $\mu^e$ , which can be written in the form

$$\bar{\mu} = J\mu^e + \mu^r, \quad (2.15)$$

where  $\mu^r$  is the reference residual lattice charge density, which is a constant. From (2.7)<sub>6</sub>, (2.8)<sub>7</sub>, (2.9)<sub>2-4</sub> and (2.12), we can write

$$\mathcal{J}_L = \hat{\mathcal{J}}_L(E_{KM}, \delta_K, G_K, \mu^e), \quad (2.16)$$

where

$$G_K = \mu_{, K}^e. \quad (2.17)$$

If we define  $\hat{\chi}$  by<sup>13</sup>

$$\hat{\chi} = \chi - \epsilon_0 \mathbf{J} \mathbf{E}_k \mathbf{E}_k / 2\rho_0, \quad (2.18)$$

then by virtue of (2.8)<sub>4-7</sub>, (2.9)<sub>1</sub>, the chain rule of differentiation and the well-known relations

$$J = \sqrt{\det C_{LM}}, \quad C_{KL}^{-1} = X_{K,i} X_{L,i}, \quad (2.19)$$

we can write

$$\hat{\chi} = \hat{\chi}(E_{KL}, \delta_L). \quad (2.20)$$

Now, from (2.7)<sub>1-5</sub>, (2.8), (2.9)<sub>1</sub>, (2.14), (2.19) and (2.20), with the aid of the differential relations

$$\frac{\partial J}{\partial (y_{i,K})} = J X_{K,i}, \quad \frac{\partial X_{N,j}}{\partial (y_{i,M})} = -X_{N,i} X_{M,j}, \quad (2.21)$$

and the chain rule of differentiation, we obtain

$$S_{Lj} = \rho_0 y_{j,K} \frac{\partial \hat{\chi}}{\partial E_{LK}} - J X_{L,j} p^e, \quad \mathcal{D}_L = -\rho_0 \frac{\partial \hat{\chi}}{\partial \delta_L}. \quad (2.22)$$

It is clear from (2.7)<sub>5</sub>, (2.8)<sub>1,5-6</sub>, (2.20) and (2.22) that we may write

$$S_{Lj} = \hat{S}_{Lj}(y_{j,K}, \delta_M, \mu^e), \quad \mathcal{D}_L = \hat{\mathcal{D}}_L(E_{KM}, \delta_L), \quad (2.23)$$

and for later use we note that the constitutive response functions (2.16) and (2.23) as well as all the others are  $C^2$ -functions of their arguments.

### 3. General Properties of Acceleration Waves

Let  $\sigma$  be a propagating surface which may be represented in  $(\underline{y}, t)$  space by the equation

$$f(\underline{y}, t) = 0. \quad (3.1)$$

The unit vector normal  $\underline{n}$  to  $\sigma$  and its speed of displacement  $u_n$  are given by

$$n_i = \frac{f_{,i}}{|f_{,k}|}, \quad u_n = - \frac{\partial f / \partial t}{|f_{,k}|}. \quad (3.2)$$

Corresponding to  $\sigma$ , we have the alternative representation of the surface  $\Sigma$  in  $(\underline{X}, t)$  space by means of the equation

$$\mathfrak{F}(\underline{X}, t) = f(\underline{y}(\underline{X}, t), t) = 0. \quad (3.3)$$

The unit vector  $\underline{N}$  normal to  $\Sigma$  and its speed of propagation  $U_N$  are given by

$$N_K = \frac{\mathfrak{F}_{,K}}{|\mathfrak{F}_{,L}|}, \quad U_N = - \frac{\dot{\mathfrak{F}}}{|\mathfrak{F}_{,L}|}. \quad (3.4)$$

It is a simple matter using (3.2) - (3.4) and the chain rule of differentiation to show that

$$N_K = \frac{F_{ik} n_i}{|F_{kL} n_k|}, \quad U_N = U \frac{|f_{,k}|}{|F_{,k}|} = U / |F_{kL} n_k|, \quad (3.5)$$

where

$$F_{iK} \equiv y_{i,K},$$

and

$$U = u_n - \underline{v} \cdot \underline{n}, \quad (3.6)$$

is the local speed of propagation of the surface.

Let  $\psi(\underline{X}, t)$  be a function which suffers a jump discontinuity across the surface  $\Sigma$ , but is a continuous function everywhere else. We define the jump  $[\psi]$  in the function  $\psi$  to be

$$[\psi] = \psi^- - \psi^+, \quad (3.7)$$

where  $\psi^-$  and  $\psi^+$  are the limiting values of  $\psi$  immediately behind and just in front of a point lying on  $\Sigma$ , respectively. The surface  $\Sigma$  is said to be an acceleration wave if the fields  $y_i(\underline{X}, t)$ ,  $v_i(\underline{X}, t)$  and  $F_{iL}(\underline{X}, t)$  are continuous everywhere but  $\dot{v}_i(\underline{X}, t)$ ,  $\dot{F}_{iL}(\underline{X}, t)$  and  $F_{iL,M}(\underline{X}, t)$ , as well as all

higher order partial derivatives of  $y_i(X, t)$ , suffer jump discontinuities across  $\Sigma$ , but are continuous functions everywhere else. From the geometric conditions of compatibility<sup>14</sup> on the jump in the gradient of a continuous function and the kinematic condition of compatibility<sup>15</sup>, we can obtain<sup>16</sup>

$$\begin{aligned} [F_{iK, L}] &= s_i N_K N_L = a_i F_{pK} F_{qL} n_p n_q, \quad s_i = [N_K N_R F_{iM, R}], \\ [F_{iK}] &= -U_N s_i N_K = -U a_i F_{pK} n_p, \\ [\dot{v}_i] &= U_N^2 s_i = U^2 a_i, \quad s_i = B_n^2 a_i, B_n^2 = F_{iK} F_{jK} n_i n_j. \end{aligned} \quad (3.8)$$

At this point it should be noted that we need make no assumptions with regard to the continuity properties enjoyed by the electric potential  $\varphi(X, t) = \varphi(y(X, t), t)$  or the free electronic charge density  $\mu^e(X, t)$  apart from assuming that at points not on  $\Sigma$  these functions together with their partial derivatives of all orders are continuous. The vector  $\underline{a}$  is called the amplitude vector of the acceleration wave. If we write  $\underline{a} = a \underline{r}$ , where  $\underline{r} \cdot \underline{n} \geq 0$ ,  $|\underline{r}| = 1$ , then if  $a > 0$  the wave is said to be expansive, while a wave for which  $a < 0$  is said to be compressive. If  $\underline{r} = \underline{n}$  the wave is longitudinal, while if  $\underline{r} \cdot \underline{n} = 0$  it is transverse.

