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THE HAAR TRANSFORM:

ITS THEORY AND COMPUTER IMPLEMENTATION

GARY SIVAK

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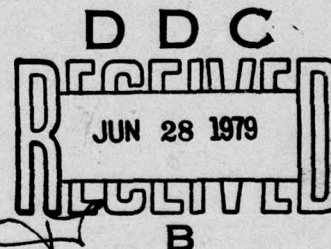
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Haar functions, square wave step functions of various rates of value
change or "sequency", take on the values +1, -1, and 0 in the interval (0, 1).
Since Haar functions form an orthogonal system, any digital function or set of
digital data can be represented as a linear combination of these functions,
just as Fourier analysis represents data as a linear combination of sines and
cosines.

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20. (Cont'd)

The Fast Haar Transform algorithm developed in this paper drastically reduces the number of operations required to transform a set of 16 data elements, i.e., from 256 to only 30. Therefore, one of the chief advantages that the Fast Haar Transform has over the Fast Fourier Transform is that it is between four and five times faster in terms of the number of computer calculations required, thereby reducing the cost of the computer time needed by 80 percent.

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INTRODUCTION

This report documents research in the development of a Fast Haar Transform algorithm and its application to digital data analysis, as performed by the author during the period June-to-October of 1976. It is an outgrowth of previous research in the area of transform application to image processing.

Just as the Fourier Transform represents data as a linear combination of sines and cosines, the Haar Transform represents a function as a linear combination of square-wave step functions called Haar functions. These functions are defined and grouped into matrices from which a transform is developed. Then the fast transform algorithm is presented with a flowchart and examples. A summary of the advantages that the Fast Haar Transform (FHT) has over the Fast Fourier Transform (FFT) is presented at the end of the report.

THEORY OF HAAR FUNCTIONS

It is instructive to examine the Haar functions, to order them into matrices, and to formulate a fast transform algorithm that can be computer-implemented for digital data analysis.

Around 1900, Haar defined the set of discontinuous step functions which are used to obtain the transform that today bears his name (ref 1). The Haar functions (step functions of various rates of value change or "sequency") take on the values +1, 0, and -1 in the closed interval,

$$[0, 1] \text{ i.e. } 0 \le x \le 1$$

The first and simplest Haar function is:

$$H_0^0(x) = 1, \qquad x \in [0, 1] \tag{1}$$

The second Haar function is:

$$H_0^1(x) = \begin{cases} 1 & x \in [0, 1/2) \\ 0 & x = 1/2 \\ -1 & x \in (1/2, 1] \end{cases} \tag{2}$$

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Finally, in general, for $m \geq 1$, and $1 \leq k \leq 2^m$

$$H_0^1(x) = \begin{cases} \sqrt{2^m} & x \; \varepsilon \; (\dfrac{k-1}{2^m}, \dfrac{k-\frac{1}{2}}{2^m}) \\[2ex] -\sqrt{2^m} & x \; \varepsilon \; (\dfrac{k-\frac{1}{2}}{2^m}, \dfrac{k}{2^m}) \\[2ex] 0 & x \; \varepsilon \; (\dfrac{\ell-1}{2^m}, \dfrac{\ell}{2^m}) \end{cases} \tag{3}$$

for $\ell \neq k$ and $1 \leq \ell \leq 2^m$

By choosing any integral value of m, one can vary k and construct any of the 2^m Haar functions of the mth order desired.

At points of discontinuity, let $H_m^k(x)$ be the mean on either side of adjoining intervals, i.e., at the points 0, and 1, let $H_m^k(x)$ take on its values as in the intervals $(0, \dfrac{1}{2^{m+1}})$ and $(1- \dfrac{1}{2^{m+1}}, 1)$, respectively.

The total collection of the Haar functions is a complete set and an orthonormal system, that is one in which the functions are normalized and orthogonal. Clearly, $H_m^k(x)$ is normalized. Also, the $H_o^k(x)$ functions are orthogonal to all others, that is, the integral over the domain of definition of their product vanishes. Finally, in general, for $m \geq 1$ and $1 \leq i, j \leq 2^m$, for $i \neq j$:

$$\int_0^1 H_m^i(x) \; H_m^j(x) \; dx = 0 \tag{4}$$

while for $n > m$, the interval in which $H_n^i(x)$ does not vanish is contained in an interval of constant length of $H_m^j(x)$ and therefore:

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$$\int_0^1 H_n^i (x) H_m^j (x)\, dx = \pm \sqrt{2} \int_0^1 H_n^i (x)\, dx = 0 \qquad (5)$$

Thus, the existence of an orthonormal set of functions is established.

MATRICES AND BASIS OF TRANSFORM

The fact of orthogonality is important and means, therefore, that a function can be expressed in terms of or represented by a linear combination of Haar functions. This means that for a given digital function, $F_d (x_i)$, for $i = 0, 1, 2 \ldots, N-1$, that is, over the range of the index i for which $F_d (x_i)$ is valid, one can write a Haar function decomposition as follows:

$$F_d (x_i) = \sum_{j=1}^{m} \sum_{k=1}^{2j} c_{jk} H_j^k (x_i) \qquad (6)$$

where j represents the particular order of the given Haar function under consideration of which 2^j or k individual cases exist. The quantity C_{jk} is the weighting factor, the real number designating how much of a contribution $H_j^k (x_i)$ is giving to the digital function or data string series $F_d (x_i)$.

In general, for functions in the complex plane, one simply treats the real and imaginary parts separately and writes:

$$F_d (x_i) = R_d (x_i) + jI_d (x_i) \qquad (7)$$

where $j = \sqrt{-1}$ and $R_d (x_i)$ and $I_d (x_i)$ are the real and imaginary parts that are expanded in terms of Haar functions as in equation 6.

Thus, any set of digital data can be expressed in terms of Haar function weighting coefficients, just as N data points can be expressed as a linear combination of sines and cosines in the Fourier Transform representation.

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For Haar functions, a transform is most easily motivated by arranging the functions in order of increasing sequency, or frequency of value change, into one or another of the N by N Haar matrices with:

$$N = 2^n \tag{8}$$

where n is a positive integer and where N and n are analogous to k and m, respectively, of equation 6.

Note, in equation 3, the presence of the bothersome square roots $\pm \sqrt{2^m}$ in the Haar functions and also in the matrix elements. If the Haar Transform is represented by an N by N matrix T and its inverse by T^I, the following condition is desired:

$$T \, T^I = T^I T = I_N \tag{9}$$

where I_N is the N by N identity matrix with 1's on the main diagonal and 0's everywhere else. The factors $\pm \sqrt{2^m}$, from both the Haar transform matrix and its inverse, can be grouped onto the inverse; therefore, for a given order of Haar functions, simply divide by $N = 2^n$. Thus modified, the Haar Transform matrix for the N = 16 case is given in figure 1. Its inverse, except for the outside factor of 1/N, is given in figure 2, which was computer generated by program HIMAT. (See appendix A.)

To perform actual processing, take an input data string with elements IN_j, the Haar Transform matrix with elements T_{ij} with the inverse transform matrix T^I_{ki}, the output data string with elements OUT_k, and a storage string with elements X_i, where, i and j range from 1 to N. Multiply the input data by the transform matrix, proceeding down the rows and storing the sums in X. Multiply X by the inverse transform matrix, proceeding down the rows and storing the results in the output data string. In equation form:

$$X_i = \sum_{j=1}^{N} T_{ij} \, IN_j \qquad\qquad OUT_k = \sum_{i=1}^{N} T^I_{ki} \, X_i \tag{10}$$

where in T_{ij} and T^I_{ki}, the first index designates the row, and the second the column position for the given matrix element.

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$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Figure 1. The Haar 16-by-16 transform matrix.

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$$\begin{bmatrix}
1 & 1 & 2 & 0 & 4 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & 4 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & -4 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 0 & -4 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
1 & -1 & 0 & -2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
1 & -1 & 0 & -2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
1 & -1 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\
1 & -1 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{bmatrix}$$

Figure 2.   The inverse Haar Transform matrix, N = 16 case.

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During processing, the Haar Transform matrix operates on each element of the data string, sampling at skip frequencies differing by powers of 2 from coarse to fine. The lowest is a DC term, the highest being value changes between adjacent data elements, i.e., the bandwidth limit of the digital input. Specifically, for the 16 by 16 transform matrix case, the DC level and spatial frequency alternations of 1, 2, 4, and 8 cycles per the data field length of 16 points occur. In this case, the Haar Transform represents each data string of equation 6 as equation 11:

$$F_d (x_i) = F_8 + F_4 + F_2 + F_1 + DC \text{ term} \qquad (11)$$

where F_8, F_4, F_1, and F , are square wave spatial frequency contributions of 8, 4, 2, and 1 cycle per data field length, respectively. (See ref 2.) These are square wave functions that take on the values $\pm a$, $\pm b$, $\pm c$, $\pm d$, and e, for the DC term as shown in figure 3.

Equation 11 is a restatement of equation 6, and shows how a data set is constructed from its individual Haar components.

A demonstration of orthogonality is instructive to show the existence of the Haar components in equation 11, by inverting it to obtain F_8, F_4, F_2, and F_1 as functions of the input data points $F_d (x_i)$. One therefore writes out the Haar component contributions in equation 11 as specified in figure 3 for all 16 digital data points $F_d (x_i)$

$$F_8 \;=\; -a+a-a+a-a+a-a+a-a+a-a+a-a+a-a+a$$

$$F_4 \;=\; -b-b+b+b-b-b+b+b-b-b+b+b-b-b+b+b$$

$$F_2 \;=\; -c-c-c-c+c+c+c+c-c-c-c-c+c+c+c+c$$

$$F_1 \;=\; -d-d-d-d-d-d-d-d+d+d+d+d+d+d+d+d$$

$$DC \;=\; e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e\;e$$

$$P_i \;=\; 1\;2\;3\;4\;5\;6\;7\;8\;9\;10\;11\;12\;13\;14\;15\;16$$

(12)

