A FINITE-DIFFERENCE FORMULATION OF
THE MAGNETIC VECTOR POTENTIAL

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This interim report represents the first phase in the investigation of the computer analysis of nonlinear electromagnetic circuits using magnetic field vector quantities. The finite difference method is formulated and boundary conditions are discussed. Solution procedures are stated.

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INTRODUCTION

Magnetic circuits are currently used in projectile fuzing circuits. Specifically, magnetic setback generators are used to power fuzing circuits. Computer analysis, using magnetic field vector quantities, can be a valuable tool in the optimization of such systems. A theoretical basis of the finite difference method used to calculate the magnetic vector potential in a nonlinear magnetic circuit is presented herein.

A typical magnetic setback generator is shown in figure 1. This configuration will be used to derive results that will be generally applicable.

The determination of the magnetic field distribution of a circuit containing iron, current densities, and permanent magnets is not amenable to linear analysis. The nonlinear, partial differential equations and boundary conditions for the cross-sectional region of a machine can be defined, but analytical solutions to the defined problem are only possible for the most simple cases. The difficulty is due to:

Material inhomogeneity - There are large differences in the reluctivities of the magnetic material (iron) and nonmagnetic materials (air, copper).

Geometrically complex media interfaces

Material nonlinearity - The reluctivity is a function of the flux density in the magnetic material.

The approach presented here is to transform the system equations into a finite difference formulation, defined at a finite number of discrete points within the circuit. The resulting set of simultaneous equations can then be solved by an iterative procedure, while taking into account the material's nonlinearity.

The following assumptions are made:

- The field distribution is two-dimensional.

- The field is static (stationary).

- The machine is infinitely long (in the Z direction) and end effects can be neglected.

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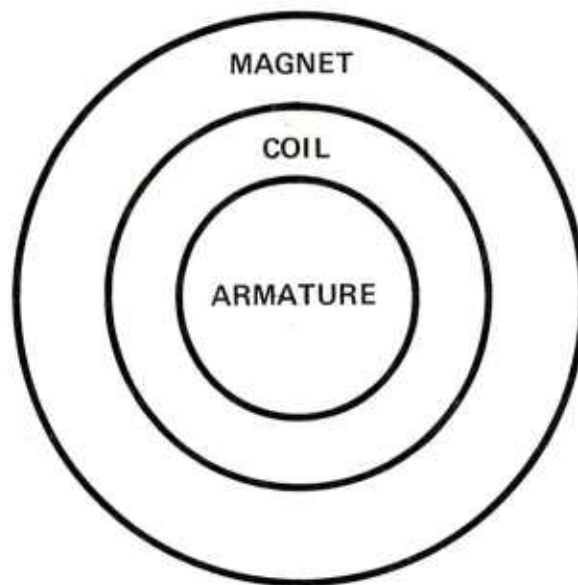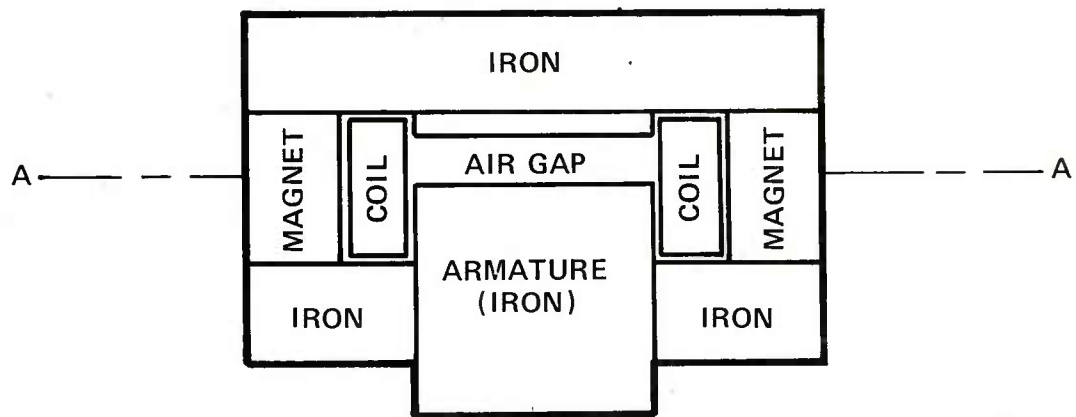
Figure 1. Magnetic setback generator

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- The magnetization characteristic of the materials is represented by a single valued function, i.e. hysteresis is neglected.

- The material is isotropic.

- The eddy current losses may be neglected (see appendix A).

The assumptions allow the performance of a two-dimensional analysis. Of course, a three-dimensional model would be more representative of the physical problem (at the cost of increased model complexity and computation time). Nevertheless, the model presented should yield some useful results.

The following notations will be used throughout this report.

$\vec{A}$ — Magnetic vector potential

$\vec{B}$ — Magnetic flux density (magnetic induction)

$\vec{H}$ — Magnetic field intensity

I — Current

J — Current density

L — Path segment

$\vec{M}$ — Magnetization vector

S — Surface area

$\vec{a}_X$ — Unit vector in the X direction

h — Mesh cell dimension

μ — Permeability

ν — Reluctivity

ρ — Charge density

∇ — The vector operator ($\nabla = \frac{\partial}{\partial X} \vec{a}_X + \frac{\partial}{\partial Y} \vec{a}_Y$

$+ \frac{\partial}{\partial Z} \vec{a}_Z$)

THE NONLINEAR VECTOR POTENTIAL FORMULATION

The Basic Equation

Electromagnetic field problems can be solved through solution of Maxwell's equations with appropriate boundary conditions. Unfortunately, in many cases the mathematics become difficult when using Maxwell's equations directly. The magnetic vector potential (A) is used in this formulation to simplify the mathematics.

The partial differential equation describing the vector potential for steady magnetic fields in two dimensions using rectangular coordinates is:

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y} \right) = -J \qquad (1)$$

where ν, A, and J are the reluctivity (see appendix B), the Z component of the vector potential, and the Z component of the current density, respectively.

Equation (1) will now be derived. The magnetic field intensity ($\vec{H}$) is defined to be

$$\vec{H} = \frac{1}{\mu} \vec{B} = \nu \vec{B} \qquad (2)$$

where μ is the permeability

ν is reluctivity

$\vec{B}$ is the magnetic flux density

The point form of the Maxwell-Ampere CURL equation may be stated:

$$\text{CURL } \vec{H} = \nabla \times \vec{H} = \vec{J} \qquad (3)$$

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The magnetic vector potential ($\vec{A}$) is defined by:

$$\text{CURL } \vec{A} = \nabla \times \vec{A} = \vec{B} \qquad (4)$$

and from Coulomb's convention

$$\text{DIV } \vec{A} = \nabla \cdot \vec{A} = 0 \qquad (5)$$

Since we are only considering direct currents, for which the charge density (ρ) is constant, the continuity equation for the current density vector ($\vec{J}$) is expressed as:

$$\text{DIV } \vec{J} = \nabla \cdot \vec{J} = -\frac{d\rho}{dt} = 0 \qquad (6)$$

That is, the current or charge per second, diverging from a small volume, per unit volume is equal to the time rate of decrease of charge per unit volume.

Substitution of (2) and (4) into (3) results in:

$$\text{CURL } \vec{H} = \text{CURL } (\nu\vec{B}) = \text{CURL } (\nu \, \text{CURL } \vec{A})$$

therefore,

$$\text{CURL } (\nu \, \text{CURL } \vec{A}) = \vec{J} \qquad (7)$$

It is now assumed that the system under study possesses axial symmetry, thereby allowing a description of the magnetic field in two dimensions, i.e. a cross section of the system (appendix C). It is also assumed that there is no magnetic flux in the Z direction, i.e. $\vec{B}_Z = 0$, $\frac{\partial \vec{B}}{\partial Z} = 0$. In addition, it is assumed that current will only flow perpendicularly through the cross section; i.e. $\vec{J}_X = 0$, $\vec{J}_Y = 0$, $\vec{J}_Z = \vec{J}$. Since the vector potential is in the same direction as the current density, $\vec{A}_X = \vec{A}_Y = 0$ and $\vec{A} = \vec{A}_Z$. Equation 7 can be written in its explicit form by definition of CURL

$$\vec{B} = \text{CURL } (\vec{A})$$

$$= \left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right)\vec{a}_X + \left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right)\vec{a}_Y + \left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)\vec{a}_Z$$

$$(8)$$

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where $\vec{a}_X$, $\vec{a}_Y$, $\vec{a}_Z$ are unit vectors in the X, Y, Z directions.

Therefore,
$$\left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right) = B_X$$

$$\left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right) = B_Y$$

$$\left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right) = B_Z$$

Therefore
$$\vec{J} = \text{CURL}\,(\nu \vec{B}) = \text{CURL}\,(\nu\ \text{CURL}\ \vec{A})$$

$$= \left[\frac{\partial(\nu B_Z)}{\partial Y} - \frac{\partial(\nu B_Y)}{\partial Z}\right]\vec{a}_X + \left[\frac{\partial(\nu B_X)}{\partial Z} - \frac{\partial(\nu B_Z)}{\partial X}\right]\vec{a}_Y$$

$$+ \left[\frac{\partial(\nu B_Y)}{\partial X} - \frac{\partial(\nu B_X)}{\partial Y}\right]\vec{a}_Z$$

$$= \left\{\frac{\partial}{\partial Y}\left[\nu\left(\frac{\partial}{\partial X}A_Y - \frac{\partial A_X}{\partial Y}\right)\right] - \frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right)\right]\right\}\vec{a}_X$$

$$+ \left\{\frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right)\right] - \frac{\partial}{\partial X}\left[\nu\left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)\right]\right\}\vec{a}_Y$$

$$+ \left\{\frac{\partial}{\partial X}\left[\nu\left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right)\right] - \frac{\partial}{\partial Y}\left[\nu\left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right)\right]\right\}\vec{a}_Z$$

$$(9)$$

Substituting the values $A_X = A_Y = 0$ and $\vec{A} = A_Z$ into equation (9) yields

$$\vec{J} = \left\{-\frac{\partial}{\partial Z}\left[\nu\left(-\frac{\partial A_Z}{\partial X}\right)\right]\right\}\vec{a}_X + \left\{\frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial A_Z}{\partial Y}\right)\right]\right\}\vec{a}_Y$$

$$+ \left\{\frac{\partial}{\partial X}\left[\nu\left(-\frac{\partial A_Z}{\partial X}\right)\right] - \frac{\partial}{\partial Y}\left[\nu\left(\frac{\partial A_Z}{\partial Y}\right)\right]\right\}\vec{a}_Z$$

$$= \frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial \vec{A}}{\partial X}\right)\right] + \frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial \vec{A}}{\partial Y}\right)\right] + \left\{\frac{\partial}{\partial X}\left[\nu\left(-\frac{\partial \vec{A}}{\partial X}\right)\right]\right.$$

$$\left. - \frac{\partial}{\partial Y}\left[\nu\left(\frac{\partial \vec{A}}{\partial Y}\right)\right]\right\}$$

$$(10)$$

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It is assumed that $\vec{J} = J_Z$ and $J_X = J_Y = 0$.

