

AD-A070 470

CALIFORNIA UNIV IRVINE SCHOOL OF ENGINEERING

F/G 9/2

TRANSIENT BEHAVIOR OF THE LMS ADAPTIVE FILTER-RESPONSE TO VARIA--ETC(U)

APR 79 N J BERSHAD, P L FEINTUCH, F A REED

N00014-75-C-0682

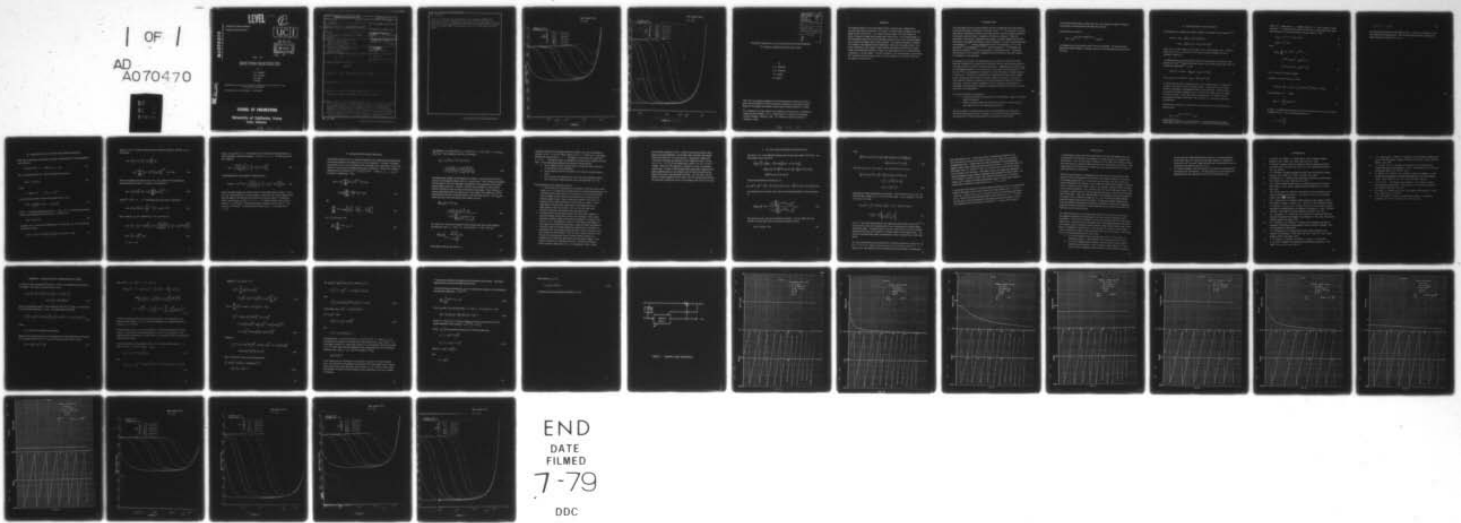
UNCLASSIFIED

TR-79-1-ONR

NL

| OF |

AD
A070470



END
DATE
FILMED
7-79
DDC

LEVEL

B

STOCHASTIC SYSTEMS RESEARCH

TECHNICAL REPORT ONR-79-1

UCI

DDC
RECEIVED
JUN 26 1979
C

ADA070470

APRIL 1979

TRANSIENT BEHAVIOR OF THE LMS ADAPTIVE FILTER-
RESPONSE TO VARIABLE FREQUENCY SPECTRAL LINES

By

N.J. Bershad

P.L. Feintuch

F.A. Reed

B. Fisher

Reproduction in whole or in part is permitted for any purpose of the
United States Government.

Distribution of this document is unlimited.

DDC FILE COPY

SCHOOL OF ENGINEERING

University of California, Irvine
Irvine, California

79 06 25 037

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR-ONR 79-1	2. GOVT ACCESSION NO	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6. TRANSIENT BEHAVIOR OF THE LMS ADAPTIVE FILTER- RESPONSE TO VARIABLE FREQUENCY SPECTRAL LINES		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10. N.J. Bershad, P.L. Feintuch, F.A. Reed, B. Fisher		8. CONTRACT OR GRANT NUMBER(s) 15. N00014-75-C-0682 N00024-77-C-6251
9. PERFORMING ORGANIZATION NAME AND ADDRESS School of Engineering University of California, Irvine Irvine, California 92717		11. PROGRAM ELEMENT PROJECT TASK AREA & WORK UNIT NUMBERS NR-042-290 (12) 44/A
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics and Probability Program Office of Naval Research Arlington, Virginia		12. REPORT DATE 17 April 1979
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 14. TR-79-1-ONR		13. NUMBER OF PAGES 42
		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) unlimited		
17. DISTRIBUTION STATEMENT (of abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) LMS adaptive filter, LMS algorithm, LMS filter configuration.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The transient behavior of the LMS adaptive filter is studied when configured as a canceller operating in the presence of a fixed or variable complex frequency sine-wave signal buried in white noise. For a fixed frequency signal, the mean weights are shown to respond to signal more rapidly than to noise alone. For a chirped signal, a fixed parameter matrix first-order difference equation is derived for the mean weights and a closed-form steady-state solution obtained. The transient response is obtained as a function of the eigenvectors and eigenvalues of the input covariance matrix.		

406 322

1073

20. continued)

Sufficient conditions for the stability of the transient response are derived and an upper bound on the eigenvalues obtained. Finally, the mean-square error is evaluated when responding to a chirped signal. The gain coefficient of the LMS algorithm is determined that minimizes the mean-square error for chirped signals as a function of chirp rate and signal and noise powers.