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$F_8 =$

$F_4 =$

$F_2 =$

$F_1 =$

dc-term =
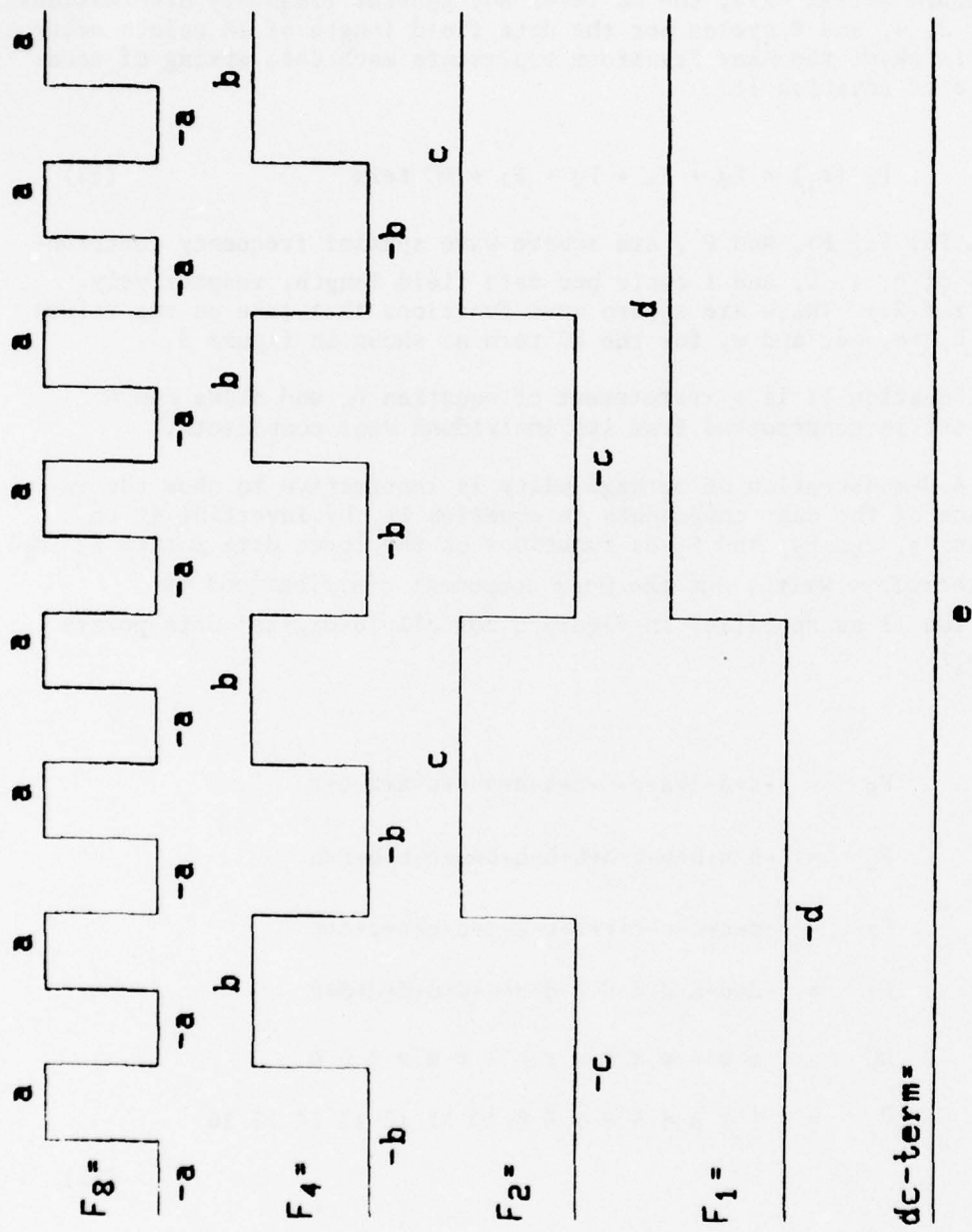
Figure 3. Square wave patterns for Haar Transform.

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where P_i is the data point in question. One adds the corresponding inputs for each of the data points P_i in $F_d(x_i)$ to get:

$$P_1 = -a\ -b\ -c\ -d\ +e \qquad P_2 = a\ -b\ -c\ -d\ +e$$

$$P_3 = -a\ +b\ -c\ -d\ +e \qquad P_4 = a\ +b\ -c\ -d\ +e$$

$$P_5 = -a\ -b\ +c\ -d\ +e \qquad P_6 = a\ -b\ +c\ -d\ +e$$

$$P_7 = -a\ +b\ +c\ -d\ +e \qquad P_8 = a\ +b\ +c\ -d\ +e$$

$$P_9 = -a\ -b\ -c\ +d\ +e \qquad P_{10} = a\ -b\ -c\ +d\ +e$$

$$P_{11} = -a\ +b\ -c\ +d\ +e \qquad P_{12} = a\ +b\ -c\ +d\ +e$$

$$P_{13} = -a\ -b\ +c\ +d\ +e \qquad P_{14} = a\ -b\ +c\ +d\ +e$$

$$P_{15} = -a\ +b\ +c\ +d\ +e \qquad P_{16} = a\ +b\ +c\ +d\ +e$$

$$(13)$$

Let:

$$x_1 = a$$

$$x_2 = b$$

$$x_3 = c$$

$$x_4 = d$$

$$x_5 = e$$

$$(14)$$

9

With 5 inputs, this system of equations determines 16 outputs.
Thus, to invert it, one has the choice of which outputs to use. Try:

$$y_1 = P_{16} = x_1 + x_2 + x_3 + x_4 + x_5$$

$$y_2 = P_{14} = x_1 - x_2 + x_3 + x_4 + x_5$$

$$y_3 = P_{12} = x_1 + x_2 - x_3 + x_4 + x_5$$

$$y_4 = P_8 = x_1 + x_2 + x_3 - x_4 + x_5$$

$$y_5 = P_{10} = x_1 - x_2 - x_3 + x_4 + x_5$$

(15)

The matrix for this system is:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

(16)

Notice that columns 1 and 5 are identical, the determinant is 0, and
the system cannot be inverted. So let:

$$x_1 = x_1 + x_5$$

$$x_2 = x_2 \ , \ x_3 = x_3 \ , \ x_4 = x_4 \ , \ x_5 = x_5$$

$$y_1 = y_1 \ , \ y_2 = y_2 \ , \ y_3 = y_3 \ , \ y_4 = y_4 \ , \ y_5 = y_5$$

(17)

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Then choose the equations for y_1 through y_4 and write the matrix for the 4-by- system:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

(18)

As found by computer generation, the inverse of the matrix is:

$$M^I = \begin{bmatrix} -.5 & .5 & .5 & .5 \\ .5 & -.5 & 0 & 0 \\ .5 & 0 & -.5 & 0 \\ .5 & 0 & 0 & -.5 \end{bmatrix}$$

(19)

Thus, one obtains equation 20:

$$X_1 = (-Y_1 + Y_2 + Y_3 + Y_4)/2$$

$$x_2 = (y_1 - y_2)/2$$

$$x_3 = (y_1 - y_3)/2$$

$$x_4 = (y_1 - y_4)/2$$

(20)

and therefore from equations 14, 15, and 17:

$$a + e = (p_{14} + p_{12} + p_8 - p_{16})/2$$

$$b = (p_{16} - p_{14})/2$$

$$c = (p_{16} - p_{12})/2$$

$$d = (p_{16} - p_8)/2$$

(21)

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Note that not all square wave components in equation 21 are obtained as independent expressions, i.e., only the sum of a and e is present. The reason is that the signs of a, b, c, and d have not been specified. This is equivalent to not having specified the phases of the contributing square wave oscillations. An ambiguity of a 180-degree phase angle in each one occurs because a statement has not been made as to whether the oscillation should begin positive or negative. The Haar transform, however, does this, and, therefore, accurately represents the original input data.

Comparing figure 1 with equation 10, note that for the transform data storage X, X_1 corresponds to the matrix elements in group 1 of the figure since only these elements were used to compute it. Similarly, X_2 corresponds to group 2, X_3 and X_4 to group 3, X_5 through X_8 to group 4, X_9 through X_{16} to group 5. The magnitude of the elements in group 1 of the Haar Transform matrix affect the strength of the DC term. Similarly, the magnitude of the elements in groups 2, 3, 4, and 5 affect the relative weights of the Haar contributions of F_1, F_2 F_4 and F_8 respectively, of equation 11. That is, for $1 \leq i \leq 5$, multiplying the elements of group i by an enhancement factor $\overline{m}_i > 1$ will enhance the presence of the ith spatial frequency component. Conversely, incorporating factors $m_i < 1$ will suppress the presence of the given square wave spatial frequency pattern. This has applications to bandwidth reduction and noise-suppression capability.

THE FAST HAAR TRANSFORM

Motivation

Now that the nature and effect of the Haar Transform are understood, working out the N = 16 case in detail will demonstrate the motivation for the algorithm obtained. The author-designed and tested Fast Haar Transform is more efficient for machine computation than is the indexing of a computer through the numerous rows and columns of extensive, memory-consuming N by N matrices.

The N = 16 case was found to be the optimum for illustrative purposes because the N = 32 case is too unwieldy to be written; every case smaller is too trivial and has too few details and operations to clearly demonstrate all the procedures involved.

As dictated by equation 10, one writes out the 16 components of the Haar Transform T as a function of the components of a digital input vector I. This matrix multiplication of each of 16 elements in

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each of 16 rows requires 256 operations. By regrouping the computations into recursive patterns of accumulating sums and differences, one will obtain the same results in 30 operations. This computational reorganization is the Fast Haar Transform. The details follow:

$$T_1 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 +$$
$$I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} +$$
$$I_{13} + I_{14} + I_{15} + I_{16}$$

$$T_2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 +$$
$$I_7 + I_8 - I_9 - I_{10} - I_{11} - I_{12} -$$
$$I_{13} - I_{14} - I_{15} - I_{16}$$

$$T_3 = I_1 + I_2 + I_3 + I_4 -$$
$$I_5 - I_6 - I_7 - I_8$$

$$T_4 = I_9 + I_{10} + I_{11} + I_{12} -$$
$$I_{13} - I_{14} - I_{15} - I_{16}$$

$$T_5 = I_1 + I_2 - I_3 - I_4$$

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$$T_6 = I_5 + I_6 - I_7 - I_8$$

$$T_7 = I_9 + I_{10} - I_{11} - I_{12}$$

$$T_8 = I_{13} + I_{14} - I_{15} - I_{16}$$

$$T_9 = I_1 - I_2$$

$$T_{10} = I_3 - I_4$$

$$T_{11} = I_5 - I_6$$

$$T_{12} = I_7 - I_8$$

$$T_{13} = I_9 - I_{10}$$

$$T_{14} = I_{11} - I_{12}$$

$$T_{15} = I_{13} - I_{14}$$

$$T_{16} = I_{15} - I_{16}$$

(22)

Working through the input data, one groups the components into the pattern of accumulating sums and differences into which they naturally fall as follows:

$$A_1 = I_1 + I_2 \qquad A_5 = I_9 + I_{10}$$

$$A_2 = I_3 + I_4 \qquad A_6 = I_{11} + I_{12}$$

$$A_3 = I_5 + I_6 \qquad A_7 = I_{13} + I_{14}$$

$$A_4 = I_7 + I_8 \qquad A_8 = I_{15} + I_{16}$$

(23)

One continues this procedure:

$$B_1 = A_1 + A_2 \qquad B_3 = A_5 + A_6$$

$$B_2 = A_3 + A_4 \qquad B_4 = A_7 + A_8$$

(24)

And finally one has:

$$C_1 = B_1 + B_2 \qquad C_2 = B_3 + B_4$$

(25)

15

Note that equation 22 states the last N/2 Haar components directly, as differences between adjacent signal elements. By comparing the results of equations 23, 24, and 25, with equation 22, one finds that the first N/2 Haar Transform components of the input signal can also be expressed as differences. They are now generated in a reverse order, the first Haar Transform components last as follows:

$$T_5 = A_1 - A_2 \qquad T_7 = A_5 - A_6$$

$$T_6 \quad A_3 - A_4 \qquad T_8 = A_7 - A_8$$

$$(26)$$

Also:

$$T_3 = B_1 - B_2 \qquad T_4 = B_3 - B_4$$

$$(27)$$

And finally:

$$T_1 = C_1 + C_2 \qquad T_2 = C_1 - C_2$$

$$(28)$$

Thus the Haar Transform has been implemented in a fast form.