Therefore, $J_Z = \dfrac{\partial}{\partial X}\left[\nu\left(-\dfrac{\partial A_Z}{\partial X}\right)\right] - \dfrac{\partial}{\partial Y}\left[\nu\left(\dfrac{\partial A_Z}{\partial Y}\right)\right]$

$$J_X = \frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial A_Z}{\partial X}\right)\right] = 0$$

$$J_Y = \frac{\partial}{\partial Z}\left[\nu\left(\frac{\partial A_Z}{\partial Y}\right)\right] = 0$$

Therefore, $\dfrac{\partial}{\partial X}\ \nu\dfrac{\partial A}{\partial X}\ +\dfrac{\partial}{\partial Y}\ \nu\dfrac{\partial A}{\partial Y}\ = -J$ (1)

where $\quad J = J_Z\,\vec{a}_Z$

$$A = A_Z\,\vec{a}_Z$$

Equation (1) is the nonlinear equivalent of Poisson's equation.

For air regions which do not have current density, equation (1) reduces to Laplace's equation.

$$\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} = 0 \tag{11}$$

For air regions which include the current density fields, equation (1) reduces to the linear Poisson equation:

$$\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} = \frac{-J}{\nu_0} \tag{12}$$

where ν_0 is the reluctivity of free space.

For the iron regions of the magnetic circuit, it will be assumed that the current density is zero, i.e., no eddy currents. This is realistic if we consider the iron to be made of thin insulated laminations lying

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normal to the current density vector $(\vec{J}_Z)$. In these regions, the reluctivity is a function of magnetic induction (B) so that equation (1) becomes:

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X}\right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y}\right) = 0 \tag{13}$$

Permanent Magnet Sources

In order to include permanet magnets in the vector potential formulation, the concept of a magnetization vector $(\vec{M})$ is used. Briefly, a magnetized body behaves as though there were an internal current with density (CURL $\vec{M}$) analagous to $\vec{J}$ (see appendix D). Under the previously stated assumptions, $\vec{M}$ is in the X-Y plane and perpendicular to $\vec{J}_Z$.

For a system including permanent magnet sources, the magnetic flux density $(\vec{B})$ is defined to be

$$\vec{B} = \mu(\vec{H} + \vec{M}) \tag{14}$$

Rearranging this equation produces:

$$\vec{H} = \left(\frac{1}{\mu}\right) \vec{B} - \vec{M} = \nu \vec{B} - \vec{M} \tag{15}$$

where $\vec{H}$ is the magnetic field intensity

$\vec{B}$ is the magnetic flux density

$\vec{M}$ is the magnetization.

Restating the point form of the Maxwell Ampere CURL equation (3) shows:

$$\text{CURL } \vec{H} = \nabla \times \vec{H} = \vec{J}$$

Substituting equation (15) into equation (3) results in:

$$\text{CURL } [\nu \vec{B} - \vec{M}] = \vec{J} \tag{16}$$

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Since $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$, equation (16) can be arranged:

$$CURL\ (\nu\vec{B}) = \vec{J} + CURL\ (\vec{M}) \tag{17}$$

By substituting equation (4) into equation (17) :

$$CURL\ (\nu\ CURL\ \vec{A}) = \vec{J} + (CURL\ \bar{M}) \tag{18}$$

Previously it was found that, under our assumptions

$$CURL\ (\nu\ CURL\ \vec{A}) = \frac{\partial}{\partial X}\ (\nu\frac{\partial A}{\partial X}) + \frac{\partial}{\partial Y}\ (\nu\frac{\partial A}{\partial Y})$$

Therefore, by substitution:

$$\frac{\partial}{\partial X}\ (\nu\frac{\partial A}{\partial X}) + \frac{\partial}{\partial Y}\ (\nu\frac{\partial A}{\partial Y}) = -[\vec{J} + (CURL\ \vec{M})] \tag{19}$$

Next, CURL $\vec{M}$ is explicitly expressed as:

$$CURL\ \vec{M} \equiv \left(\frac{\partial M_Z}{\partial Y} - \frac{\partial M_Y}{\partial Z}\right)\vec{a}_X + \left(\frac{\partial M_X}{\partial Z} - \frac{\partial M_Z}{\partial X}\right)\vec{a}_Y + \left(\frac{\partial M_Y}{\partial X} - \frac{\partial M_X}{\partial Y}\right)\vec{a}_Z$$
$$\tag{20}$$

For the circuit under consideration

$$M_Z = 0 \text{ and } \frac{\partial M_Z}{\partial Y} = 0,\ \frac{\partial M_Z}{\partial X} = 0$$

Also, considering only the X-Y plane

$$\frac{\partial M_Y}{\partial Z} = 0\ ,\quad \frac{\partial M_X}{\partial Z} = 0$$

therefore, by substitution

$$CURL\ \vec{M} = \left(\frac{\partial M_Y}{\partial X} - \frac{\partial M_X}{\partial Y}\right)\vec{a}_Z \tag{21}$$

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the total expression becomes:

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y} \right) = -J + \frac{\partial M_X}{\partial Y} - \frac{\partial M_Y}{\partial X} \qquad (22)$$

For air regions not having current density, equation (22) reduces to Laplace's equation:

$$\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} = 0$$

For air regions, including current density, equation (22) reduces to the linear Poisson equation

$$\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} = \frac{-J}{\nu_0}$$

For the iron regions, it is assumed that the current density and the magnetization vector are both zero. Therefore, equation (22) reduces to

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y} \right) = 0$$

For the permanent magnet regions assume that the current density vector is zero, and therefore, equation 22 reduces to

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y} \right) = \frac{\partial M_X}{\partial Y} - \frac{\partial M_Y}{\partial X} \qquad (23)$$

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DISCRETIZATION

Grid System

In order to use numerical solution methods, the nonlinear partial differential equation is transformed into a difference equation. In order to do this, the magnetic circuit cross section is subdivided by horizontal and vertical gridlines. The vector potential is then evaluated at the intersections of the gridlines, yielding an array of discrete point values. The rectangular area, formed by two adjacent horizontal gridlines and two adjacent vertical gridlines, forms a mesh. Nonuniformly sized meshes are used to account for irregular boundaries. That is, a fine mesh is used where accuracy is needed, such as boundaries. Conversely, a coarse mesh is used where reluctivity variations are small, thereby minimizing the number of gridpoints and resultant computations.

The grid system is constructed so that each row is cut by the same columns and each column is cut by the same rows. All nonorthogonal boundaries are approximated by stair-type contours. A basic mesh, formed by the grid system, is shown in figure 2.

Reluctivity

The vector potential is defined at the grid points of figure 2, and the reluctivities are defined to be constant over the area of each of the four meshes meeting at a typical mesh point (0).

In order to determine the reluctivity of the rectangles, $\overline{0184}$, $\overline{0251}$, $\overline{0362}$, $\overline{0473}$, the average flux density (magnetic induction) at the center of each rectangle is calculated. The X & Y components of the flux density at the center of rectangle $\overline{0184}$ are calculated as

$$B_X = 1/2 \left(\frac{A_1 - A_8 + A_0 - A_4}{h_4} \right) \tag{24}$$

$$B_Y = 1/2 \left(\frac{A_1 - A_0 + A_8 - A_4}{h_1} \right) \tag{25}$$

(see appendix E) where h_4 is the distance between meshpoints 0 and 4, and h_1 is the distance between meshpoints 0 and 1. The vector potentials

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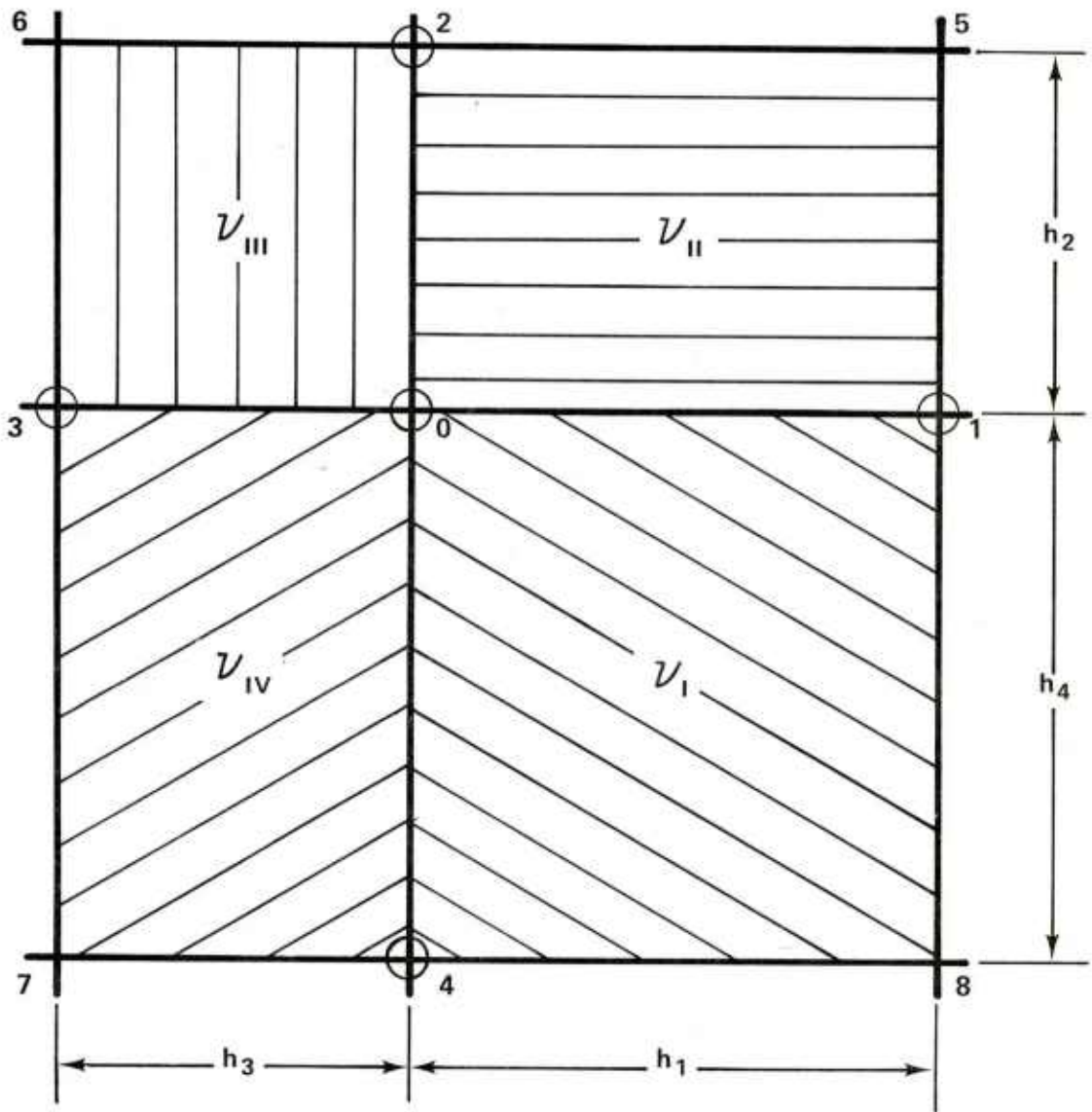
Figure 2. The basic mesh

at points 0, 1, 8, 4 are denoted as A_0, A_1, A_8, A_4, respectively. The absolute value of the flux density will be

$$|\vec{B}| = \sqrt{\left(\frac{A_1 - A_8 + A_0 - A_4}{2h_4}\right)^2 + \left[-\left(\frac{A_1 - A_0 + A_8 - A_4}{2h_1}\right)\right]^2}$$

$$(26)$$

This value of $|\vec{B}|$ is used to calculate the reluctivity by using an approximation to the nonlinear magnetization characteristic (see appendix B).

The reluctivities in the other three rectangles having gridpoint 0 as a joint corner are determined similarly.