The jump conditions across a surface of discontinuity  $\Sigma$  can readily be obtained from the integral forms in (2.3) - (2.6) along with the fact that  $E_i$  remains bounded. The resulting jump conditions thus obtained consist of Eqs. (2.46)<sub>1-2</sub>, (2.47)<sub>1</sub>, (2.49) and (2.50) of Ref.3, which are required in this work and we reproduce here in the form

$$N_{K\sim} [\delta_{K\sim}] - U_{N\sim} [\bar{\mu}] = 0, \quad (3.9)$$

$$N_{K\sim} [S_{Kj\sim}] + \rho_o U_{N\sim} [v_{j\sim}] = 0, \quad (3.10)$$

$$[\varphi^e]_{\sim} = 0, \quad [\varphi]_{\sim} = 0, \quad (3.11)$$

$$N_{K\sim} [\delta_{K\sim}] = 0. \quad (3.12)$$

In view of (2.9)<sub>3</sub>, (3.11)<sub>1</sub> may be written

$$\hat{\varphi}^e(\mu^{e-}) - \hat{\varphi}^e(\mu^{e+}) = 0, \quad (3.13)$$

and if we assume that  $\partial\hat{\varphi}^e(\mu^e)/\partial\mu^e \neq 0$  it follows (the argument is given by Coleman and Gurtin<sup>6</sup>) that  $\mu^e$  is continuous across  $\Sigma$ , i.e.,

$$[\mu^e] = 0. \quad (3.14)$$

Furthermore, since (2.15) is of the form

$$\bar{\mu} = J\mu^e + \mu^r, \quad \mu^r = \text{constant} \quad (3.15)$$

it follows, since  $J = \det F_{iK}$  is continuous across  $\Sigma$ , that

$$[\bar{\mu}] = 0. \quad (3.16)$$

Next, since  $\varphi$  is continuous across  $\Sigma$ , from the geometric condition of compatibility<sup>14</sup> and (2.8)<sub>7</sub>, we have

$$[\delta_{K\sim}] = \bar{\alpha}N_K, \quad \bar{\alpha} = -[N_L\varphi, L\sim]. \quad (3.17)$$

Thus, in view of (2.23)<sub>2</sub>, we may write (3.12) in the form

$$N_K \hat{\varphi}_K(E_{RL}, \delta_M^+ + \bar{\alpha}N_M) - N_K \hat{\varphi}_K(E_{RL}, \delta_M^+) = 0, \quad (3.18)$$

and if we assume that  $\partial\hat{\varphi}_K/\partial\delta_L \neq 0$  it follows from (3.18) that  $\bar{\alpha} = 0$  and, hence, from (3.17) that

$$[\delta_{K\sim}] = 0, \quad (3.19)$$

and thus the reference electric field  $\delta$  is continuous across  $\Sigma$ . Now, in view of (3.14), from (2.17) and the geometric condition of compatibility we have

$$[G_{K\sim}] = \bar{\omega}N_K, \quad \bar{\omega} = [N_L\mu^e, L\sim], \quad (3.20)$$

so that, on using (2.16), (3.14), (3.16), (3.19) and (3.20), we may rewrite (3.9) in the form

$$N_K \hat{\varphi}_K(E_{RL}, \delta_M^+, G_L^+ + \bar{\omega}N_L, \mu^e) - N_K \hat{\varphi}_K(E_{RL}, \delta_M^+, G_L^+, \mu^e) = 0. \quad (3.21)$$

From the assumption that  $\partial \hat{f}_K / \partial G_L \neq 0$  it follows from (3.21) that  $\bar{\omega} = 0$  so that

$$[G_K] = 0, \quad (3.22)$$

which means that  $G_K$  is continuous across  $\Sigma$ . As a consequence of (2.16), (2.8)<sub>5-6</sub>, (2.23), (3.14), (3.19), (3.22) and the definition of an acceleration wave, we have

$$[\hat{S}_{Kj}] = [\hat{D}_{Kj}] = [\hat{g}_{Kj}] = 0, \quad (3.23)$$

and thus Eqs. (3.9) - (3.12) are satisfied identically. Other important conditions resulting from (3.19), (3.22) and the conditions of compatibility<sup>14-15</sup> are

$$\begin{aligned} [\delta_{K,L}] = -[\varphi_{,KL}] = -\alpha N_{KN}, [\dot{\delta}_{Kj}] = U_N \alpha N_K, \quad \alpha = [N_R N_S \varphi_{,RS}], \\ [G_{K,L}] = [\mu^e_{,KL}] = \omega N_{KN}, [\dot{G}_{Kj}] = -U_N \omega N_K, \quad \omega = [N_R N_S \mu^e_{,RS}]. \end{aligned} \quad (3.24)$$

From (2.10), (2.22), (2.7)<sub>3,5</sub>, (2.8)<sub>5-6</sub>, (2.9)<sub>1-2</sub> and (2.21), we obtain

$$A_{jKpL} F_{pL,K} + B_{jKL} \delta_{L,K} + H_{jK} G_K = \rho_0 \dot{v}_j, \quad (3.25)$$

where

$$\begin{aligned} A_{jKpL} = \frac{\partial \hat{S}_{Kj}}{\partial F_{pL}} = \rho_0 \frac{\partial^2 \hat{\chi}}{\partial F_{jK} \partial F_{pL}} - \frac{\partial}{\partial F_{pL}} (J X_{K,j}) \hat{p}^e(\mu^e) = \delta_{jp} S_{KL} + \\ + F_{jN}^F F_{pS}^C C_{KNLS} - J (X_{K,j} X_{L,p} - X_{L,j} X_{K,p}) \hat{p}^e(\mu^e), \end{aligned} \quad (3.26)$$

$$B_{jKL} = \frac{\partial \hat{S}_{Kj}}{\partial \delta_L} = \rho_0 \frac{\partial^2 \hat{\chi}}{\partial F_{jK} \partial \delta_L} = - \frac{\partial \hat{D}_{Lj}}{\partial F_{jK}} = F_{jN}^B B_{KNL}, \quad (3.27)$$

$$H_{jK} = \frac{\partial \hat{S}_{Kj}}{\partial \mu^e} = - J X_{K,j} \frac{\partial p^e}{\partial \mu^e}(\mu^e), \quad (3.28)$$

and

$$S_{KL} = \rho_0 \frac{\partial \hat{\chi}}{\partial E_{KL}}, \quad C_{KNLS} = \rho_0 \frac{\partial^2 \hat{\chi}}{\partial E_{KN} \partial E_{LS}}, \quad B_{KNL} = \rho_0 \frac{\partial^2 \hat{\chi}}{\partial \delta_L \partial E_{KN}}. \quad (3.29)$$

Since  $\hat{S}_{Kj}(\dots)$  is by hypothesis a  $C^2$ -function, it follows that the coefficients in (3.26) - (3.28) are continuous across the singular surface  $\Sigma$ .