Diagramatic Representation

The tree diagram of figure 4 is a computer generated signal flow graph for the N = 16 case of the Haar Transform. (See the listing of Program FIG4, appendix B.) It shows the flow of operations in computing the Haar components T_1 through T_{16} from the digital input signal components I_1 through I_{16}. The elements are generated in reverse order in aggregates corresponding to the groups in figure 1.

To take the Fast Haar Transform, compute the last 8 or N/2 elements, i.e., T_9 through T_{16}, or $T_{(N/2+1)}$ through T_N directly by taking adjacent input signal component differences. Then, sum element pairs to form the A array, take differences between adjacent elements; thus, obtaining transform components T_5 through T_8 or $T_{(N/4+1)}$ through $T_{N/2}$. Then, sum the elements of the A array to form the B's and take adjacent element differences to form components T_3 and T_4 or $T_{(N/8+1)}$ and $T_{N/4}$ Finally, sum again to form C_1 and C_2, and add and add and subtract to compute elements T_1 and T_2, respectively.

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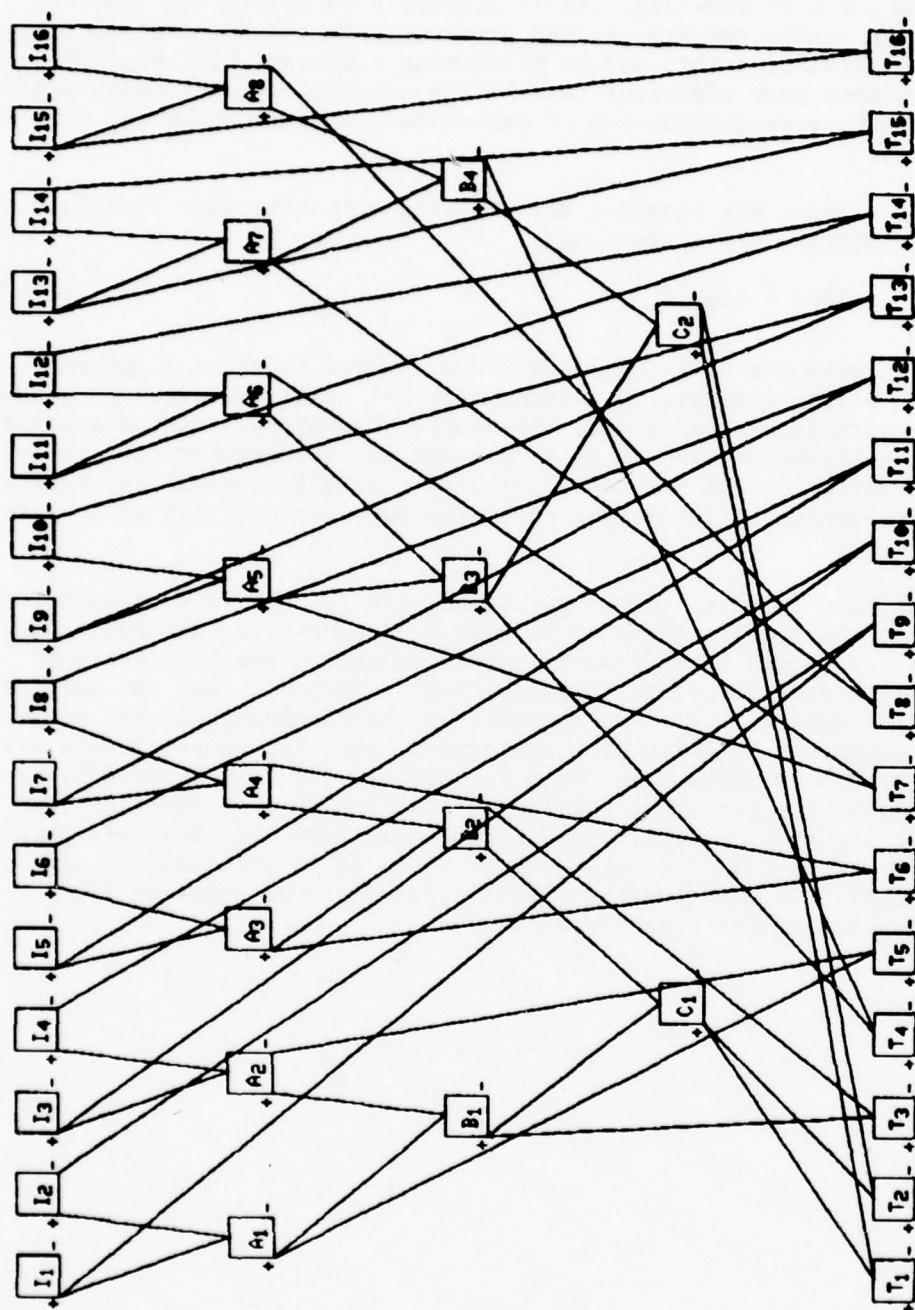
Figure 4. Haar Transform diagram, N = 16 case.

In taking the Haar Transform for the N = 16 case, note that 30 operations, 16 subtractions and 14 additions, are required. In general, for $N = 2^n$ elements, $2(N-1)$ algebraic additions are required. Thus, the reader can see, as was previously mentioned, that to obtain the Haar Transform this way be performing a General Fast Haar Transform is much more efficient than having a computer laboriously index through the rows and columns of cumbersome, memory-consuming, N by N matrices.

The reader may remember that to take a Fast Fourier Transform (FFT) requires more operations:

$$OP = N \log_2 N \tag{29}$$

Thus, to take the Haar Transform for 32 points requires 30 percent, for 128 points requires 29 percent, for 512 points requires 22 percent of the work necessary to take the Fourier Transform. Stated another way, the 512-point case saves 78 percent of the computer calculation time required by the FFT. Also, the Haar Transform does not require the time-consuming generation of trigonometric quantities as does the FFT.

At this point in the computations, one is free to work in Haar space and manipulate the Haar Transform components as desired. One can multiply some by factors > 1, thus enhancing the prominence of the square-wave functions to which they correspond. One can multiply others by factors < 1, thus suppressing the prominence of the square-wave components to which they correspond. Lastly, as previously stated, one can zero out some components to compress the data bandwidth. By performing a procedure similar to that employed in equations 22 through 28, one can develop an inverse transform and return to physical space to obtain the output data values as in equations 22 and 26 through 28. By adopting the specific procedure of equation 10, one computes the output data components:

18

$$O_1 = T_1 + T_2 + 2T_3 + 4T_5 + 8T_9$$

$$O_2 = T_1 + T_2 + 2T_3 + 4T_5 - 8T_9$$

$$O_3 = T_1 + T_2 + 2T_3 - 4T_5 + 8T_{10}$$

$$O_4 = T_1 + T_2 + 2T_3 - 4T_5 - 8T_{10}$$

$$O_5 = T_1 + T_2 - 2T_3 + 4T_6 + 8T_{11}$$

$$O_6 = T_1 + T_2 - 2T_3 + 4T_6 - 8T_{11}$$

$$O_7 = T_1 + T_2 - 2T_3 - 4T_6 + 8T_{12}$$

$$O_8 = T_1 + T_2 - 2T_3 - 4T_6 - 8T_{12}$$

$$O_9 = T_1 - T_2 + 2T_4 + 4T_7 + 8T_{13}$$

$$O_{10} = T_1 - T_2 + 2T_4 + 4T_7 - 8T_{13}$$

$$O_{11} = T_1 - T_2 + 2T_4 - 4T_7 + 8T_{14}$$

$$O_{12} = T_1 - T_2 + 2T_4 - 4T_7 - 8T_{14}$$

$$O_{13} = T_1 - T_2 - 2T_4 + 4T_8 + 8T_{15}$$

$$O_{14} = T_1 - T_2 - 2T_4 + 4T_8 - 8T_{15}$$

$$O_{15} = T_1 - T_2 - 2T_4 - 4T_8 + 8T_{16}$$

$$O_{16} = T_1 - T_2 - 2T_4 - 4T_8 - 8T_{16}$$

(30)

19

Note that to be properly scaled, these output results must be divided by a factor of N.

In a manner similar to equations 23 through 25, one forms groups of partial sums as intermediate steps in the computation of the output of equation 30 as follows:

$$A_1 = T_1 + T_2 \qquad\qquad A_2 = T_1 - T_2$$

$$\text{(31)}$$

$$B_1 = A_1 + 2T_3 \qquad\qquad B_3 = A_2 + 2T_4$$

$$B_2 = A_1 - 2T_3 \qquad\qquad B_4 = A_2 - 2T_4$$

$$\text{(32)}$$

$$C_1 = B_1 + 4T_5 \qquad\qquad C_5 = B_3 + 4T_7$$

$$C_2 = B_1 - 4T_5 \qquad\qquad C_6 = B_3 - 4T_7$$

$$C_3 = B_2 + 4T_6 \qquad\qquad C_7 = B_4 + 4T_8$$

$$C_4 = B_2 - 4T_6 \qquad\qquad C_8 = B_4 - 4T_8$$

$$\text{(33)}$$

20

Finally, one can construct the output data as follows:

$$O_1 = C_1 + 8T_9 \qquad O_9 = C_5 + 8T_{13}$$

$$O_2 = C_1 - 8T_9 \qquad O_{10} = C_5 - 8T_{13}$$

$$O_3 = C_2 + 8T_{10} \qquad O_{11} = C_6 + 8T_{14}$$

$$O_4 = C_2 - 8T_{10} \qquad O_{12} = C_6 - 8T_{14}$$

$$O_5 = C_3 + 8T_{11} \qquad O_{13} = C_7 + 8T_{15}$$

$$O_6 = C_3 - 8T_{11} \qquad O_{14} = C_7 - 8T_{15}$$

$$O_7 = C_4 + 8T_{12} \qquad O_{15} = C_8 + 8T_{16}$$

$$O_8 = C_4 - 8T_{12} \qquad O_{16} = C_8 - 8T_{16}$$

(34)

As one can therefore see, the output components are computed from a recursive pattern of accumulating sums and differences. Haar components are added to the various A's, B's, and son on, according to the groups of figure 1 to which they correspond. The signs alternate in a regular fashion. As the index of the Haar components added to the sums of equations 31 through 34 runs from 1 through N, the factors that multiply them range from 1 through N/2.

Flowchart

This detailed analysis of the N = 16 case of the Haar Transform, although perhaps seemingly cumbersome, suggests a generalization of the FHT for N = 32, 64, or greater. The FHT is conveniently representable by a flowchart as in figure 5, that can be used to implement a computer routine for its calculation.