FINITE DIFFERENCE FORM OF THE VECTOR POTENTIAL

In this section, the finite difference form of the vector potential will be derived using two approaches, the point form and the integral form of the Maxwell-Ampere circuital law. Although different forms are obtained in each case, the results will be shown to be equivalent.

Point Form

In order to obtain the finite difference form of equation (1)

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X}\right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y}\right) = -J$$

In terms of the basic mesh, the differentials will be replaced with differences (refer to figure 3).

Let

$$\left(\frac{\partial A}{\partial X_{1-0}}\right) = \frac{A_1 - A_0}{h_1} \qquad (27)$$

$$\left(\frac{\partial A}{\partial X_{0-3}}\right) = \frac{A_0 - A_3}{h_3} \qquad (28)$$

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$$\left(\frac{\partial A}{\partial Y}\right)_{2\;0} = \frac{A_2 - A_0}{h_2} \tag{29}$$

$$\left(\frac{\partial A}{\partial Y}\right)_{0\;4} = \frac{A_0 - A_4}{h_4} \tag{30}$$

Also let

ν_{ab} be the weighted average of ν_I and ν_{II} along a-b

ν_{bc} be the weighted average of ν_{II} and ν_{III} along b-c

ν_{cd} be the weighted average of ν_{III} and ν_{IV} along c-d

ν_{da} be the weighted average of ν_{IV} and ν_I along d-a

Therefore,

$$\frac{\partial}{\partial X}\left(\nu\frac{\partial A}{\partial X}\right) = \frac{\nu_{ab}\left.\left(\frac{\partial A}{\partial X}\right)\right|_{1-0} - \nu_{cd}\left.\left(\frac{\partial A}{\partial X}\right)\right|_{0-3}}{\dfrac{h_1 + h_3}{2}}$$

$$= \frac{2\left[\nu_{ab}\left(\dfrac{A_1 - A_0}{h_1}\right) - \nu_{cd}\dfrac{A_1 - A_3}{h_3}\right]}{h_1 + h_3} \tag{31}$$

$$\frac{\partial}{\partial Y}\left(\nu\frac{\partial A}{\partial Y}\right) = \frac{\nu_{bc}\left.\left(\frac{\partial A}{\partial Y}\right)\right|_{2-0} - \nu_{da}\left.\left(\frac{\partial A}{\partial Y}\right)\right|_{0-4}}{\dfrac{h_2 + h_4}{2}}$$

$$= \frac{2\left[\nu_{bc}\left(\dfrac{A_2 - A_0}{h_2}\right) - \nu_{da}\left(\dfrac{A_0 - A_4}{h_4}\right)\right]}{h_2 + h_4} \tag{32}$$

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Figure 3. Mesh cell

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By substitution equation (1) becomes:

$$\frac{(2)\,\nu_{ab}\,(A_1 - A_0)}{h_1\,(h_1 + h_3)} - \frac{(2)\,\nu_{cd}\,(A_0 - A_3)}{h_3\,(h_1 + h_3)} + \frac{(2)\,\nu_{bc}\,(A_2 - A_0)}{h_2\,(h_2 + h_4)}$$

$$- \frac{(2)\,\nu_{da}\,(A_0 - A_4)}{h_4\,(h_2 + h_4)} = -J_T \tag{33}$$

Let

$$\alpha_1 = \frac{2}{h_1\,(h_1 + h_3)} \tag{34}$$

$$\alpha_2 = \frac{2}{h_2\,(h_2 + h_4)} \tag{35}$$

$$\alpha_3 = \frac{2}{h_3\,(h_1 + h_3)} \tag{36}$$

$$\alpha_4 = \frac{2}{h_4\,(h_2 + h_4)} \tag{37}$$

Substituting into equation (33):

$$\nu_{ab}\,\alpha_1\,(A_1 - A_0) + \nu_{bc}\,\alpha_2\,(A_2 - A_0) + \nu_{cd}\,\alpha_3\,(A_3 - A_0)$$

$$+ \nu_{da}\,\alpha_4\,(A_4 - A_0) = -J_T \tag{38}$$

Solving for A_0:

$$A_0 = \frac{J_T + A_1\alpha_1\nu_{ab} + A_2\alpha_2\nu_{bc} + A_3\alpha_3\nu_{cd} + A_4\alpha_4\nu_{da}}{\alpha_1\nu_{ab} + \alpha_2\nu_{bc} + \alpha_3\nu_{cd} + \alpha_4\nu_{da}}$$

$$\tag{39}$$

The average reluctivities can be written as:

$$\nu_{ab} = \nu_I \left(\frac{h_4}{h_2 + h_4}\right) + \nu_{II} \left(\frac{h_2}{h_2 + h_4}\right) \tag{40}$$

$$\nu_{bc} = \nu_{II} \left(\frac{h_1}{h_1 + h_3}\right) + \nu_{III} \left(\frac{h_3}{h_1 + h_3}\right) \tag{41}$$

$$\nu_{cd} = \nu_{III} \left(\frac{h_2}{h_2 + h_4}\right) + \nu_{IV} \left(\frac{h_4}{h_2 + h_4}\right) \tag{42}$$

$$\nu_{da} = \nu_{IV} \left(\frac{h_3}{h_1 + h_3}\right) + \nu_I \left(\frac{h_1}{h_1 + h_3}\right) \tag{43}$$

where ν_I is the reluctivity in region I (a constant)

ν_{II} is the reluctivity in region II (a constant)

ν_{III} is the reluctivity in region III (a constant)

ν_{IV} is the reluctivity in region IV (a constant)

Let $$\beta_1 = \frac{h_1}{h_1 + h_3} \tag{44}$$

$$\beta_2 = \frac{h_2}{h_2 + h_4} \tag{45}$$

$$\beta_3 = \frac{h_3}{h_1 + h_3} \tag{46}$$

$$\beta_4 = \frac{h_4}{h_2 + h_4} \tag{47}$$

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The expression for A_0 becomes

$$A_0 = \frac{J_T + A_1\alpha_1 (\nu_I\beta_4 + \nu_{II}\beta_2) + A_2\alpha_2 (\nu_{II}\beta_1 + \nu_{III}\beta_3) + A_3\alpha_3 (\nu_{III}\beta_2 + \nu_{IV}\beta_4) + A_4\alpha_4 (\nu_{IV}\beta_3 + \nu_I\beta_1)}{\alpha_1 (\nu_I\beta_4 + \nu_{II}\beta_2) + \alpha_2 (\nu_{II}\beta_1 + \nu_{III}\beta_3) + \alpha_3 (\nu_{III}\beta_2 + \nu_{IV}\beta_4) + \alpha_4 (\nu_{IV}\beta_3 + \nu_I\beta_1)}$$

$$(48)$$

If the constant current density in the mesh cells is J_I, J_{II}, J_{III}, J_{IV}, then J_T may be written:

$$J_T = J_I \frac{h_1 h_4}{(h_1 + h_3)(h_2 + h_4)} + J_{II} \frac{h_1 h_2}{(h_1 + h_3)(h_2 + h_4)}$$

$$+ J_{III} \frac{h_2 h_3}{(h_1 + h_3)(h_2 \ h_4)} + J_{IV} \frac{h_3 h_4}{(h_1 + h_3)(h_2 + h_4)}$$

$$= J_I \beta_1\beta_4 + J_{II} \beta_1\beta_2 + J_{III} \beta_2\beta_3 + J_{IV} \beta_3\beta_4 \qquad (49)$$

Next, permanent magnets are included in the formulation. To do this, the concept of the magnetization vector $(\vec{M})$, which is explained in appendix D is used. Figure 4 represents the basic mesh and includes the magnetization vector $(\vec{M})$ for each cell. The magnetization vector is assumed to be a constant value throughout the mesh cell and, for the given assumptions, it has only X and Y components.

In the previously derived expression (equation (22)) which included permanent magnets,:

$$\frac{\partial}{\partial X} \left(\nu \frac{\partial A}{\partial X}\right) + \frac{\partial}{\partial Y} \left(\nu \frac{\partial A}{\partial Y}\right) = -J + \left(\frac{\partial M_X}{\partial Y} - \frac{\partial M_Y}{\partial X}\right)$$

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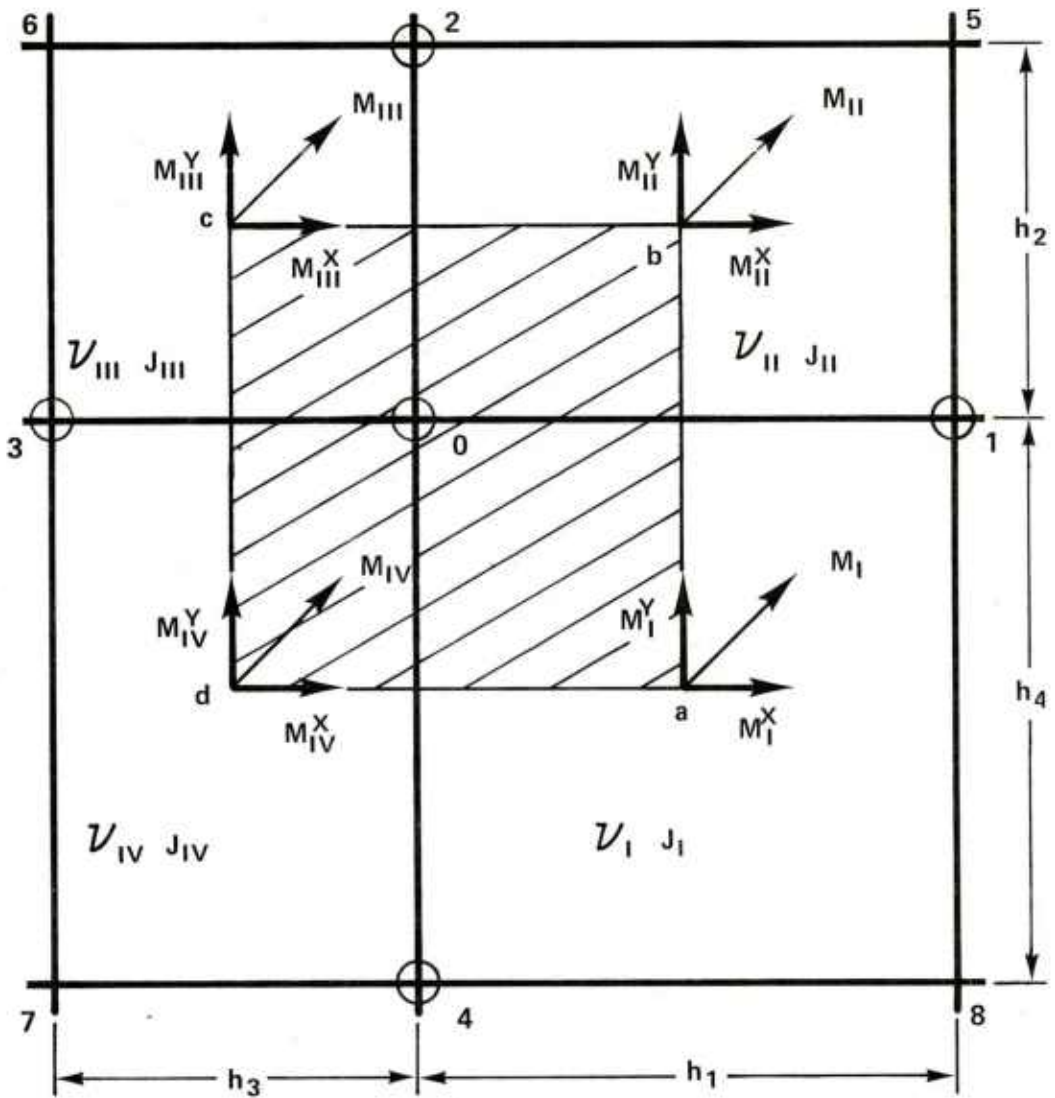
Figure 4.   Mesh cell with magnetization vector ($\vec{M}$)