Thus, taking the jump in Eq. (3.25) across  $\Sigma$  and employing (3.8)<sub>1</sub>, (3.22) and (3.24)<sub>1</sub> we obtain

$$A_{jKpL} N_K N_L s_p - \alpha B_{jKL} N_K N_L = \rho_0 U_N^2 s_j. \quad (3.30)$$

We now need the expression for  $\alpha$  in terms of  $s$ . From (2.11) and (3.27) we have

$$-B_{iLK} F_{iL,K} + \kappa_{KL} \delta_{L,K} = \bar{\mu}, \quad (3.31)$$

where

$$\kappa_{KL} = \frac{\partial \hat{\delta}_K}{\partial \delta_L} = -\rho_0 \frac{\partial^2 \hat{\chi}}{\partial \delta_K \partial \delta_L}. \quad (3.32)$$

Since  $\hat{\delta}_K(\dots)$  is a  $C^2$ -function, it follows that the coefficients in (3.31) are continuous across the singular surface  $\Sigma$ . Thus, on taking the jump in (3.31) across  $\Sigma$  and using (3.8)<sub>1</sub>, (3.16) and (3.24)<sub>1</sub>, we find

$$\alpha = -\zeta L_i s_i, \quad (3.33)$$

where

$$L_i = B_{iLK} N_L N_K = F_{iM} B_{MLK} N_L N_K, \quad \zeta = (\kappa_{KL} N_K N_L)^{-1}. \quad (3.34)$$

The substitution of (3.33) into (3.30) with the aid of (3.26) and (3.34)<sub>1</sub> yields the following propagation condition

$$(\hat{Q}_{jp} - \rho_0 U_N^2 \delta_{jp}) s_p = 0, \quad (3.35)$$

where

$$\begin{aligned} \hat{Q}_{jp} &= \hat{Q}_{jp}(F_{rL}, \delta_K, N_M) = A_{jKpL} N_K N_L + \zeta L_j L_p = \delta_{jp} S_{KL} N_K N_L + \\ &F_{jL}^F F_{pM}^C KLRM N_K N_R + \zeta L_j L_p = \hat{Q}_{jp} \end{aligned} \quad (3.36)$$

is the acoustic tensor. We note that  $\hat{Q}_{jp}$  is symmetric and for fixed  $n$  it is a function of the deformation gradient  $F_{rL}$ , the reference electric field  $\delta_K$  at the wavefront, but is independent of  $\mu^e$ , the density of free electronic charge at the wavefront. It follows from (3.35) that the amplitude  $a$  of

an acceleration wave traveling in the direction  $\underline{n}$  in a piezoelectric semiconductor must be a proper eigenvector of the symmetric acoustic tensor  $\hat{Q}_{jp}$  and the speed of propagation  $U_N$  must be such that  $\rho_0 U_N^2$  is the corresponding eigenvalue of  $\hat{Q}_{jp}$ .

The equation (3.35) was derived by Truesdell<sup>17</sup> for acceleration waves in elastic media. It has since been derived by a number of authors for acceleration waves in a variety of media<sup>7-9</sup>. We note in particular that the acoustic tensor (3.36) has precisely the same form as the corresponding acoustic tensor which occurs in the theory of wave propagation in elastic dielectrics<sup>10</sup>.

Equations (3.35) admit a nontrivial solution if and only if

$$\det (\hat{Q}_{jp} - \rho_0 U_N^2 \delta_{jp}) = 0, \quad (3.37)$$

and this equation determines the possible speeds of propagation for a given direction of propagation  $\underline{n}$ . On the other hand, if the amplitude  $\underline{a}$  of a wave is known, then the corresponding speed of propagation is determined by the formula

$$\rho_0 U_N^2 = \hat{Q}_{jp} a_j a_p / a_i a_i. \quad (3.38)$$

Since the acoustic tensor  $\hat{Q}_{jp}$  is symmetric, it has three real eigenvalues. However, at this stage it is possible that all of these eigenvalues may be negative in which case no real waves will exist at all. We now wish to record the conditions which guarantee the possible existence of at least some real waves. A detailed analysis of the situation for purely elastic materials has been given by Truesdell<sup>18</sup>, Truesdell and Noll<sup>19</sup>, Wang and Truesdell<sup>20</sup>, Chadwick and Currie<sup>21</sup>.

Once the deformation gradient and electric field ahead of the wave are known, it follows from (3.5)<sub>1</sub>, (3.29), (3.34), (3.36), (2.8)<sub>5-6</sub> and (2.9)<sub>1</sub>

that the acoustic tensor depends on  $\underline{n}$  only. That is, we have

$$\hat{Q}_{jp}(\underline{F}_{rL}, \underline{\phi}_K, N_M) = \bar{Q}_{jp}(\underline{n}), \quad (3.39)$$

for fixed  $\underline{F}_{rL}$  and  $\underline{\phi}_K$ . If

$$\bar{Q}_{ij}(\underline{n})n_i n_j > 0 \quad (3.40)$$

for all unit vectors  $\underline{n}$ , the material may be said to have positive longitudinal piezoelectricity. Truesdell<sup>18</sup> has shown that when (3.40) is satisfied there exists at least one direction in which a longitudinal wave may exist and propagate. If the acoustic tensor is strongly elliptic in the sense that

$$\bar{Q}_{ij}(\underline{n})\mu_i \mu_j > 0, \quad (3.41)$$

for all unit vectors  $\underline{n}$  and  $\underline{\mu}$ , it then follows (Truesdell<sup>18</sup>) that there is at least one direction of propagation in which a longitudinal wave and two transverse waves with orthogonal amplitudes may exist and propagate. In particular, it should be noted that if the strong ellipticity condition (3.41) is satisfied and if the deformation and electric field are uniform ahead of the wave then there exists at least one direction in which a plane longitudinal and two plane transverse waves may exist and propagate for all times. For propagation in all other directions under the above-mentioned circumstances the three plane waves are, of course, not necessarily either purely longitudinal or purely transverse, but may consist of an admixture of all mechanical displacement components. Nevertheless, in the most general case if the propagation velocities are distinct, the three plane waves have mutually orthogonal mechanical displacement fields. In the next section, we examine the manner in which the amplitudes of such waves vary as they traverse the material.

#### 4. Growth and Decay of Plane Acceleration Waves

In this section we derive the differential equation which determines the manner in which the amplitude of a plane acceleration wave varies as it traverses the material. It is assumed that the material ahead of the plane wavefront is at rest in a state of homogeneous strain, is subject to a uniform electric field and that the charge density of the free electronic fluid is uniform and constant prior to the arrival of the wavefront.