Mean Square Error
vs. $\mu\sigma_n^2$

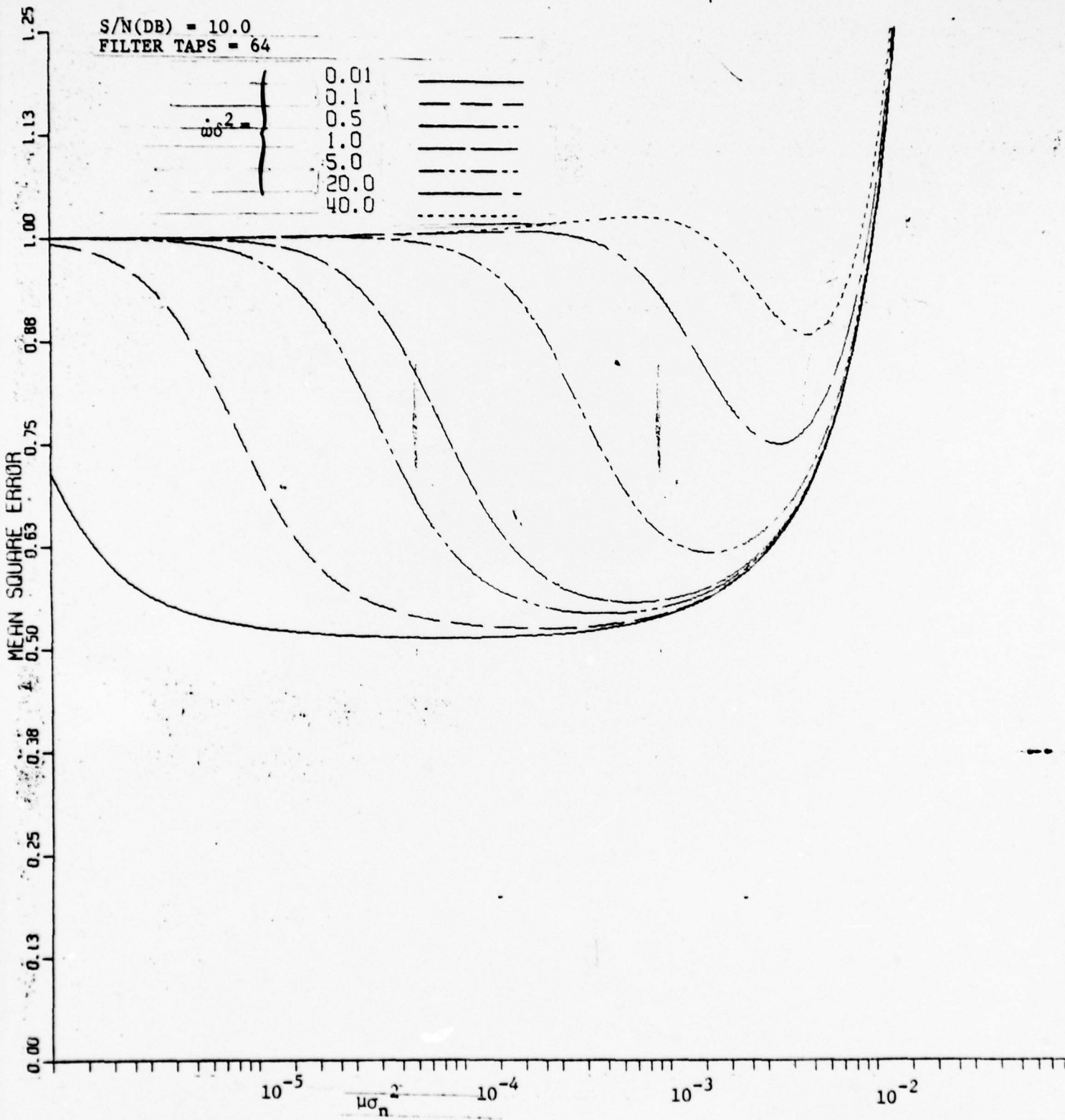


FIGURE 14

Mean Square Error

vs. $\mu\sigma_n^2$

S/N(DB) = 0.0
FILTER TAPS = 64

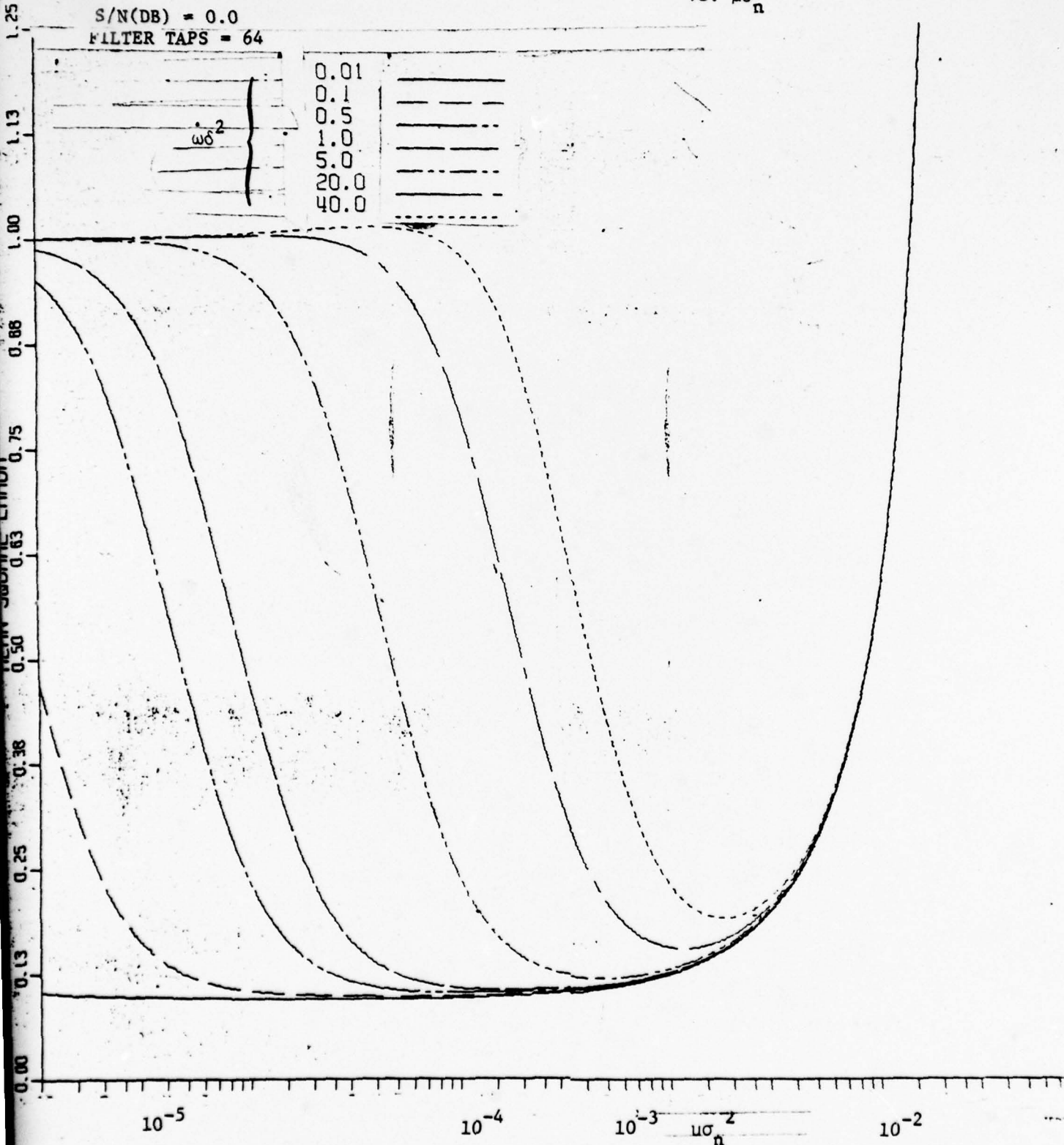


FIGURE 15

Accession For	
NTIS GMA&I	<input checked="" type="checkbox"/>
DDC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Avail and/or special
A	

TRANSIENT BEHAVIOR OF THE LMS ADAPTIVE FILTER-RESPONSE
TO VARIABLE FREQUENCY SPECTRAL LINES*

by

N. J. Bershad¹

P. L. Feintuch²

F. A. Reed³

B. Fisher⁴

*This work was partially supported by Naval Sea Systems Command Code 06H1-1 under Contract N00024-77-C-6251 and partially supported by the Office of Naval Research, Probability and Statistics Branch under Contract N00014-75-C-0682.

