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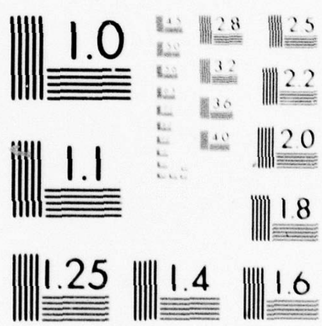

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COMPUTER TECHNIQUES FOR PLASTICITY

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Implicit in the development of any plasticity model are assumptions associated with the behavior of the actual material. Assumptions generally employed are listed with a discussion of their implications. Several models that incorporate these assumptions or attempt to overcome their restrictions are presented. Several alternative algorithms have evolved and are currently being used in finite element programs for plastic analysis. Each of these has computational advantages and liabilities so that the analyst is presented with choices

amont acceptable procedures. These procedures are reviewed in order to facilitate a rational selection among competing procedures that best suit the needs of a particular problem class. The contents of this memorandum will appear as a chapter in the ASME Pressure Vessel and Piping Division Vol. I - Decade of Progress in Design and Analysis (1980).

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COMPUTER TECHNIQUES FOR PLASTICITY

H. Armen* and A. Pifko+

INTRODUCTION

Advances in fields of scientific endeavor are typically made in response to new performance requirements. This certainly has been the case in the area of computational plasticity where there have been increasing demands from government and industrial organizations for analytically determining accurate stress, strain, and displacement fields in a wide range of complex structures subjected to severe environmental and mechanical loading conditions. These considerations have been a motivating force behind the development of general purpose programs for the plastic analysis of structures. The "Decade of Progress" in computational plasticity is represented by the general availability and acceptance of these multi-purpose computer programs for nonlinear structural analysis. This achievement has been the direct result of advances in the areas of structural mechanics and computer sciences. Specifically, the burgeoning developments in finite element methods, made possible by advances in computing hardware and software, formed the foundation from which efficient and accurate algorithms for plastic analysis could be developed.

Several alternative algorithms have evolved and are currently being used in programs for plastic analysis. Each of these has computational advantages and liabilities so that the analyst is presented with choices among acceptable procedures.

These procedures are reviewed to facilitate a rational selection among competing procedures that best suit the needs of a particular problem class.

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The development of algorithms within the framework of finite element computational techniques for plastic analysis was accomplished independently from the development of appropriate constitutive relations. Initially, only the simplest plasticity theories were used. With the availability of a computational tool that could treat problems of a more complex nature than previously practical (within constraints of computer time) analysts could turn their attention towards the development of plasticity theories that could more accurately predict the essential features of experimentally observed behavior without the penalties of introducing unrealistic simplifications.

Implicit in the development of any plasticity model are assumptions associated with the behavior of the material. The number and degree of these assumptions effect the generality of the resulting model and its compatibility with actual material behavior. Some assumptions generally employed are listed with a discussion of their implications. This is followed by a synoptic presentation of several models that incorporate these assumptions or attempt to overcome their restrictions.

The models discussed have either been incorporated into available general purpose plastic analysis programs or have been proposed for possible implementation.

ASSUMPTIONS OF PLASTIC BEHAVIOR

1. The existence of an initial yield condition defining the elastic limit of the material in a multiaxial stress state. This assumption is most often used and represents a mathematical convenience that facilitates computational efficiency. Its applicability varies with the material under consideration.

The two popular yield criteria for structural materials are those attributed to von Mises and Tresca. The former implies that yielding begins at an arbitrary material point whenever the combination of stresses is such that the strain energy of distortion per unit volume at the point is equal to the corresponding energy developed in a bar uniaxially stressed to the elastic limit.

The Tresca condition states that inelastic action at any point in a body begins only when the maximum shearing stress on some plane through the point reaches a value equal to the maximum shearing stress in a tension specimen.

2. The existence of loading surfaces that define the limits of elastic and plastic behavior beyond initial yielding --- the response after initial yielding differs among various plasticity theories. This post yielding response, called the hardening rule, is described by specifying a subsequent yield surface, termed the "loading surface," which is a convenient mathematical idealization of some macroscopically observed behavior. The "consistency" condition requires that the stress state at any point remain on the loading surface.

3. Plastic strain rates are linearly related to their corresponding stress rates by means of a flow rule (the basis of constitutive relations in the treatment of plasticity). A flow rule that is generally used to describe elastic-ideally plastic behavior is the Prandtl-Reuss relation, which is a generalization of the Levy-Mises equations. The Prandtl-Reuss assumption is that the plastic strain increments, $d\varepsilon_{ij}^{p}$, is proportional to the corresponding stress deviator, σ_{ij}, where the instantaneous non-negative value of the constant of proportionality is left to the inventiveness of the user of these equations. The concept of an effective stress or effective plastic strain (generally in terms of one or a combination of their corresponding invariants) is in itself an assumption that is usually introduced to reduce the complexity of a multiaxial situation to one that can be related to uniaxial behavior [1]. Thus, the proportionality parameter can be the ratio of the effective plastic strain increment to the prevailing effective stress.

A more general approach to determining a flow rule is the use of the concept of a plastic potential. The assumption is made that there exists a scalar function of stress, say $f(\sigma_{ij})$, from which the component of plastic strain increments may be determined (e.g., proportional to $\partial f/\partial\sigma_{ij}$). If $f(\sigma_{ij})$ represents the yield surface in stress space, then the above assumption represents a result of Drucker's postulate [2], which states that the work done by an external agency during a complete cycle of loading and unloading must be non-negative. Furthermore, this assumption leads to an incremental, or associated linear flow theory of plasticity, in which the increment of plastic strain (strain rate) is in the direction of the outward normal to the surface represented by $f(\sigma_{ij})$ in stress space, at the current value of stress. A strain rate vector deviating from the outward normal to the yield surface in a direction independent of the stress rate vector constitutes non-associated flow theories of plasticity. Non-associated flow theories are particularly suitable for work-softening materials and can reasonably fit the behavior observed in some soils. Associated and non-associated flow theories are, in general, distinct from the deformation theory of plasticity, in which the total plastic strains are related to the final stress state. According to this latter theory, a relationship between final states of stress and strain exists for any given loading process --- unloading being specified by a separate law.

4. Materials are isotropic with respect to initial yielding --- whether induced by previous cold-working or as a result of an anisotropic distribution of crystals, most structural materials exhibit some form of initial anisotropy. The von Mises and Tresca criteria assume isotropy with respect to the orientation of the stresses and their sense (tensile or compressive).

5. Plastic incompressibility --- incompressibility assumptions are generally employed in most nonlinear structural analysis programs. Investigations of the influence of hydrostatic pressure on the plastic response of metals have been considered by Bridgman [3]. Despite some evidence to the contrary [4], most investigations suggest that hydrostatic pressure has little or no effect on the initial and subsequent behavior of metals under quasi-static loading rates. This has been most recently confirmed by Fung, et al. [5].

The use of plastic incompressibility requires treating a variable Poisson ratio, i.e. an elastic value, and a value equal to one-half in the plastic range. The implications of this are discussed in assumption 8.

The treatment of the plastic yielding of nonmetals, such as clay, ice, and concrete, represents an area of investigation in which the effects of hydrostatic pressure are significant. The plasticity theories employed for the treatment of these materials are non-associated linear flow types.