The differentiation of the equation of motion (3.25) with respect to  $t$ , with  $X$  fixed, yields

$$A_{jKpL} \dot{F}_{pL, K} + B_{jKL} \dot{\delta}_{L, K} + H_{jK} \dot{G}_K + C_j = \rho_0 \ddot{v}_j, \quad (4.1)$$

where

$$\begin{aligned} C_j = & A_{jKpLqM} \dot{F}_{pL, K} \dot{F}_{qM} + B_{jKpLM} \dot{F}_{pL, K} \dot{\delta}_M \\ & + H_{jKpL} \dot{F}_{pL, K} \dot{\mu}^e + B_{jKpML} \dot{\delta}_{L, K} \dot{F}_{pM} \\ & + B_{jKLM} \dot{\delta}_{L, K} \dot{\delta}_M + H_{jKpL} \dot{G}_K \dot{F}_{pL} \\ & + H_{jK} \dot{G}_K \dot{\mu}^e, \end{aligned} \quad (4.2)$$

with

$$\begin{aligned} A_{jKpLqM} = & \frac{\partial^2 \hat{S}_{Kj}}{\partial F_{pL} \partial F_{qM}} = \delta_{jp} F_{qN}^C KLMN + \delta_{jq} F_{pN}^C KMLN \\ & + \delta_{pq} F_{jN}^C KNLM + F_{jN}^F pT^F qR^C KNLTMR - JX_{M, q} (X_{K, j} X_{L, p} \\ & - X_{L, j} X_{K, p}) p^e + J (X_{K, q} X_{M, j} X_{L, p} + X_{K, j} X_{M, p} X_{L, q} \\ & - X_{L, q} X_{M, j} X_{K, p} - X_{L, j} X_{K, q} X_{M, p}) p^e, \end{aligned} \quad (4.3)$$

$$B_{jKpLM} = \frac{\partial^2 \hat{S}_{Kj}}{\partial F_{pL} \partial \delta_M} = \delta_{jp} B_{KLM} + F_{jN}^F pS^B KNLSM,$$

$$H_{jKpL} = \frac{\partial^2 \hat{S}_{Kj}}{\partial \mu^e \partial F_{pL}} = -J (X_{K, j} X_{L, p} - X_{L, j} X_{K, p}) \frac{\partial p^e}{\partial \mu^e},$$

$$B_{jKLM} = \frac{\partial^2 \hat{S}_{Kj}}{\partial \delta_L \partial \delta_M} = F_{jN}^B NKLM, \quad \tilde{H}_{jK} = -JX_{K, j} \frac{\partial^2 p^e}{\partial (\mu^e)^2},$$

and

$$\begin{aligned}
 C_{KLMNRT} &= \frac{\partial^2 \hat{S}_{KL}}{\partial E_{MN} \partial E_{RT}} = \rho_0 \frac{\partial^3 \hat{\chi}}{\partial E_{KL} \partial E_{MN} \partial E_{RT}}, \\
 B_{KNLSM} &= \frac{\partial^2 \hat{S}_{KN}}{\partial E_{LS} \partial \delta_M} = \rho_0 \frac{\partial^3 \hat{\chi}}{\partial E_{KN} \partial E_{LS} \partial \delta_M}, \\
 B_{NKLM} &= \frac{\partial^2 \hat{S}_{NK}}{\partial \delta_L \partial \delta_M} = \rho_0 \frac{\partial^3 \hat{\chi}}{\partial E_{NK} \partial \delta_L \partial \delta_M}.
 \end{aligned} \tag{4.4}$$

In view of our earlier assumptions on the continuity of the response functions  $\hat{S}_{KJ}(\dots)$ , the coefficients in (4.3) are continuous across the singular surface  $\Sigma$ . On taking the jump in (4.1) across  $\Sigma$ , keeping in mind our assumptions concerning the uniformity of the rest state of the material ahead of the wavefront and making use of the relation

$$[\varphi \psi] = \varphi^+ [\psi] + \psi^+ [\varphi] + [\varphi] [\psi], \tag{4.5}$$

and employing (3.22) we obtain

$$A_{jKpL} [\dot{F}_{pL, K}] + B_{jKL} [\dot{\delta}_{L, K}] + H_{jK} [\dot{G}_K] + [C_j] = \rho_0 [\ddot{v}_j], \tag{4.6}$$

where

$$\begin{aligned}
 [C_j] &= A_{jKpLqM} [F_{pL, K}] [\dot{F}_{qM}] + B_{jKpLM} [F_{pL, K}] [\dot{\delta}_M] \\
 &\quad + [\dot{F}_{pL}] [\delta_{M, K}] + B_{jKLM} [\delta_{L, K}] [\dot{\delta}_M].
 \end{aligned} \tag{4.7}$$

Since we are dealing with a plane wavefront entering a uniform state, from the compatibility conditions and the definition of an acceleration wave, we have<sup>22</sup>

$$\begin{aligned}
 [\dot{F}_{rL, K}] &= b_{rL, K} N_r N_K, \quad b_r = [N_r N_s \dot{F}_{rM, S}], \\
 [\ddot{v}_i] &= U_N^2 b_i + 2U_N^2 \frac{\delta s_i}{\delta t},
 \end{aligned} \tag{4.8}$$

$$[\dot{\delta}_{L, K}] = \beta N_L N_K, \quad \beta = - [N_r N_s \varphi_{, RS}],$$

where  $\delta f/\delta t$  is the displacement derivative<sup>23</sup> of  $f$ . Substituting from (3.8), (3.24) and (4.8) into (4.6), we obtain

$$\begin{aligned} & (A_{jKpL} N_K N_L - \rho_0 U_N^2 \delta_{jp}) b_p + \beta B_{jKL} N_K N_L - U_N \omega H_{jK} N_K - 2f_0 U_N^2 \frac{\delta s_j}{\delta t} \\ & - U_N A_{NjKpLqM} N_K N_L N_M N_P N_Q + 2U_N \alpha B_{NjKpLM} N_K N_L N_M N_P - U_N \alpha^2 B_{jKLM} N_K N_L N_M = 0. \end{aligned} \quad (4.9)$$

We now need the expression for  $\omega$  in terms of  $s$ . To this end, with the aid of (2.16) and (2.8)<sub>5-6</sub> we rewrite Eq. (2.13) in the form

$$\Sigma_{KrL} F_{rL,K} + \Sigma_{KL} \delta_{L,K} + \Omega_{KL} G_{L,K} + \epsilon_K G_K + \dot{\mu} = 0, \quad (4.10)$$

where

$$\begin{aligned} \Sigma_{KrL} &= \frac{\partial \hat{g}_K}{\partial F_{rL}} = F_{rN} \Sigma_{KLN}, \quad \Sigma_{KMN} = \frac{\partial \hat{g}_K}{\partial E_{MN}}, \quad \Sigma_{KL} = \frac{\partial \hat{g}_K}{\partial \delta_L}, \\ \Omega_{KL} &= \frac{\partial \hat{g}_K}{\partial G_L}, \quad \epsilon_K = \frac{\partial \hat{g}_K}{\partial \mu^e}. \end{aligned} \quad (4.11)$$