21

INPUT

LOG$_2$ N $\rightarrow$ NU
N $\rightarrow$ NO
NU + 2 $\rightarrow$ ME
$-$ 2N $\rightarrow$ IP
$\emptyset \rightarrow$ IR

2
NO/2 $\rightarrow$ NO
IP + 4NO $\rightarrow$ IP
IR + 2NO $\rightarrow$ IR
ME $-$ 1 $\rightarrow$ ME
$\emptyset \rightarrow$ J

3
J + 1 $\rightarrow$ J
2J $\rightarrow$ L
L $-$ 1 $\rightarrow$ K
NO + J $\rightarrow$ NOJ
E$_{ME}$(Q$_{IP+K}$ $-$ Q$_{IP+L}$) $\rightarrow$ T$_{NOJ}$
Q$_{IP+K}$ + Q$_{IP+L}$ $\rightarrow$ Q$_{IR+J}$

4
IS
J = NO
?
NO

YES

5
IS
NO > 2
?
YES

NO

6
E$_2$(Q$_{IR+1}$ $-$ Q$_{IR+2}$) $\rightarrow$ T$_2$
E$_1$(Q$_{IR+1}$ + Q$_{IR+2}$) $\rightarrow$ T$_1$
(T$_1$ + T$_2$) / N $\rightarrow$ Q$_1$
(T$_1$ $-$ T$_2$) / N $\rightarrow$ Q$_2$
$-$ 1 $\rightarrow$ IP
$\emptyset \rightarrow$ IR

7
NO $\rightarrow$ N$_2$
N$_2$ / N $\rightarrow$ X
IP + NO/4 $\rightarrow$ IP
IR + NO/2 $\rightarrow$ IR
$\emptyset \rightarrow$ J

8
J + 1 $\rightarrow$ J
INTEGER PART (J+1)/2 $\rightarrow$ ND
N$_2$ + ND $\rightarrow$ ID
$(-1)^{(J+1)} \cdot$ X $\rightarrow$ Y
Q$_{IP+ND}$ + Y $\cdot$ T$_{ID}$ $\rightarrow$ Q$_{IR+J}$

9
IS
J = NO
?
NO

YES

10
IS
NO < N
?
YES

NO

11
OUTPUT   Q
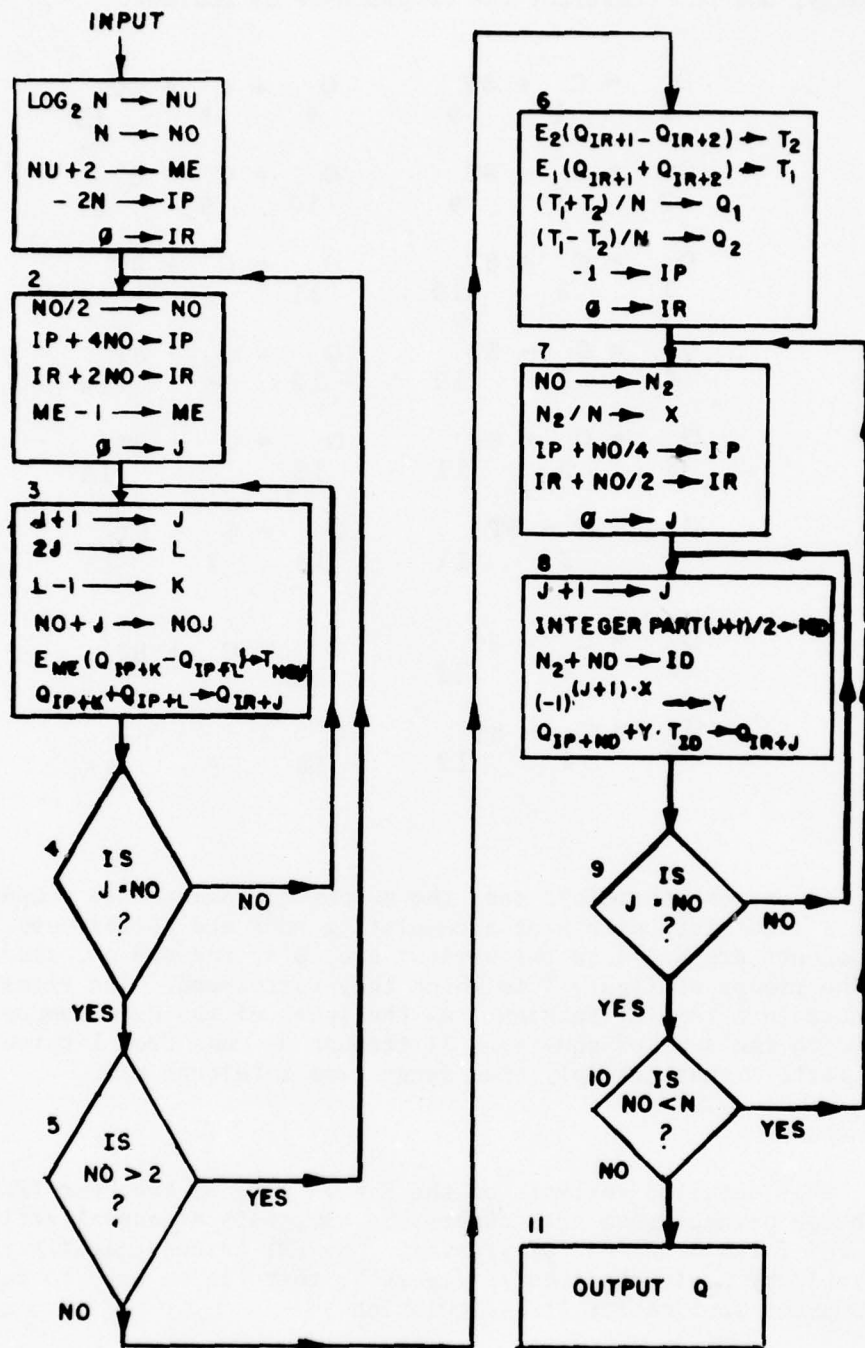
Figure 5.   Fast Haar Transform flow chart.

22

In block 1, N is the number of points to transform, that must equal an integer power of 2. In block 2, IP and IR are the positions one counts from when desiring the current and soon-to-be computed values. The variable ME indicates which group of the transform one is currently in, to use to keep track of the E array, the enhancement or suppression multipliers. For N = 16, it is initially 5. In block 3, NOJ designates which transform component one currently wishes to compute. In the first pass, for the N = 16 case, it ranges from 9 through 16 as in the second half of equation 22, where one computes T_9 and Q_{IR+1}, which represents A_1. The indices L and K keep track of which transform components enter these computations. One tests for the end of the loop, and if it is reached, goes to block 5. If the variable NO > 2, one goes back to block 2, where the counters are reset, ME is decreased, and one is in the next group, i.e., 4. For the N = 16 case, one is now computing transform elements T_5 through T_8 and the B array.

When the loops break, one computes T_1 and T_2 directly. All this time, the multiplier factors are being factored into the transform components, doing all the data filtering and manipulation. Then in block 6, one computes Q_1 and Q_2, the A array of equation 31. Then, in block 7, NO is set to 4 and one computes the B array of equation 32. Finally, on the last pass, when NO = N, the output data points are computed as in equation 34. These values are the final answers and need not be scaled, for that was accomplished by the variable X in block 7. Block 11 contains 2N-2 entries, in the Q array, of which the last N are the outputs.

Fortran Subroutine

Table 1 is a listing of a FORTRAN subroutine which will implement the Haar Transform and return the output data of up through 64 points. The line numbers start with No. 610, to allow room for the placement of a main data-producing program ahead of the routine to form one file. The user is required to furnish the Q array of 126 or 2N-2 positions, the first 64 or N of which are the input data, while the last 62 may contain computer "garbage", since they will be overwritten by the partial sum arrays.

The user must also specify NU, the log to the base 2 of N, which for 64 points is of course 6. Giving a number smaller than this will implement the transform of correspondingly fewer points: 32, 16, etc. Finally, the user must furnish the array E, the NU+1 multipliers or enhancement factors that are incorporated into the transform to modify the data under analysis.

23

Table 1.   FORTRAN subroutine to implement Haar Transform

```
      SUBROUTINE HAAR(Q,E,NU)                                   000610
      DIMENSION T (64), Q (126), E (7)                          000620
      N =2**NU                                                  000630
      NO =N                                                     000640
      ME = NU+2                                                 000650
      IP = (-2)*N                                               000660
      IR = 0                                                    000670
5     NO = NO/2                                                 000680
      IP = IP + 4*NO                                            000690
      IR = IR + 2*NO                                            000700
      ME = ME-1                                                 000710
      DO 6 J =1,NO                                              000720
      L = 2*J                                                   000730
      K = L-1                                                   000740
      NOJ = NO+J                                                000750
      T(NOJ) = E(ME)*(Q(IP+K)-Q(IP+L))                          000760
6     Q(IR+J) =Q(IP+K)+Q(IP+L)                                  000770
      IF (NO.GT.2) GO TO 5                                      000780
      T(2)  =E(2)*(Q(IR+1)-Q(IR+2))                             000790
      T(1)  =E(1)*(Q(IR+1)+Q(IR+2))                             000800
      Q(1)  =(T(1)+T(2))/FLOAT(N)                               000810
      Q(2)  =(T(1)-T(2))/FLOAT(N)                               000820
      IP= -1                                                    000830
      IR = 0                                                    000840
7     N2 = NO                                                   000850
      X = FLOAT(N2)/FLOAT(N)                                    000860
      NO = NO*2                                                 000870
      IP = IP + NO/4                                            000880
      IR = IR + NO/2                                            000890
      DO 8 J =1,NO                                              000900
      ND = (J+1)/2                                              000910
      ID = N2 + ND                                              000920
      Y = (-1.0)**(J+1)*X                                       000930
8     Q(IR+J) = Q(IP+ND)+Y*T(ID)                                000940
      IF (NO.LT.N) GO TO 7                                      000950
      RETURN                                                    000960
      END                                                       000970
```

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APPLICATIONS

Signal Decomposition

The Haar Transform, besides being more efficient in terms of
computer time than the Fast Fourier Transform, does not contain its
inherent aliasing problem, which is that the last N/2 spatial frequency
elements that the Fast Fourier Transform generates are completely
superfluous data. By virtue of its construction, the highest spatial
frequency that the Haar Transform algorithm samples is N/2 complete
square-wave oscillations per N data points. For NU = $\log_2$ N, just
NU+1 output weights exist, one for the DC term and one each for the
NU Haar functions of the NUth order being employed. Therefore, the
Haar transform is capable of filtering and reconstructing any set of
digital data, as has been shown, even if they are not explicitly peri-
odic or continuous. The transform is widely applicable to the analy-
sis, filtering, and bandwidth compression of any pulsed data. It can
be employed for image processing and enhancement of video data (ref 2).
Given a set of quantized video data, the Haar Transform can bring out
edge and feature detail, i.e., improve the image contrast. It can
suppress noise, i.e., increase the signal-to-noise ratio. Finally,
it can be employed to decrease the amount of video data, i.e., com-
press the TV transmission bandwidth to speed up video data transmis-
sion, or alleviate the very likely possibility of channel crowding.
Compression is done by zeroing out one or more of the highest square-
wave components of portions of an image that contain little high-fre-
quency information. Zeroing one component produces output with ad-
jacent elements doubled, as in the digital output function:

$$OUT_D = 1, 1, \quad 2, 2, \quad 3, 3, \quad 2, 2, \quad 1, 1, \tag{35}$$

One then selects every second point, cutting the data volume in half.

Table 1 is constructed from the output of computer program HARTST
(app C). It presents an example of Haar Transform signal decomposition
and shows the spatial frequency contributions to the nonperiodic digital
ramp: 1, 2, 3, ... 16. This is the digital function:

$$F_d (x_i) = i + 1 \tag{36}$$

where i = 0, 1, 2, ... 15. The table shows values ranging from the
large DC term, through the last, the highest frequency of oscillation,
i.e., 8 cycles in the data field length of 16 points.

25

Waveform Synthesis

Just as in the Fourier representation where one can construct a square-wave by the superpositioning of sine waves of designated frequency and amplitude, one can perfectly construct a sine wave by the superpositioning of its Haar Transform square-wave components.