¹N. J. Bershad is with the University of California, Irvine and also a Consultant to Hughes Aircraft Company. ²P. L. Feintuch and ³F. A. Reed are with Hughes Aircraft Company, Fullerton, Calif. ⁴B. Fisher is with the University of California, Irvine.

ABSTRACT

The transient behavior of the LMS adaptive filter is studied when configured as a canceller operating in the presence of a fixed or variable complex frequency sine-wave signal buried in white noise. For a fixed frequency signal, the mean weights are shown to respond to signal more rapidly than to noise alone. For a chirped signal, a fixed parameter matrix first-order difference equation is derived for the mean weights and a closed-form steady-state solution obtained. The transient response is obtained as a function of the eigenvectors and eigenvalues of the input covariance matrix. Sufficient conditions for the stability of the transient response are derived and an upper bound on the eigenvalues obtained. Finally, the mean-square error is evaluated when responding to a chirped signal. The gain coefficient of the LMS algorithm is determined that minimizes the mean-square error for chirped signals as a function of chirp rate and signal and noise powers.

I INTRODUCTION

The LMS adaptive filter has been proposed and used in situations where the statistics of the input processes are unknown or partially known [1-3]. The structure of the LMS algorithm for adjusting the weights of the adaptive filter requires quadratic operations on stochastic input data which, in general, are difficult to analyze. Under the assumption of statistically independent data samples, the mean weight vector and the covariance of the weight fluctuations have been obtained for a variety of stationary input data statistics [1-11]. Special configurations of the LMS algorithm, such as noise cancelling [4] line enhancing [4, 6, 8, 9, 11], spectral analysis, [5, 12] and single frequency line detection [7-10], have been studied in considerable detail. The special characteristics of the LMS filter configuration have been used to aid in the analysis of the behavior of the algorithm.

The purpose of this paper is to present some exact analytical results for the LMS algorithm configured as an adaptive noise canceller when the input process consists of a chirped sine wave in additive stationary white noise. Although some previous work on LMS algorithm behavior in a non-stationary environment has been published [13-16], only one [16] has investigated the response of the LMS algorithm to chirped sinusoids in white noise. The analysis is performed by assuming the chirping is slow enough so that a quasi-stationary model for the mean weights can be used. In this paper, exact analytical results are obtained for the chirped sinusoidal signal with arbitrary chirp rate. Since the adaptive cancelling of dynamic signals is a key element in cancelling, line enhancing and frequency tracking, the analytical results for the above model have wide applicability.

Two principal results of this paper are

1. A closed form analytical expression for the LMS mean weights in a dynamic signal environment.
2. Explicit trade-off results between filter parameters, weight variances, mean-square-error, and input signal dynamics.

The latter result is of special interest since it shows explicitly the compromise between fast adaptation in order to respond to variations in the input statistics and slow adaptation to reduce the fluctuations in the adaptation process itself.

For the narrowband signal in white-noise case, the configuration shown in Figure 1 can be used to model the above LMS algorithm functions.

With reference to Figure 1,

$$d(n) = \sigma_s e^{j(\omega_0 n\Delta t + \dot{\omega}(n\Delta t)^2/2 + \phi)} + n_1(n\Delta t)$$

Δ is chosen so that $n_1(n\Delta t)$ and $n_1(n\Delta t - \Delta)$ are un-correlated. On the other hand, because the desired signal is a chirped sine-wave, it decorrelates more slowly than the noise.

II. DYNAMIC MODEL FOR THE INPUTS

The algorithm for changing the complex weights of the adaptive filter is given by ^[17],

$$\begin{aligned} W(n+1) &= W(n) + \mu \left[d(n) - X^T(n) W(n) \right] X^*(n) \\ &= W(n) + \mu \left[d(n) X^*(n) - X^*(n) X^T(n) W(n) \right] \end{aligned} \quad (1)$$

where $W(n)$ = filter weight vector at time n , $d(n)$ = desired signal, $X(n)$ = observed data vector at time n , and where $*$ and T denote complex conjugate and vector transpose respectively.

Averaging equation (1) and assuming 1) the data sequence $X(n)$ is statistically independent over time ^[1-4] and 2) the present weight vector and the present data vector are statistically independent ^[11], yields

$$E\{W(n+1)\} = E\{W(n)\} + \mu \left[R_{dx}(n) - R_{xx}(n) E\{W(n)\} \right] \quad (2)$$

where $R_{dx}(n) = E\{d(n)X^*(n)\}$, $R_{xx}(n) = E\{X^*(n)X^T(n)\}$.

In practice the algorithm sampling interval (Δt) is usually chosen to correspond to the delay δ between the taps of the adaptive filter. Furthermore (Δt) is usually chosen to correspond to independent samples of the noisy data. Hence the delay Δ is chosen to be integer multiples of (Δt) in order for the noises in the two inputs to be un-correlated. On the other hand, the longer that Δ is chosen, the less correlated is the signal component. Thus choice of $\Delta = \delta$ is the best that can be accomplished.*

When the input consists of a complex sine-wave with linearly-varying frequency in additive noise,

$$d(t) = \sigma_s e^{j(\omega_0 t + \dot{\omega} t^2/2 + \phi)} + n(t) \quad (3)$$

*Other integer values of δ for the bulk delay Δ can be studied using the subsequent analysis and the results show that $\Delta = \delta$ yields the best filter performance.

where σ_s^2 = signal power, ω_o = signal frequency, $\dot{\omega}$ = rate of change of signal frequency, ϕ = random phase of signal and the noise is independent of the signal with noise power σ_n^2 and normalized covariance matrix G, then

$$R_{xx}(n) = \sigma_n^2 G + \sigma_s^2 D(n) D^*(n)^T \quad (4)$$

$$R_{dx}(n) = \sigma_s^2 D(n) \quad (5)$$

where

$$D^T(n) = \begin{pmatrix} e^{j\omega_o \delta} & e^{j\dot{\omega} \delta^2 n} & e^{-j\delta^2 \dot{\omega}/2} & \dots \\ e^{j\omega_o m \delta} & e^{j\dot{\omega} m \delta^2 n} & e^{-jm^2 \delta^2 \dot{\omega}/2} & \dots \\ e^{j\omega_o M \delta} & e^{j\dot{\omega} M \delta^2 n} & e^{-jM^2 \delta^2 \dot{\omega}/2} & \dots \end{pmatrix} \quad (6)$$

with M = number of complex weights.