In view of the continuity assumptions on  $\hat{g}_K(\dots)$ , it follows that the coefficients in (4.11) are continuous across the singular surface. Hence, on taking the jump in (4.10) across  $\Sigma$  and employing (3.8), (3.22) and (3.24)<sub>1,4</sub>, we have

$$\omega = v \left( -\Sigma_{KrL} N_K N_L N_R + \alpha \Sigma_{KL} N_K N_L - [\dot{\mu}] \right) \quad (4.12)$$

where

$$v = (\Omega_{KL} N_L N_K)^{-1}. \quad (4.13)$$

Taking the material time derivative of (2.15), we obtain

$$\dot{\mu} = J\dot{\mu}^e + \dot{J}\mu^e, \quad (4.14)$$

the jump in which, with (2.8)<sub>1</sub>, (3.8)<sub>3</sub>, (3.14), (3.22), the kinematic condition of compatibility<sup>15</sup> and the definition of an acceleration wave, yields

$$[\dot{\mu}] = -\mu^e J X_{K,i} U_N s_i N_K, \quad (4.15)$$

where we have employed the well-known relation

$$\dot{J} = JX_{K,i} \dot{F}_{iK}, \quad (4.16)$$

in obtaining (4.15). The substitution of (3.33) and (4.15) into (4.12) enables us to write

$$\omega = v \Sigma_r s_r, \quad (4.17)$$

where

$$\Sigma_r = -\Sigma_{KrL} N_L N_K - (\Sigma_{KL} N_K N_L)_r + J U_N^\mu N_K X_{K,r}. \quad (4.18)$$

We now must express  $\beta$  in terms of  $\underline{s}$  and  $\underline{b}$ . To this end we take the material time derivative of (3.31) to obtain

$$\begin{aligned} & -B_{iLK} \dot{F}_{iL,K} + \kappa_{KL} \dot{\delta}_{L,K} - B_{iLrSK} \dot{F}_{rS} F_{iL,K} \\ & - B_{iLKM} (\dot{\delta}_M^F F_{iL,K} + \dot{F}_{iL} \delta_{M,K}) \\ & + \kappa_{KLM} \dot{\delta}_M^{\delta}_{L,K} = \dot{\mu} \end{aligned} \quad (4.19)$$

where

$$\kappa_{KLM} = \frac{\partial^2 \hat{\delta}_K}{\partial \delta_L \partial \delta_M} = -\rho_0 \frac{\partial^3 \hat{\chi}}{\partial \delta_K \partial \delta_L \partial \delta_M}, \quad (4.20)$$

which is continuous across the singular surface. Clearly, all the coefficients occurring in Eq. (4.19) are continuous across the singular surface  $\Sigma$ . Thus, on taking the jump in (4.19) across  $\Sigma$ , recalling the time independent uniform state ahead of the wavefront and employing (4.15) we have

$$\begin{aligned} & -B_{iLK} [\dot{F}_{iL,K}] + \kappa_{KL} [\dot{\delta}_{L,K}] - B_{iLrSK} [\dot{F}_{rS}] [F_{iL,K}] \\ & - B_{iLKM} ([\dot{\delta}_M^F] [F_{iL,K}] + [\dot{F}_{iL}] [\delta_{M,K}]) \\ & + \kappa_{KLM} [\dot{\delta}_M^{\delta}] [\delta_{L,K}] = [\dot{\mu}]. \end{aligned} \quad (4.21)$$

Substituting from (3.8), (3.24), (3.33), (3.34), (4.8) and (4.15) into (4.21), we obtain

$$\beta = \zeta_{L_p} b_p - \zeta_{U_N} \zeta_{\mu}^e X_{K,i} N_{K i}^s + U_N \zeta_{M_{ij}} s_i s_j, \quad (4.22)$$

where

$$M_{ij} = (-B_{iLjMK} + 2\zeta_{B_{iLKM}} L_j + \zeta_{\kappa_{LMN}}^2 L_i L_j) N_{K N} N_{L M}. \quad (4.23)$$

We now substitute from (3.33), (4.17) and (4.22) into (4.9) and employ (3.34)<sub>1</sub> and (3.36) to obtain

$$(\hat{Q}_{jp} - \rho_o U_N^2 \delta_{jp}) b_p = 2\rho_o U_N^2 \frac{\delta s_j}{\delta t} + \alpha_{jp} s_p + \alpha_{j pq} s_p s_q, \quad (4.24)$$

where

$$\alpha_{jp} = \zeta_{U_N} \zeta_{\mu}^e L_j X_{K,p} N_K + \nu U_N H_{jK} N_K \Sigma_p, \quad (4.25)$$

and

$$\alpha_{j pq} = U_N \zeta_{L_j M} p_q + U_N [A_{jKpLqM} + 2\zeta_{B_{jKpLM}} L_q + \zeta_{B_{jKLM}}^2 L_p L_q] N_{K N} N_{L M}. \quad (4.26)$$

Note that Eqs. (4.24) serve to determine the components of the amplitude  $\tilde{b}$  of the third order discontinuity induced by the acceleration wave. However, for the moment, our primary objective is to use Eqs. (4.24) to obtain the differential equation governing the evolutionary behavior of the amplitude  $\tilde{a}$  of the acceleration wave. In Sec.6 we study the solution of (4.24) in greater detail and we discuss both induced discontinuities and higher order waves.

Using Eqs. (3.5), (3.8)<sub>4-6</sub>, (4.23), (4.25) and (4.26), we may rewrite Eq. (4.24) in the form

$$(\hat{Q}_{jp} - \rho_o U_N^2 \delta_{jp}) b_p = 2\rho_o U^2 \frac{\delta a_j}{\delta t} + \bar{\alpha}_{jp} a_p + \bar{\alpha}_{j pq} a_p a_q, \quad (4.27)$$

where

$$\bar{\alpha}_{jp} = \zeta_{U_N} \zeta_{\mu}^e L_j n_p + \nu U_N H_{jK} F_{sK}^n s_p, \quad (4.28)$$

$$\bar{\alpha}_{j pq} = U \zeta_L \bar{M}_{j pq} + U [A_{j K p L q M} + 2B_{j K p L M} L_q + \zeta^2 B_{j K L M} L_p L_q] F_r K^r F_s L^s F_t M^t n_r n_s n_t, \quad (4.29)$$

and

$$\bar{M}_{p q} = [-B_{p K q L M} + 2\zeta B_{p K L M} L_q + \zeta^2 \kappa_{K L M}^2] F_r K^r F_s L^s F_t M^t n_r n_s n_t. \quad (4.30)$$

We may write

$$a_j = a r_j, \quad |r| = 1, \quad (4.31)$$

where  $\tilde{r}$  is the unit eigenvector of  $\hat{Q}_{j p}$  corresponding to the eigenvalue  $\rho_0 U_N^2$ , and recall that, since we are dealing with a plane wavefront propagating into a uniform region, the components of  $\tilde{r}$  are constant. Equation (4.27) may now be written

$$(\hat{Q}_{j p} - \rho_0 U_N^2 \delta_{j p}) b_p = 2\rho_0 U^2 r_j \frac{\delta a}{\delta t} + (\bar{\alpha}_{j p} r_p) a + (\bar{\alpha}_{j p q} r_p r_q) a^2. \quad (4.32)$$

If we now contract (4.32) with  $r_j$  and employ (3.35) and (3.36), we find that the amplitude  $a$  of the acceleration wave satisfies the equation

$$\frac{\delta a}{\delta t} = -\omega_0 a + \beta_0 a^2, \quad (4.33)$$

where

$$\omega_0 = \frac{\bar{\alpha}_{j p} r_j r_p}{2\rho_0 U^2}, \quad (4.34)$$

$$\beta_0 = \frac{-\bar{\alpha}_{j p q} r_j r_p r_q}{2\rho_0 U^2}. \quad (4.35)$$

The implications of Eq. (4.33) are examined in detail in the next section.