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The only term not evaluated was:

$$\left(\frac{\partial M_X}{\partial Y} - \frac{\partial M_Y}{\partial X} \right).$$

Let

$(M)_{ab}^{Y}$ be the weighted average of the Y component along ab

$(M)_{cd}^{Y}$ be the weighted average of the Y component along cd

$(M)_{bc}^{X}$ be the weighted average of the X component along bc

$(M)_{da}^{X}$ be the weighted average of the X component along da

Therefore at point 0

$$\left(\frac{\partial M_X}{\partial X} \right)_0 = \frac{(M)_{bc}^{X} - (M)_{da}^{X}}{\dfrac{h_4 + h_2}{2}} \tag{50}$$

$$\left(\frac{\partial M_Y}{\partial X} \right)_0 = \frac{(M)_{ab}^{Y} - (M)_{cd}^{Y}}{\dfrac{h_1 + h_3}{2}} \tag{51}$$

By definition

$$(M)_{bc}^{X} = \frac{M_{II}^{X} h_1 + M_{III} h_3}{h_1 + h_3} \tag{52}$$

$$(M)_{da}^{X} = \frac{M_{IV}^{X} h_3 + M_{I}^{X} h_1}{h_1 + h_3} \tag{53}$$

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$$(M)^Y_{ab} = \frac{M^Y_I h_4 + M^Y_{II} h_2}{h_2 + h_4} \tag{54}$$

$$(M)^Y_{cd} = \frac{M^Y_{III} h_2 + M^Y_{IV} h_4}{h_2 + h_4} \tag{55}$$

Therefore by substitution

$$\frac{\partial M_X}{\partial Y} = \frac{\left[\dfrac{M^X_{II} h_1 + M^X_{III} h_3}{h_1 + h_3}\right] - \left[\dfrac{M^X_{IV} h_3 + M^X_I h_1}{h_1 + h_3}\right]}{\dfrac{h_4 + h_2}{2}}$$

$$= \frac{2\left[\left(M^X_{II} - M^X_I\right)\beta_1 + \left(M^X_{III} - M^X_{IV}\right)\beta_3\right]}{h_2 + h_4} \tag{56}$$

$$\frac{\partial M_Y}{\partial X} = \frac{\left[\dfrac{M^Y_I h_4 + M^Y_{II} h_2}{h_2 + h_4}\right] - \left[\dfrac{M^Y_{III} h_2 + M^Y_{IV} h_4}{h_2 + h_4}\right]}{\left(\dfrac{h_1 + h_3}{2}\right)}$$

$$= \frac{2\left[\left(M^Y_I - M^Y_{IV}\right)\beta_4 + \left(M^Y_{II} - M^Y_{III}\right)\beta_2\right]}{(h_1 + h_3)} \tag{57}$$

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Therefore, the complete expression is:

$$A_0 = \frac{A_1 \alpha_1 (\nu_I \beta_4 + \nu_{II} \beta_2) + A_2 \alpha_2 (\nu_{II} \beta_1 + \nu_{III} \beta_3) + A_3 \alpha_3 (\nu_{III} \beta_2 + \nu_{IV} \beta_4) + A_4 \alpha_4 (\nu_{IV} \beta_3 + \nu_I \beta_1) + J_T + M_T}{\alpha_1 (\nu_I \beta_4 + \nu_{II} \beta_2) + \alpha_2 (\nu_{II} \beta_1 + \nu_{III} \beta_3) + \alpha_3 (\nu_{III} \beta_2 + \nu_{IV} \beta_4) + \alpha_4 (\nu_{IV} \beta_3 + \nu_I \beta_1)}$$

(58)

where,

$$\alpha_1 = \frac{2}{h_1 (h_1 + h_3)}$$

$$\alpha_2 = \frac{2}{h_2 (h_2 + h_4)}$$

$$\alpha_3 = \frac{2}{h_3 (h_1 + h_3)}$$

$$\alpha_4 = \frac{2}{h_4 (h_2 + h_4)}$$

$$\beta_1 = \frac{h_1}{h_1 + h_3}$$

$$\beta_2 = \frac{h_2}{h_2 + h_4}$$

$$\beta_3 = \frac{h_3}{h_1 + h_3}$$

$$\beta_4 = \frac{h_4}{h_2 + h_4}$$

$$J_T = J_I \beta_1 \beta_4 + J_{II} \beta_1 \beta_2 + J_{III} \beta_2 \beta_3 + J_{IV} \beta_3 \beta_4$$

$$M_T = 2 \left\{ \frac{[(M_I^Y - M_{IV}^Y)\beta_4 + (M_{II}^Y - M_{III}^Y)\beta_2]}{h_1 + h_3} \right.$$

$$\left. - \frac{[(M_{II}^X - M_I^X)\beta_1 + (M_{III}^X - M_{IV}^X)\beta_3]}{(h_2 + h_4)} \right\}$$

Integral Form

Ampere's circuital law may be used to derive the finite difference form of the vector potential. Ampere's circuital law may be stated: The line integral of the magnetic field intensity (H) about any closed path is exactly equal to the current enclosed by that path i.e.

$$\oint \vec{H} \cdot d\vec{L} = I \tag{59}$$

Positive current is defined as following in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Since

$$\vec{B} \equiv \text{CURL } \vec{A} \tag{60}$$

and $\quad \vec{H} \equiv \nu \vec{B}$ (61)

$$\tag{62}$$

Therefore, $\quad \vec{H} = \nu \vec{B} = \nu \text{ CURL } \vec{A}$

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Substituting into equation (59):

$$I = \oint \nu \, CURL \, \vec{A} \cdot d\vec{L} \tag{63}$$

Next, apply Ampere's law along the path a-b-c-d-a as shown in figure 3. Therefore,

$$\oint \nu \, CURL \, \vec{A} \cdot d\vec{L} = \int_a^b \nu_{ab} \, CURL \, \vec{A} \cdot d\vec{L} + \int_b^c \nu_{bc} \, CURL \, \vec{A} \cdot d\vec{L}$$

$$+ \int_c^d \nu_{cd} \, CURL \, \vec{A} \cdot d\vec{L} + \int_d^a \nu_{da} \, CURL \, \vec{A} \cdot d\vec{L} \tag{64}$$

The weighted average reluctance along each of the path segments is:

$$\nu_{ab} = \frac{\nu_I h_4 + \nu_{II} h_2}{h_2 + h_4} \tag{65}$$

$$\nu_{bc} = \frac{\nu_{II} h_1 + \nu_{III} h_3}{h_1 + h_3} \tag{66}$$

$$\nu_{cd} = \frac{\nu_{III} h_2 + \nu_{IV} h_4}{h_2 + h_4} \tag{67}$$

$$\nu_{da} = \frac{\nu_{IV} h_3 + \nu_I h_1}{h_1 + h_3} \tag{68}$$

Since the dot product between two orthogonal unit vectors is zero, the following products are zero:

$$\vec{a}_X \cdot \vec{L}_{ab} = \vec{a}_Y \cdot \vec{L}_{bc} = \vec{a}_X \cdot \vec{L}_{cd} = \vec{a}_Y \cdot \vec{L}_{da} = 0$$

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Therefore, after replacing the differentials with differences, the path segments become:

$$\Delta L_{ab} = \frac{h_2 + h_4}{2}$$

$$\Delta L_{bc} = - \frac{h_1 + h_3}{2}$$

$$\Delta L_{cd} = - \frac{h_2 + h_4}{2}$$

$$\Delta L_{da} = \frac{h_1 + h_3}{2}$$

and

$$CURL\ \vec{A} = \left(\frac{\partial \vec{A}_Z}{\partial Y}\right) \vec{a}_Y + \left(\frac{-\partial A_Z}{\partial X}\right) \vec{a}_Y$$

(See appendix E.)

Therefore,

$$\int_a^b \nu_{ab}\ CURL\ \vec{A} \cdot d\vec{L}_{ab} = \int_a^b \nu_{ab} \left[-\frac{\partial A_Z}{\partial X}\ \vec{a}_Y \right] \cdot d\vec{L}_{ab}$$

$$= \nu_{ab} \left[-\frac{\Delta A_Z}{\Delta X} \right]\ \Delta L_{ab}$$

$$= \left[\frac{\nu_I h_4 + \nu_{II} h_2}{h_2 + h_4} \right] - \left[\frac{A_1 - A_0}{h_1} \right] \left[\frac{h_2 + h_4}{2} \right]$$

$$= - \frac{(A_1 - A_0)}{h_1} \left[\frac{\nu_I h_4}{2} + \frac{\nu_{II} h_2}{2} \right]$$

$$(69)$$

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$$\int_b^c \nu_{bc} \text{ CURL } \vec{A} \cdot d\vec{L}_{bc} = \int_b^c \nu_{bc} \left[\frac{\partial A_Z}{\partial Y} \vec{a}_X \right] \cdot d\vec{L}_{bc} = \nu_{bc} \left[\frac{\Delta A_Z}{\Delta Y} \right] \Delta L_{bc}$$

$$= \left[\frac{\nu_{II} h_1 + \nu_{III} h_3}{h_1 + h_3} \right] \left[\frac{A_2 - A_0}{h_2} \right] \left[-\left(\frac{h_1 + h_3}{2} \right) \right]$$

$$= - \frac{(A_2 - A_0)}{h_2} \left[\frac{\nu_{II} h_1}{2} + \frac{\nu_{III} h_3}{2} \right] \qquad (70)$$

$$\int_c^d \nu_{cd} \text{ CURL } \vec{A} \cdot d\vec{L}_{cd} = \int_c^d \nu_{cd} \left[- \frac{\partial A_Z}{\partial X} \vec{a}_Y \right] \cdot d\vec{L}_{cd} = \nu_{cd} \left[- \frac{\Delta A_Z}{\Delta X} \right] \Delta L_{cd}$$

$$= \left[\frac{\nu_{III} h_2 + \nu_{IV} h_4}{h_2 + h_4} \right] \left[- \frac{A_0 - A_3}{h_3} \right] \left[\frac{-(h_2 + h_4)}{2} \right]$$

$$= \frac{(A_0 - A_3)}{h_3} \left[\frac{\nu_{III} h_2}{2} + \frac{\nu_{IV} h_4}{2} \right] \qquad (71)$$

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$$\int_d^a \nu_{da} \ \text{CURL} \ \vec{A} \cdot d\vec{L}_{da} = \int_d^a \nu_{da} \left[\frac{\partial A_Z}{\partial Y} \ \vec{a}_X \right] \cdot d\vec{L}_{da} = \nu_{da} \left[\frac{\Delta A_Z}{\Delta Y} \right] \left[\Delta L_{da} \right]$$

$$= \left[\frac{\nu_{IV} h_3 + \nu_I h_1}{h_1 + h_3} \right] \left[\frac{A_0 - A_4}{h_4} \right] \left[\frac{h_1 + h_3}{2} \right]$$

$$= \frac{A_0 - A_4}{h_4} \left[\frac{\nu_{IV} h_3}{2} + \frac{\nu_I h_1}{2} \right] \tag{72}$$