This is demonstrated in the graphs of figures 6 through 12, where a 128-point, 1 cycle sine wave is approximated by 2, 3, 4, 5, 6, 7, and finally all of its Haar components as presented by Program FIG. 6, (app D.) The figures are scaled to the same size. No DC-term is present, because the average value of sine x is, of course, 0.

The weights of the various Haar components are:

$$F_1 = .63649$$

$$F_2 = -.01562$$

$$F_4 = -.27012$$

$$F_8 = -.18087$$

$$F_{16} = -.09643$$

$$F_{32} = -.04892$$

$$F_{64} = -.02453$$

$$(37)$$

where the subscript on each variable is indicative of the number of cycles of oscillation of that given Haar component per data field length of 128 points.

26

Table 2. Square wave components of a Haar Transform signal decomposition

Input	DC-term	F_1	F_2	F_4	f_8
1	8.5	-4	-2	-1	-.5
2	8.5	-4	-2	-1	.5
3	8.5	-4	-2	1	-.5
4	8.5	-4	-2	1	.5
5	8.5	-4	2	-1	-.5
6	8.5	-4	2	-1	.5
7	8.5	-4	2	1	-.5
8	8.5	-4	2	1	.5
9	8.5	4	-2	-1	-.5
10	8.5	4	-2	-1	.5
11	8.5	4	-2	1	-.5
12	8.5	4	-2	1	.5
13	8.5	4	2	1	.5
14	8.5	4	2	-1	.5
15	8.5	4	2	1	-.5
16	8.5	4	2	1	.5

27
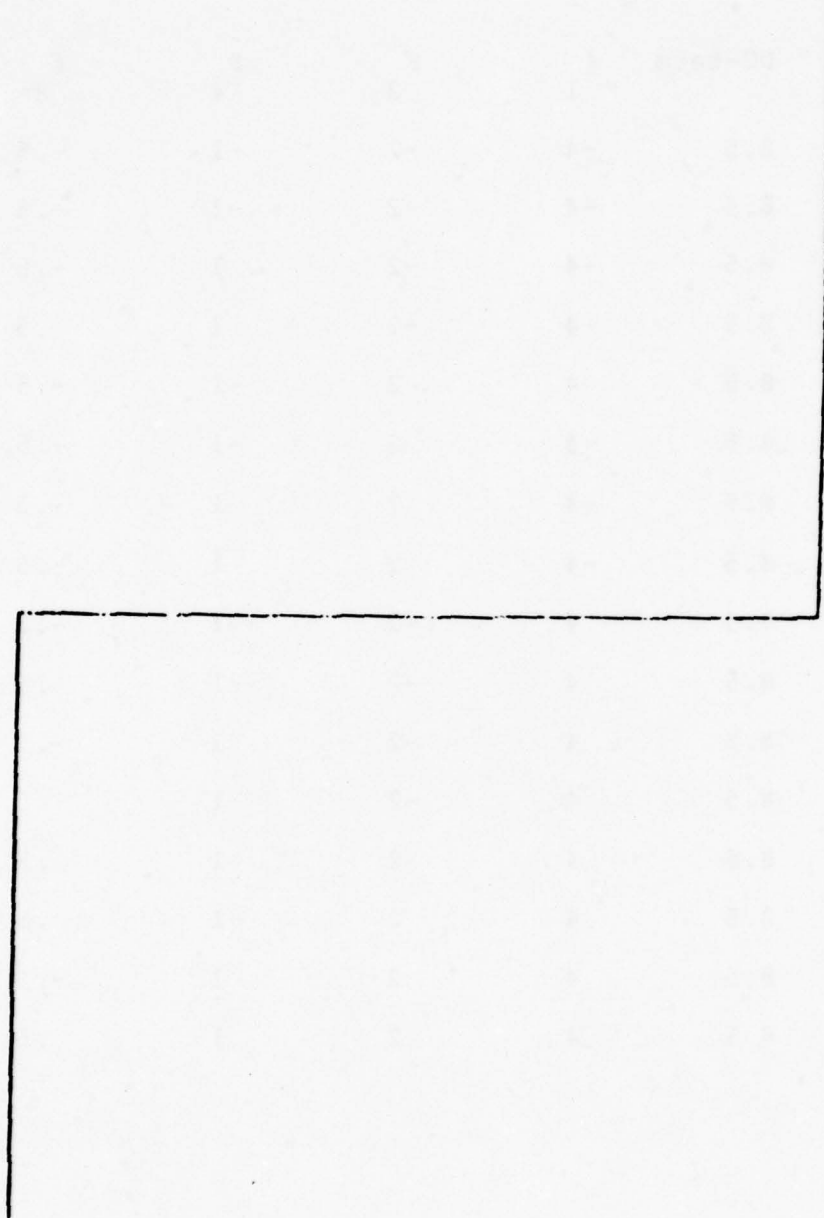
Figure 6.   2 Haar components for sine synthesis.
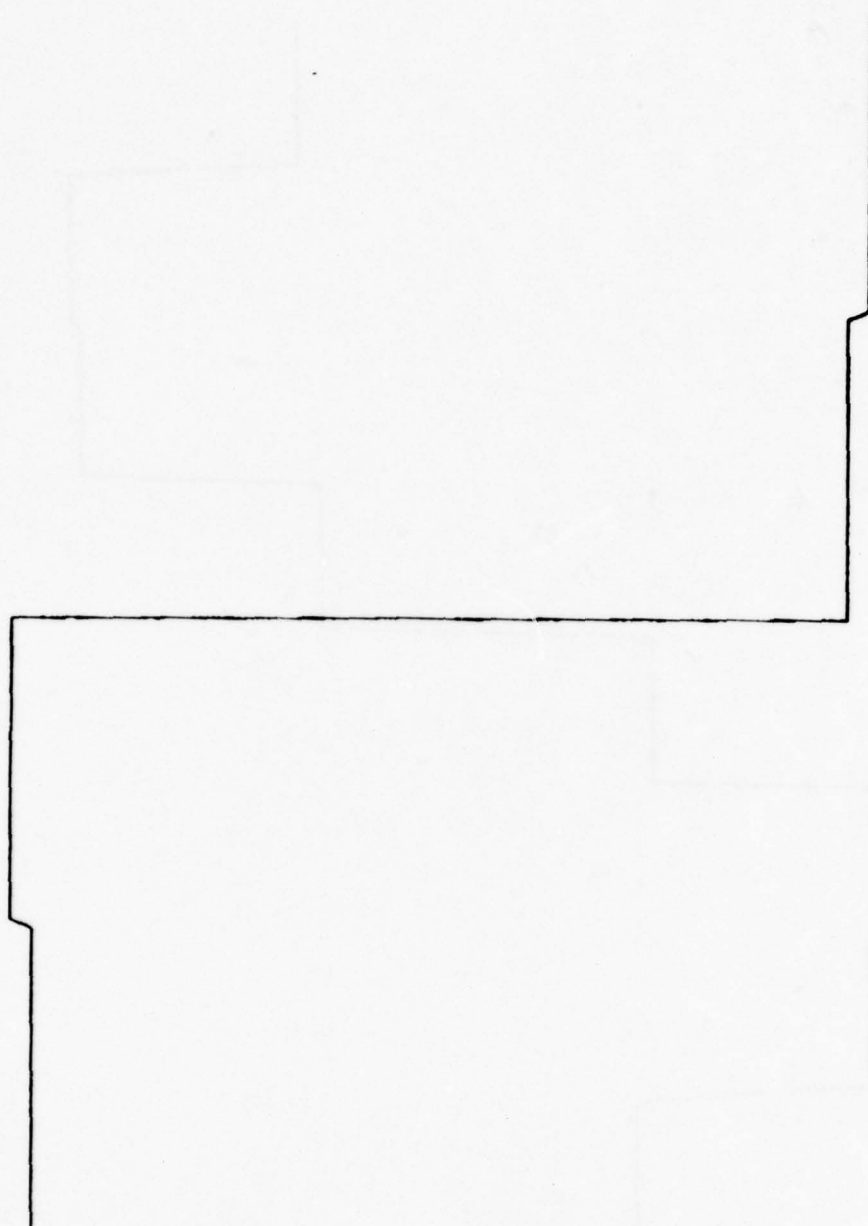
Figure 7.   3 Haar components for sine wave synthesis.
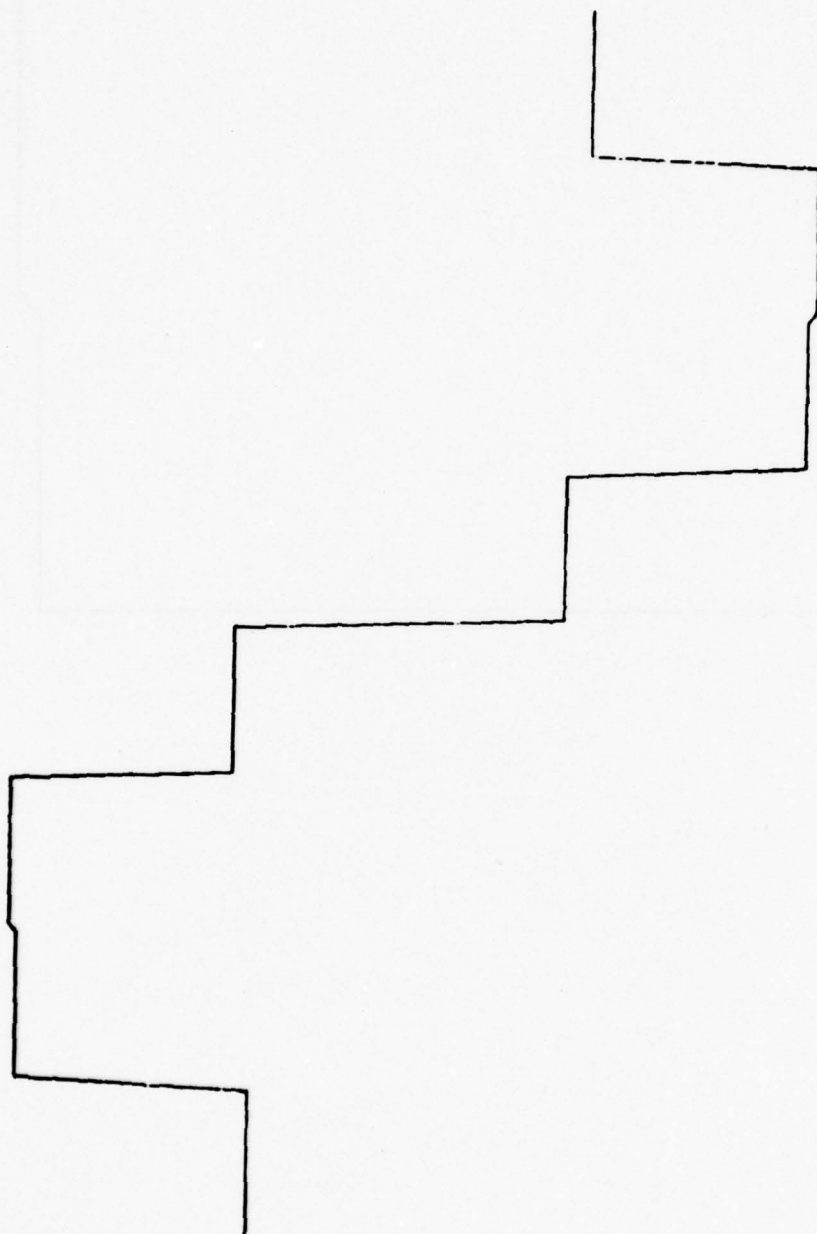
Figure 8. 4 Haar components for sine wave synthesis.