### 5. Implications of the Growth Equation

Equation (4.34), which governs the evolutionary behavior of plane acceleration waves oriented arbitrarily and propagating into a region subject to a time independent state of homogeneous strain and uniform electric field in a deformable semiconductor, is of the same form as Eq. (5.6) of Ref.1. Clearly, we expect the behavior of the amplitude of a plane acceleration wave to be, at least qualitatively, similar to that of a purely longitudinal wave. This is for the most part the case but, as expected, in contrast to the situation prevailing in the one-dimensional case discussed in Ref.1, the coefficients  $\omega_0$  and  $\beta_0$ , defined by (4.34) and (4.35), respectively, are not absolute constants for a given material and state even though  $\underline{\xi}$ ,  $\underline{F}$  and  $\mu^e$  are uniform ahead of the wavefront, but vary with the propagation direction  $\underline{n}$ . Of course, once  $\underline{\xi}$ ,  $\underline{F}$  and  $\mu^e$  are prescribed ahead of the wave then for a given  $\underline{n}$ ,  $\underline{x}$  is determined by  $\underline{n}$  through Eqs. (3.35). After the unit vector  $\underline{x}$  has been determined from Eqs. (3.35), the coefficients  $\omega_0$  and  $\beta_0$  are fixed. Thus, for a given state ahead of the wavefront, i.e., values of  $\underline{F}$ ,  $\underline{\xi}$  and  $\mu^e$ ,  $\omega_0$  and  $\beta_0$  are constants for a given  $\underline{n}$ .

When neither of the quantities  $\omega_0, \beta_0$  vanishes then the solution of Eq. (4.33) is

$$a(t) = \lambda_0 / [(\lambda_0/a_0) - 1] e^{\omega_0 t} + 1, \quad (5.1)$$

where

$$\lambda_0 = \omega_0 / \beta_0, \quad (5.2)$$

and  $a(0)$  is the value of the amplitude of the wave at time  $t=0$ . It is clear that the behavior of the amplitude of a given plane wave is determined by the coefficients  $\omega_0$  and  $\beta_0$  as well as by the initial amplitude  $a_0$ . In order to discuss all possible cases which may arise we first suppose that

$$\omega_0 > 0 \text{ and } \beta_0 \neq 0, \quad (5.4)$$

then, in view of (5.2), we have

$$\text{sgn } \lambda_0 = \text{sgn } \beta_0, \quad (5.4)$$

and then from (5.1) three distinct possibilities arise:

- i) If  $\text{sgn } a(0) = \pm \text{sgn } \beta_0$  and  $|a_0| < |\lambda_0|$ , then  $a(t) \rightarrow 0$  monotonically as  $t \rightarrow \infty$ .
- ii) If  $a(0) = \lambda_0$ , then  $a(t) = a(0)$ .
- iii) If  $\text{sgn } a(0) = \text{sgn } \beta_0$  and  $|a(0)| > |\lambda_0|$ , then  $a(t) \rightarrow \infty$  monotonically within a finite time  $t_\infty$  given by

$$t_\infty = - (1/\omega_0) \ln [1 - (\lambda_0/a(0))]. \quad (5.5)$$

We now suppose that

$$\omega_0 < 0 \text{ and } \beta_0 \neq 0, \quad (5.6)$$

then we have

$$\text{sgn } \lambda_0 = - \text{sgn } \beta_0, \quad (5.7)$$

and again from (5.1) three distinct possibilities arise:

- i)\* If  $\text{sgn } a(0) = - \text{sgn } \beta_0$  and  $|\lambda_0| > |a(0)|$ , then  $a(t) \rightarrow \lambda_0$  monotonically as  $t \rightarrow \infty$ .
- ii)\* If  $a(0) = \lambda_0$ , then  $a(t) = a(0)$ .
- iii)\* If  $\text{sgn } a(0) = \text{sgn } \beta_0$ , then  $a(t) \rightarrow \infty$  within a finite time  $\bar{t}_\infty$  given by

$$\bar{t}_\infty = - (1/\omega_0) \ln [1 + |\lambda_0/a_0|]. \quad (5.8)$$

It is clear from the foregoing results that the number  $\lambda_0$  plays a fundamental role in determining whether the amplitude of an acceleration wave will grow or decay as the wave traverses the material. For this reason we follow the usual custom and call  $\lambda_0$  the critical amplitude for acceleration waves encountering a homogeneous steady state. We note that if  $\omega_0 > 0$ ,

then the behavior of an acceleration wave propagating into a piezoelectric semiconductor which is in a uniform steady state is precisely the same as that of an acceleration wave propagating into a homogeneously deformed material with memory<sup>6,7</sup>. In particular, if the initial amplitude of the wave is less in absolute value than the critical initial amplitude, the amplitude decreases to zero as the wave propagates. On the other hand, if the initial amplitude is greater in absolute value than the critical initial amplitude, the amplitude of the wave becomes unbounded in a finite time. This, of course, suggests the formation of a shock. As noted in Ref.1, the case  $\omega_0 < 0$ , which has no mechanical analogue, is the case of primary interest and importance. The foregoing analytical treatment shows that in this case the amplitude of the wave either tends to  $\lambda_0$  eventually or else becomes unbounded in a finite time. Furthermore, note that in this case if  $a(0)$  and  $\beta_0$  have the same sign, the amplitude of the wave always becomes unbounded in a finite time. Moreover, since  $a(0)$  arises from the thermal noise, there are always some  $a(0)$  with the same sign as  $\beta_0$ .

Let us now consider the behavior of a wave for which  $\beta_0$  vanishes. In our earlier treatment of one-dimensional acceleration waves<sup>1</sup> we noted that the vanishing of  $\beta_0$  corresponded to a linear material. Indeed, while  $\beta_0$  also vanishes identically here if the response of the material is linear, it may also vanish because of a combination of other factors even though the response of the material is nonlinear. For example, in a given material, once  $\underline{\xi}$  and  $\underline{F}$  are prescribed it may be possible to choose  $\underline{n}$  [and hence  $\underline{r}$  through Eq. (3.35)] in such a way that  $\beta_0$  vanishes. In particular it is easily verified in the relatively simple case of a purely transverse acceleration wave propagating in the direction of the applied electric field and of a principal axis of homogeneous deformation in an isotropic material that  $\beta_0$  vanishes even though

the response of the material is nonlinear. In this simplest but extremely important case Eq. (4.33) has the solution

$$a(t) = a_0 e^{-\omega_0 t}, \quad (5.9)$$

which means that if  $\omega_0 > 0$ ,  $a(t)$  is a monotonically decreasing function of time and the amplitude of the wave decreases as the wave traverses the material. On the other hand, if  $\omega_0 < 0$ ,  $a(t)$  is a monotonically increasing function of  $t$  and the amplitude of the wave increases without bound as the wave traverses the material. Of course, if  $\omega_0 = 0$ , then  $a(t) = a_0$  so that the wave propagates at constant amplitude.

