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AXIALLY-DEPENDENT EQUILIBRIUM OF AN UNNEUTRALIZED ELECTRON BEAM--ETC(U)
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Axially-Dependent Equilibrium of an Unneutralized Electron Beam in an Axial Magnetic Field

J. Guillory
Department of Physics and Astronomy
University of Maryland
College Park, Maryland
20742

JUN 1979

DRAFT - MS IN PREPARATION

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I. INTRODUCTION

Virtual cathode formation in vacuum-propagating beams may be more widespread than previously recognized. The "spin death" of parallel beam energy by $E_r \times B_z$ rotation in front of a density pileup, may assit or steepen the axial density gradient, since the rotation speed tends toward Brillouin flow, $\omega = \omega_c/2$. It may be that the propagation limit (on the beam density) is not $\omega_p^2 \sim \omega_c^2$ in the applied or total magnetic field, but rather $\omega_{p\ max}^2$ (in the presence of self-field pileup) small enough that $cE_r/B_z < V_0$ (v_0=beam electron speed), with E_r given self-consistently from the charge distribution.

In what follows, self-consistent BGK-like axially dependent equilibria will be described, in which the space charge effects determine the axial extent and profile of the beam. Generalization to axially nonuniform magnetic fields is relatively straightforward. Sections III and IV treat the problem in a purely one-dimensional way, keeping the electrostatic effects of E_z electric fields but not the effects of beam rotation caused by E_r. Section V generalizes these results to include E_r. Section VI summarizes the conclusions of this introductory work.

II. APPROXIMATIONS AND ASSUMPTIONS

To calculate the electron density n(s) at a point s along magnetic field lines in terms of an effective potential U(s) for the parallel motion, it is assumed that all changes of U(s) and n(s) with time occur much slower than the time required for a typical particle transit from s=0 to the end of the system and back, and that the beam is on, supplying electrons at a constant rate, for a time much longer than the transit time. It is also assumed that the beam source absorbs all reflected electrons, that is, reflexing is neglected. This "fast bounce approximation" then allows use of the steady state Vlasov equation, in its integrated form, to calculate n(s) in terms of the momentum distribution $f_0(p)$ at s=0 and the generalized potential U(s).

The potential for electron parallel motion is taken to have the form $U(s)=\mu B(s)-e\Phi(s)$, where μ is the magnetic moment invariant, B(s) is the magnetic field strength, and $\Phi(s)$ includes all effects due to the electric field. $\Phi(s)$ may or may not be approximated as a purely electrostatic potential. The electrostatic potential $\phi(s)$ defined by $E_{\parallel}=-\partial\phi/\partial s$ enters into $\Phi(s)$, but $e\Phi(s)$ in general also contains the energy in E×B drift motion (taken out of parallel motion.)

It will be assumed that there is some point, s=M, such that for s>M, no particles are reflected at s. This point may be at infinity, but it is convenient to keep it in the analysis.

For simplicity in evaluating the integrals involving $f(\vec{p},s=0)$, the electron beam distribution in momentum will be assumed monoenergetic with a rectangular distribution in pitch angles. (More general angle distributions are also tractable.) Also for simplicity, we take the case where $\partial B/\partial s=0$, i.e., where the reflection of beam electrons is entirely due to electric field effects and not to magnetic mirroring. (This is easily generalized for nonrelativistic motion, and changes only the boundary of integration of the integral expression for n(s).)

Finally, since we want only to show the basic physics of beam self-reflection, we consider nonrelativistic beam momenta. Generalization to relativistic motion is straightforward but mixes $p_{\parallel}$ and $p_{\perp}$ through the factor

$$\gamma = \sqrt{1 + (p_{\parallel}^2 + p_{\perp}^2)/m_0^2c^2} \ ,$$

which occurs in the n(s) integrals.

The collisionless, magnetic-moment-preserving behavior assumed here implies that instabilities do not disrupt the electron motion. This requirement may be satisfied since the virtual cathode formation will cause inhomogeneity of the density on the scale of one beam Debye length. Thus, even though reflection of a beam with small parallel momentum spread generally allows counterstreaming instability in homogeneous plasmas, this instability may not be able to grow to significant amplitude in the space provided, all the more because density gradients absorb the unstable waves. Radial motion due to $E_z \times B_\theta$ is neglected. This additional self-pinching due to E_z must be offset by centrifugal effects arising from the curvature of the helical magnetic field lines.