Using Eqs. (4) and (5) in Eq. (2) yields

$$E\{W(n+1)\} = \left[I - \mu (\sigma_n^2 G + \sigma_s^2 D(n) D^*(n)^T) \right] E\{W(n)\} + \mu \sigma_s^2 D(n) \quad (7)$$

For white noise, $G = I$. Define

$$M(n) = I + \frac{\sigma_s^2}{\sigma_n^2} D(n) D^*(n)^T \quad (8)$$

For any n, the eigenvectors of M(n) are the vector D(n) and any set of (M-1) vectors orthogonal to D(n). The associated eigenvalues are

$$\lambda_1 = 1 + M \frac{\sigma_s^2}{\sigma_n^2}$$

$$\lambda_2 = \lambda_3 = \dots = \lambda_M = 1 \quad (9)$$

Note that the eigenvalues are independent of time. All the time variations in $M(n)$ are contained in the eigenvectors. This special property of $M(n)$ is exploited to obtain closed form solutions for Eq. (7).

III. SOLUTION OF EQ (7) FOR THE MEAN WEIGHT BEHAVIOR

Since $M(n)$ is Hermitian, there exists a unitary transformation $P(n)$ which diagonalizes $M(n)$ for each n ,

$$P(n) M(n) P^{-1}(n) = \lambda = \text{Diag} (\lambda_1, \lambda_2 \dots \lambda_m) \quad (10)$$

The λ_i are not functions of n . Due to the special form of $D(n)$,

$$D^T(n) = V^n D^T(0) \quad (11)$$

where

$$V = \text{Diag} (a, a^2, \dots, a^M), \quad a = e^{j\omega\delta^2}$$

Also $P(n)$ can be written in terms of the eigenvectors of $M(n)$,

$$P^+(n) = \frac{1}{\sqrt{M}} \left[D(n), R_1(n), \dots, R_{M-1}(n) \right] \quad (12)$$

where $^+$ = conjugate transpose and $R_1, R_2 \dots R_{M-1}$ are $M-1$ mutually ortho-normal vectors, also orthogonal to $D(n)$ for each n . Using Eq. (11),

$$P(n) = P(0) (V^*)^n \quad (13)$$

Using Eqs. (8), (11) and (13) and defining $Z(n) = P(n) E[W(n)]$, Eq. (7) can be written in terms of Z only as

$$Z(n+1) = P(0) V^* P^{-1}(0) \left[I - \mu \sigma_n^2 \lambda \right] Z(n) + \mu \sigma_s^2 P(0) V^* D(0) \quad (14)$$

Since Eq. (14) is a constant coefficient linear difference equation, with $P(0) = P_0$, it follows that

$$Z(n) = \left\{ P_0 V^* P_0^{-1} [I - \mu \sigma_n^2 \lambda] \right\}^n Z(0) + \mu \sigma_s^2 \sum_{m=1}^n \left\{ P_0 V^* P_0^{-1} [I - \mu \sigma_n^2 \lambda] \right\}^{m-1} P_0 V^* D(0) \quad (15)$$

Before investigating the general case of Eq. (15), consider the fixed frequency sinusoid signal case when $V = I$ and Eq. (15) simplifies to

$$Z(n) = [I - \mu \sigma_n^2 \lambda]^n Z(0) + \mu \sigma_s^2 \sum_{m=1}^n [I - \mu \sigma_n^2 \lambda]^{m-1} S \quad (16)$$

where $S^T = (\sqrt{M}, 0, 0, \dots, 0)$. Expressing the matrix sum in closed form

$$Z(n) = [I - \mu \sigma_n^2 \lambda] Z(0) + \frac{\sigma_s^2}{\sigma_n^2} \lambda^{-1} S [I - (I - \mu \sigma_n^2 \lambda)^n] \quad (17)$$

Thus, using Eq. (9), the components of $Z(n)$ are given by

$$z_1(n) = \left[1 - \mu (\sigma_n^2 + M \sigma_s^2) \right]^n z_1(0) + \frac{\sqrt{M} \sigma_s^2 / \sigma_n^2}{1 + M \sigma_s^2 / \sigma_n^2} \left\{ 1 - \left[1 - \mu (\sigma_n^2 + M \sigma_s^2) \right]^n \right\}$$

$$z_j(n) = \left[1 - \mu \sigma_n^2 \right]^n z_j(0) \quad (18)$$

$$j = 2, 3, \dots, M$$

Hence, for $\mu(\sigma_n^2 + M\sigma_s^2) < 1$, the response of the weights to the signal frequency is more rapid than to any other frequency. If $z_j(0) = 0$, $j = 1, 2, \dots, M$, then $z_1(n)$ is the only response,

$$z_1(n) = \frac{\sqrt{M}\sigma_s^2 / \sigma_n^2}{1 + M\sigma_s^2 / \sigma_n^2} \left\{ 1 - \left[1 - \mu(\sigma_n^2 + M\sigma_s^2) \right]^n \right\} \quad (19)$$

Transforming back to the original coordinate system,

$$E[W(n)] = P_o^{-1} Z(n) = \frac{\sigma_s^2 / \sigma_s^2}{1 + M\sigma_s^2 / \sigma_n^2} \left\{ 1 - \left[1 - \mu(\sigma_n^2 + M\sigma_s^2) \right]^n \right\} D(0) \quad (20)$$

Hence, the mean weights are scaled versions of the desired signal response. From Eq. (18), note that the time it takes the filter to adapt from zero initial conditions and learn the signal is less than the time required to forget the signal if it disappears. That is, from Eq (19), if $z_j(0) = 0$, $j = 1, 2, \dots, M$, signal response time is proportional to $1 - \mu(\sigma_n^2 + M\sigma_s^2)$. If the signal suddenly disappears so that $z_1(0) \neq 0$, then from Eq. (17) with $\sigma_s^2 = 0$, the decay time towards $z_1(n) = 0$, is proportional to $1 - \mu\sigma_n^2$.

IV. STEADY-STATE WEIGHT BEHAVIOR

The explicit solution of Eq. (14) requires evaluation of the eigenvalues and eigenvectors of the matrix operator in brackets in Eq. (15) (see Appendix I). However, the steady-state solution to Eq. (14) is obtainable without knowledge of the eigenvalues. In Eq. (15), set $Z(0) = 0$ (zero initial conditions) without loss of generality. Let Q be the matrix of eigenvectors of the matrix $P_o V^* P_o^{-1} \left\{ I - \mu \sigma_n^2 \lambda \right\}$ and $\Lambda = \text{Diag} (\Lambda_1, \Lambda_2 \dots \Lambda_m)$ be the matrix of eigenvalues. Thus

$$\begin{aligned} Z(n) &= \mu \sigma_s^2 \sum_{m=1}^n [Q \Lambda Q^{-1}]^{m-1} P_o V^* D(o) \\ &= \mu \sigma_s^2 Q \left(\sum_{m=0}^{n-1} \Lambda^m \right) Q^{-1} P_o V^* D(o) \end{aligned} \quad (21)$$

But

$$\sum_{m=0}^{n-1} \Lambda^m = \text{Diag} \left[\frac{1 - \Lambda_1^n}{1 - \Lambda_1}, \frac{1 - \Lambda_2^n}{1 - \Lambda_2}, \dots, \frac{1 - \Lambda_m^n}{1 - \Lambda_m} \right] \quad (22)$$

for $|\Lambda_i| < 1$ for all i , and

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{n-1} \Lambda^m = (I - \Lambda)^{-1} \quad (23)$$