30
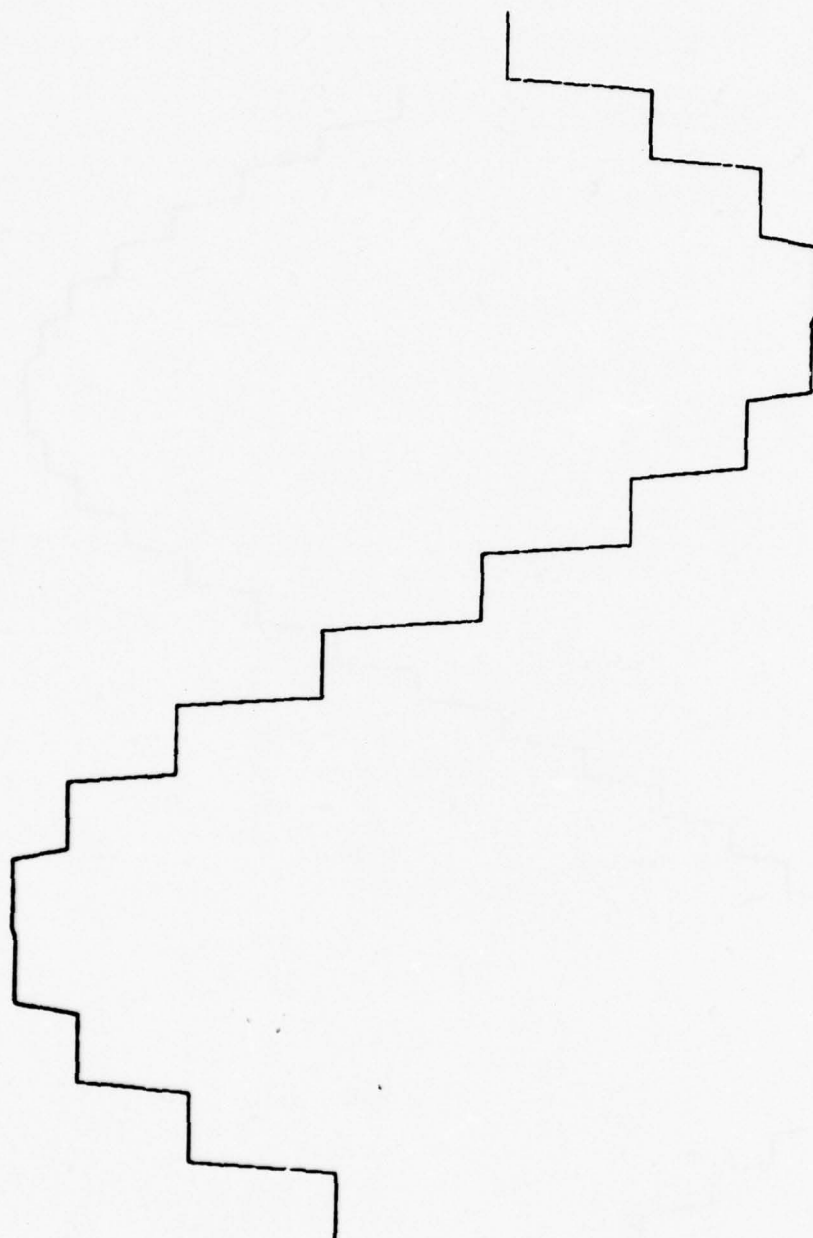
Figure 9.   5 Haar components for sine wave synthesis.

31
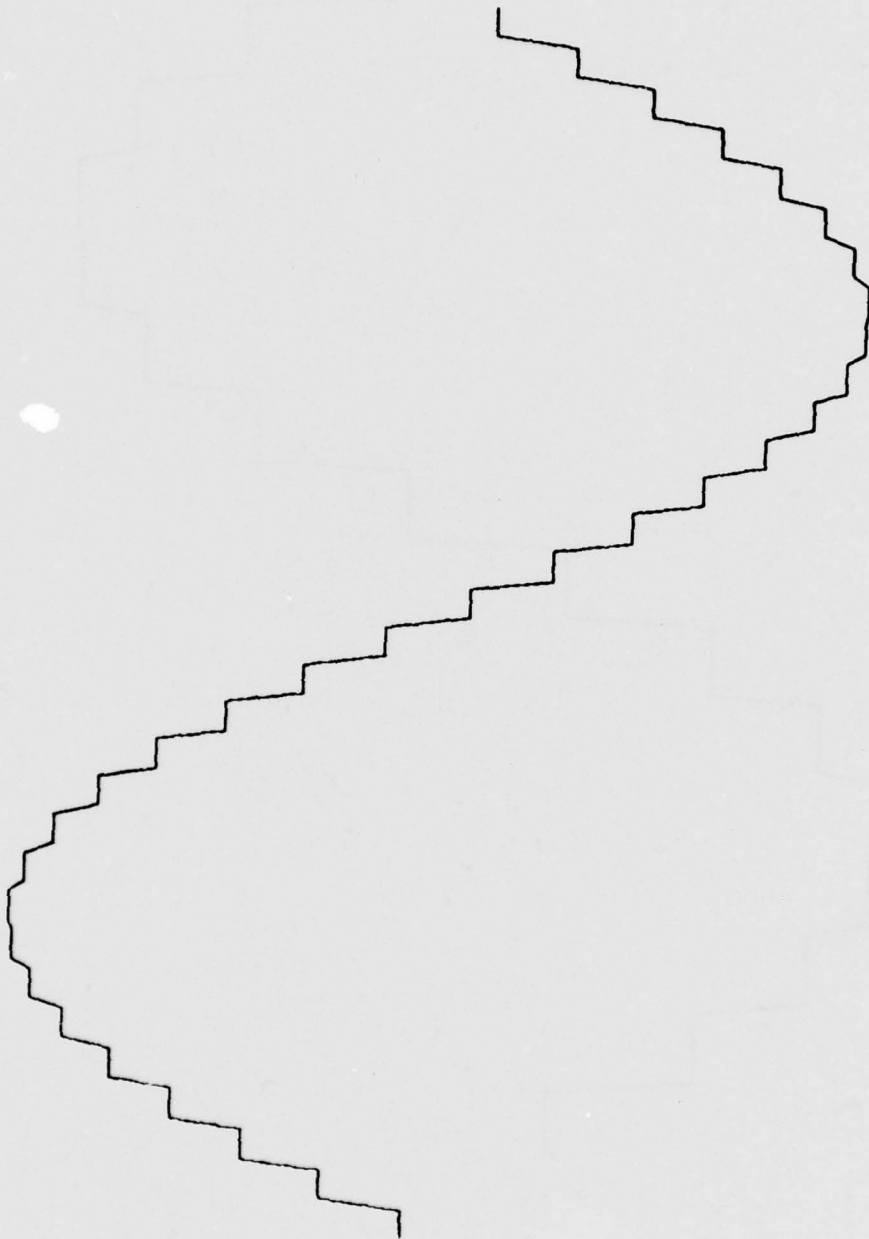
Figure 10. 6 Haar components for sine wave synthesis.

32
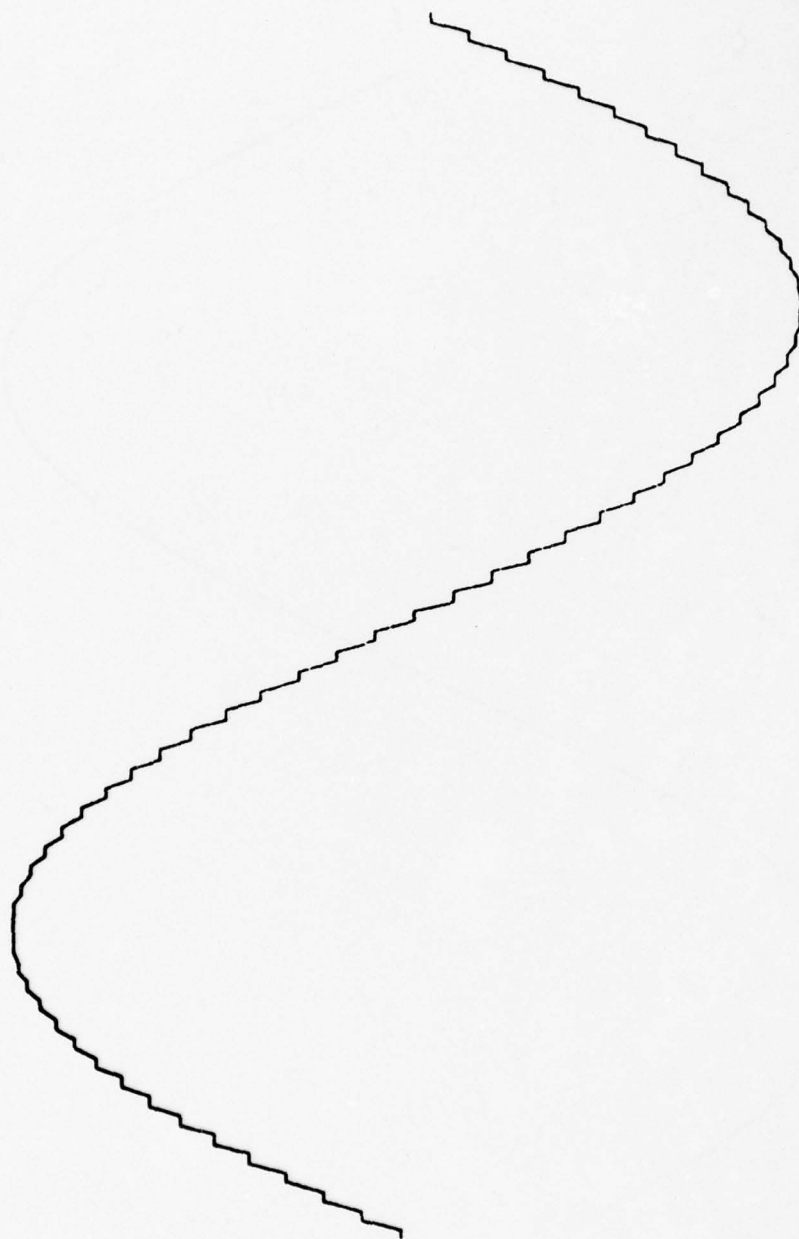
Figure 11.   7 Haar components for sine wave synthesis.

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Figure 12. All 8 Haar components: synthesized sine wave.

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CONCLUSIONS

A new powerful and versatile tool is now available for digital data analysis. Some of the advantages that the Fast Haar Transform has over the Fast Fourier Transform are:

     1. Computations are done between 4 and 5 times faster in terms of the computer time required, thereby realizing an 80 percent savings in computer time and cost.

     2. No requirements to generate or store in memory, tables of trigonometric values.

     3. Aliasing ambiguities are eliminated.

     4. It is optimally suited for pulsed data, and image analysis and enhancement.

Although available in the literature, material on the theory and applications of the Haar Transform is not prolific. Hopefully the presentation of this original work has provided the needed information in a clear and logical manner, and will be of value as an aid in digital data processing applications.