6. Yielding and subsequent response are insensitive to rate of deformation --- the nature of the constitutive equations of plasticity for rate-sensitive material behavior has been the subject of several notable investigations (see survey paper of [6]). These investigations have shown that limits on dislocation velocity and rate of dislocation density for most structural materials are responsible for the generally observed fact that plastic flow is retarded with increasing strain rate.

7. Elastic unloading and coincidence of yield and loading surfaces --- this is an idealization of material behavior, that implies that unloading from some plastic state to neighboring state results in a change in the elastic state only. Furthermore, implicit in most analyses are the assumptions that additional plastic strains can occur only upon reloading to a stress state beyond that of initial unloading and subsequent behavior (assuming no reverse yielding has occurred) is identical to that which would have been obtained had unloading never occurred.

8. The total kinematic strain may be decomposed into elastic and plastic components --- this assumption is employed in the linear flow theories of plasticity and is used in developing many of the incremental constitutive equations. The assumption is mathematically convenient when used in conjunction with the concept of a well-defined yield surface and is generally valid for small deformations. In addition, this assumption facilitates the use of a variable Poisson ratio when plastic incompressibility is employed.

In addition to the above assumptions, most successful applications of classical small strain flow theories of plasticity have been generally limited to situations in which the loading is monotonically increasing and the ratio of the various load components are held constant (proportional loading). Applications of flow theory to problems involving severe changes in stress ratio among the various components of stress, such as occurs during plastic buckling, have been shown to be unconservative because the reduction in the in-plane shear stiffness is generally underestimated. Hutchinson has discussed this problem extensively in [7]. Deformation theory, on the other hand, has been shown to provide consistently better correlation with results for bifurcation problems for plates and shells, [8, 9]. This situation illustrates a significant fact that, with few exceptions, appears to have been overlooked in many experimental investigations. While the size, shape, and definition of yield and subsequent loading surfaces are of considerable importance, equal emphasis should be given to a proper description of the plastic hardening modulus under a variety of conditions.

Cyclic loading situations involving reversed plastic flow during which the material response may exhibit cyclic hardening or softening behavior falls into the category of problems of general interest in which classical plasticity theories have met with limited success. Applications of cyclic loading conditions to several problems ranging from membrane stressed sheets to plates and shells have been presented in [10, 11]. In these studies, "stabilized" material behavior was assumed.

Finally, most applications of plastic analysis have been limited to isothermal conditions. The influence of elevated temperatures on material response (elastic and inelastic) is generally treated within the framework of theories of viscous strain (creep). For a great many problem areas, ranging from metal forming processes to the analysis of nuclear reactor components, elevated stress and temperature levels of short duration must be tolerated. In these problems plastic strain development may be quite significant. Thus, non-isothermal plasticity theory is required. Furthermore, it is desirable that such a theory include the effects of temperature on the elastic, as well as the plastic properties of the materials.

MODELS OF PLASTIC BEHAVIOR

The following is a brief description of some models for plastic behavior. Several of these models have been developed within the framework of the assumptions mentioned; others have been developed specifically to overcome one or more of the restrictions associated with these assumptions. The primary function of the models is to provide a set of constitutive relations that can be used to describe the response history of nonlinearly deforming media.

ISOTROPIC HARDENING

This theory, proposed by Hill [12] and Hodge [13], assumes that during plastic flow the loading surface expands uniformly about the origin in stress space, maintaining the same shape, center, and orientation as the yield surface. Figure 1 illustrates, on the basis of a simplification to a two-dimensional plot, the yield and loading surfaces when the stress state shifts from point 1 to 2. Unloading and subsequent reloading in the reverse direction will result in yielding at the stress state represented by point 3. The path 2-3 will be elastic, and 0-2 is equal to 0-3.

Isotropic representation of work hardening does not account for the Bauschinger effect exhibited by most structural materials. In fact, contrary to observations, this theory implies that, because of work hardening, the material will exhibit an increase in the compressive yield stress equal to the increase in the tensile yield stress. Furthermore, since plastic deformation is an anisotropic process, it cannot be expected that a theory that predicts isotropy in the plastic range will lead to realistic results when complex loading paths, involving changes in direction of the stress vector in stress space (not necessarily completely reversed), are considered.

SLIP THEORY

Utilizing the physical concept of slip surfaces in crystals, Batdorf and Budiansky [14] have developed a theory that describes a loading surface that is distorted relative to the yield surface and previous loading surfaces. This theory predicts the formation of corners at the instantaneous stress state on the loading surface during plastic deformation. After some prestrain the yield locus is the minimum surface through the point of prestrain and the initial yield locus. A representation of the growth of the yield function in going from a stress state at the origin 0 to the final state represented by point 3 is given in Fig. 2. In this figure, the unshaded region is that enclosed by the yield surface, and the various shaded regions indicate the stages (I, II and III) in the formation of the loading surfaces in going from stress state 0 to 3.

Since the stress state is almost always in a corner, the resulting constitutive relation between stresses and strains becomes quite complex. For this reason this theory is rarely selected for application.

PIECEWISE LINEAR PLASTICITY

In this representation, the yield surface consists of a finite number of plane surfaces whose intersections constitute corners. The oldest and most widely used piecewise linear yield surface is that associated with the Tresca yield condition. The loading surface is assumed also to consist of plane surfaces, and the subsequent hardening behavior can be classified as:

1. The hardening rule of independent plane loading surface --- one of the earliest discussions of this representation of the hardening behavior is given in [15] and is illustrated in Fig. 3(A). As seen from this figure, in which q_1 and q_2 are the only nonzero stress components, a loading path, 0-1-2, in any quadrant of the stress plane does not affect the loading surface in the remaining quadrants. Thus, this hardening rule does not take the Bauschinger effect into account.

2. The hardening rule of interdependent loading surface --- this type of hardening rule, originally proposed by Hodge [15], is a generalization of the hardening rule described in the previous paragraph. By specifying a dependence between the planes that compose the loading surface, a loading path intersecting any one plane of this surface may effect changes in each of the remaining planes. As illustrated in Fig. 3(B), this hardening rule can be used to specify any piecewise linear loading surface and is capable of taking the Bauschinger effect into account.

A special case of the interdependent loading surfaces is considered in [16]. It is assumed that plastic strain is due to slipping along three independent slip planes, along any one of which the shear is a maximum. Piecewise linear stress-strain relations are written in terms of coefficients representing the hardening behavior of the material. These coefficients are functions of stress and are dependent upon a linear strain-hardening rule employed in the analysis. By specifying the correspondence between various segments of the yield surface and the slip planes, total plastic strains for any loading are computed as the sum of the contributions from the three independent sets of slip planes. It is further assumed that the corresponding segments of the yield surface must maintain a constant elastic range from positive to negative yielding. An illustration of the subsequent loading surfaces determined in this way is shown in Fig. 3(C). It is seen from this figure that the Bauschinger effect can be taken into account.

A comprehensive review of the piecewise linear strain-hardening theory of plasticity is presented in [17].

KINEMATIC HARDENING

The hardening behavior postulated in this theory assumes that during plastic deformation the loading surface translates as a rigid body in stress space, maintaining the size, shape, and orientation of the yield surface. The primary aim of this theory, due to Prager [18, 19], is to provide a means of accounting for the Bauschinger effect.

For piecewise linear yield surfaces, kinematic hardening may be considered to be a special case of the hardening rule of interdependent loading surfaces. However, it is not limited to piecewise linear yield surfaces.