Let us recall from Eqs. (3.28), (4.28) and (4.34) that

$$\omega_0 = \frac{1}{2\rho U} \left[ \rho \mu^e L_i r_i - v \frac{\partial p^e}{\partial \mu^e} \Sigma_i r_i \right] n_j r_j. \quad (5.10)$$

In particular, Eq. (5.10) shows that  $\omega_0$  vanishes whenever  $\underline{r}$  is orthogonal to  $\underline{n}$ . Thus, if  $\beta_0$  vanishes, either because the response of the material is linear or because  $\underline{\delta}$ ,  $\underline{F}$  and  $\underline{n}$  have appropriate values, then purely transverse acceleration waves will propagate at constant amplitude and not grow and, of course, purely transverse shocks will not form. On the other hand, in this very special case of purely transverse acceleration waves, suppose that  $\beta_0$  does not vanish for plane wave propagation in a prescribed direction  $\underline{n}$ , then Eq. (4.33) reduces to

$$\frac{\delta a}{\delta t} = \beta_0 a^2, \quad (5.11)$$

so that

$$a = a_0 [1 - \beta_0 a_0 t]^{-1}. \quad (5.12)$$

Of course, the solution (5.12) has the same form as the corresponding solution for acceleration waves in nonheat conducting elastic media. Note that if  $a_0$  and  $\beta_0$  have the same sign then a shock will form after a time

$$\bar{t}_\infty = 1/\beta_0 a_0, \quad (5.13)$$

and, as already noted, since  $a(0)$  arises from the thermal noise, a shock will always form for nonzero  $\beta_0$  when  $\omega_0$  is zero.

When  $\beta_0$  vanishes and the acceleration wave is not purely transverse, it should be clear from the above discussion that the threshold condition, at which the amplitude  $a(t)$  just begins to grow, may be defined by

$$\hat{\omega}_0 = 0 \quad (5.14)$$

where

$$\hat{\omega}_0 = \frac{1}{2\rho U} \left[ \zeta \mu^e L_i r_i - v \frac{\partial p^e}{\partial \mu^e} \Sigma_i r_i \right]. \quad (5.15)$$

A most important limiting form of (2.16) [or (2.7)<sub>6</sub> with (2.12), (2.8)<sub>7</sub>, (2.9)<sub>3</sub> and (2.7)<sub>5</sub>] is

$$j_K = - J \mu^e m_{KL} \delta_L - J D_{KL}^C G_L, \quad (5.16)$$

where  $m_{KL}$  is the mobility tensor and  $D_{KL}^C$  is the diffusivity tensor, which may be written in the form

$$D_{KL}^C = - m_{KL} \partial p^e / \partial \mu^e, \quad (5.17)$$

and, of course, we can have  $m_{KL} = m_{KL}(E_{KL}, \delta_L)$ . In this simple but important limiting case, in which the current is given by Eq. (5.16) with the mobility tensor  $m_{KL}$  and diffusivity tensor  $D_{KL}^C$  constant, from (5.15), (4.18), (4.11), (4.13), (5.16), (2.21), (5.17) and the fact that  $G_K$  vanishes ahead of the wave front we obtain

$$\hat{\omega}_0 = - \frac{1}{2\rho U} v \frac{\partial p^e}{\partial \mu^e} \mu^e N_K J_{K,i} r_i [N_L m_{LR} \delta_R + U_N], \quad (5.18)$$

from which, with (5.14), we find that the threshold relation is given by

$$N_L m_{LR} \delta_R + U_N = 0. \quad (5.19)$$

When the deformation is infinitesimal, we have

$$F_{rL} \approx \delta_{rL}, \quad \delta_R \approx \delta_{Rj} E_j = E_R, \quad U_N \approx U = U_0, \quad (5.20)$$

which enables us to write (5.19) in the form

$$N_L m_{LR} \delta_R^T = N_L m_{LR} E_R^T = -U_0, \quad (5.21)$$

where it should be recalled that  $\underline{N}$  (or  $\underline{n}$ ) is the normal to the plane wave surface. Equation (5.21) is the generalization of the well-known relation for the threshold field obtained in Eq. (5.13) of Ref. 1 for the one-dimensional case to the arbitrarily anisotropic three-dimensional case treated here for the restricted limiting form (5.16) of the current equation (2.16).

#### 6. Weak Waves and Induced Discontinuities

Following Coleman and Gurtin<sup>24</sup>, we define a wave of order  $N$  as follows:

A propagating singular surface  $\Sigma$  is a wave of order  $N$  if the field  $\underline{y}(\underline{X}, t)$  and its first  $N-1$  partial derivatives with respect to  $\underline{X}$  and  $t$  are continuous everywhere but the  $N$ th order partial derivatives suffer jump discontinuities at  $\Sigma$ , but are continuous functions everywhere else.

In particular, we note the case  $N=2$  represents an acceleration wave. If  $N > 2$ , the wave is said to be a weak wave.

Our object here is to study the propagation and growth of weak waves in a piezoelectric semiconducting material. It suffices to consider waves of order 3. We confine our attention to the study of plane waves and we assume that the material ahead of the wavefront is in a state of homogeneous strain, is subject to a uniform electric field and that the charge density of the free electronic fluid is uniform and constant prior to the arrival of the wavefront.

Since we are dealing with plane waves of order 3, from the compatibility conditions across  $\Sigma$  we have

$$\begin{aligned}
[\ddot{F}_{rL, K}] &= [\dot{v}_{r, LK}] = b_r N_L N_K, \\
[\ddot{F}_{rL, KM}] &= -U_N^{-1} b_r N_L N_K N_M, \\
[\ddot{F}_{rL}] &= [\dot{v}_{r, L}] = -U_N b_r N_L, \\
[\ddot{v}_{r}] &= U_N^2 b_r
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
[\ddot{F}_{rK, L}] &= [\dot{v}_{r, KL}] = d_r N_K N_L, \quad d_r = [N_M N_S \ddot{\phi}_{rS, M}], \\
[\ddot{v}_{r}] &= U_N^2 d_r + 2U_N^2 \frac{\delta b_r}{\delta t}.
\end{aligned}$$

It follows from (2.11) and (3.16) in essentially the same way that (3.19) followed from (3.18) that  $\delta_{L, K}$  is continuous across  $\Sigma$  and since  $\delta_L$  is continuous across  $\Sigma$  also, from the kinematic condition of compatibility  $\dot{\delta}_L$  is continuous across  $\Sigma$ . From the geometric condition of compatibility, we have

$$[\dot{\delta}_{L, K}] = \beta N_L N_K, \quad [\ddot{\delta}_{L, K}] = \bar{\beta} N_L N_K, \quad \bar{\beta} = -[N_R N_S \ddot{\phi}_{RS}]. \tag{6.2}$$