In Appendix I, it is shown that $|\Lambda_i| < 1$, for all i if $0 < \mu(\sigma_n^2 + M\sigma_s^2) < 2$. Let $Z_{ss} = \lim_{n \rightarrow \infty} Z(n)$. Then, using Eq. (23), Eq. (21) becomes

$$\begin{aligned} Z_{ss} &= \mu\sigma_s^2 Q(I-\Lambda)^{-1} Q^{-1} P_o V^* D(o) \\ &= \frac{\mu\sigma_s^2 P_o [V - (1 - \mu\sigma_n^2) I]^{-1} D(o)}{1 + \mu\sigma_s^2 D_o + [V - (1 - \mu\sigma_n^2) I]^{-1} D(o)} \end{aligned} \quad (24)$$

The steady-state weights are the quantities of interest. Note that they will be time varying, even though the adaptive filter is in steady-state. Here, steady-state implies that the adaptive filter has converged, interpreted as the convergence of the transformed weight vector $Z(n)$. However, the filter has converged to a time-varying solution to follow the time-varying, non-stationary input signal. Thus, when Z_{ss} is inverse transformed back to the mean value of the weights, the transform is via the eigenvectors of the input covariance matrix, which are time-varying. Let $E[W_{ss}(n)]$ denote the mean value of the steady-state weights at time n .

$$\begin{aligned} E[W_{ss}(n)] &= P^{-1}(n) Z_{ss} \\ &= \frac{\mu\sigma_s^2 [V - (1 - \mu\sigma_n^2) I]^{-1} D(n)}{1 + \mu\sigma_s^2 \sum_{k=1}^M \frac{1}{e^{j\omega k \delta^2} - (1 - \mu\sigma_n^2)}} \end{aligned} \quad (25)$$

As a check, Eq. (25) can be compared with the steady-state value of the weights in the stationary case, i. e., with $\dot{\omega} = 0$. For that case $V = I$, $D(n) = D(o)$ and

$$E[W_{ss}(n)]_{\dot{\omega}=0} = \frac{\frac{\sigma_s^2 / \sigma_n^2}{2} D(o)}{1 + M \frac{\sigma_s^2}{\sigma_n^2}} \quad (26)$$

which agrees with Eq. (20) when $n \rightarrow \infty$.

Computer evaluation of the steady-state mean weights in Eq. (25) is presented in Figures 2-5 for $\dot{f} = 5 \text{ Hz/sec}^2$ and in Figure 6-9 for $\dot{f} = 1.25 \text{ Hz/sec}^2$. In all cases, the filter has 128 taps with $\mu = .1$. The signal-to-noise ratios are varied from 10 to 10^{-1} . The figures display the magnitude and phase of the weights across the filter. Three interesting phenomena are displayed in these figures:

1. As the signal-to-noise ratio decreases, the adaptive filter uses more of the taps but at lower amplitudes,
2. The tap phases follow the movement of the linearly varying frequency input,
3. As \dot{f} increases, the taps at the far end of the line contribute relatively less to the filter output than those taps at the beginning of the line.

These phenomena can be explained as follows:

1. The two sources of randomness that contribute to the filter output mean-square-error, are input noise and algorithm noise (weight misadjustment). The contribution of the input noise to the mean square error decreases linearly with the number of taps whereas the algorithm noise increases linearly with the number of taps. Thus, at high input signal-to-noise ratios, the algorithm noise is the limiting factor and few taps are needed. At low input signal-to-noise ratios, input noise is the limiting factor and a large number of taps are needed in order to reject the input noise. Eventually algorithm noise becomes the significant factor.
2. The figures show only the mean values of the steady-state weights at a particular instant of time after the filter has converged. Hence, there should be a quadratic phase shift with the tap number in accordance with $D(n)$ in Eq. (6). Comparison of Figure 3-5 with Figure 2 and Figure 7-9 with Figure 6 shows that the steady-state weights do display this behavior.
3. The filter trades off coherent integration (proportional to the number of significantly non-zero weights) against the phase changes required at each tap to follow the chirped signal. Since the phase change required at each iteration for each tap grows linearly with tap number (entries in V), weights at the far end of the line must make large phase changes in comparison to those at the beginning of the line. Note that the quadratic phase correction along the line, $D(o)$, is independent of time. Hence,

once the filter estimates f_0 and \dot{f} , it knows $D(\omega)$ and can introduce these phase corrections statically. On the other hand, the filter must change phase by the entries in V at each iteration. Large phase changes are most easily made when the magnitude of the weights are small. In Figure 3-5, $\Delta\phi = \dot{\omega}\delta^2 = 2\pi/10M$ radians and in Figure 7-9, $\Delta\phi = 2\pi/40M$ radians. Hence the M^{th} weight has to change by $\pi/5$ and $\pi/20$ radians, respectively. In order to accommodate these large phase changes for the same algorithm step size, the weights of the far end of the line must be smaller than those at the beginning of the line. As \dot{f} decreases, the difference in phase changes at the two ends of the line decreases and the filter can make use of significant values for the weights at the far end of the line.

V. THE MEAN SQUARE ERROR IN STEADY-STATE

The error, $\epsilon(n)$, is the difference between $d(n)$ and the filter output, $W^T(n) X(n)$. Its mean square value is given by

$$\begin{aligned} E[|\epsilon(n)|^2] &= E\left\{ [d(n) - W^T(n) X(n)] [d^*(n) - W^+(n) X^*(n)] \right\} \\ &= E[d(n) d(n)^*] - E[W^T(n) X(n) d^*(n)] - E[d(n) W^+(n) X^*(n)] \\ &\quad + E[W^T(n) X(n) W^+(n) X^*(n)] \end{aligned}$$

Using the assumptions preceding Eq. (2),

$$E[|\epsilon(n)|^2] = (\sigma_s^2 + \sigma_n^2) - \sigma_s^2 2 \operatorname{Re} \left\{ E[W(n)]^+ D(n) \right\} + E[W^T(n) X(n) W^+(n) X^*(n)] \quad (28)$$

The middle term in brackets in Eq. (28) can be evaluated using Eq. (25) and is given by

$$E[W_{ss}(n)]^+ D(n) = \frac{\mu \sigma_s^2 \sum_{k=1}^M \left(e^{j\omega k \delta^2} - (1 - \mu \sigma_n^2) \right)^{-1}}{1 + \mu \sigma_s^2 \sum_{k=1}^M \left(e^{j\omega k \delta^2} - (1 - \mu \sigma_n^2) \right)^{-1}} \quad (29)$$

The last term in Eq. (28) can be evaluated as follows. Let the weight vector be written as a mean value plus a zero-mean fluctuation process.