An illustration of kinematic hardening, as applied in conjunction with the von Mises yield curve in the q_1, q_2 plane, is provided in Fig. 4. The yield surface and loading surface are shown in this figure for a shift of the stress state from point 1 to point 2. The translation of the center of the yield surface is denoted by α_{ij}.

As a consequence of assuming a rigid translation of the loading surface, kinematic hardening predicts an ideal Bauschinger effect for completely reversed loading conditions. A modification to this theory, proposed by Ziegler [20], eliminates inconsistencies that arise when Prager's original model is used in a subspace of stress.

Although originally devised to be used in conjunction with linear strain hardening behavior, this model has been used for materials exhibiting nonlinear hardening [10, 11], and has been further generalized to cyclic loading involving work-hardening and work-softening behavior [21].

A model of combined kinematic and isotropic hardening in which the subsequent loading surfaces expand and translate is presented by Hodge [22].

MECHANICAL SUBLAYER AND FIELDS OF WORKHARDENING MODULI MODELS

A technique to model the arbitrary nonlinear mechanical behavior of a solid by means of a parallel assemblage of elastic ideally plastic solids can be traced to Duwez [23], with extensions by White [24] and Besseling [25]. This modeling concept equates the integrated effect of a network of ideally plastic solids to the actual behavior. An extension of this model, proposed by Mroz [26, 27] to account for the work-hardening behavior of metals under cyclic loading conditions, introduces the notion of a field of work-hardening moduli and the variation of this field during the course of plastic deformation.

In this proposed model, a stress-strain curve of an initially isotropic material is represented by n linear segments of constant tangent plastic moduli, as shown in Fig. 5. In stress space, this approximation can be represented by n hypersurfaces f_0, f_1 ..., f_n, where f_0 is the initial yield surfaces, and f_1 to f_n define regions of constant work-hardening moduli.

Figure 6 illustrates these hypersurfaces in the σ_1, σ_2 plane for an initially isotropic material. As seen in this figure, the surfaces f_0, f_1 ... f_n are similar and concentric, and for simplicity are schematically represented by a family of circles. If we consider porportional loading in the σ_2 direction, corresponding to σ in Fig. 6, and if we assume that the surfaces can experience a rigid translation without experiencing a change of size or orientation, then when the stress state reaches point A on Fig. 6, the surface f_0 will translate until it reaches the circle f_1 at the stress corresponding to point B. The circles f_0 and f_1 translate together until point C is reached, where now f_0, f_1 and f_2 are attached at a common point of contact. For unloading and subsequent reversed loading, when the stress reaches a point corresponding to point E (Fig. 6), reverse plastic flow occurs and the surface f_0 translates downward along the σ_2 axis until it reaches the surface f_1 at F. Mroz further proposes that the curve of reverse loading in Fig. 5 join the curve OA'B'G that is obtained by symmetry with respect to the origin from OABC. Thus, the curve of reverse loading EFG is uniquely defined by the curve of primary loading, represented by an equation of the form $\sigma = f(e)$. If a new coordinate system (σ, $\bar{e}$), with origin at C is used, we have for the curve CEFG, $\sigma/2 = f(\bar{e}/2)$. This relation is usually referred to as the Masing relation [28], and is a useful rule for describing steady cyclic behavior.

In the generalization of this model to nonproportional loading, it is assumed that during translation of the hypersurfaces the individual surfaces do not intersect but consecutively contact and push each other. It should be noted that when f_1 tends to infinity, in which case the work-hardening modulus is constant (the work-hardening curve being represented by a straight line), the theory proposed by Mroz is identical to Prager's kinematic hardening model.

The further generalization of the theory of work-hardening moduli is associated with an expansion or contraction of the surfaces f_0, f_1 ..., f_n so that transitory phenomena (work-stiffening, work-softening, or non-isothermal conditions) can be treated. Thus, the hypersurfaces f_K are not constants but functions of a monotonically increasing scalar parameter during plastic flow. One suggestion for the scalar is presented in [27].

TWO SURFACE THEORIES

Eisenberg and Phillips [29] presented an early generalization of conventional plasticity theories to account for the phenomenon of noncoincident yield and reloading stress states. To account for this behavior a two-surface plasticity theory was proposed involving a yield surface completely enclosed by a prescribed loading surface. The size of the yield surface always remains unchanged. The loading surface on the other hand varies in size, coinciding with the yield surface for elastic behavior and containing the stress point for loading beyond initial yielding. During unloading the yield surface remains unchanged and the loading surface shrinks to accommodate the stress state until it coincides with the yield surface. Upon unloading and subsequent reloading the two surfaces separate.

The incorporation of this behavior, should it be a significant factor for a material under consideration, does not appear to pose any difficulties beyond those normally associated with conventional theories.

A comprehensive and satisfying generalization of the concept of a two-surface plasticity theory was proposed by Dafalias and Popov [30]. In this theory the concept of a bounding surface is introduced. This surface, always enclosing the yield and subsequent loading surface in stress space, is used to model complex loading situations, including cyclic loading involving hardening and softening behavior. This work appears to have been motivated by the general observation of the noncoincidence of the yield and loading stresses previously discussed. The approach to multiaxial loading as presented by Dafalias and Popov is not tied to any hardening law and appears to be sufficiently general and sound to warrant further examination.

A modification of Mroz's fields of work-hardening moduli model has been recently proposed by Krieg [31]. This modification, similar to the model proposed by Dafalias and Popov [30], replaces all but two of the discrete surfaces specified by a Mroz model by a continuum of intermediate loading surfaces whose distribution is prescribed. The two surfaces, are represented by an inner curve labelled by Krieg as the loading surface, and an outer curve, termed the limit surface. These two surfaces separate the material behavior into three distinct zones: an elastic zone contained within the loading surface, an asymptotic plastic zone outside the limit surface, and a so-called "metaelastic" zone between the two surfaces. On the basis of a uniaxial stress-strain curve, these zones are joined by a continuous function, generated for a variety of situations including reversed loading. Both the loading and limit surface can vary according to a combined kinematic-isotropic hardening behavior. The motion of the loading surface is identical to that assumed by Mroz. For the general multiaxial case the theory requires the retention of three vectors and three scalars, a small increase over the two vectors required for kinematic hardening alone.

ANISOTROPIC THEORIES

One of the first treatments of the plastic flow of an initially anisotropic metal was suggested by Hill [12]. In this theory an orthotropic yield criterion was assumed to be quadratic in the stress components, and to reduce to the von Mises law when the degree of anisotropy was small. In Cartesian coordinates the function, suggested by Hill, takes the following form

$$f(\sigma_k) = N_{ij}\sigma_i\sigma_j - 1 = 0 \qquad (1)$$

where the contracted tensor notation is used, i, j, k = 1, 2, ... 6, and the parameter N_{ij} are constants related to the six yield stresses in the principal directions of anisotropy. A consequence of this assumed function is an initial rotation of the von Mises ellipse in the $\sigma_1 - \sigma_2$ space.

Similar anisotropic theories have been suggested by several investigators, most notably Jackson, Smith and Lankford [32], Dorn [33] and Hu [34]. Implicit in Hill's theory are the following assumptions:

Orthotropic anisotropy

The principal axes of anisotropy either coincide with the principal stress axes and the principal strain axes or the transformation between the axes are known

The principal axes of anisotropy do not rotate during plastic flow

No distinction is made between tensile and compressive stresses

The anisotropic coefficients (N_{ij}) remain unchanged during plastic flow.