III. DENSITY IN TERMS OF POTENTIAL

The density at s is comprised of forward-going and reflected electrons:

$$n(s) = n_+(s) + n_-(s) \ . \tag{1}$$

The velocity distribution of the forward-going portion is specified at s=0:

$$f_+(v_1, v_\perp, s=0) = N\delta(|v| - v_0)H(v)H(|y_0 - v_\perp|) \tag{2}$$

where H(x) is the Heaviside function: H(x)=1 for x>0, =0 for x<0, and

y_0 is a maximum value of $v_\perp$. This distribution is nonzero on an arc of a circle in v space at s=0. See Fig. 1.

The velocity distribution of the reflected electrons, $f_-(v_\parallel, v_\perp, 0)$ follows from reflection conditions and so is not specified a priori.

One can now use the integrated moment form of the one-dimensional steady state Vlasov equation to express n(s) in terms of $f_+(\vec{v},0)$ and $\Phi(s)$:

$$n(s) = 2\pi[B(s)/B(0)] \int_0 ydy \int_\Sigma xdx(x^2-x_s^2)^{-1/2} f(x,y,0) \tag{3}$$

$$x \equiv v_\parallel, \; y \equiv v_\perp,$$

where

$$x_s^2 \equiv y_0^2 \left(\frac{B(s)}{B(0)} - 1\right) + \frac{2e}{m} [\Phi(0) - \Phi(s)] \tag{4}$$

is the square of the 'escape' velocity required to reach point s, and

$$\Sigma \equiv \operatorname*{Sup}_{s'<s} x_{s'}(y) \tag{5}$$

is the maximum value taken on by $x_{s'}$ at points between s'=0 and s'=s.

Particles with midplane parallel velocity $x > x_s$ contribute to the density at s; those with $x < x_s$ or Σ have been reflected at some s'<s and do not appear at s. Because of the beamlike nature of $f_+(\vec{v},0)$, there are no electrons with x less than some x_0 (i.e., with y greater than some y_0); thus for s less than some s_0 there are no turning points. For $s > s_0$ there are turning points until s>M. The point s_0 is specified by its potential, i.e.,

$$x_{s_0}^2 = x_0^2$$

with $x_{s_0}^2$ given by Eq. (4) with s replaced by s_0. The minimum parallel velocity at the reference plane, x_0, is a parameter of the distribution function.

For points $s>s_0$, the lower boundary of the x-integration in Eq. (3) is Σ, as given in Eq. (4). For $s<s_0$, the minimum is simply x_0. In both cases the maximum can be taken as v_0, as is clear from Fig. 1. Thus, one has when $B(s) \equiv B(0)$

$$n_+(s) = \frac{\pi}{2} N \int_0^{y_s^2} d(y^2) \int_{\Sigma^2 \text{ or } x_0^2}^{v_0^2} \frac{d(x^2)}{\sqrt{x^2-x_s^2}} \, \delta(\sqrt{x^2+y^2}-v_0) \tag{6}$$

$$= 2\pi V_0 N \sqrt{v_0^2-x_s^2} \quad \text{for} \quad s>s_0 \;, \tag{7}$$

$$= 2\pi V_0 N [\sqrt{v_0^2-x_s^2} - \sqrt{v_0^2-y_0^2-x_s^2}] \quad \text{for} \quad s<s_0 \;. \tag{8}$$

In Fig. 2, the forward-going density $n_+(s)$ is shown vs x_s^2, i.e., vs. the potential difference

$$\Delta\varphi \equiv (2e/m)[\Phi(0)-\Phi(s)] \;. \tag{9}$$

The density of reflected particles is calculated similarly. One assumes there is an s=M such that no reflections occur beyond M. Then $n_-(s)$ is comprised of particles that go past s but not past M:

$$n_-(s) = \frac{\pi}{2} N \int_{y_m^2}^{\mathrm{Min}\; y_s^2, y_0^2} d(y^2) \int_{x_s^2}^{x_m^2} \frac{d(x^2)}{\sqrt{x^2-x_s^2}} \, \delta(\sqrt{x^2+y^2}-v_0) \tag{10}$$

$$= 2\pi V_0 N \sqrt{x_m^2-x_s^2} \quad \text{for} \quad s_0<s<M \tag{11}$$

$$= 2\pi V_0 N [\sqrt{x_m^2-x_s^2} - \sqrt{x_0^2-x_s^2}] \quad \text{for} \quad s<s_0 \;. \tag{12}$$

Note that

$$x_m^2-x_s^2 = (2e/m)[\Phi(M)-\Phi(s)] \;. \tag{13}$$

The density at s is then

$$n(s) = n_+(s)+n_-(s) = 2\pi N v_0 [\sqrt{v_0^2-\Delta\varphi} - \sqrt{x_0^2-\Delta\varphi} + \sqrt{\Delta\varphi_m-\Delta\varphi} - \sqrt{x_0^2-\Delta\varphi}] \qquad (14)$$

with any radical dropped once it becomes imaginary. For convenience, let $\psi=\Delta\varphi/v_0^2$. The physical range of the "independent variable" ψ is then from zero ($\psi(0)=0$) to ψ_M ($\psi(M)=\psi_m$). The density n(s) is shown vs. ψ in Fig. 3.