Assuming that the current density (J) in each region of the mesh is constant, the total current enclosed by rectangle a-b-c-d-a in figure 3 is

$$I_0 = \iint \vec{J} \cdot d\vec{S} = J_Z (\Delta S) = \frac{J_I h_1 h_4}{4} + \frac{J_{II} h_1 h_2}{4} + \frac{J_{III} h_2 h_3}{4} + \frac{J_{IV} h_3 h_4}{4}$$

$$= \frac{1}{4} (J_I h_1 h_4 + J_{II} h_1 h_2 + J_{III} h_2 h_3 + J_{IV} h_3 h_4) \tag{73}$$

Combining previous results

$$I_0 = \oint \vec{H} \cdot d\vec{L} = - \frac{(A_1 - A_0)}{h_1} \left[\frac{\nu_I h_4}{2} + \frac{\nu_{II} h_2}{2} \right] - \frac{(A_2 - A_0)}{h_2} \left[\frac{\nu_{II} h_1}{2} + \frac{\nu_{III} h_3}{2} \right]$$

$$+ \frac{(A_0 - A_3)}{h_3} \left[\frac{\nu_{III} h_2}{2} + \frac{\nu_{IV} h_4}{2} \right] + \frac{(A_0 - A_4)}{h_4} \left[\frac{\nu_{IV} h_3}{2} + \frac{\nu_I h_1}{2} \right] \tag{74}$$

Solving for the vector potential at point 0:

$$A_0 = \frac{I_0 + \sum\limits_{i=1}^{4} A_i \alpha'_i}{\sum\limits_{i=1}^{4} \alpha'_i} \tag{75}$$

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where

$$I_0 = \frac{1}{4} (J_I h_1 h_4 + J_{II} h_1 h_2 + J_{III} h_2 h_3 + J_{IV} h_3 h_4)$$

$$\alpha_1' = \frac{\nu_I h_4 + \nu_{II} h_2}{2 h_1}$$

$$\alpha_2' = \frac{\nu_{II} h_1 + \nu_{III} h_3}{2 h_2}$$

$$\alpha_3' = \frac{\nu_{III} h_2 + \nu_{IV} h_4}{2 h_3}$$

$$\alpha_4' = \frac{\nu_{IV} h_3 + \nu_I h_1}{2 h_4}$$

To include permanent magnets in the formulation, the concept of the magnetization vector is used.

The magnetic induction $(\vec{B})$ in a system including air, iron, currents, and permanent magnets is described by

$$\vec{B} = \mu (\vec{H} + \vec{M}) \tag{76}$$

which, rearranged is

$$\vec{H} = \frac{1}{\mu} (\vec{B}) - \vec{M} = \nu \vec{B} - \vec{M} \tag{77}$$

Substituting equation (77) into equation (59)

$$\oint (\nu \vec{B} - \vec{M}) \cdot d\vec{L} = I \tag{78}$$

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or

$$\oint \nu \vec{B} \cdot d\vec{L} - \oint \vec{M} \cdot d\vec{L} = I \qquad (79)$$

Rearranged

$$\oint \nu \vec{B} \cdot d\vec{L} = I + \oint \vec{M} \cdot d\vec{L} \qquad (80)$$

Since

$$B \equiv CURL \ \vec{A} \qquad (81)$$

therefore, $\quad \oint (\nu \ CURL \ \vec{A}) \cdot d\vec{L} = I + \oint \vec{M} \cdot d\vec{L} \qquad (82)$

The only term not previously evaluated is $\oint \vec{M} \cdot d\vec{L}$. Using figure 4 as a basis,

$$\oint \vec{M} \cdot d\vec{L} = \int_a^b \vec{M} \cdot d\vec{L}_{ab} + \int_b^c \vec{M} \cdot d\vec{L}_{bc} + \int_c^d \vec{M} \cdot d\vec{L}_{cd}$$

$$+ \int_d^a \vec{M} \cdot d\vec{L}_{da} \qquad (83)$$

Note, that $\vec{M}$ in the two-dimensional case is located in the X-Y plane, i.e., $\vec{M}_X = 0$.

Since the dot product between two orthogonal unit vectors is zero; the following products are zero:

$$\vec{M}_{ab}^x \cdot \vec{L}_{ab} = \vec{M}_{bc}^y \cdot \vec{L}_{bc} = \vec{M}_{cd}^x \cdot \vec{L}_{cd} = \vec{M}_{da} \cdot \vec{L}_{da} = 0$$

$$(84)$$

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The weighted average of $\vec{M}$ along each path segment is:

$$\vec{M}_{ab}^{Y} = \frac{M_{I}^{Y} h_4 + M_{II}^{Y} h_2}{h_2 + h_4} \tag{85}$$

$$\vec{M}_{bc}^{X} = \frac{M_{II}^{X} h_1 + M_{III}^{X} h_3}{h_1 + h_3} \tag{86}$$

$$\vec{M}_{cd}^{Y} = \frac{M_{III}^{Y} h_2 + M_{IV}^{Y} h_4}{h_2 + h_4} \tag{87}$$

$$\vec{M}_{da}^{X} = \frac{M_{IV}^{X} h_3 + M_{I}^{X} h_1}{h_1 + h_3} \tag{88}$$

Therefore,

$$\int_{a}^{b} \vec{M}_{ab}^{Y} \cdot d\vec{L}_{ab} = \left[\frac{M_{I}^{Y} h_4 + M_{II}^{Y} h_2}{h_2 + h_4}\right] \left[\frac{h_2 + h_4}{2}\right] \tag{89}$$

$$\int_{b}^{c} \vec{M}_{bc}^{X} \cdot d\vec{L}_{bc} = \left[\frac{M_{II}^{X} h_1 + M_{III}^{X} h_3}{h_1 + h_3}\right] \left[\frac{-(h_1 + h_3)}{2}\right] \tag{90}$$

$$\int_{c}^{d} \vec{M}_{cd}^{Y} \cdot d\vec{L}_{cd} = \left[\frac{M_{III}^{Y} h_2 + M_{IV}^{Y} h_4}{h_2 + h_4}\right] \left[\frac{-(h_2 + h_4)}{2}\right] \tag{91}$$

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$$\int_d^a \overline{M}_{da}^X \cdot d\overline{L}_{da} = \left[\frac{M_{IV}^X h_3 + M_I^X h_1}{h_1 + h_3} \right] \left[\frac{h_1 + h_3}{2} \right] \tag{92}$$

and,

$$\oint \overline{M} \cdot d\overline{L} = 1/2 \left[\left(M_I^X - M_{II}^X \right) h_1 + \left(M_{II}^Y - M_{III}^Y \right) h_2 \right.$$
$$\left. + \left(M_{IV}^X - M_{III}^X \right) h_3 + \left(M_I^Y - M_{IV}^Y \right) h_4 \right] \tag{93}$$

Substituting this result into equation (75), the final expression for the vector potential becomes

$$A_0 = \frac{\sum\limits_{i=1}^{4} A_i \, \alpha_i' + I_0 + M_0}{\sum\limits_{i=1}^{4} \alpha_i'} \tag{94}$$

where

$$\alpha_1' = \frac{\nu_I h_4 + \nu_{II} h_2}{2h_1} \quad , \quad \alpha_2' = \frac{\nu_{II} h_1 + \nu_{III} h_3}{2h_2}$$

$$\alpha_3' = \frac{\nu_{III} h_2 + \nu_{IV} h_4}{2h_3} \quad , \quad \alpha_4' = \frac{\nu_{IV} h_3 + \nu_I h_1}{2h_4}$$

$$I_0 = 1/4 \, (J_I h_1 h_4 + J_{II} h_1 h_2 + J_{III} h_2 h_3 + J_{IV} h_3 h_4)$$

$$M_0 = 1/2 \left[\left(M_I^X - M_{II}^X \right) h_1 + \left(M_{II}^Y - M_{III}^Y \right) h_2 \right.$$
$$\left. + \left(M_{IV}^X - M_{III}^X \right) h_3 + \left(M_I^Y - M_{IV}^Y \right) h_4 \right]$$

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The formulation derived using the point form of the Maxwell Ampere law is equivalent to the equation just derived.

From previously defined variables,

$$\alpha_1 \ (\nu_I \beta_4 + \nu_{II} \beta_2) = \left(\frac{2}{h_1(h_1+h_3)}\right)\left[\nu_I\left(\frac{h_4}{(h_2+h_4)}\right) + \nu_{II}\left(\frac{h_2}{h_2+h_4}\right)\right]$$

$$= \frac{2(\nu_I h_4 + \nu_{II} h_2)}{h_1(h_1+h_3)(h_2+h_4)}$$

$$= \frac{4}{(h_1+h_3)(h_2+h_4)} \left[\frac{\nu_I h_4 + \nu_{II} h_2}{2h_1}\right]$$

$$= \frac{4}{(h_1+h_3)(h_2+h_4)} \quad (\alpha_1') \tag{95}$$

Similarly,

$$\alpha_2 \ (\nu_{II} \beta_1 + \nu_{III} \beta_3) = \frac{4}{(h_1+h_3)(h_2+h_4)} \ (\alpha_2') \tag{96}$$

$$\alpha_3 \ (\nu_{III} \beta_2 + \nu_{IV} \beta_4) = \frac{4}{(h_1+h_3)(h_2+h_4)} \ (\alpha_3') \tag{97}$$

$$\alpha_4 \ (\nu_{IV} \beta_3 + \nu_I \beta_1) = \frac{4}{(h_1+h_3)(h_2+h_4)} \ (\alpha_4') \tag{98}$$

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Substituting into equation (58),

$$A_0 = \frac{\sum\limits_{i=1}^{4} A_i \alpha_i' + I_0 + \left[\dfrac{(h_1+h_3)(h_2+h_4)}{4}\right] M_T}{\sum\limits_{i=1}^{4} \alpha_i'}$$ (99)

Next, the magnetization term is evaluated.

$$\left[\frac{(h_1+h_3)(h_2+h_4)}{4}\right] M_T = \left[\frac{(h_1+h_3)(h_2+h_4)}{4}\right]\left[\frac{2\left[\left(M_I^Y - M_{IV}^Y\right)\beta_4 + \left(M_{II}^Y - M_{III}^Y\right)\beta_2\right]}{(h_1+h_3)}\right.$$

$$\left.\frac{-2\left[\left(M_{II}^X - M_I^X\right)\beta_1 + \left(M_{III}^X - M_{IV}^X\right)\beta_3\right]}{(h_2+h_4)}\right]$$

$$= (1/2)\left[(h_1+h_3)(h_2+h_4)\right]\left\{\frac{\left[\left(M_I^Y - M_{IV}^Y\right)\beta_4 + \left(M_{II}^Y - M_{III}^Y\right)\beta_2\right](h_2+h_4)}{(h_1+h_3)(h_2+h_4)}\right.$$
$$\left.\frac{+\left[\left(M_I^X - M_{II}^X\right)\beta_1 + \left(M_{III}^X - M_{IV}^X\right)\beta_3\right](h_1+h_3)}{(h_1+h_3)(h_2+h_4)}\right\}$$

$$= (1/2)\left\{\left[\left(M_I^Y - M_{IV}^Y\right)\frac{h_4}{(h_2+h_4)} + \left(M_{II}^Y - M_{III}^Y\right)\frac{h_2}{(h_2+h_4)}\right](h_2+h_4)\right.$$
$$\left.+\left[\left(M_I^X - M_{II}^X\right)\frac{h_1}{(h_1+h_3)} + \left(M_{IV}^X - M_{III}^X\right)\frac{h_3}{(h_1+h_3)}\right](h_1+h_3)\right\}$$