A generalization of Hill's equation for anisotropic plasticity, and one that combines isotropic and kinematic hardening is presented by Baltov and Sawczuk [35]. In compact notation this theory assumes a yield condition in the following form

$$f(\sigma_k) = N_{ij}\ (\sigma_i - \alpha_j) - 1 = 0 \qquad (2)$$

Unlike previous investigations, the fourth order tensor of anisotropic coefficient, N_{ij}, is a prescribed function of the plastic strains, and hence changes during the course of plastic flow. The kinematic hardening parameter, α_i, is also a function of the plastic strains. In a comparison of experimental results for combined tension and torsion it was found that the proposed theory better suited the experimental data than the conventional kinematic hardening theory.

More recently, several investigations have been concerned with defining a strength criterion for orthotropic materials with specific reference to advanced filamentary composites. Generally, a criterion of this type is used to define a "failure surface" in stress space. These failure surfaces may be useful for defining yield surfaces of orthotropic metals. Notable among these proposed theories is the work of Tsai and Wu [36]. The basic assumption associated with this anisotropic strength criterion is a failure surface in the following form

$$f(\sigma_k) = N_i\sigma_i + N_{ij}\sigma_i\sigma_j - 1 = 0 \qquad (3)$$

The parameters N_i and N_{ij} are strength tensors of the second and fourth rank respectively and are functions of an appropriate number of independent material strengths. The linear term in the above equation can be used to account for the difference between tensile and compressive induced "failure".

RATE-SENSITIVE MODELS

The survey paper of Lee [6] lists some 200 references associated with the investigation and application of dynamic plasticity. Various functional representations and varying degrees of complexity have been proposed. Many of these are impractical for use, either because of their extreme complexity or because they have been simplified beyond the point of usefulness.

The most popular constitutive relation, suggested by Malvern [37], employs the concept of a reference, or static, stress-strain function. The dynamic stress-strain behavior if determined from the static curve in some prescribed manner as a function of the strain rate. For uniaxial conditions this relation is in the following form

$$\frac{\sigma}{\sigma_0} = 1 + \left(\frac{\dot{\epsilon}}{D}\right)^{1/n} \qquad (4)$$

where n and D are material constants, σ_0 and σ are the static and dynamic yield stresses, respectively, and $\dot{e}$ is the strain rate.

A generalization of the above equation, as suggested by Perzyna [38], replaces the uniaxial strain rate component by the tensor invariant of the plastic deformation rate

$$D_2^p = (\dot{e}_{ij}^p \, \dot{e}_{ij}^p)/2 \qquad (5)$$

Concepts associated with time-independent plastic models may be extended within the framework of the above assumptions to treat rate-sensitive yielding. On this basis the rate-dependent loading function may be written in the following form

$$f(\sigma_{ij}, \dot{e}_{ij}) = \frac{J_2^{\frac{1}{2}}}{k} - \left[1 + \left(\frac{D_2{}^{p^{\frac{1}{2}}}}{D}\right)^{1/n}\right] \qquad (6)$$

where J_2 is the second stress invariant, and k is the yield stress in shear.

Another aspect of rate sensitivity, in addition to its effect on yield stress levels, is the variation of the strain-hardening behavior of the material with varying levels of strain rate. In conventional flow theories this dependence must be incorporated in the plastic modulus which will now be a function of strain rate as well as stress level.

NON-ISOTHERMAL MODELS

A treatment of non-isothermal response of structures experiencing simultaneous changes in load and temperature clearly requires a substantial modification of existing isothermal procedures. Without consideration for irreversible thermodynamics and for the uncoupled, quasi-static problem, the treatment of thermoplasticity requires the elastic-plastic constitutive equations to account for the influence of temperature on the elastic coefficients (primarily restricted to Young's modulus), yield stress, plastic hardening coefficient, and rate of thermal expansion.

A theory of non-isothermal deformation of rigid, work hardening solids is presented by Prager [39]. In this model a general constitutive law for plastic flow is developed that is homogeneous of order one in the rates of temperature, stress, and strain. A yield criterion is chosen to be a function of the state variables of stress, plastic strain, degree of hardening and temperatures, i.e. $f = f(\sigma, e^P, h, T)$. The motivation for including the degree of hardening (or hardening modulus) term, h, is to provide the material with a "memory" of some previous plastic deformation that will affect subsequent behavior. Conditions for loading, neutral loading and unloading from some plastic state are represented by the following conditions

$$\frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial T} dT \begin{cases} > 0 & \text{loading} \\ = 0 & \text{neutral loading} \\ < 0 & \text{unloading} \end{cases} \tag{7}$$

If the problem is discretized with respect to space and some monotonically increasing parameter that can be used to prescribe the applied mechanical and thermal history of loading, then an incremental relation between plastic strain, stress and temperature can be represented by the following linear relation

$$\Delta e^P = C_i \, \Delta\sigma_i + \bar{C}\Delta T \tag{8}$$

where the contracted tensor notation is used and C is the conventional plasticity compliance matrix, modified to account for the effect of temperature on the hardening characteristics of material; and $\bar{C}$ is a function of stress and accounts for the influence of temperature varying yield stresses.

Examples of the treatment of thermoplastic behavior formulated within the framework of finite element methods is presented by Ueda and Yamakawa

[41]. Problems in both geometric and material nonlinearities are considered. The treatment of changing elastic-plastic properties with temperature is of particular interest. Accounting for varying material properties is a formidable computational problem since it may lead to nonsymmetric stiffness influence coefficients. This situation is discussed within the context of two general solution procedures presented in the next section, e.g. the initial strain approach and the tangent modulus method.

MODELS WITHOUT A YIELD SURFACE

A theory that attempts to circumvent the ambiguities associated with defining a specific yield surface and prescribing hardening as the subsequent translation, expansion, or distortion of that surface has been proposed by Valanis [42, 43]. In his theory of plasticity an intrinsic time parameter, independent of external clock time, is chosen to be a monotonically increasing function of deformation. One obtains the stress response by monitoring this history of strain associated with this deformation.

From a physical viewpoint the theory has many advantages. Phenomena such as cross-hardening, noncoincident yield and loading points and cyclic hardening are capable of being described. On the other hand, the absence of a yield surface does not facilitate the computational effort in a general purpose discrete model analysis. In fact, it may introduce additional effort.

IDEALLY PLASTIC BEHAVIOR

The plasticity models discussed thus far are used to describe the hardening behavior of materials subsequent to initial yielding. They attempt to describe the process of hardening graphically as an expansion, translation, distortion, etc., of the initial yield surface. For the case of elastic-ideally plastic behavior the yield surface is assumed to remain unchanged. In [11], the treatment of multiaxial ideally plastic behavior requires that the stress increment vector be tangent to the yield surface and the plastic strain increment vector be normal to the loading surface. The conditions on the stress rate is the consistency condition ensuring the stress state to remain on the yield surface. This condition provides a linear relationship among the various components of stress rate. The condition on the strain rate provides a linear relation among the various components of plastic strain rates. The combination of independent increments of stress and plastic strain are subsequently determined as a function of the increment of total strain as outlined in [11].