IV. SELF CONSISTENT SOLUTION FOR $\Delta B=0$

The equation for n(s) in terms of $\psi(s)$ is now coupled to a Poisson-like equation for $\psi(s)$ whose right-hand side is n(s). The resulting equation has the form

$$\mathcal{L}(\psi) = \omega_p^2(\psi,\partial\psi/\partial r) \qquad (15)$$

where ω_p^2 is a functional of ψ (and in general, its derivatives) and $\mathcal{L}$ is the Laplacian operator. The simplest model is a purely one-dimensional one with $\Phi(s)=\phi(s)$ (i.e., purely electrostatic potential) and $\mathcal{L}=d^2/ds^2$. In terms of the scaled distance $z\equiv s\cdot(2\omega_{p0}/v_0)$ the equation is

$$\frac{d^2\psi}{dz^2} = -\frac{[\sqrt{1-\psi}-2\sqrt{\psi_0-\psi}+\sqrt{\psi_m-\psi}]}{[1-2\sqrt{\psi_0}+\sqrt{\psi_m}]} \quad , \quad 0<\psi<\psi_m \qquad (16)$$

where $\psi_0\equiv x_0^2/v_0^2$ and where the normalization N has been evaluated in terms of the density at s=0 and subsumed into the definition of z via $\omega_{p0}^2\equiv 4\pi n_0 e^2/m$. It is not known whether this nonlinear equation always has unique solutions. When the spread in parallel velocities of the beam is small we can cast this equation in a soluble form if all electrons are reflected somewhere: then $\psi_m=1$, i.e., $e[\Phi(0)-\Phi(M)]=\frac{1}{2}mv_0^2$, and the right-hand side of (16) is

$$\frac{\sqrt{1-\psi}-\sqrt{\psi_0-\psi}}{1-\sqrt{\psi_0}} .$$

For small angle spread we may write $\psi_0=1-\varepsilon$ (see Fig. 1) where in terms of the pitch angle spread θ_0, $\varepsilon=\theta_0^2/2$. Expanding in ε, the right-hand side becomes

$$\frac{\alpha}{\sqrt{1-\psi}}$$

where $\alpha\to1$ as $\varepsilon\to0$.

The resulting equation,

$$\frac{d^2\psi}{dz^2} = -\frac{\alpha}{\sqrt{1-\psi}} \tag{17}$$

is a turning point problem and can be solved by quadrature:

$$z = \frac{1}{3\alpha}\left[(1-\sqrt{1-\psi})(1+2\sqrt{1-\psi})^{1/2}\right] \tag{18}$$

giving $\psi(z)$ implicitly as $z(\psi)$. In particular, the condition that no particles are reflected beyond some point s=M (and that all are reflected somewhere) gives ψ_m or M, subject to $\psi_m<1$:

$$M = \frac{1}{3\alpha}\left[(1-\sqrt{1-\psi_m})(1+2\sqrt{1-\psi_m})^{1/2}\right] . \tag{19}$$

See Fig. 4.

In the limit of no angle spread, all the electrons are reflected at this same point. One can then use Eqs. (14) and (18) to obtain n(s):

$$n(s) = 1/u(s) \qquad u(s) = \sqrt{1-\psi} \tag{20}$$

$$u^3 - \frac{3}{2}u^2 - u + \frac{1}{2}(1-9z^2) = 0 \tag{21}$$

with $z=s\cdot2\omega_{p0}/v_0$ as before.