$$W(n) = E[W(n)] + \zeta(n) \quad (30)$$

Then

$$\begin{aligned} E\left[W^T(n) X(n) W^+(n) X^*(n)\right] &= E\left[W^+(n)\right] E\left[X^*(n) X^T(n)\right] E\left[W(n)\right] \\ &+ E\left[\zeta^+(n) X^*(n) X^T(n) \zeta(n)\right] \end{aligned} \quad (31)$$

The first term in Eq. (31) is known. The second term in Eq. (31) is

$$\begin{aligned} E\left[\zeta^+(n) X^*(n) X^T(n) \zeta(n)\right] &= E\left[\zeta^+(n) E\left[X^*(n) X^T(n)\right] \zeta(n)\right] \\ &= \left(\sigma_s^2 + \sigma_n^2\right) E\left[\zeta^+(n) \zeta(n)\right] \\ &= \left(\sigma_s^2 + \sigma_n^2\right) M \sigma_w^2, \end{aligned} \quad (32)$$

assuming the weight fluctuations are stationary, uncorrelated from tap-to-tap, and have the same variance σ_w^2 for each individual weight. Thus, using Eqs. (31) and (32) in Eq. (28) yields

$$\begin{aligned} E\left[|\epsilon(n)|^2\right] &= \sigma_s^2 \left|1 - E|W(n)|^+ D(n)\right|^2 + \sigma_n^2 \left[1 + E|W(n)|^+ E|W(n)|\right] \\ &+ M \left(\sigma_s^2 + \sigma_n^2\right) \left[\frac{\mu \sigma_n^2}{2 - (M+1) \mu \sigma_n^2}\right] \end{aligned} \quad (33)$$

where σ_w^2 has been approximated by the weight fluctuations under noise alone^[4, 5, 7]. The first term in Eq. (33) represents the error in estimating the chirped complex exponential signal. The second term is the sum of the noise power in the reference channel and the noise power passed by the mean weights of the adaptive filter. The last term represents the weight misadjustment variance multiplied by the total input power.

Eq. (33), normalized by the total input power, has been evaluated as a function of $\mu \sigma_n^2$ for $M = 16, 32, 64$, signal-to-noise ratios of 0 and +10 dB and various $\omega \delta^2$. In Figure 10-15, the trade-off can be seen between static and dynamic contributions to

total mean-square error. In each case, there is an optimum selection of $\mu\sigma_n^2$ which minimizes the deleterious effects of signal errors, input noise and weight misadjustment noise. Comparison of the filter performance for increasing input signal-to-noise ratios verifies improved system performance. On the other hand, for sufficiently large $\dot{\omega}\delta^2$ and SNR = 10 dB, it is seen that the normalized man square error increases as M increases. This effect is due to the weight mis-adjustment noise exceeding the longer coherent integration gain obtained with larger filters. As $\dot{\omega}\delta^2$ decreases, a point is reached where sufficient smoothing time is available (small $\mu\sigma_n^2$) to reduce the weight misadjustment noise to a level so that improved performance is obtained for longer filters (e.g. $\dot{\omega}\delta^2 = 4 \times 10^{-5}$).

It can be seen from Figure 10-15 that the optimum selection of $\mu\sigma_n^2$, for a given filter length M and signal-to-noise ratio, varies in the same manner as $\dot{\omega}\delta^2$. As $\dot{\omega}\delta^2$ increases, a larger value of $\mu\sigma_n^2$ is required to achieve the minimum mean-square error. However this minimum mean-square error increases as $\dot{\omega}\delta^2$. The filter has less time to learn the statistics of the signal and hence must make a larger mean-square-error as the price for responding to a faster moving signal.

CONCLUSIONS

A mathematical model of the mean weight behavior for the LMS adaptive filter has been presented when the filter is operating as a single frequency line enhancer and line follower. For a fixed frequency complex sine wave input, the LMS filter weights have been shown to respond to signal and noise more rapidly than to noise alone. This implies that the filter learns more quickly that a line has appeared than it is able to forget that the line is turned off.

When the signal frequency is changing linearly with time, the mathematical model predicts a time-varying behavior of the filter mean weights necessary to respond to the changing signal frequency. As the chirp rate increases, the filter reduces the relative amplitudes of the weights so as to adjust the effective filter length to optimally match the properties of the signal. That is, for example, suppose the filter designer selects a filter of length $M = M_1$. However, the chirp rate is sufficiently large so that the change in signal frequency, between algorithm iterations, is greater than the bandwidth of the filter. Then, the LMS algorithm will automatically scale the amplitude of its weights to have an effective length M_2 , $M_2 < M_1$, such that the signal remains inside the adaptive filter between iterations. As long as the signal frequency lies within the LMS filter bandwidth, the LMS filter algorithm can track the changing frequency since there is sufficient correlation between the two inputs to drive the LMS algorithm in the correct direction.

The mathematical model of the mean weight behavior has been used for selecting μ , the adaptation coefficient of the algorithm, for a wide variety of signal and noise parameters. The criteria of optimality was that of minimizing the filter output mean square error, since the error is the driving term in the weight adjustment algorithm. (An alternate criteria, based on a signal detection model using the filter output, could also be a candidate for optimization.) A set of curves of normalized mean-square error as a function of signal-to-noise ratio and chirp rate were obtained. From these curves, the following observations can be made:

1. For a given signal-to-noise ratio and chirp rate $\omega\delta^2$, there exists an optimum selection of μ that minimizes the overall mean square error.
2. For slowly changing signal frequency, the mean square error exhibits a relatively broad minimum. This is because a large range of $\mu\sigma_n^2$ will follow the slowly changing signal frequency yet allow sufficient smoothing

so as to keep the weight misadjustment noise below a certain minimum. On the other hand, for a rapidly moving signal frequency, mismatch in selection of $\mu\sigma_n^2$ can cause a significant increase in mean-square error as compared to the optimum selection of $\mu\sigma_n^2$.

3. As a system designer, one would choose a $\mu\sigma_n^2$ that would be optimum for the fastest chirp rate expected. The mean square error would always be upperbounded by the mean square error for the fastest chirp rate.