REVIEW OF STATIC SOLUTION PROCEDURES

In the following it is assumed that the reader is sufficiently familiar with the derivation of the necessary components of elastic stiffness influence coefficients, formulated within the framework of the displacement method of finite element analysis. The matrix equation relating generalized displacements and loads may be written in the following matrix notation

$$Ku = P \tag{9}$$

where K is the matrix of conventional elastic stiffness influence coefficients, u is the vector of generalized displacements and P is the vector of generalized loads.

Basically, the procedure used to solve the small displacement plasticity problem may be divided into two categories: in one the effects of plasticity are accounted for directly in the stiffness influence coefficients; the second treats plasticity as an effective load that is used in conjunction with the applied mechanical and thermal loads for general equilibrium. The latter is referred to as the residual force or initial strain method, and the former is termed the tangent modulus method. A derivation of their corresponding equations from virtual work principals, the relationship between the approaches, and a discussion of several variations are presented in [10], [11] and [44].

The governing equation associated with each technique is customarily written in incremental form. The tangent modulus procedure, for example, uses the following form

$$K_T \, \Delta u = \Delta P \tag{10}$$

where the prefix notation Δ denotes the rate of change of the parameter following it with respect to any monotonically increasing function of time, and K_T represents the matrix of elastic-plastic stiffness influence coefficients. The plasticity model chosen for use is explicitly contained in K_T.

The residual force method uses the following matrix relation

$$K\Delta u = \Delta P + \Delta Q \tag{11}$$

where the vector ΔQ is the plastic residual load vector. For the initial strain approach ΔQ is the product of the initial strain matrix and the increments of plastic strain; $\Delta Q = K^* \Delta e_p$. For the initial stress approach, ΔQ is the product of the initial stress matrix and the increments of plastic strain. The plasticity model explicitly enters the analysis through the calculation of the increment in plastic strain.

It should be mentioned that the treatment of ideally plastic behavior may be incorporated in the tangent modulus method or both approaches associated with the residual force method.

The most obvious way of solving either set of equations for a complete history of response is a forward Euler integration scheme in which the applied load history is divided into incremental load steps. Assuming the solution is known at some arbitrary step i, we desire the solution for the increment of displacement associated with the applied increment of load. The displacement solution is used to obtain total strain increments (from kinematic considerations). Using the elastic and plastic constitutive equations and the assumption that allows the decomposition of the total strain into elastic and plastic components, we can determine increments of plastic strain and stress. This information is used as a basis for going on to the next step, and the process is repeated until the desired history has been obtained. Thus, the solution to the nonlinear response is obtained as a sequence of linear solutions in which the set of stiffness influence coefficients are modified by a residual force to maintain equilibrium.

The advantages associated with each procedure are contained in the following synopsis:

- The tangent modulus method permits the use of larger load increments than the residual force method because no approximation need be made for the current increment of plastic strain. Step size is thus limited by the approximations inherent in the assumptions of flow theory of plasticity.
- The residual force method retains the same set of elastic stiffness coefficients throughout the analysis so that calculations associated with forming the stiffness matrix and solving the governing equations need be performed only once. Furthermore, this method may be used for the analysis of materials that exhibit strain-softening behavior for materials that require a non-associative flow rule to develop their plastic constitutive relations, and for non-isothermal plasticity problems where the material compliances vary with temperature. In each of these latter situations use of the tangent modulus approach results in a nonsymmetric stiffness matrix that is generally costlier to deal with than the symmetric case.

The disadvantages associated with each procedure are the mirror images of the above advantages; namely:

- The tangent modulus method requires a successive reformulation and redecomposition of the stiffness coefficient matrix, K_T. For optimum computational efficiency this set of coefficients should be positive definite, thus precluding materials that exhibit strain softening, or have non-associative flow rules, or consider plastic material properties that vary with temperature.

- To maintain stability and accuracy, the residual force method requires small load steps to be used in the successive linearization procedure, primarily because a predicted value of the plastic strain increments is used in Eq. (11).

The size of the load increment is an obvious quantity that is commonly used as a basis of comparison between competing successive linearization schemes. However, it is not and should not be construed as the overriding factor. For example, it is generally recognized that in using the tangent modulus approach greater load increments than those required for a comparable solution using the residual force method are possible. On the surface, this would represent a strong argument favoring the former. It is the experience of this writer, however, that the ratio of computational times required for a full decomposition and subsequent solution (as required in the tangent modulus approach) to the time required to obtain the solution alone (as in the residual force method) can be great enough to offset the gains associated with the larger load increment. The question of which approach is more cost-effective is now not quite so clear.

With either approach we can expect the successive linearization procedure to drift from a true equilibrium position for the nonlinear response. This drifting is a combined result of truncation, the successive linearization procedure and the fact that information not yet available is required for a true solution. In the case of the tangent modulus method, this information is necessary to form the updated stiffness coefficients K_T; for the residual force method, the value of the plastic strain increment is required to form ΔQ. Several techniques (iterative and non-iterative) can be applied to reduce the amount of drifting to tolerable levels. Many of these techniques are reviewed by Tillerson [45]. With respect to both the tangent modulus and residual force method the simplest corrective procedure involves the introduction of an equilibrium correction term that may be added as a load vector at regular intervals (not necessarily each step) in the incremental procedure.

The equilibrium correction term at the ith step is represented by R^i in the following for the tangent modulus approach

$$R^i = P^i - f^i \tag{12}$$

where f^i represents the internal forces at the start of the step. For the residual force method

$$R^i = Ku^i - P^i - Q^i \tag{13}$$

which can be shown to be

$$R^i = K^* (\Delta e_p^i - \Delta e_p^{i-1}) \tag{14}$$

If the system is in equilibrium at the ith step R^i is a null vector. Otherwise, it must be added to the incremental equation at the i + 1 step. This procedure represents a one-step iteration. It can also be used in an iterative manner within a load step until the correction term converges to a predetermined small value.

Several other techniques have been suggested and successfully incorporated to improve the accuracy associated with an incremental Euler integration scheme for the static plastic analysis of structures. These procedures are summarized in the following:

1. The use of mid-increment correction procedures: This technique used by Felippa [46] in conjunction with the tangent modulus method to solve problems of combined geometric and material nonlinearity requires the solution of two sets of equations within a single load step. Half the incremental load is applied and the solution used to evaluate the tangent modulus coefficients at that mid-increment. The problem is then resolved using the entire increment of load and the mid-increment tangent modulus. A two-stage iterative process that allows larger load increments to be used in conjunction with the residual force method is described by Vos [47]. This procedure involves employing an iterative solution to the problem at the beginning of the increment, assuming a fixed normal to the yield surface, and then re-evaluating the normal at the midpoint of the plastic increment.

2. Stress scaling to ensure consistency: It is possible that the successive linearization procedure results in a situation where at the end of an increment of load the stress state of an element may lie outside the yield surface. Stricklin, et al. [48] have shown

that stress scaling back to the yield surface to ensure consistency, and subsequently distributing the unbalanced nodal forces during the next load increment is a computationally efficient procedure.

3. The successive implementation of the constitutive relations on a subincremental basis within each load step [49]: This allows for a larger load step to be used to evaluate displacements and total strain. These are then divided into a number of smaller increments that are used with the constitutive relations. Subincrementation is of particular importance when using the tangent modulus method where it is imperative to limit the number of complete solutions to the governing equations without violating the plasticity theory.