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$$= 1/2 \left[\left(M^X_I - M^X_{II} \right) h_1 + \left(M^Y_{II} - M^Y_{III} \right) h_2 + \left(M^X_{IV} - M^X_{III} \right) h_3 \right.$$

$$\left. + \left(M^Y_I - M^Y_{IV} \right) h_4 \right]$$

$$= M_0 \tag{100}$$

Therefore, equations (58) and (94) are equivalent i.e.

$$A_0 = \frac{\displaystyle\sum_{i=1}^{4} A_i \alpha'_i + I_0 + M_0}{\displaystyle\sum_{i=1}^{4} \alpha'_i}$$

$$= \frac{\begin{array}{c} A_1 \alpha_1 \left(\nu_I \beta_4 + \nu_{II} \beta_2 \right) + A_2 \alpha_2 \left(\nu_{II} \beta_1 + \nu_{III} \beta_3 \right) + A_3 \alpha_3 \left(\nu_{III} \beta_2 + \nu_{IV} \beta_4 \right) \\ + A_4 \alpha_4 \left(\nu_{IV} \beta_3 + \nu_I \beta_1 \right) \quad + J_T + M_T \end{array}}{\begin{array}{c} \alpha_1 \left(\nu_I \beta_4 + \nu_{II} \beta_2 \right) + \alpha_2 \left(\nu_{II} \beta_1 + \nu_{III} \beta_3 \right) + \alpha_3 \left(\nu_{III} \beta_2 + \nu_{IV} \beta_4 \right) \\ + \alpha_4 \left(\nu_{IV} \beta_3 + \nu_I \beta_1 \right) \end{array}}$$

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BOUNDARY CONDITIONS

In order to solve the vector potential equations, it is necessary to determine boundary conditions.

Using figure 5 boundary conditions that are consistant with the assumptions can be determined. The various boundary types can be described as:

Type A - Line A-B is an axis of symmetry, and, in this case, it is also a center line of the magnetic circuit. Therefore, A-B represents a line of flux, i.e. an equipotential where

A-B = constant potential (A)

Type B - The equi-potential A-B has its return path B-C, C-D, D-E, E-F, F-A

Type C - It is assumed that the magnetic induction outside the contours of the machine is negligible.

Type D - This type of boundary exists at air-iron interfaces. At these boundaries, the tangential component of the magnet intensity vector ($\vec{H}$) and the normal component of the magnetic vector ($\vec{B}$) must be continuous. For a boundary that is parallel to the X direction, these conditions are satisfied when:

$$\nu \left(\frac{\partial A}{\partial Y}\right)_{iron} = \nu_0 \left(\frac{\partial A}{\partial Y}\right)_{air}$$

and

$$\left(\frac{\partial A}{\partial X}\right)_{iron} = \left(\frac{\partial A}{\partial X}\right)_{air}$$

At air-iron boundaries parallel to the Y-axis, these conditions will be satisfied when:

$$\nu \left(\frac{\partial A}{\partial X}\right)_{iron} = \nu_0 \left(\frac{\partial A}{\partial X}\right)_{air}$$

and

$$\left(\frac{\partial A}{\partial Y}\right)_{iron} = \left(\frac{\partial A}{\partial Y}\right)_{air}$$

Boundaries a-ℓ and b-c are of this type.

Type E - At air-iron interface corners, such as point ℓ
and point c in figure 5, all four conditions of Type D must be satis-
fied simultaneously.

Type F - This type boundary is at the air-current field
region interface, such as ℓ-c. It is assumed that the current
carrying material has the same permeability as air. At the boundary,
the tangential component of $\overline{H}$ and the normal component of $\overline{B}$ must be
continuous. Therefore, at the boundary

$$\left(\frac{\partial A}{\partial X}\right)_{\substack{\text{current}\\\text{region}}} = \left(\frac{\partial A}{\partial X}\right)_{\text{air}}$$

$$\left(\frac{\partial A}{\partial Y}\right)_{\substack{\text{current}\\\text{region}}} \quad \left(\frac{\partial A}{\partial Y}\right)_{\text{air}}$$

Type G - This type of boundary is at the iron-current field
region, such as c-d, d-e, h-k, or k-ℓ. Since the reluctance of the
current carrying region is the same as in air, the boundaries are
the same as Type D boundaries.

Type H - This boundary is at the permanent magnet-current
carrying region interface, represented by line e-h. Since, in
figure 5, this boundary is parallel to the Y axis, the following
conditions must be true at the boundary:

$$\left(\frac{\partial A}{\partial Y}\right)_{\text{magnet}} = \left(\frac{\partial A}{\partial Y}\right)_{\substack{\text{current}\\\text{region}}}$$

$$\nu\left(\frac{\partial A}{\partial X}\right)_{\text{magnet}} = \nu_0\left(\frac{\partial A}{\partial X}\right)_{\substack{\text{current}\\\text{region}}}$$
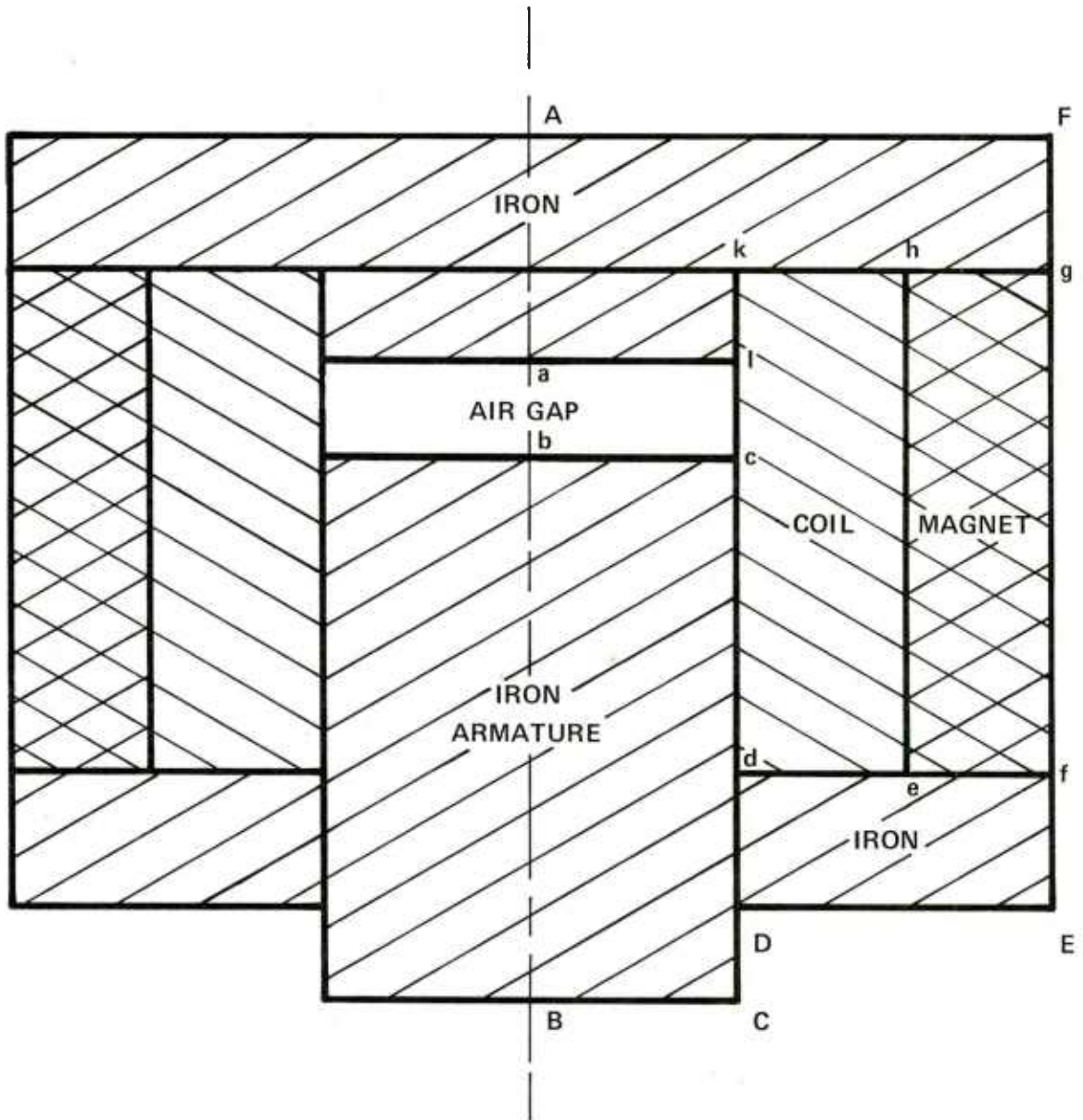
Figure 5.   Magnetic circuit with boundaries noted

Type K - This boundary is at the permanent magnet-iron interface, i.e. lines e-f and g-h. Since the boundaries are parallel to the X-axis, the following conditions must hold:

$$\nu\left(\frac{\partial A}{\partial Y}\right)_{iron} = \nu\left(\frac{\partial A}{\partial Y}\right)_{magnet}$$

$$\left(\frac{\partial A}{\partial X}\right)_{iron} = \left(\frac{\partial A}{\partial X}\right)_{magnet}$$

SOLUTION PROCEDURES

The vector potential (A_0) at a grid point not on an outer boundary is given by equation (94), evaluated after the appropriate reluctivities and α and β constants have been substituted. Thus, N simultaneous equations are obtained for the potentials at N internal grid points. The solution of the set of N simultaneous equations yields the numerical value of the vector potential at the N grid points.

The solution of the N simultaneous equations can be obtained by an iterative procedure via computer. This is necessary, because the coefficients in equation (94) are functions of the reluctivities that are in turn functions of the CURL of the vector potential calculated by equation (94). Using the relaxation method requires a two-step iterative process. In the first step, the coefficients in equation (94) are assumed to be constant and the vector potential is calculated. In the second step, the coefficients, which are functions of the reluctivity, are recalculated from the numerical values of the vector potential of step one. The new values are then used to calculate the vector potential. This process is repeated until the vector potentials have converged to values that are considered close enough to final values.

In general, the solution procedures may be summarized

1. Input of the permanent magnet data, the gridsystem, and the nonlinear relationships ν (reluctivity)= $f(\vec{B})$ and $\overline{M}$ (magnetization) = $f(\vec{B})$.

2. Calculation of the coefficients of the difference equations (initially, coefficients are assigned constant values).

3. Solution of the difference equations with point interation, line iteration, elimination (if ν and $\overline{M}$ are constant) or with the Newton-Raphson method (if ν and $\overline{M}$ depend on $\vec{B}$).

4. Acceleration of convergence if the difference equations are solved by iteration.

5. Check of convergence: Steps 2-5 are repeated until a specified convergence criterion is met.

6. Calculation of characteristic values of the magnetic field, such as the magnetic induction, the vector equi-potential lines (lines of flux), etc.

CONCLUSIONS AND RECOMMENDATIONS

The finite difference approach to analysis of nonlinear electro magnetic circuits is a viable technique. One possible formulation and its theoretical basis are presented in this report. In addition, system equations corresponding to those utilized in the existing CLYDE computer program (appendix F) are shown.

It is recommended that the investigation be continued. Specifically, it should be determined if the CLYDE program can be modified to solve nonlinear electromagnetic circuit problems, or if a new computer program must be developed.