In the case that not all electrons are reflected (i.e., U(s) has a maximum less than $mv_0^2/2$) the one-dimensional electrostatic problem can be similarly reduced to numerical quadrature:

$$\frac{d^2\psi}{dz^2} = -a[\sqrt{1-\psi}+\sqrt{\psi_m-\psi}-2\sqrt{\psi_0-\psi}] \tag{22}$$

with

$$a=[1-2\sqrt{\psi_0}+\sqrt{\psi_m}]^{-1} . \tag{23}$$

The solution is obtained by numerically integrating

$$dz = \frac{d\psi}{2[W_{max}-W(\psi)]^{1/2}} \tag{24}$$

where

$$\begin{aligned} W(\psi) &= a\int^{\psi} d\psi'[\sqrt{1-\psi'}+\sqrt{\psi_m-\psi'}-2\sqrt{\psi_0-\psi'}] \\ &= \frac{2a}{3}\,[1+\psi_m^{3/2}-2\psi_0^{3/2}-(1-\psi)^{3/2}-(\psi_m-\psi)^{3/2}+2(\psi_0-\psi)^{3/2}] \end{aligned} \tag{25}$$

Here, as before, $\psi_0=x_0^2/v_0^2$ is the $\cos^2$ of the beam angle spread; and W_{max} is the maximum value of W in the range of integration over ψ.

V. EFFECT OF RADIAL ELECTROSTATIC FIELD

Although the treatment in Eqs. (16)-(25) is strictly one-dimensional, i.e., involves only electric fields along s, a formal generalization is also possible when electron parallel streaming energy is converted to azimuthal $\vec{E}_0\times\vec{B}$ rotation due to the radial electrostatic self-field, $E_0(r,s)$. Since any increase in the energy of this azimuthal motion,

$$\frac{1}{2}\, m(cE_0/B)^2 ,$$

must come from the initial streaming energy, it can be treated as part of the effective potential for parallel motion, U(s), which now becomes

$$U(r,s) = \frac{m}{2}\, y_0^2 B_0^{-1} B(s) - e\phi(s) + \frac{m}{2}\, c^2 [E_0(r,s)/B(s)]^2 . \tag{26}$$

As mentioned in Sec. I, this can be written

$$U(r,s) = \frac{m}{2}\, y_0^2 B_0^{-1} B(s) - e\Phi(r,s) \tag{27}$$

where Φ now includes all effects due to the electric field, including the radial field, and where the magnetic moment,

$$\mu = \frac{m}{2}\, y_0^2 B_0^{-1} ,$$

is assumed conserved.

For arbitrarily large beam density, azimuthal current now makes B(s) nonuniform:

$$\nabla\times\vec{B}(r,s) = \frac{4\pi e}{c}\, n(r,s) \left(\frac{cE(r,s)}{B(r,s)}\right) \hat{e}_\theta \tag{28}$$

and the natural coordinates ($\parallel \vec{B}$, $\perp \vec{B}$) become noncylindrical, complicating the ∇^2 operator. However, to demonstrate the physics of the electrostatic effects alone, we may now restrict the remaining discussion in this section to the case of small beam radius, low density, or strong $\vec{B}_0$. Writing Eq. (28) in dimensional form with $\vec{B}(r,s)=\vec{B}_0+\vec{B}_s(r,s)$, one can show that

$$\left(\frac{eB_s}{mc}\right) \sim \frac{r_b^2}{c^2}\, \omega_{p0}^4 \left(\frac{eB_0}{mc}\right)^{-1} \tag{29}$$

with r_b the beam radius and ω_{p0} the plasma frequency based on the beam density at s=0. Thus in Eqs. (30)-(35), we shall require

$$\frac{r_b}{c}\frac{\omega_p^2}{\omega_c} << 1 \tag{30}$$

so that the magnetic self-fields are weak, while the electrostatic self-fields are still strong enough to be important. In this case, because of the electrostatic nature of the problem, one has

$$E_0(r,s) = -\frac{\partial}{\partial r}\,\phi(r,s) \tag{31}$$

and straight magnetic field lines. Thus we can write

$$\Phi(r,s)=\phi(r,s)-(mc^2/2eB_0^2)\left(\frac{\partial\phi}{\partial r}\right)^2 \tag{32}$$

for the case B(s)=const.=B_0.

Because reflection of electrons depends only on the potential for parallel motion, an azimuthally symmetric nonneutral distribution of electrons will have its density, n(r,s) given by Eq. (3), as before, except for the modification in the definition of Φ. For the distribution function of Eq. (2), the results of Eqs. (6)-(14) remain valid. A self-consistent equilibrium for straight magnetic field lines can now be obtained in the form of Eq. (15), letting

$$\psi = \frac{2e}{mv_0^2}\,\phi(r,s) \tag{33}$$

be the scaled electrostatic potential, and defining

$$[v_0/(2eB_0/mc)]^2 \equiv b\ , \tag{34}$$

we may simply replace ψ by $\Psi+b(\partial\psi/\partial r)^2$ in the right-hand side of Eq. (16). Note that b is just 1/4 of the (gyroradius)^2, so that the natural radius scaling is of the order of the gyroradius. Eq. (16) then becomes, for the 2-dimensional electrostatic problem,