4. The use of multipoint stress locations: The use of multiple points within an element for stress (or strain) evaluation assumes an important role in an elastic-plastic analysis. Since the constitutive relations are imposed at each of these points, the plastic strain variation and hence the elastic-plastic boundary within an element is dependent on the number and location of the stress points, the user in effect is modeling for the anticipated plastic response.

This is a particularly useful notion when using the residual force method since the finite element model can be formulated based on the necessary accuracy required to obtain the displacement field and the definitions of the number and location of the stress points determines the effective load vector. This procedure was followed for a three-dimensional isoparametric hexahedra element by Levy et al. [50] for several elastic and elastic-plastic problems. The cost-effectiveness of the procedure is not as clear when applied to the tangent modulus method since the number and location of the stress points are an integral part of the evaluation of the stiffness matrix.

5. The use of substructuring techniques: The objective of substructuring procedures, or static condensation, is to reduce computational costs and improve the accuracy associated with a nonlinear analysis by eliminating a substantial amount of computations. The method is particularly well suited to the initial strain method for treating problems of contained

plastic flow, and where the regions of pure elastic behavior and elastic-plastic behavior can be distinguished and estimated with a high level of confidence. In this case, since the stiffness matrix is not updated, a reduced matrix obtained by static condensation is initially formed and then used throughout the plastic analysis. Another associated saving is obtained by recovering stresses and strains only in elements in the reduced set. The substructuring technique can also be used with additional effort and complexity with the tangent stiffness method [51]. A study of substructuring as applied to three particular problems involving small contained regions of plastic flow in which the residual force method is used is presented in [52]. The results of this study may be summarized by stating that a reduction of an order of magnitude in computing time requirements may be realized per incremental step by incorporating a substructuring option.

Finally, the accuracy and efficiency of any solution to a nonlinear structural analysis problem depends upon a number of other factors, not the least of which is the user's experience in setting the finite element idealization with respect to size, arrangement, and type of elements used. These factors have been discussed by Waltz et al. [53] and Trucke [54] for linear elastic systems and are equally applicable, if not more critical for elastic-plastic analyses.

NONLINEAR DYNAMICS

A prerequisite to developing a successful nonlinear dynamic analysis capability is a sound knowledge of the options and pitfalls associated with nonlinear static analysis. Many of the comments concerning efficiency and accuracy discussed in connection with static analyses are directly applicable to nonlinear dynamics. The additional ingredients are the inclusion of an inertia term to the governing equations and the introduction of a time integration scheme to solve the equations of motion. Much attention [55-63] has been given to the development and evaluation of discrete methods to numerically integrate the equations of motion. Little will be added here to the technical content of these references, rather we will outline the various options and pitfalls that exist with respect to nonlinear dynamic analysis and make some summary and experience based comments.

At the outset it should be stated that a significant factor effecting time step size for a dynamic plastic analysis is the plasticity theory. That is, the time step must be such that the assumptions intrinsic to the plasticity theory are not violated. This is in contrast to linear dynamic analysis for which step size is controlled entirely by numerical stability and accuracy considerations.

Basically the procedures used to solve the equations of motion of both linear and non-linear dynamic analysis are divided into two categories, direct time integration and modal superposition. Some of the distinctions of the two methods are discussed below.

DIRECT INTEGRATION PROCEDURES

The coupled equations of motion for an undamped discrete system can be written as

$$M\Delta\ddot{u} + \Delta f = \Delta P + R \tag{15}$$

where M is the mass matrix, Δf, ΔP the incremental internal and external force vectors, respectively, and R the residual load vector. Indices are omitted from Eq. (15) but it is implied that the equation is written for the i + 1 increment, i.e., in passing from t_i to t_{i+1}. The two unknowns in Eq. (15) are the incremental vector of internal forces Δf, and the incremental acceleration vector, $\Delta\ddot{u}$. The mass matrix may be formulated on the basis of a "consistent" or lumped mass approach. The distinction between the two being whether the mass is represented by means of finite element interpolating functions or is directly discretized by lumping components at nodes. The former approach leads to a banded matrix while the latter leads to a diagonal matrix. Which approach to use depends on the integration scheme employed in the analysis [64]. Two approaches to the solution of Eq. (15) can be taken each effecting the form of the equation to be solved in a sequence of time steps. In the first type, referred to as explicit, Δf is obtained entirely from previous information so that elements of the mass matrix are the only coefficients of the unknowns (accelerations or displacements). In the second type, termed implicit, combinations of the mass and stiffness matrices are combined to form coefficients of the unknowns. The choice of which method to use is clearer for linear problems than for nonlinear ones, with implicit methods overwhelmingly used for structural dynamics and explicit methods for problems, where high frequency response is significant, as in the treatment of wave propagation effects.

Explicit integrators are generally conditionally stable [55] with the critical time step inversely proportional to the highest frequency in the discrete model. Implicit integrators are generally unconditionally stable and tend to filter out the higher frequency response. This allows for larger time steps, the choice of which is controlled by the modes necessary to predict the essential features of the response. For linear problems using a constant time step, both methods lead to coefficient matrices that are constant through the entire response spectrum.

The complicating factor for nonlinear problems is a consequence of the change in stiffness due to plasticity. The operational choices on which method to use in this case are not unlike the choices between using the tangent modulus or initial strain approaches for static nonlinear problems since they involve trade-offs between smaller less "costly" time steps for explicit integration versus larger but relatively more "costly" time steps for implicit integration. Reference to the term "costly" here is related to the degree of complexity and magnitude of subsidiary computations during each time step. Some of the distinctions of the two methods are outlined below.

Explicit Integrator

The explicit formulation solves Eq. (15) directly as

$$M\Delta\ddot{u} = \Delta P - \Delta f + R \tag{16}$$

with Δf obtained from past information. For example, if $\Delta f = K_T \Delta u$, then using the central difference operator $\Delta u = 2\Delta u_i - \Delta u_{i-1} + \Delta t^2 \Delta\ddot{u}_i$. Quantities without indices imply the

$i + 1^{th}$ increment. There is small distinction between the tangent stiffness and initial strain approaches for explicit integration since $\Delta f = K_T \Delta u$ for the tangent stiffness method and $\Delta f = K\Delta u + \Delta Q$ for the initial strain approach. In either case the end result is a residual internal load vector. Further, the implication is that the operations are performed on the element level rather than performing operations on the larger global system. Since, as stated above the equations of motion require only the incremental vector of internal forces, the notion of a stiffness matrix can be discarded altogether (as suggested by Belytchko [65]), with Δf obtained directly from an integral involving stress and the matrix that maps displacement to strains. In either case calculations involve successive solutions to Eq. (16) with incremental alterations to the right hand side. This becomes a simple procedure with a diagonal mass matrix since the operation involves simple division. With a consistent mass matrix an initial factorization can be performed so that subsequent solutions require only a forward and back substitution of banded triangular matrices. In passing, we remark that Kreig and Key [64] indicate that there is an improvement in accuracy when the lumped mass approach is used with an explicit integrator because the errors in the discrete system caused by this combination tend to be counterbalancing. Because of the ease in obtaining solutions to Eq. (16), the computation time for an explicit method becomes strongly dependent on the element level stress/strain recovery and the formation of Δf. Computer costs are therefore directly tied to the number of elements in the discrete model and the number of time steps necessary in the analysis. This leads to the major drawback of explicit methods, namely that more refined models have an increased frequency spectrum, which for numerical stability require a smaller time step. Consequently, there is a complementary effect caused by a larger set of elements in combination with smaller time steps. Because of this situation a break-even point occurs when the economics of simpler calculations are overridden by the requirement of ever smaller time steps.