REFERENCES

1. B. Widrow, P. Mantey, L. Griffiths and B. Goode "Adaptive Antenna Systems," Proc. IEEE, Vol. 55 pp. 2143-2159, Dec. 1967.
2. B. Widrow "Adaptive Filters" in Aspects of Network and System Theory, R. Kalman and N. DeClaris Eds, New York: Holt, Rinehart and Winston, 1971, pp. 563-587.
3. L.J. Griffiths "A Simple Algorithm for Real-Time Processing in Antenna Arrays," Proc. IEEE, Vol. 56 pp. 1696-1704, Oct. 1969.
4. B. Widrow et al., "Adaptive Noise Cancelling Principles and Applications," Proc. IEEE Vol. 63 pp. 1692-1716, Dec. 1975.
5. L.J. Griffiths "Rapid Measurement of a Digital Instantaneous Frequency," IEEE Trans. Acoust. Speech, Signal Processing ASSP-23, pp. 207-222, Apr. 1975.
6. J.R. Zeidler and D.M. Chabries "An Analysis of the LMS Adaptive Filter used as a Spectral Line Enhancer," TN 1476 Feb. 1975, Naval Undersea Center, San Diego, California.
7. P.L. Feintuch and N.J. Bershad "Signal Detection Using Adaptive Filters," 1976 IEEE Int. Symp. on Inform. Th., June 21-24, 1976, Ronneby, Sweden.
8. L. Griffiths, J. Keeler and R. Medaugh "Detection and Convergence Results Relating to the Performance of an Adaptive Line Enhancer", Dept. of Electrical Engr., University of Colorado, Boulder, Colorado, Dec. 1976.
9. S.T. Alexander, J.R. Zeidler and P.M. Reeves "An Analysis of ROC Performance Evaluation of the Adaptive Line Enhancer," Tech. Rpt. Naval Ocean Systems Center, San Diego, Calif.
10. S.T. Alexander, J.R. Zeidler, M. Shensa and R. Medaugh "Detection of Sinusoids in White Noise Using Adaptive Linear Prediction Filtering," May, 1978 (submitted for publication).
11. J.T. Rickard and J.R. Zeidler "Second Order Output Statistics of the Adaptive Line Enhancer," NOSC Tech. Rpt. 202, 1 Dec 1977, Naval Ocean Systems Center, San Diego, Calif.
12. D.W. Tufts, L.J. Griffiths, B. Widrow, J. Glover, J. McCool and J. Treichler "Adaptive Line Enhancement and Spectrum Analysis," Proc IEEE, Vol. 65, No. 1, pp. 169-173.

13. T. P. Daniell and J. E. Brown III "Adaptation in Non-Stationary Applications," in Proc 1970 IEEE Symp. Adaptive Processes (9th) Austin, Texas, Paper No. XXIV-4, Dec. 1970.
14. Y. T. Chien, K. S. Fu "Learning in Non-Stationary Environment Using Dynamic Stochastic Approximation," in Proc. 5th Allerton Conf. on Circuits and Systems Theory, pp. 337-345, 1967.
15. B. Widrow, J. M. McCool, M. G. Larimore, and C. R. Johnson Jr., "Stationary and Non-Stationary Learning Characteristics of the LMS Adaptive Filter," Proc. IEEE Vol. 64, No. 8, pp. 1151-1162, Aug 1976.
16. J. R. Treichler, "The Spectral Line Enhancer - The Concept, an Implementation and an Application," PhD Thesis, Dept. of Electrical Engineering, Stanford University, May 1977.
17. B. Widrow, J. McCool and M. Ball "The Complex LMS Algorithm," IEEE Proc. April 1975, pp. 719-720.
18. L. A. Zadeh and C. A. Desoer "Linear System Theory - A State Space Approach," pp. 575-576, McGraw Hill, New York 1963.

APPENDIX I. EIGENVALUES OF TRANSFORMATION MATRIX

In order to easily evaluate the m^{th} power of a matrix, the eigenvalues of the matrix are needed. The matrix in brackets in Eq. (15) is

$$\begin{aligned} P_o V^* P_o^{-1} [I - \mu \sigma_n^2 \lambda] &= P_o V^* [P_o^{-1} - \mu \sigma_n^2 P_o^{-1} \lambda] \\ &= P_o V^* [I - \mu \sigma_n^2 M(0)] P_o^{-1} \end{aligned} \quad (\text{I-1})$$

Since P_o premultiplies and P_o^{-1} post-multiplies $R = V^* [I - \mu \sigma_n^2 M(0)]$, it is only necessary to find the eigenvalues, λ , of R . The eigenvalues of R satisfy:

$$\left| V^* \left[I - \mu \left(\sigma_n^2 I + \sigma_s^2 D_o D_o^+ \right) \right] - \lambda I \right| = \left| (1 - \mu \sigma_n^2) I - \lambda V - \mu \sigma_s^2 D_o D_o^+ \right| = 0 \quad (\text{I-2})$$

where

$D_o = D(0)$ and $|\quad|$ denotes the determinant.

Because of the simple structure to R , an expression for the eigenvalues can be found. Given a matrix of the form $B = A + a_1 b_1^+$ where a_1 and b_1 are column vectors,

$$|B| = |A| \left[1 + b_1^+ A^{-1} a_1 \right] \quad (\text{I-3})$$

Thus, with $A = (1 - \mu\sigma_n^2) I - \lambda V$, $a_1 = D_o = b_1$,

$$\begin{aligned} \left| (1 - \mu\sigma_n^2) I - \lambda V - \mu\sigma_s^2 D_o D_o^+ \right| &= \left| (1 - \mu\sigma_n^2) I - \lambda V \left\{ I - \mu\sigma_s^2 D_o^+ \right. \right. \\ &\quad \left. \left. \text{Diag} \left(\frac{a}{(1 - \mu\sigma_n^2) a - \lambda}, \frac{a^2}{(1 - \mu\sigma_n^2) a^2 - \lambda}, \dots, \frac{a^M}{(1 - \mu\sigma_n^2) a^M - \lambda} \right) D_o \right\} \right| \\ &= \left[1 - \mu\sigma_n^2 \right]^M \left| I - \frac{V}{1 - \mu\sigma_n^2} \left\{ 1 - \mu\sigma_s^2 \sum_{m=1}^M \frac{a^m}{(1 - \mu\sigma_n^2) a^m - \lambda} \right\} \right| = 0 \end{aligned} \quad (I-4)$$

The term in brackets yields an M 'th order polynomial in λ for which there is no general analytic solution. Eq. (I-4) must be programmed on a digital computer for various μ , σ_s^2 , σ_n^2 and a .

Although explicit values of λ are not obtainable, a simple upper bound on the eigenvalues of λ and hence on the transient behavior of Eq. (21) are obtainable. This upper bound on the eigenvalues is useful since it is an indication of the slowest possible response of the system.

Let \underline{u} be an eigenvector of the matrix in Eq. (I-1) with associated eigenvalue, λ_i . Then, with $\underline{u}^+ \underline{u} = 1$, and $\underline{u}^T = [u_1 u_2 \dots u_m]$

$$\lambda_i \underline{u} = P_o V^* P_o^{-1} (I - \mu\sigma_n^2 \lambda) \underline{u} \quad (I-5)$$

Now,

$$\lambda_i^* \underline{u}^+ \underline{u} \lambda_i = \left| \lambda_i \right|^2 = \underline{u}^+ (I - \mu\sigma_n^2 \lambda)^+ P_o^{-1+} V^{*+} P_o^+ P_o V^* P_o^{-1} (I - \mu\sigma_n^2 \lambda) \underline{u} \quad (I-6)$$

Using $P_o^{-1} = P_o^+$ and $V^+ = V^{-1}$

$$\begin{aligned} |\Lambda_i|^2 &= \sum_{j=1}^M |u_j|^2 |1 - \mu \sigma_n^2 \lambda_j|^2 \\ &= |u_1|^2 \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 + |1 - \mu \sigma_n^2|^2 \sum_{j=2}^M |u_j|^2 \end{aligned} \quad (I-7)$$