REFERENCES

1. G. Alexits, Convergence Problems of Orthogonal Series, Pergamon, NY, 1961, p 46-72

2. Gary Sivak, "Advanced Video Signal Processing for Image Enhancement," Frankford Arsenal 1976 Technical Symposium, p 95-114

APPENDIX A

PROGRAM HIMAT

```
      PROGRAM HIMAT(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)      000010
      DIMENSION MAT(16,16)                                      000020
      DO 1 I = 1,16                                             000030
      DO 1 J = 1,16                                             000040
    1 MAT(I,J) = 0                                              000050
      DO 2 I = 1,16                                             000060
      MAT(I,1) = 1                                              000070
      IF(I .LE. 8) MAT(I,2) = 1                                 000080
    2 IF(I .GT. 8) MAT(I,2) = -1                                000090
      DO 3 J = 3,4                                              000100
      DO 3 I = 1,8                                              000110
      K1 = (I-1) /4                                             000120
      K = J-3                                                   000130
      L = 2*(-1)**K1                                            000140
      I1 = I + 8*K                                              000150
    3 MAT(I1,J) = L                                             000160
      DO 4 J = 5,8                                              000170
      DO 4 I = 1,4                                              000180
      K1 = (I-1) /2                                             000190
      K = J-5                                                   000200
      L = 4*(-1)**K1                                            000210
      I1 = I + 4*K                                              000220
    4 MAT(I1,J) = L                                             000230
      DO 5 J = 9,16                                             000240
      DO 5 I = 1,2                                              000250
      K1 = I-1                                                  000260
      K = J-9                                                   000270
      L = 8*(-1)**K1                                            000280
      I1 = I + 2*K                                              000290
    5 MAT(I1,J) = L                                             000300
      DO 6 I = 1,16                                             000310
    6 WRITE (6,500) (MAT(I,J),J=1,16)                           000320
  500 FORMAT (12X,16I3)                                         000330
      READ *,DUMMY                                              000340
      STOP                                                      000350
      END                                                       000360
```

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APPENDIX B

PROGRAM FIG4

```
      PROGRAM FIG4(INPUT,OUTPUT,TAPE61=100,TAPE62=100)              000010
      DIMENSION LETTER(3),ITAB(138),XCO(46),YCO(46),MSG(42)         000020
      DATA ITAB/1HI,1H1,1H ,1HI,1H2,1H ,1HI,1H3,1H ,1HI,1H4,1H ,1HI,1H5,000030
     &1H ,1HI,1H6,1H ,1HI,1H7,1H ,1HI,1H8,1H ,1HI,1H9,1H ,1HI,1H1,1H0, 000040
     &1HI,1H1,1H1,1HI,1H1,1H2,1HI,1H1,1H3,1HI,1H1,1H4,1HI,1H1,1H5,    000050
     &1HI,1H1,1H6,1HA,1H1,1H ,1HA,1H2,1H ,1HA,1H3,1H ,1HA,1H4,1H ,   000060
     &1HA,1H5,1H ,1HA,1H6,1H ,1HA,1H7,1H ,1HA,1H8,1H ,1HB,1H1,1H ,   000070
     &1HB,1H2,1H ,1HB,1H3,1H ,1HB,1H4,1H ,1HC,1H1,1H ,1HC,1H2,1H ,   000080
     &1HT,1H1,1H ,1HT,1H2,1H ,1HT,1H3,1H ,1HT,1H4,1H ,1HT,1H5,1H ,   000090
     &1HT,1H6,1H ,1HT,1H7,1H ,1HT,1H8,1H ,1HT,1H9,1H ,1HT,1H1,1H0,   000100
     &1HT,1H1,1H1,1HT,1H1,1H2,1HT,1H1,1H3,1HT,1H1,1H4,1HT,1H1,1H5,   000110
     &1HT,1H1,1H6/                                                   000120
      DATA MSG/70,105,103,32,52,2*32,72,2*97,114,32,                 000130
     &116,114,97,110,115,102,111,114,109,32,100,105,97,103,114,      000140
     &97,109,44,32,78,32,61,32,49,54,32,99,97,115,101/               000150
      CALL CONNEC(5LINPUT,0)                                         000160
      CALL CONNEC(6LOUTPUT,0)                                        000170
      XLM = .1                                                       000180
      RM = .1                                                        000190
      TM = .1                                                        000200
      BM = .2                                                        000210
      YFIG = 250.                                                    000220
      CALL INITT(30)                                                 000230
      CALL TERM(2,4096)                                              000240
      CALL DWINDO(0.,4096.,0.,3120.)                                 000250
      CALL CSIZE(IXSIZ,IYSIZ)                                        000260
      XNUMLTS=42.                                                    000270
      XFIG=2048.-XNUMLTS/2.*IXSIZ                                    000280
      DIM = 110.                                                     000290
      DIM2 = DIM / 2.                                                000300
      XS = 40.                                                       000310
      YS = 24.                                                       000320
      SPACE = 31.                                                    000330
      SLOW = 10.                                                     000340
      PS = 31.                                                       000350
      HDIST = (1. - XLM - RM )*4096./15.                             000360
      VDIST = 3120.*(1. - TM - BM ) / 4.                             000370
      HMARG = XLM * 4096.                                            000380
      VSTART = 3120. * (1. - TM )                                    000390
      ICO = 0                                                        000400
      JLET = -2                                                      000410
      DO 1 IROW = 1,5                                                000420
      IF ( IROW .NE. 5 ) NCOL = 2**(5 - IROW)                        000430
      IF ( IROW .EQ. 5) NCOL = 16                                    000440
      SKIP = FLOAT(16 / NCOL )                                       000450
      DO 1 ICOL = 1,NCOL                                             000460
      JLET = JLET + 3                                                000470
```

```
      DO 2 ILET = 1,3                                          000480
   2 LETTER ( ILET ) = ITAB(JLET + ILET - 1)                   000490
      ICO = ICO + 1                                            000500
      XC = HMARG + SKIP*(HDIST/2.+HDIST*(ICOL-1))              000510
      YC = VSTART- FLOAT(IROW-1) * VDIST                       000520
      IF ( ICO .EQ. 19)   XC = XC - .5 * DIM                   000530
      IF ( ICO .EQ. 20) XC = XC - DIM                          000540
      IF ( ICO .EQ. 22) XC = XC + .5*DIM                       000550
      IF( ICO .EQ. 23) XC = XC + .25 * DIM                     000560
      IF ( ICO .EQ. 25) XC = XC + DIM                          000570
      IF ( ICO .EQ. 27) XC = XC - 2. * DIM                     000580
      IF(ICO .EQ. 28) XC = XC -.25*DIM                         000590
      IF ( ICO .EQ. 29) XC = XC - .25 * DIM                    000600
      IF ( ICO .EQ. 30) XC = XC + .25 * DIM                    000610
      XCO(ICO) = XC                                            000620
      YCO(ICO) = YC                                            000630
   1 CALL DRAW(LETTER,DIM,XC,YC,XS,YS,SPACE,SLOW,PS)           000640
      NCO1 = -32                                               000650
      NCO2 = 0                                                 000660
      DO 3 I1 = 1,3                                            000670
      NDRAW = 2**(5-I1)                                        000680
      NCO1 = NCO1 + 2*NDRAW                                    000690
      NCO2 = NCO2 + NDRAW                                      000700
      DO 3 IDRAW = 1,NDRAW                                     000710
      JDRAW = (IDRAW+1) / 2                                    000720
      ICO = NCO1 + IDRAW                                       000730
      JCO = NCO2 + JDRAW                                       000740
      XL1 = XCO(ICO) - DIM2                                    000750
      YL1 = YCO(ICO) - DIM2                                    000760
      XL2 = XCO(JCO)                                           000770
      YL2 = YCO(JCO) + DIM2                                    000780
      CALL MOVEA(XL1, YL1)                                     000790
   3 CALL DRAWA(XL2,YL2)                                       000800
      NCO1 = -32                                               000810
      DO 4 I1 = 1,3                                            000820
      NDRAW = 2**(5-I1)                                        000830
      NCO1 = NCO1 + 2 * NDRAW                                  000840
      NCO2 = NDRAW / 2 + 30                                    000850
      DO 4 IDRAW = 1,NDRAW                                     000860
      Q = DIM2 * FLOAT((-1)**IDRAW)                            000870
      JDRAW = (IDRAW + 1) / 2                                  000880
      ICO = NCO1 + IDRAW                                       000890
      JCO = NCO2 + JDRAW                                       000900
      XL1 = XCO(ICO) + Q                                       000910
      YL1 = YCO(ICO) - DIM2                                    000920
      XL2 = XCO(JCO)                                           000930
      YL2 = YCO(JCO) + DIM2                                    000940
      CALL MOVEA(XL1, YL1)                                     000950
   4 CALL DRAWA(XL2, YL2)                                      000960
      DO 5 IT = 1,2                                            000970
      DO 5 IC = 1,2                                            000980
```

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```
      ICO = 28 + IC                                                  000990
      JCO = 30 + IT                                                  001000
      K = 1 +   (IT+IC) / 4                                          001010
      XL1 = XCO(ICO) + FLOAT((-1)**K) * DIM2                         001020
      YL1 = YCO(ICO) - DIM2                                          001030
      XL2 = XCO(JCO)                                                 001040
      YL2 = YCO(JCO) + DIM2                                          001050
      CALL MOVEA(XL1, YL1)                                           001060
    5 CALL DRAWA(XL2, YL2)                                           001070
      CALL CHRSIZ(1)                                                 001080
      CALL MOVEA(XFIG,YFIG)                                          001090
      CALL ANSTR(42,MSG)                                             001100
      CALL ANMODE                                                    001110
      READ *,DUMMY                                                   001120
      STOP                                                           001130
      END                                                            001140
      SUBROUTINE DRAW(LETTER,DIM,XC,YC,XS,YS,SPACE,SLOW,PS)          001150
      DIMENSION LETTER(3), X(4),Y(4),ISIGN(2)                        001160
      DATA ISIGN/1H+,1H-/                                            001170
      XMIN = XC - DIM / 2.                                           001180
      XMAX = XC + DIM / 2.                                           001190
      YMIN =  YC - DIM / 2.                                          001200
      YMAX = YC + DIM / 2.                                           001210
      X(1) = XMIN                                                    001220
      X(2) = XMAX                                                    001230
      X(3) = XMAX                                                    001240
      X(4) = XMIN                                                    001250
      Y(1) = YMIN                                                    001260
      Y(2) = YMIN                                                    001270
      Y(3) = YMAX                                                    001280
      Y(4) = YMAX                                                    001290
      CALL CHRSIZ(3)                                                 001300
      DO 1 I = 1,2                                                   001310
      XL = XC - XS + FLOAT(I-1) * SPACE                              001320
      IF(LETTER(3).EQ." ") XL = XL + 10.                             001330
      YL = YC - YS - FLOAT(I-1) * SLOW                               001340
      CALL MOVEA(XL, YL)                                             001350
      IF(I.EQ.2) CALL CHRSIZ(4)                                      001360
    1 CALL A1OUT(I,LETTER(I))                                        001370
      CALL MOVEA(X(4), Y(4))                                         001380
      DO 2 I = 1,4                                                   001390
    2 CALL DRAWA(X(I), Y(I))                                         001400
      DO 3 I = 1,2                                                   001410
      XL = XMIN + FLOAT(I-1)*DIM + FLOAT(I-2)*PS                     001420
      IF(I.EQ.2) XL = XL + 15.                                       001430
      YL = YMIN                                                      001440
      CALL MOVEA(XL, YL)                                             001450
    3 CALL A1OUT(1,ISIGN(I))                                         001460
      CALL ANMODE                                                    001470
      RETURN                                                         001480
      END                                                            001490
```

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APPENDIX C

PROGRAM HARTST