The most popular explicit technique currently in use is the constant step central difference operator [55]. It has been our preference, however, to use a variable time step

Modified Adams Predictor-Corrector method [66] that is explicit in the predictor and implicit in the corrector solutions. The advantage of this method is that the time step is automatically chosen to reflect current system stiffness and dynamic response. Our experience has been that this method automatically choses time steps near those required by the central difference method.

Use of an explicit integrator fits naturally into a nonlinear analysis since there are very little additional calculations required compared to a linear analysis. The smaller time steps are consistent with the initial premise that the assumptions of plasticity theory are not to be violated. However, as the frequency spectrum increases, stability requirements cause the critical time step to be reduced so that ultimately the method becomes uneconomical. We conclude this discussion with a direction for future research that leads to an explicit integrator capable of filtering out higher frequency response while maintaining the desired accuracy in the lower frequency regime.

Implicit Integrator

The implicit formulation is based on difference operators that contain both the currently unknown acceleration and displacement. When an operator of this form is substituted into Eq. (15) this yields

$$\bar{K}\Delta u = \Delta P + Q_d + R \tag{17}$$

where $\bar{K} = K_T + \frac{M}{\beta\Delta t^2}$, Q_d is a dynamic load vector that involves products of the mass matrix and vectors of known quantities such as displacements, velocities, and accelerations, and Δt and β are the time step and a parameter arising from the particular integrator used, respectively.

Unlike an explicit integrator there is a distinction between the tangent modulus and initial strain approaches when using an implicit integrator. This distinction is in the same sense as that existing for static problems. That is, the effect of plasticity can enter directly into the stiffness matrix and thereby the coefficient matrix, $\tilde{K}$, or it can appear as an effective load vector. Equation (17) represents the tangent stiffness approach since the coefficient matrix explicitly contains the tangent stiffness matrix K_T. In this case the matrix $\tilde{K}$ must be reassembled and a complete solution obtained in each time step. As the mesh size increases this becomes increasingly expensive and ultimately dominates the calculations.

Using the initial strain approach leads to

$$\tilde{K}\Delta u = \Delta P + Q_d + \Delta Q + R \tag{18}$$

where $\tilde{K} = K + \frac{M}{\beta\Delta t^2}$, K is the elastic stiffness matrix, and ΔQ is the plastic pseudoload vector. If a constant time step approach is used the coefficient matrix remains constant so that subsequent solutions require only forward and back substitutions of triangular matrices. This procedure is competitive with an explicit technique since after the first step, computations involve only solutions of the factored coefficient matrix, the formation of the dynamic load vector Q_d and the formation of ΔQ. It must be noted that the time step used in this procedure must be controlled by the linearization implicit in using estimated values of plastic strain increments, rather than accuracy consideration defined entirely by the integrator.

An inner loop iterative procedure can be developed on the basis of Eqs. (17) or (18). Iterations are performed until the structure is in equilibrium to within some predetermined tolerance. This tolerance can be based on a measure of the change of displacement increment [66] or on the work done by the residual load vector [63], In either case, if the procedure does not converge within a prescribed number of iterations the time step is reduced. Thus, the iterations can define a variable time step approach in which the criterion for reducing the time step is based on the system nonlinearities. It is our opinion that accuracy checks should also be included based on the current system dynamics in the same manner as used in predictor-corrector methods. This has been suggested in [59] but to our knowledge has not been implemented and tested in a general purpose program.

There are currently a number of implicit operators that are frequently used for structural analysis. Among these are the Newmark-β family [67], Wilson-θ [68], Houbolt [69] and the more recently developed stiffly stable methods by Park [70]. No one method at this point seems to have been accepted as the "best" one for all problems.

SPECIAL CONSIDERATIONS

One of the distinguishing features of many practical problems associated with plastic analyses is that the nonlinearities can be contained in localized regions. Because of this, substructuring in a number of forms can be an important factor in reducing computational costs. The most straightforward approach is to simply use a Guyan [71] reduction technique to reduce the number of degrees of freedom in the entire model, particularly in the elastic region and then to recover tresses and strains only in the limited region where plastic flow is postulated. Mixed methods [72, 73] are also used, where modal techniques are utilized in some well defined lastic region with direct integration used in the plastic region. Other methods are currently being developed [74-77] for use in mixed media problems such as fluid structure interaction and hold promise for dynamic plastic analysis. Hughes [74] has demonstrated a mixed implicit-explicit method with a predictor corrector explicit operator and a Newmark family implicit integrator. Within our context this method may be useful for elastic-plastic dynamic analysis with the implicit integrator used in the elastic region and the explicit used in the elastic-plastic region. Other techniques that allow different time steps in different regions [75, 76, 77], warrant investigation since simple explicit integration with time steps consistent with the path dependence attributes of plasticity theory could be combined with larger time step implicit integration in a larger elastic region.

MODAL METHODS

Modal superposition for linear elastic systems requires the equations of motion to be transformed into an uncoupled form in which only a portion of the eigenvalues and eigenvectors are retained. To reduce computational costs this transformation may be preceded by a condensation procedure that eliminates a prescribed set of degrees of freedom. One of two procedures is usually employed to eliminate the unwanted information; the first is the Guyan reduction scheme [71], and the second procedure is a static (zero mass) condensation method. Both procedures result in an approximate representation of the mass and stiffness characteristics of the discrete model. As in the case of substructuring in a static analysis, care must be taken to avoid eliminating regions where plasticity may develop. The effects of plasticity may be treated within the framework of the tangent modulus or residual force method.

The use of modal superposition for nonlinear problems appears, at first glance, to violate the well-known fact that superposition principles are not applicable to nonlinear systems. However, nonlinear dynamic analysis by modal superposition requires some additional considerations. When used in conjunction with the tangent modulus method the procedure requires that subsequent modes, developed beyond those associated with the elastic state, be obtained during regular intervals of the load-time history. Nickell [78] has discussed this procedure for the case of combined geometric and material nonlinearities. Although Nickell's formulation is sufficiently general for combined plasticity and large deflection, examples involving only geometric nonlinearities are considered in [78]. The subsequent modal spectrum for nonlinear states is determined by an iterative procedure, using the most recently determined spectrum as an initial estimate. A direct time integration scheme is then employed to solve the reduced set of uncoupled equations.

The use of a modal method in conjunction with the residual force approach does not appear to have been given much consideration. On the surface it appears particularly attractive since a single set of modes (based on the elastic behavior) could be used throughout the analysis. Only the residual force due to plasticity would have to be transformed in each time step.

CLOSING COMMENTS

The computational capability available for the plastic analysis of structures has experienced a tremendous growth during the past decade. Indeed, the level of structural analysis capability that has been achieved has at times outstripped our ability to describe accurately complex material behavior such as cyclic, time, and temperature dependent plasticity. Prior to the development of the programs now available, the designer or analyst confronted with a problem involving material nonlinearities was left with a choice of using his engineering judgment alone or in conjunction with potentially expensive laboratory tests. He now has the further option of performing numerical analysis to gain insight into the behavior of the structure.