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2} = -\Big\{&\sqrt{1-\psi-b(\partial\psi/\partial r)^2}\\ &-2\sqrt{\cos^2\theta_0-\psi-b(\partial\psi/\partial r)^2}\\ &+\sqrt{\mathrm{Max}_z[\psi+b(\partial\psi/\partial r)^2]-[\psi+b(\partial\psi/\partial r)^2]}\Big\}\\ &\times\Big\{1-2\cos\theta_0+\sqrt{\mathrm{Max}_z[\psi+b(\partial\psi/\partial r)^2]}\Big\}^{-1}\end{aligned} \tag{35}$$

where θ_0 is the initial spread in pitch angles of beam electrons, and where the operation Max_z is taken at fixed r. The nonlinear partial differential equation (35) appears suitable for moderately simple computer solution by overrelaxation methods.

The condition (30), together with the requirement that the electron gyroradius be small compared with the beam size r_b, give the requirement

$$v_0/c \ll (\omega_c/\omega_p)^2 \; , \; \frac{r_b\omega_c}{c}$$

where $\omega_c \equiv eB_0/mc$.

VI. CONCLUSIONS

In the one-dimensional case of purely electrostatic repulsion ($E_z>0$, $E_r=0$), self reflection of the entire electron population occurs in a finite distance even if there is no pitch angle spread. For small or zero spread, this virtual cathode occurs at a distance $L \approx v_0/6\omega_{p0}$, i.e., at less than one beam Debye length based on the initial kinetic energy per particle. For a cold beam, all the beam electrons are reflected at the same point, namely at this distance L. The density profile is then given by $n(s)=1/u$ with u the solution of the cubic equation (21); this profile is shown in Fig. 5. The purely electrostatic one-dimensional problem with angle spread has been reduced to quadrature, as given by Eqs. (24) and (25).

For the case with $E_r \neq 0$, the scaled electrostatic potential $\psi=2e\phi/mv_0^2$ is given by the parabolic equation (35). This equation is suitable for moderately simple computer solution by the method of overrelaxation.

FIGURE CAPTIONS

Fig. 1 Reference- ~~Mid~~plane velocity space, $x=v_{\parallel}$, $y=v_{\perp}$, at spatial point s=0. Beam input velocity distribution is nonzero on the spherical cap with $x>x_0$. Beam electrons with $x_0<x<x_s$ are reflected somewhere in the interval (0,s) and do not contribute to n(s). Reflected particles (x<0) also contribute to n(s). There may be an $x_M<v_0$ such that electrons with $x_M<x<v_0$ are never reflected. In this case, $-v_0<x<-x_M$ does not contribute to n(s).

Fig. 2 Forward-going electron density $n_+(s)$ at s [Eqs. (7) and (8)], vs. the potential difference [Eq. (9)] between point s and s=0; shown for beam pitch angle spread θ_0 at s=0 given by $\sin^2\theta_0 \equiv y_0^2/v_0^2=0$, 0.1, 0.3, 0.5.

Fig. 3 Total electron density n(s) at s, vs. the normalized potential difference ψ between point s and s=0 [Eq. (14)]. Shown for $\psi_0=0.7$, $\psi_M=0.95$. The lower, dashed curve describes the never-reflected electrons at s>M, i.e., beyond the potential maximum.

Fig. 4 $\psi(z)$ for z<M, from Eq. (18).