BIBLIOGRAPHY

1. S.V. Ahamed and E.A. Erdelyi, "Nonlinear Vector Potential Equations for Highly Saturated Heteropolar Electrical Machines," IEEE Trans on Aerospace, Vol 2, No. 2 (1964).

2. S.V. Ahamed and E.A. Erdelyi, "Nonlinear Theory of Salient Pole Machines," IEEE Trans on Power Apparatus and Systems, Vol PAS-85, No. 1, pp 61-70 (1966).

3. E.A. Erdelyi, S.V. Ahamed, and R.D. Burtness, "Flux Distribution in Saturated DC Machines at No Load," IEEE Trans Power Apparatus and Systems, Vol PAS-84, pp 375-381 (1965).

4. E.A. Erdelyi and E.F. Fuchs, "Nonlinear Magnetic Field Analysis of DC Machines, Part I: Theoretical Fundamentals," IEEE Trans on Power Apparatus and Systems, Vol PAS-89, No. 7, pp 1546-1554 (1970).

5. E.A. Erdelyi and E.F. Fuchs, "Nonlinear Magnetic Field Analysis of DC Machines, Part II: Application of the Improved Treatment," IEEE Trans on Power Apparatus and Systems, Vol PAS-89, No. 7, pp. 1555-1564 (1970).

6. E.A. Erdelyi, E.F. Fuchs, and D.H. Binkley, "Nonlinear Magnetic Field Analysis of DC Machines, Part III: Equipotential Plots Drawn by Computer," IEEE Trans on Power Apparatus and Systems, Vol PAS-89, No. 7, pp 1565-1583 (1970).

7. R.H. Haveson, "A Method to Analyze Nonlinear Magnetic Systems," E.C.E. 699 Research Project, Department of Electrical and Computer Engineering, University of Michigan (1973).

8. W.H. Hayt, Engineering Electromagnetics, McGraw-Hill, New York (1958).

9. C.A. Holt, Introduction to Electromagnetic Fields and Waves, John Wiley & Sons, New York (1963).

10. R.F. Jackson and E.A. Erdelyi, "Combination and Separation of Coordinates and Modular Programming for DC Machine Fields," IEEE Trans on Power Apparatus and Systems, Vol PAS-87, No. 3, pp 659-664 (1968).

11. R.W. Landgraff and E.A. Erdelyi, "Influence of Airgap Curvature on the Flux Distribution in Saturated Homopolar Alternators," IEEE Trans on Aerospace, Vol 2, No. 2, pp 904-912 (1964).

12. K. Reichert, "The Calculation of Magnetic Circuits with Permanent Magnets by Digital Computer," IEEE Trans on Magnetics, Vol MAG-6, No. 4, pp 283-288 (1970).

13. K.E. Schenk, E.F. Fuchs, and F.C. Truit, "Load Analysis of Nonlinear Homopolar Inductor Alternators," IEEE Trans on Power Apparatus and Systems, Vol PAS-92, pp 442-448 (1973).

14. E.D. Torre and C.V. Longo, The Electromagnetic Field, Allyn & Bacon, Boston (1969).

15. F.C. Truit and E.A. Erdelyi, "No-Load Flux Distribution in Saturated High-Speed Homopolar Inductor Alternators," IEEE Trans on Aerospace-Support Conf Proc, pp 417-429 (1963).

16. F.C. Truit, E.A. Erdelyi, and R.E. Hopkins, "Representation of the Magnetization Characteristic of DC Machines for Computer Use," IEEE Trans on Power Apparatus and Systems, Vol PAS-87, No. 3, pp 665-669 (1968).

17. F.C. Truit, E.A. Erdelyi, and R.F. Jackson, "The Nonlinear Potential Equation and It's Numerical Solution for Highly Saturated Electrical Machines," IEEE Trans on Aerospace-Support Conference Proc, pp 430-440 (1963).

18. P. Unterweger, "Computation of Magnetic Fields in Electrical Apparatus," IEEE Trans on Power Apparatus and Systems, Vol PAS-93, pp 991-999 (1974).

19. R.I. Isakower and R.E. Barnas, "Computergraphics Language for Your Design Equations - CLYDE," Information Report 73-1, Scientific and Engineering Applications Div, Management Information Systems Directorate, Picatinny Arsenal, Dover, NJ (1973).

APPENDIX A

EDDY CURRENT LOSSES

In the two-dimensional model, infinite length in the Z direction is assumed as well as homogeneity in the Z direction. If no additional assumptions are made a loss of energy in the iron regions due to eddy currents (circulating in the X-Z plane) must be taken into account. The introduction of laminations reduces this energy loss, but also reduces the effective cross-sectional area of the magnetic material.
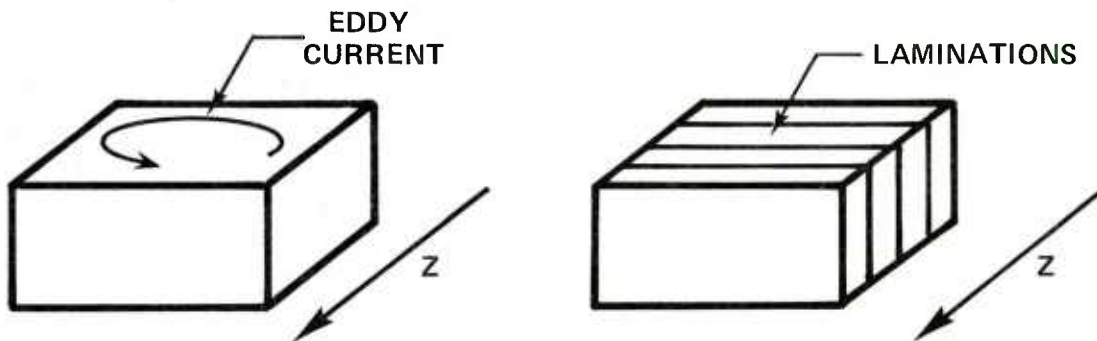
Therefore, the insulation thickness between laminations is assumed to be zero. Then, the only effect will be to eliminate eddy currents.

APPENDIX B

APPROXIMATION FOR RELUCTIVITY (ν)

In order to utilize the finite difference method for calculating the vector potential, the nonlinear magnetization curves of the materials under consideration must be represented.

The magnetization curve is the well-known hysteresis loop shown in figure B-1. This curve is nonlinear and multivalued. Since, for most materials, this hysteresis loop is rather thin, it is assumed that the magnetization curve is monotonic and single valued. By using average values of H for each value of B (or average values of B for each value of H), average curves that closely follow the so-called virgin magnetization characteristic can be obtained. In addition, it is assumed that the magnetization curves pass through the origin.

Therefore,

$$H_{AVE} = \frac{H_u + H_\ell}{2} \quad \text{or} \quad B_{AVE} = \frac{B_u + B_\ell}{2}$$

A plot of the average magnetization curve is shown in figure B-2.

In order to minimize the calculation time for the vector potential, a piecewise linear approximation is used for the average magnetization curve. As shown in a figure B-3, it is assumed that beyond a saturation magnetic induction (B_s), the permeability is that of free space. The distance between B_s and the origin is subdivided into n equal parts, denoted as ΔB. Next, all H_j's are read and stored. The slope between all P_j's are read and stored. The slope between all P_j's and P_{j+1}'s is calculated as:

$$m_j = \frac{H_{j+1} - H_j}{\Delta B}$$

45

In order to calculate the reluctivity for a flux density less than B_S and in the interval j to j+1, the following expression is used.

$$\nu = \frac{H}{B} = \frac{(B-B_j)m_j + H_j}{B}$$

To calculate reluctivity at a point $B > B_S$:

$$\nu = \frac{H_S + (B-B_S)\nu_0}{B}$$

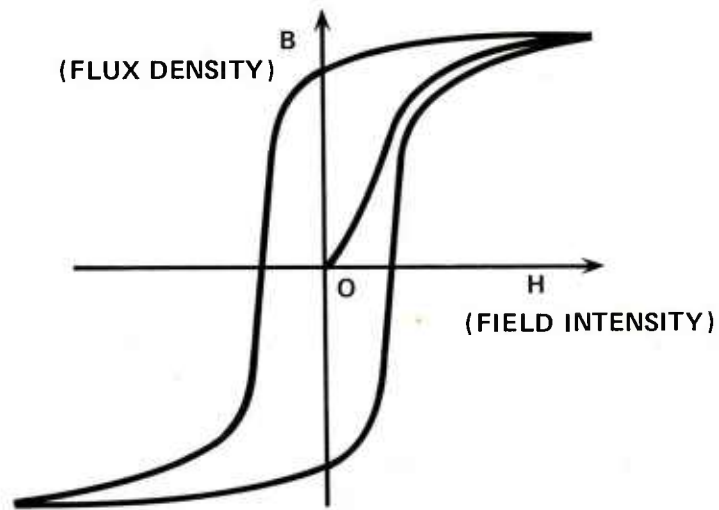where ν_0 is the reluctivity of free space.
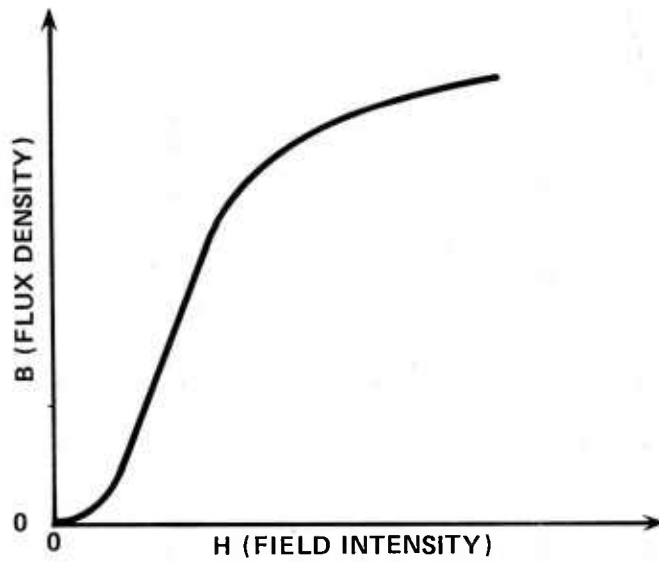
Figure B-1.   Magnetization curve (hysteresis).
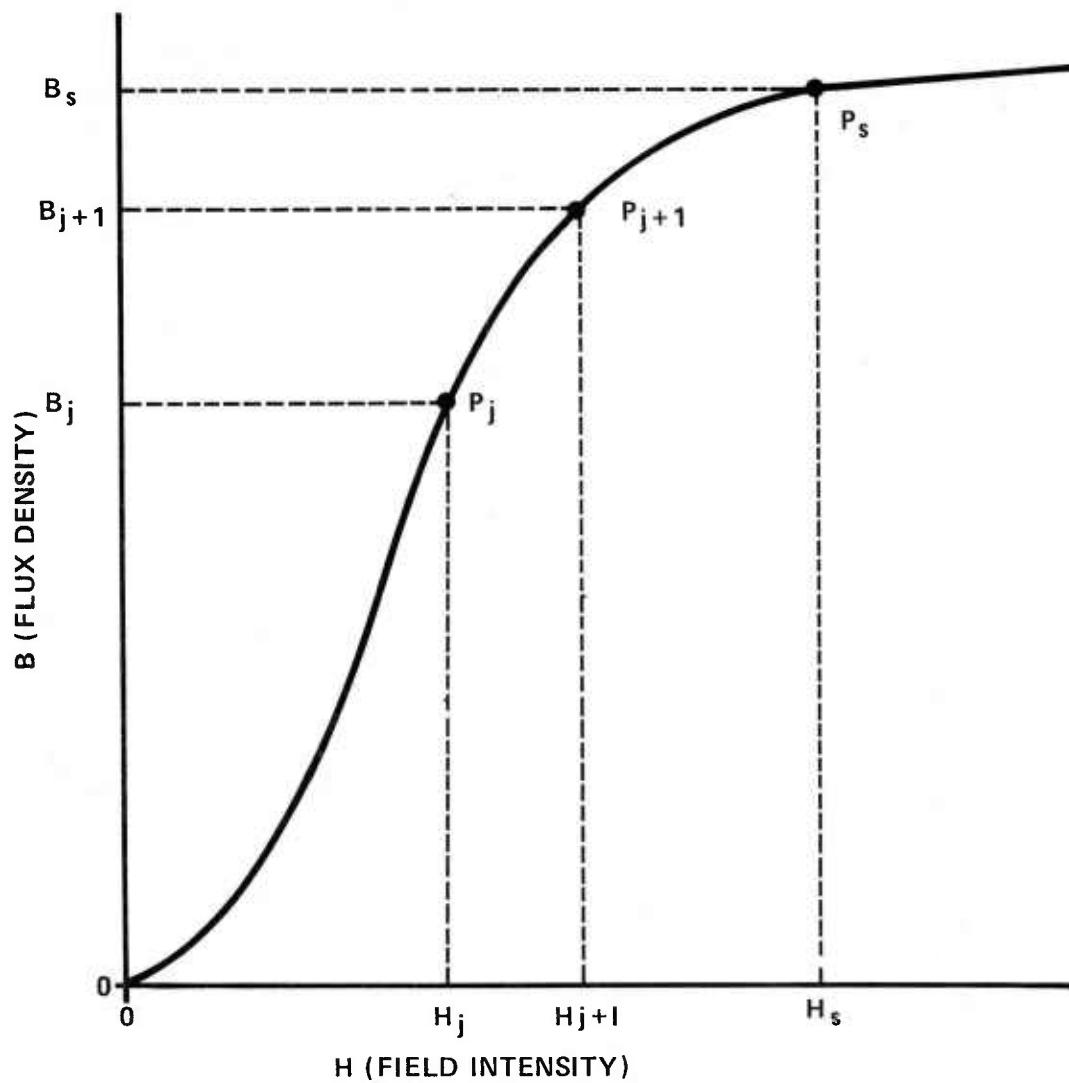