Most nonlinear analysis programs, with the exception of a few, have been developed as a spin-off of existing programs that were originally designed for linear structural analysis. Although this development is a natural one, this added dimension of generality has placed great responsibilities on the user of such programs. Perhaps the greatest asset of these programs, i.e., their ability to solve sophisticated problems also represents a potential liability, i.e., they always produce numbers. The user must still exercise engineering judgment in order to interpret the results meaningfully. Hopefully, the analytic results will confirm these feelings and provide him with additional insight. However, he now has the luxury of having his intuition fail him without suffering the consequences of a catastrophic failure or an overdesigned system.

For the analyst and researcher of fundamental structural and material phenomena, the combination of advanced numerical analysis procedures and the appropriate modeling of plasticity will continue to be used to gain insight into the significance of some of the many factors associated with plastic deformation. A perusal through the preceedings of a symposium dedicated to the numerical modeling of manufacturing processes [79] and [80] should provide the reader with an appreciation of the application of some of the models and procedures discussed here. These models have contributed significantly toward the realistic treatment of such problems as welding and metal forming.

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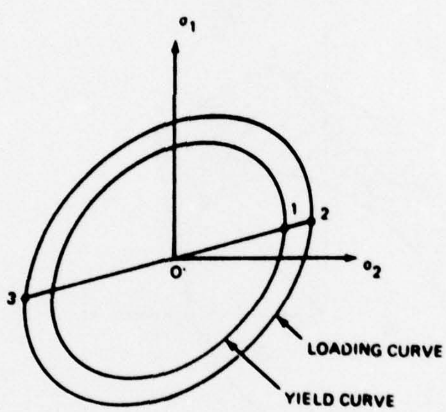

Fig. 1 Isotropic hardening.

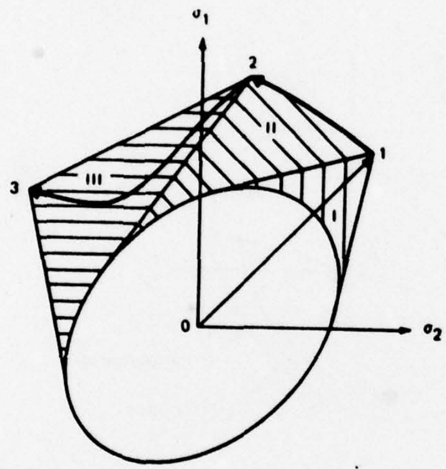

Fig. 2 Slip theory hardening.

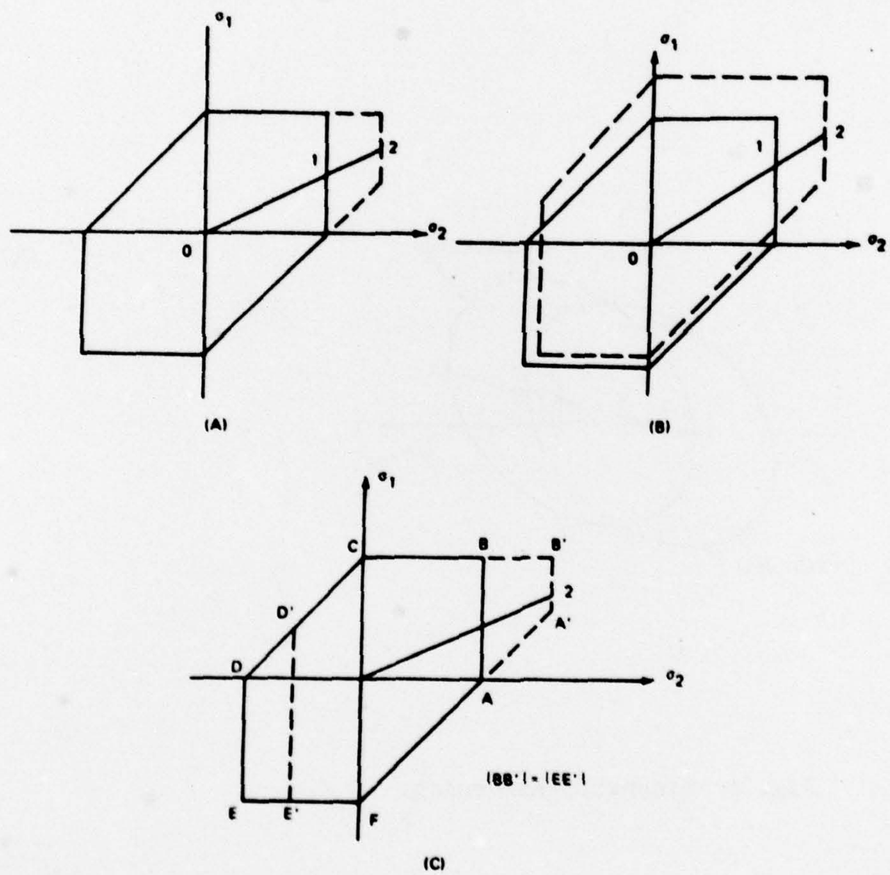

Fig. 3 Piecewise linear hardening (a) Independent loading surfaces (b) Independent loading surfaces (c) Special case of (b).

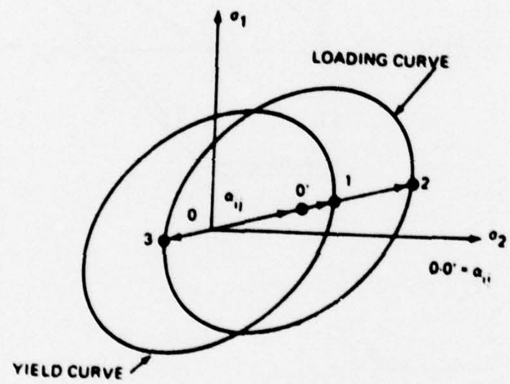

Fig. 4 Kinematic hardening.

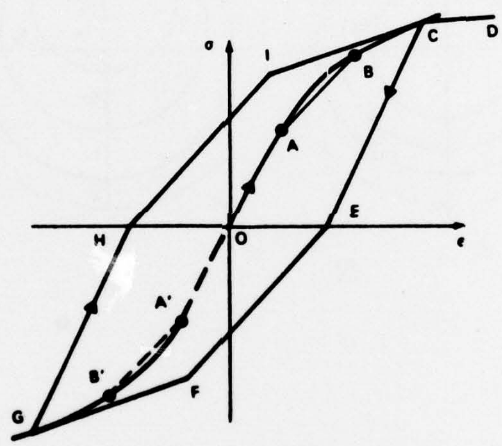

Fig. 5 Representation typical cyclic stress-strain curve by tangent moduli.

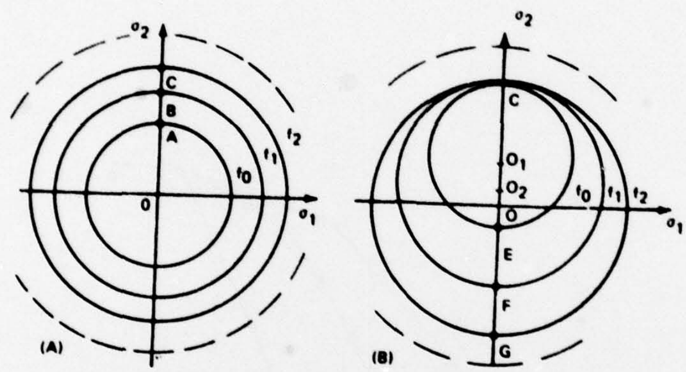

Fig. 6 Representation of hypersurfaces for mechanical sublayer model and Mroz model.