Reference Plane (s=o) Velocity Space

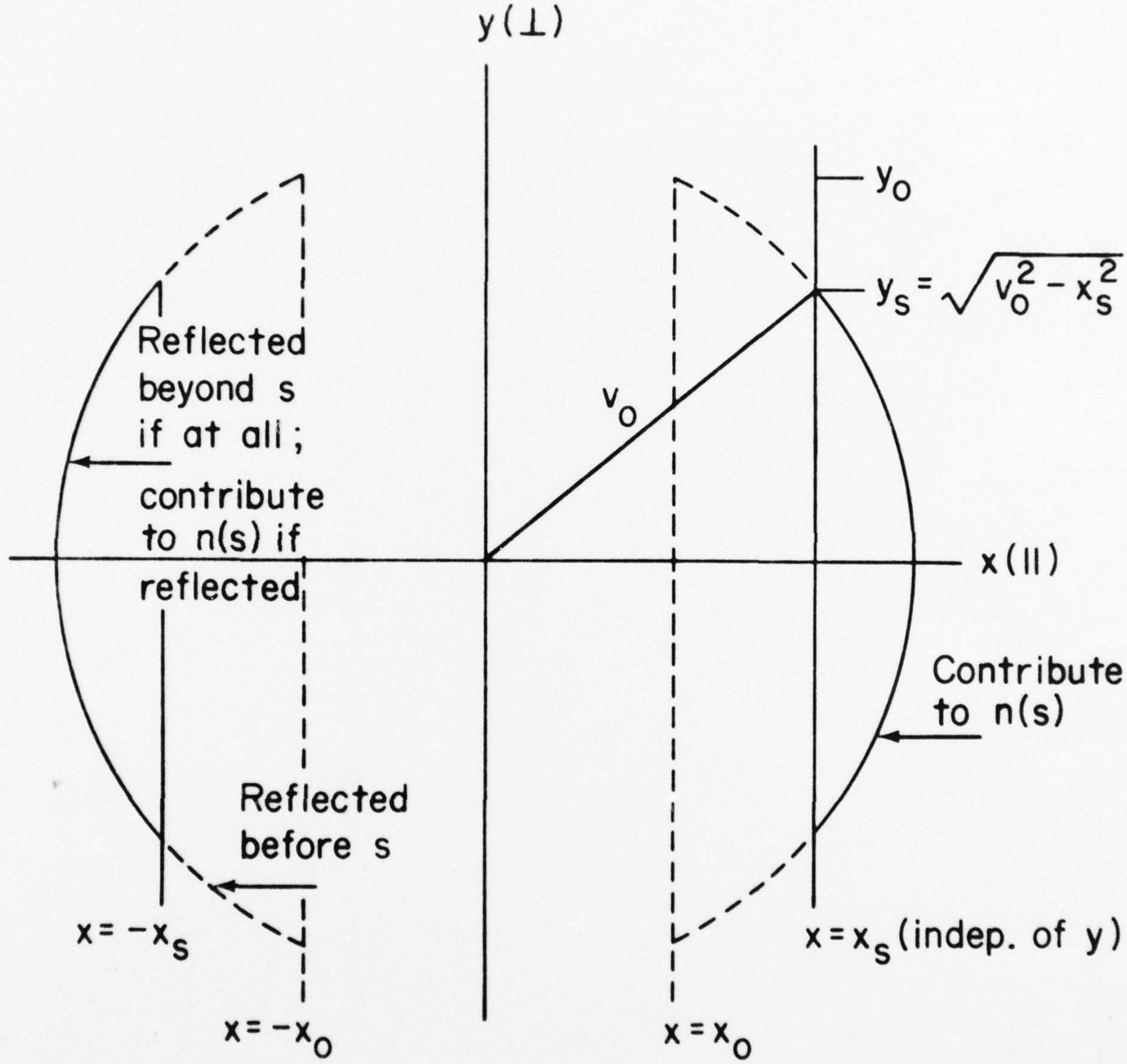