Figure B-2.   Average magnetization curve.

47

Figure B-3. Piecewise linear approximation of the magnetization curve.

THE LAPLACIAN OPERATOR

In Cartesian coordinates

$$\nabla^2 A = \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} + \frac{\partial^2 A}{\partial Z^2} = f(X, Y, Z)$$

In two dimensions this reduces to

$$\nabla^2 A = \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} \mp f(X, Y)$$

In cylindrical coordinates

$$\nabla^2 A = \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial A}{\partial R}\right) + \frac{1}{R^2}\left(\frac{\partial^2 A}{\partial \phi^2}\right) + \frac{\partial^2 A}{\partial Z^2}$$

By expanding the first term:

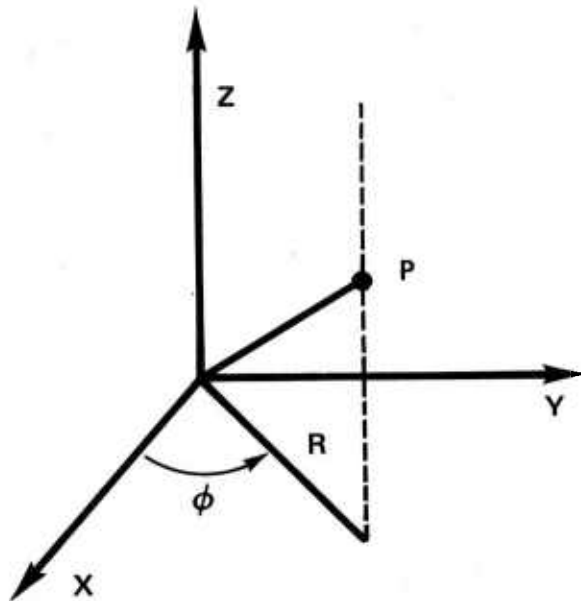$$\frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial A}{\partial R}\right) = \frac{1}{R}\left[R \frac{\partial^2 A}{\partial R^2} + \frac{\partial A}{\partial R}\right] = \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R}$$

Therefore, $\nabla^2 A = \dfrac{\partial^2 A}{\partial Z^2} + \dfrac{\partial^2 A}{\partial R^2} + \dfrac{1}{R} \dfrac{\partial A}{\partial R} + \dfrac{1}{R^2}\left(\dfrac{\partial^2 A}{\partial \phi^2}\right) = f(R, Z, \phi)$

In two dimensions this is:

$$\nabla^2 A \atop R,Z = \frac{\partial^2 A}{\partial Z^2} + \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R} = f(R, Z)$$

The relationship between coordinate systems is defined by the following representation:

$$R = \sqrt{X^2 + Y^2}$$

$$Z = Z$$

$$\phi = \text{Tan}^{-1} \frac{Y}{X}$$

$$X = R \; \text{Cos} \; \phi$$

$$Y = R \; \text{Sin} \; \phi$$

$$Z = Z$$

50

Therefore, in general,

$$\nabla^2 A = \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} + \frac{\partial^2 A}{\partial Z^2} = \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R} + \frac{1}{R^2} \frac{\partial^2 A}{\partial \phi^2} + \frac{\partial^2 A}{\partial Z^2}$$

Note:

If $Z = 0$, $\nabla^2 A = \dfrac{\partial^2 A}{\partial X^2} + \dfrac{\partial^2 A}{\partial Y^2}$

$$\neq \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R} + \frac{1}{R^2} \frac{\partial^2 A}{\partial \phi^2}$$

If $\phi = 0$, then $Y = 0$

$$\nabla^2 A = \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Z^2}$$

$$= \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R} + \frac{\partial^2 A}{\partial Z^2}$$

If $\phi = 90^{\circ}$, then $X = 0$

$$\nabla^2 A = \frac{\partial^2 A}{\partial Y^2} + \frac{\partial^2 A}{\partial Z^2}$$

$$= \frac{\partial^2 A}{\partial R^2} + \frac{1}{R} \frac{\partial A}{\partial R} + \frac{\partial^2 A}{\partial Z^2}$$

APPENDIX D

THE MAGNETIZATION VECTOR

The basic source of a magnetic field is electrical current. The magnetic field, due to electrical current flowing in a loop and observed at large distances compared with the dimensions of the loop, is called a dipole field. The small current loop is referred to as a magnetic dipole. The magnetic dipole moment ($\vec{m}$) is defined as:

$$\vec{m} = I \ S \ \vec{n}$$

where:

I is the loop current

S is the area of the loop

$\vec{n}$ is a unit vector in the direction determined by the right hand rule.

Electrons surrounding the nuclei of atoms in matter have spin and orbital motion. In effect, the electrons form current loops. When viewed macroscopically, these current loops appear to be a continuum of magnetic dipoles.

When the individual dipole moments in a material are randomly oriented, their net effect is zero. However, if there is a coherence to the orientation there will be a net effect, and the material is said to be magnetized.

The magnetization vector ($\vec{M}$) can be conceived as the net magnetic dipole moment per unit volume, i.e.,

$$\vec{M} \ = \ \lim_{\Delta V \to 0} \ \frac{\Sigma \ \vec{m}}{\Delta V}$$

The unit for magnetization is amperes per meter and is the same dimensionally as the magnetic field intensity (H). The vector $\vec{M}$ will be zero if the dipole moments are random and nonzero if there is some coherence of the dipole moments.

A magnetized body behaves as though there were an internal current density equal to CURL $\vec{M}$ and analagous to a current density vector $(\vec{J})$. CURL $\vec{M}$ is usually referred to as the atomic or molecular current density. That is because the field contributions of CURL $\vec{M}$ result from spin or orbital motions that are localized on the atomic scale.

The general definition for B in terms of its current sources is:

$$\vec{B} = \mu(\vec{H} + \vec{M}) \ .$$

APPENDIX E

X AND Y COMPONENTS OF THE MAGNETIC INDUCTION

$$\vec{B} = \text{CURL } \vec{A} = \nabla\times\vec{A}$$

$$= \left(\frac{\partial A_Z}{\partial Y} - \frac{\partial A_Y}{\partial Z}\right)\vec{a}_X + \left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right)\vec{a}_Y + \left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)\vec{a}_Z$$

Since it was assumed that $A_Y = A_X = 0$

$$B = \text{CURL } \vec{A} = \left(\frac{\partial A_Z}{\partial Y}\right)\vec{a}_X + \left(\frac{-\partial A_Z}{\partial X}\right)\vec{a}_Y$$

Therefore,

$$B_X = \frac{\partial A_Z}{\partial Y} \quad \text{and} \quad B_Y = -\frac{\partial A_Z}{\partial X}$$

As an approximation the average magnetic induction in the center of the mesh cell can be calculated:

$$B_X = \frac{\partial A_Z}{\partial X} \cong \frac{\Delta A_Z}{\Delta Y} = \frac{\left(\frac{A_0 + A_1}{2}\right) - \left(\frac{A_4 + A_8}{2}\right)}{h_4} = \frac{A_1 - A_8 + A_0 - A_4}{2h_4}$$

$$B_Y = \frac{-\partial A_Z}{\partial X} \cong \frac{-\Delta A_Z}{\Delta X} = \left[\frac{\left(\frac{A_1 + A_8}{2}\right) + \left(\frac{A_0 + A_4}{2}\right)}{h_1}\right] = -\left(\frac{A_1 - A_0 + A_8 - A_4}{2h_1}\right)$$

Therefore, $$|\vec{B}| = \sqrt{B_X{}^2 + B_Y{}^2} = \sqrt{\left(\frac{A_1-A_8+A_0-A_4}{2h_4}\right)^2 + \left[-\left(\frac{A_1-A_0+A_8-A_4}{2h_1}\right)\right]^2}$$

APPENDIX F

CLYDE SYSTEM EQUATIONS

Consideration is being given to adapting the computer program CLYDE[1] to handle nonlinear electromagnetic circuits.

The formulation of the Harmonic operator for an irregular grid in the CLYDE program is:

$$h^2\nabla^2 f_0 = \left[\frac{2A}{b_1(b_1+b_3)}\right]f_1 + \left[\frac{2B}{b_2(b_2+b_4)}\right]f_2 + \left[\frac{2A}{b_3(b_1+b_3)}\right]f_3 + \left[\frac{2B}{b_4(b_2+b_4)}\right]f_4$$

$$-\left[\frac{2A}{b_1 b_3} + \frac{2B}{b_2 b_4}\right]f_0 = h^2 D$$

A corresponding equation can be obtained by rearranging

$$A_0 = \frac{\sum_{i=1}^{4} A_i \alpha_i' + I_0 + M_0}{\sum_{i=1}^{4} \alpha_i'}$$

and then multiplying by h^2:

$$h^2\nabla^2 A_0 = \left[h^2\alpha_1'\right]A_1 + \left[h^2\alpha_2'\right]A_2 + \left[h^2\alpha_3'\right]A_3 + \left[h^2\alpha_4'\right]A$$

$$-\left[h^2\left(\sum_{i=1}^{4}\alpha_i'\right)\right]A_0 = h^2\left[I_0 + M_0\right]$$

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