In a similar manner (2.13), (3.8)<sub>2</sub> and (4.15) imply that  $G_{K, L}$  is continuous across  $\Sigma$  and since  $G_K$  is continuous across  $\Sigma$  also, from the kinematic condition of compatibility  $\dot{G}_K$  is continuous across  $\Sigma$ , but from the geometric condition of compatibility and (2.17), we have

$$[\dot{G}_{K}] = \bar{\omega} N_K, \quad \bar{\omega} = [N_R \dot{G}_R]. \tag{6.3}$$

Since for weak waves  $a_j$  vanishes, an immediate consequence of (4.28) is that

$$(\hat{Q}_{jp} - \rho_0 U_N^2 \delta_{jp}) b_p = 0, \tag{6.4}$$

so that the propagation condition for weak waves is precisely the same as that which governs the propagation of acceleration waves.

In order to obtain the differential equation which governs the evolutionary behavior of the amplitudes of weak waves, we differentiate Eq. (4.1)

with respect to  $t$  holding  $\tilde{x}$  fixed, and on taking the jump across  $\Sigma$  in the resulting equation and employing (3.34), (4.5), (6.1)<sub>2</sub>, (6.3) and recalling the steady uniform state ahead of the wavefront, we obtain

$$(A_{jKpL} N_K N_L - \rho_o U_N^2 \delta_{jp}) d_p + \bar{\beta} L_j + \bar{\omega} H_{jK} N_K - 2\rho_o U_N^2 \frac{\delta b_j}{\delta t} = 0. \quad (6.5)$$

Similarly, taking the material time derivatives of (4.10) and (4.19), respectively, taking the jumps across  $\Sigma$  in the resulting equations and employing (2.21)<sub>1</sub>, (3.34), (4.5), (4.8), (4.13), (4.14), (4.22), (6.1) - (6.3) and the compatibility conditions and recalling the steady uniform state ahead of the wavefront, we find that

$$\bar{\beta} = \zeta L_i d_i - \zeta U_N J_{\mu}^e X_{K,i} N_K b_i, \quad \bar{\omega} = -U_N v \Sigma_i b_i. \quad (6.6)$$

Substituting from (6.6) into (6.5) and employing (3.36) and (4.25), we obtain

$$(\hat{Q}_{jp} - \rho_o U_N^2 \delta_{jp}) d_p = \alpha_{jp} b_p + 2\rho_o U_N^2 \frac{\delta b_p}{\delta t}. \quad (6.7)$$

Let us write

$$b_j = U_N^2 c_j, \quad c_j = c r_j, \quad |\tilde{r}| = 1, \quad (6.8)$$

where  $\tilde{r}$  is the unit eigenvector of  $Q_{jp}$  corresponding to the eigenvalue  $\rho_o U_N^2$ . If we now contract (6.7) with  $r_j$  and employ (3.5), (3.8)<sub>6</sub>, (3.35), (3.36), (4.28) and (4.34) we find that the amplitude  $c$  of the third-order wave satisfies the equation

$$\frac{\delta c}{\delta t} = -\omega_o c, \quad (6.9)$$

which admits the solution

$$c(t) = c_o e^{-\omega_o t}. \quad (6.10)$$

It is now clear that the evolutionary behavior of the amplitude of a weak plane wave is somewhat different from that of a plane acceleration wave which propagates in the same direction at the same speed. In particular,

the evolutionary behavior of a weak wave is determined solely by the sign of  $\omega_0$  and is independent of the initial value of the amplitude of the wave. As we have noted earlier,  $\omega_0$  may be negative in certain important circumstances. When  $\omega_0$  is negative the amplitude of the weak wave will increase without bound and become an acceleration wave as the wave traverses the material. This behavior should be contrasted with the manner in which weak waves behave in other media (see e.g., Ref.6). However, if the weak wave is purely transverse, the amplitude will remain constant as the wave traverses the material in accordance with the relevant portion of the discussion in Sec.5.

Let us now suppose that there exists a particular direction  $\underline{n}$  in which three real plane acceleration waves may propagate. Let us denote the amplitudes, unit amplitude vectors and speeds of propagation of these waves by  $a^{(i)}$ ,  $\underline{r}^{(i)}$  and  $U_N^{(i)}$ ,  $i=1,2,3$ , respectively. It follows from Eq. (4.32) that the amplitude  $\underline{b}^{(i)}$  of the third order discontinuity induced by the acceleration wave of amplitude  $a^{(1)}$  is determined by the equations

$$(\hat{Q}_{jp} - \rho_0 U_N^{(1)2} \delta_{jp}) b_p^{(1)} = d_j^{(1)}, \quad (6.11)$$

with

$$d_j^{(1)} = 2\rho_0 U^{(1)2} r_j^{(1)} \frac{\delta a^{(1)}}{\delta t} + \bar{\alpha}_{jp}^{(1)} r_p^{(1)} a^{(1)} + \bar{\alpha}_{jppq}^{(1)} r_p^{(1)} r_q^{(1)} a^{(1)2}, \quad (6.12)$$

where  $\bar{\alpha}_{jp}^{(1)}$  and  $\bar{\alpha}_{jppq}^{(1)}$  are given by (4.28) and (4.29) with  $U$  replaced by  $U^{(1)} = U_N^{(1)}$ .

Now suppose that the eigenvectors of  $\hat{Q}_{ij}(\underline{n})$  are distinct so that the vectors  $\underline{r}^{(i)}$  form an orthogonal triad. We may now write  $\underline{b}^{(1)}$  in the form

$$\underline{b}^{(1)} = \sum_{\alpha=1}^3 b_{\alpha}^{(1)} \underline{r}^{(\alpha)}. \quad (6.13)$$

Since as a consequence of (4.34)  $d_j^{(1)}$  is orthogonal to  $r_j^{(1)}$ , even though  $U_N^{(1)2}$  is a root of Eq. (3.37) Eqs. (6.11) are consistent but do not determine the component of  $\tilde{b}^{(1)}$  in the direction of  $\tilde{r}^{(1)}$  uniquely. Nevertheless, when Eqs. (6.11) are contracted successively with  $r_j^{(2)}$  and  $r_j^{(3)}$ , respectively, and (3.37) is employed, it follows that

$$\begin{aligned} \rho_0 (U_N^{(2)2} - U_N^{(1)2}) b_2^{(1)} &= r_j^{(2)} d_j^{(1)}, \\ \rho_0 (U_N^{(3)2} - U_N^{(1)2}) b_3^{(1)} &= r_j^{(3)} d_j^{(1)}. \end{aligned} \quad (6.14)$$

Of course Eqs. (6.14) determine uniquely the components of the induced discontinuity in the two mutually orthogonal directions which are also orthogonal to the direction of the amplitude vector of the primary acceleration wave.

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