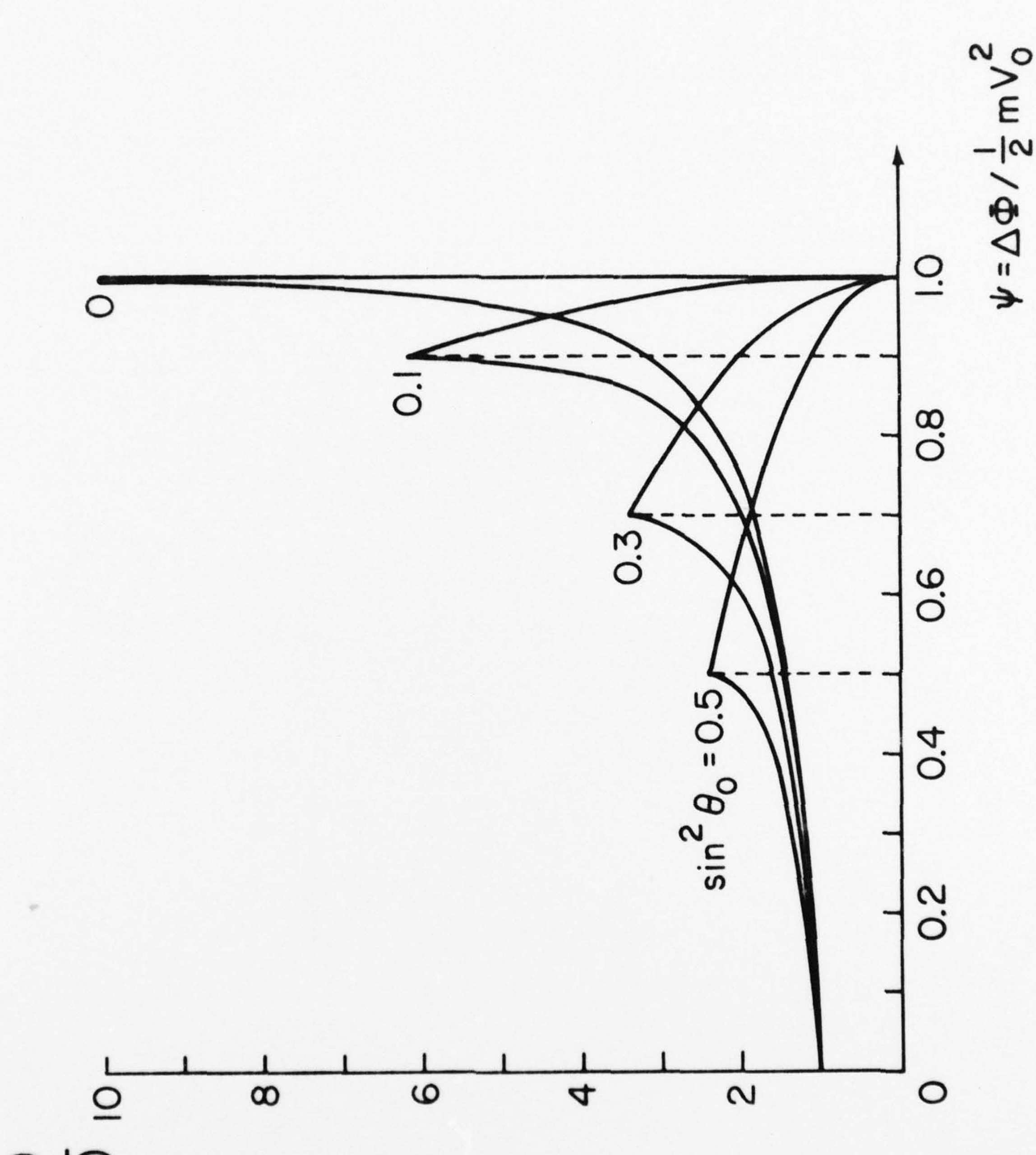