```
      PROGRAM HARTST(INPUT,OUTPUT)                                    000010
      DIMENSION X(64),Y(64),Q(126),E(7),VAL(13)                      000020
      CALL CONNEC(5LINPUT,0)                                         000030
      CALL CONNEC(6LOUTPUT,0)                                        000040
    1 PRINT *,"ENTER THE LOG TO THE BASE 2, THE NUMBER OF POINTS. "  000050
      READ *,NU                                                      000060
      NU1 = NU + 1                                                   000070
      N = 2**NU                                                      000080
      NPC = 2*N                                                      000090
      M = N-2                                                        000100
      DO 2 I = 1,N                                                   000110
    2 X(I) = 0.                                                      000120
      PRINT *,"ENTER ",NU1," MULTIPLIERS. "                          000130
      READ *,(E(J),J=1,NU1)                                         000140
      PRINT *,"FOR SYNTHESIS OR ANALYSIS, TYPE 1 OR 2. "            000150
      READ *,IPICK                                                   000160
      IF(IPICK .NE. 1) GO TO 3                                       000170
      PRINT *,"ENTER DC-LEVEL. "                                     000180
      READ *,VAL(1)                                                  000190
      DO 4 I = 1,NU                                                  000200
      IC = 2**(I-1)                                                  000210
      IL = 2*I                                                       000220
      IH = IL + 1                                                    000230
      PRINT *,"FOR ",IC," CY.S OF SPFQ. ",I,", ENTER LOW AND HIGH. " 000240
    4 READ *,VAL(IL),VAL(IH)                                         000250
      DO 5 I = 1,NU1                                                 000260
      NPC = NPC /2                                                   000270
      DO 5 J = 1,N                                                   000280
      L = 2*I-2 + MOD( (J-1) / NPC ,2)                              000290
      IF(L .EQ. 0) L = 1                                             000300
    5 X(J) = X(J) + VAL(L)                                           000310
      GO TO 6                                                        000320
    3 PRINT *,"ENTER ",N," INPUTS. "                                 000330
      READ *,(X(J),J=1,N)                                           000340
    6 DO 7 I = 1,N                                                   000350
    7 Q(I) = X(I)                                                    000360
      CALL HAAR(Q,E,NU)                                              000370
      DO 8 I = 1,N                                                   000380
    8 Y(I) = Q(M+I)                                                  000390
      PRINT *,"INPUT    OUTPUT"                                      000400
      DO 9 I = 1,N                                                   000410
    9 PRINT 430,X(I),Y(I)                                            000420
  430 FORMAT (1H ,2(F8.3,2X))                                        000430
      READ *,DUMMY                                                   000440
      IF(DUMMY .EQ. 1) GO TO 1                                       000450
      STOP                                                           000460
      END                                                            000470
```

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```
      SUBROUTINE HAAR(Q,E,NU)                                    000480
      DIMENSION T(1024),Q(2046),E(11)                            000490
      N =2**NU                                                   000500
      NO =N                                                      000510
      ME = NU+2                                                  000520
      IP = (-2)*N                                                000530
      IR = 0                                                     000540
    5 NO = NO/2                                                  000550
      IP = IP + 4*NO                                             000560
      IR = IR + 2*NO                                             000570
      ME = ME-1                                                  000580
      DO 6 J =1,NO                                               000590
      L = 2*J                                                    000600
      K = L-1                                                    000610
      NOJ = NO+J                                                 000620
      T(NOJ) = E(ME)*(Q(IP+K)-Q(IP+L))                          000630
    6 Q(IR+J) =Q(IP+K)+Q(IP+L)                                   000640
      IF (NO.GT.2) GO TO 5                                       000650
      T(2) =E(2)*(Q(IR+1)-Q(IR+2))                               000660
      T(1) =E(1)*(Q(IR+1)+Q(IR+2))                               000670
      Q(1) =(T(1)+T(2))/FLOAT(N)                                 000680
      Q(2) =(T(1)-T(2))/FLOAT(N)                                 000690
      IP= -1                                                     000700
      IR = 0                                                     000710
    7 N2 = NO                                                    000720
      X = FLOAT(N2)/FLOAT(N)                                     000730
      NO = NO*2                                                  000740
      IP = IP + NO/4                                             000750
      IR = IR + NO/2                                             000760
      DO 8 J =1,NO                                               000770
      ND = (J+1)/2                                               000780
      ID = N2 + ND                                               000790
      Y = (-1.0)**(J+1)*X                                        000800
    8 Q(IR+J) = Q(IP+ND)+Y*T(ID)                                 000810
      IF (NO.LT.N) GO TO 7                                       000820
      RETURN                                                     000830
      END                                                        000840
```

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PROGRAM FIG6

```
      PROGRAM FIG6(INPUT,OUTPUT,TAPE61=100,TAPE62=100)              000010
      DIMENSION X(128),Y(128),Q(254),E(8),MSG(50),MSG8(53)         000020
      DATA MSG/70,105,103,2*32,54,46,2*32,50,32,72,2*97,114,32,    000030
     &99,111,109,112,111,110,101,110,116,115,32,102,111,114,32,    000040
     &115,105,110,101,32,119,97,118,101,32,                        000050
     &115,121,110,116,104,101,115,105,115/                         000060
      DATA MSG8/70,105,103,32,49,50,46,2*32,65,2*108,32,56,32,     000070
     &72,2*97,114,32,99,111,109,112,111,110,101,110,116,115,58,32, 000080
     &115,121,110,116,104,101,115,105,115,122,101,100,32,          000090
     &115,105,110,101,32,119,97,118,101/                           000100
      CALL CONNEC(5LINPUT,0)                                       000110
      CALL CONNEC(6LOUTPUT,0)                                      000120
      CALL INITT(30)                                               000130
      CALL TERM(2,4096)                                            000140
      XLM = .1                                                     000150
      RM = .1                                                      000160
      TM = .1                                                      000170
      BM = .2                                                      000180
      YFIG = 250.                                                  000190
      XNMSG = 50.                                                  000200
      CALL CSIZE(IXSIZ,IYSIZ)                                      000210
      XFIG = 2048.-XNMSG/2.*IXSIZ                                  000220
      PI = 4. * ATAN(1.)                                           000230
      NU = 7                                                       000240
      NU1 = NU+1                                                   000250
      N = 2**NU                                                    000260
      M = N-2                                                      000270
      NC = 1                                                       000280
      DX = 2. * PI * FLOAT(NC) / FLOAT(N)                          000290
      DO 1 NCOMP = 2,8                                             000300
      DO  2 I=1,NU1                                                000310
    2 E(I) = 0.0                                                   000320
      DO 3 ICOMP = 1,NCOMP                                         000330
    3 E(ICOMP) = 1.0                                               000340
      XMIN = YMIN = YMAX = 0.                                      000350
      DO 4 I = 1,N                                                 000360
      X(I) = DX * FLOAT(I-1)                                       000370
    4 Q(I) = SIN(X(I))                                             000380
      XMAX = X(N)                                                  000390
      CALL HAAR(Q, E, NU)                                          000400
      DO 5 I = 1,N                                                 000410
      Y(I) = Q(M+I)                                                000420
      IF ( Y(I) .GT. YMAX) YMAX = Y(I)                             000430
    5 IF ( Y(I) .LT. YMIN ) YMIN = Y(I)                            000440
      HH = (XMAX-XMIN) / (1. - XLM - RM)                           000450
      XMIN = XMIN - HH*XLM                                         000460
      XMAX = XMAX + HH * RM                                        000470
```

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```
      HV = (YMAX - YMIN ) / (1. - TM - BM)                            000480
      YMAX = YMAX + HV * TM                                           000490
      YMIN = YMIN - HV * BM                                           000500
      CALL DWINDO(XMIN,XMAX,YMIN,YMAX)                                000510
      CALL MOVEA(X(1), Y(1))                                          000520
      DO 6 I = 2,N                                                    000530
    6 CALL DRAWA(X(I), Y(I))                                          000540
      IF(NCOMP .EQ. 8) XFIG = XFIG - 2.*IXSIZ                         000550
      CALL DWINDO(0.,4096.,0.,3120.)                                  000560
      CALL MOVEA(XFIG,YFIG)                                           000570
      IF(NCOMP .EQ. 8) GO TO 7                                        000580
      IF(NCOMP .EQ. 6) MSG(5) = 49                                    000590
      IF(NCOMP .EQ. 6) MSG(6) = 47                                    000600
      IF(NCOMP .NE. 2) MSG(6) = MSG(6) +1                             000610
      IF(NCOMP .NE. 2) MSG(10) = MSG(10) + 1                          000620
    7 IF(NCOMP .NE. 8) CALL ANSTR(50,MSG)                             000630
      IF(NCOMP .EQ. 8) CALL ANSTR(55,MSG8)                            000640
      CALL ANMODE                                                     000650
    1 READ 100,ICHAR                                                  000660
  100 FORMAT (A1)                                                     000670
      END                                                             000680
      SUBROUTINE HAAR(Q,E,NU)                                         000690
      DIMENSION T(1024),Q(2046),E(11)                                 000700
      N =2**NU                                                        000710
      NO =N                                                           000720
      ME = NU+2                                                       000730
      IP = (-2)*N                                                     000740
      IR = 0                                                          000750
    5 NO = NO/2                                                       000760
      IP = IP + 4*NO                                                  000770
      IR = IR + 2*NO                                                  000780
      ME = ME-1                                                       000790
      DO 6 J =1,NO                                                    000800
      L = 2*J                                                         000810
      K = L-1                                                         000820
      NOJ = NO+J                                                      000830
      T(NOJ) = E(ME)*(Q(IP+K)-Q(IP+L))                               000840
    6 Q(IR+J) =Q(IP+K)+Q(IP+L)                                        000850
      IF (NO.GT.2) GO TO 5                                            000860
      T(2) =E(2)*(Q(IR+1)-Q(IR+2))                                   000870
      T(1) =E(1)*(Q(IR+1)+Q(IR+2))                                   000880
      Q(1) =(T(1)+T(2))/FLOAT(N)                                      000890
      Q(2) =(T(1)-T(2))/FLOAT(N)                                      000900
      IP= -1                                                          000910
      IR = 0                                                          000920
    7 N2 = NO                                                         000930
      X = FLOAT(N2)/FLOAT(N)                                          000940
      NO = NO*2                                                       000950
      IP = IP + NO/4                                                  000960
      IR = IR + NO/2                                                  000970
      DO 8 J =1,NO                                                    000980
```

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```
      ND = (J+1)/2                                      000990
      ID = N2 + ND                                      001000
      Y = (-1.0)**(J+1)*X                               001010
    8 Q(IR+J) = Q(IP+ND)+Y*T(ID)                        001020
      IF (NO.LT.N) GO TO 7                              001030
      RETURN                                            001040
      END                                               001050
```

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