Fig. 1.



$\frac{n_+(s)}{n_+(o)}$
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1.0
$\psi = \Delta\Phi / \frac{1}{2} m V_0^2$
$\sin^2\theta_0 = 0.5$
0.3
0.1
0

Fig. 2.

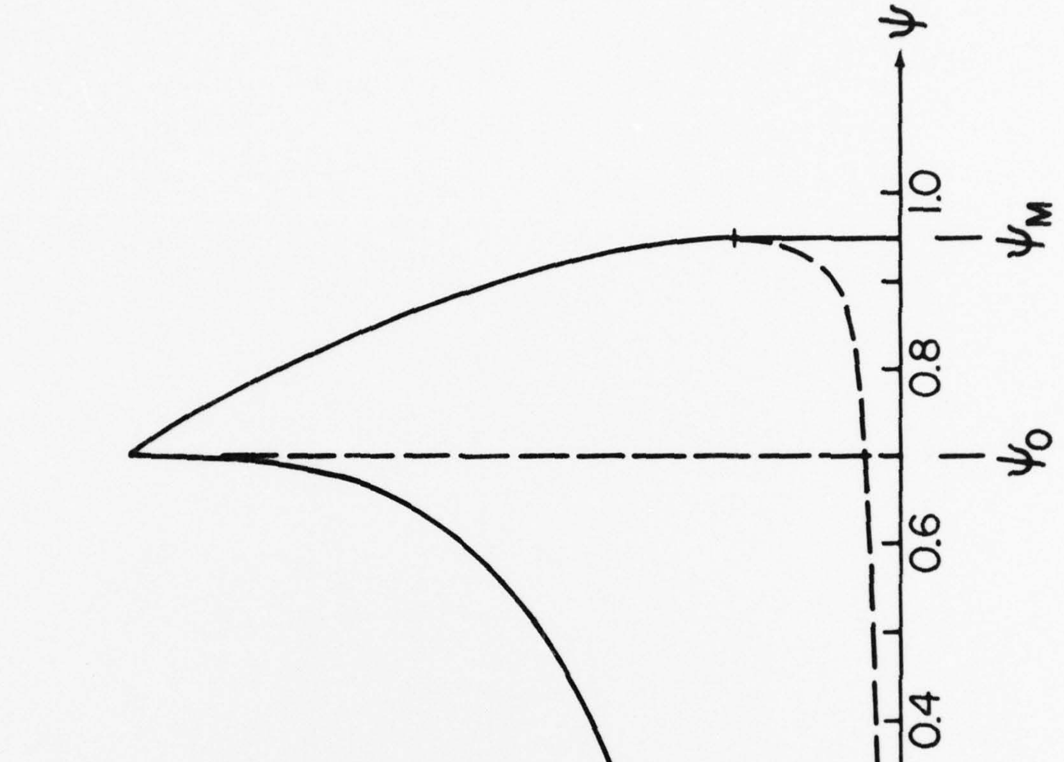


Fig.3

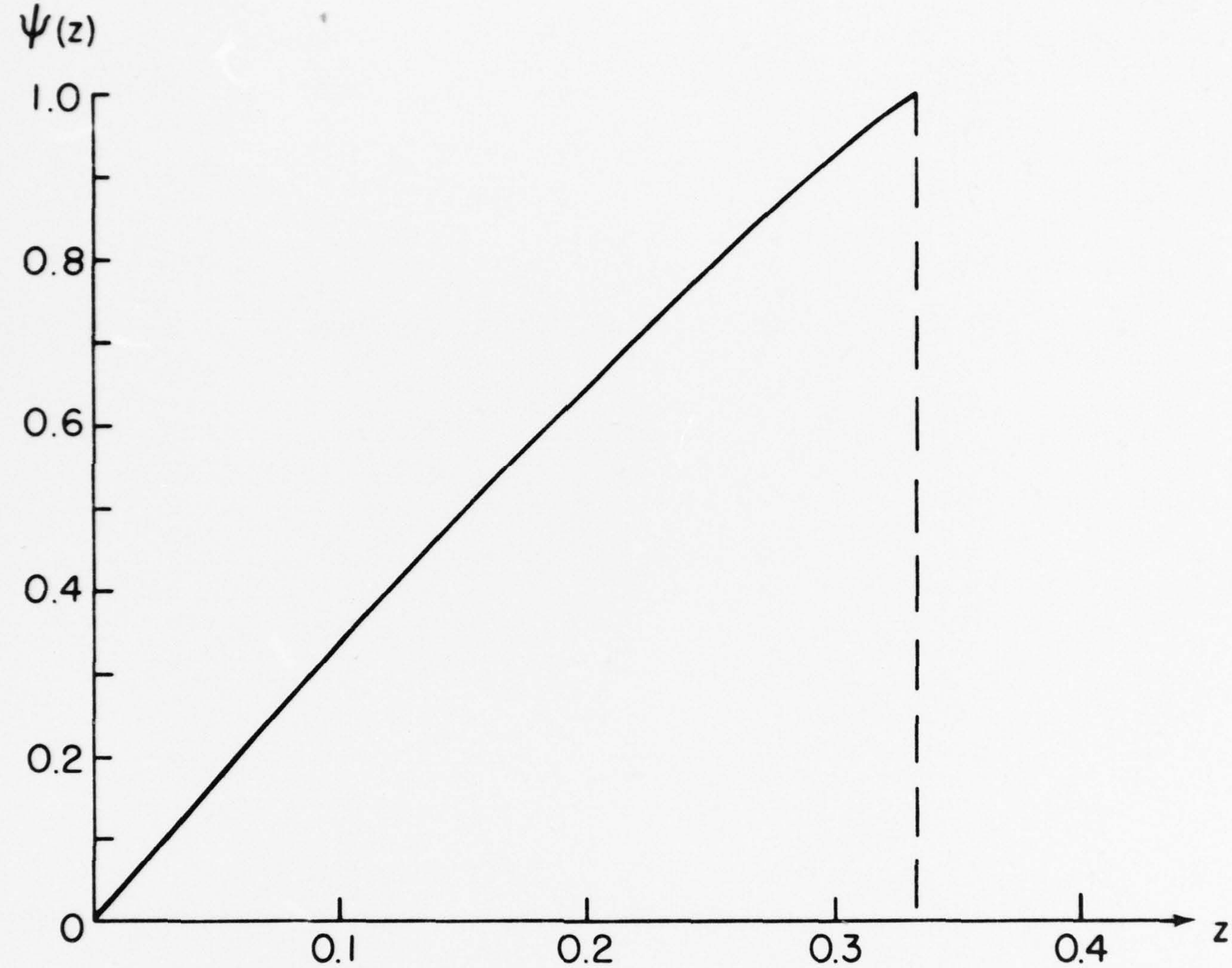


Fig 4.