Let $q = \sum_{i=2}^M |u_i|^2 \leq 1$ and $p = 1 - q = |u_1|^2 \leq 1$. Then

$$\begin{aligned} |\Lambda_i|^2 &= (1-q) \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 + q |1 - \mu \sigma_n^2|^2 \\ &= \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 + q \left(|1 - \mu \sigma_n^2|^2 - \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 \right) \\ &\leq |1 - \mu \sigma_n^2|^2 \text{ for } |1 - \mu \sigma_n^2| \geq \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 \end{aligned} \quad (I-8)$$

Similarly

$$\begin{aligned} |\Lambda_i|^2 &= p \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 + (1-p) |1 - \mu \sigma_n^2|^2 \leq \left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right|^2 \\ &\text{for } |1 - \mu (\sigma_n^2 + M \sigma_s^2)| \geq |1 - \mu \sigma_n^2| \end{aligned} \quad (I-9)$$

Eqs. (I-8) and (I-9) lead to the following bounds:

① If $\mu (\sigma_n^2 + M \sigma_s^2) \leq 1$, assuming $\sigma_n^2 \neq 0$

$$|\Lambda_i|^2 \leq |1 - \mu \sigma_n^2|^2 < 1 \quad (I-10)$$

② If $\mu(\sigma_n^2 + M\sigma_s^2) > 1$ but $\mu\sigma_n^2 \leq 1$, then for $\sigma_n^2 \neq 0$

$$|\Lambda_i|^2 \leq |1 - \mu\sigma_n^2|^2 < 1 \quad \text{for } \mu M\sigma_s^2 \leq 2(1 - \mu\sigma_n^2)$$

and

$$|\Lambda_i|^2 \leq |1 - \mu(\sigma_n^2 + M\sigma_s^2)|^2 \quad \text{for } \mu M\sigma_s^2 > 2(1 - \mu\sigma_n^2) \quad (\text{I-11})$$

In the latter case, $|\Lambda_i|^2 < 1$ if $\mu(\sigma_n^2 + M\sigma_s^2) < 2$

③ If $\mu\sigma_n^2 > 1$ then

$$|\Lambda_i|^2 \leq |1 - \mu(\sigma_n^2 + M\sigma_s^2)|^2 \quad (\text{I-12})$$

and

$$|\Lambda_i|^2 < 1 \quad \text{if } \mu(\sigma_n^2 + M\sigma_s^2) < 2$$

Combining these, it can be seen that $|\Lambda_i| < 1$ if $\mu(\sigma_n^2 + M\sigma_s^2) < 2$. Since $|\Lambda_i| < 1$ is the condition for existence of a steady state mean weight vector, $\mu(\sigma_n^2 + M\sigma_s^2) < 2$ is a sufficient condition for a steady state solution. It is interesting to note that the bound on $|\Lambda_i|^2$ in each case is just the magnitude of the largest eigenvalue in the stationary frequency case, with σ_n^2 , σ_s^2 , μ and M unchanged. Further

$$\mu(\sigma_n^2 + M\sigma_s^2) < 2$$

is the condition for the convergence of the adaptive canceller in the fixed sinusoid case. This leads to the surprising conclusion that if the mean weight vector achieves a steady state value in the stationary case for given σ_s^2 , σ_n^2 , M and μ , then a steady state solution will exist for those parameter values regardless of the rate of change of frequency.

A more direct result for convergence can be obtained as shown below. This latter approach does not bound the eigenvalues however.

From the discussion following Eq. (I-1) it is seen that R is similar to Q and therefore has the same eigenvalues. From [18],

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n A^{m-1} = [I - A]^{-1} \quad (I-13)$$

if the L_2 norm of A is less than unity, i.e., $\|A\| < 1$. In our case, $R = A$ and

$$\|R\| = \|V^* [I - \mu \sigma_n^2 (\lambda)]\| \leq \|V^*\| \|I - \mu \sigma_n^2 \lambda\| \quad (I-14)$$

But $\|V^*\| = 1$ and $\|I - \mu \sigma_n^2 \lambda\|$ can be evaluated explicitly as the square root of the largest eigenvalue of the matrix $[I - \mu \sigma_n^2 \lambda]^* [I - \mu \sigma_n^2 \lambda]$.

But $[I - \mu \sigma_n^2 \lambda]$ is self-adjoint and has only two distinct eigenvalues.

$$\Lambda_1 = 1 - \mu (\sigma_n^2 + M \sigma_s^2)$$

$$\Lambda_2 = \Lambda_3 = \lambda_M = 1 - \mu \sigma_n^2 \quad (I-15)$$

Hence $\left| 1 - \mu (\sigma_n^2 + M \sigma_s^2) \right| < 1$

and

$$\left| 1 - \mu \sigma_n^2 \right| < 1$$

which implies $|\Lambda_i| < 1$ if

$$0 < \mu (\sigma_n^2 + M\sigma_s^2) < 2$$

(I-16)

in agreement with the discussion following Eq. (I-12).

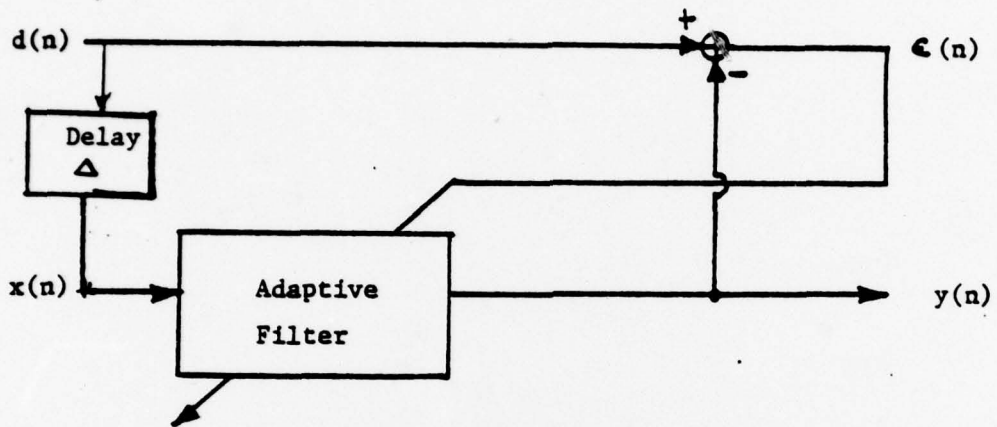


FIGURE 1. ADAPTIVE FILTER CONFIGURATION

2

FIGURE 2

UNDISTORTED SIGNAL, D(n)

$f_0 = 20 \text{ Hz}$

$f = 5 \text{ Hz/kc}^2$

$\delta = \delta t = .0125 \text{ kc}$

$M = 128$

$\Delta = 0$

$\mu = .1$

MAGNITUDE

1

0

PHASE (°)

0

0

180

20 40 60 80 100 120

40 80

TAP NO.

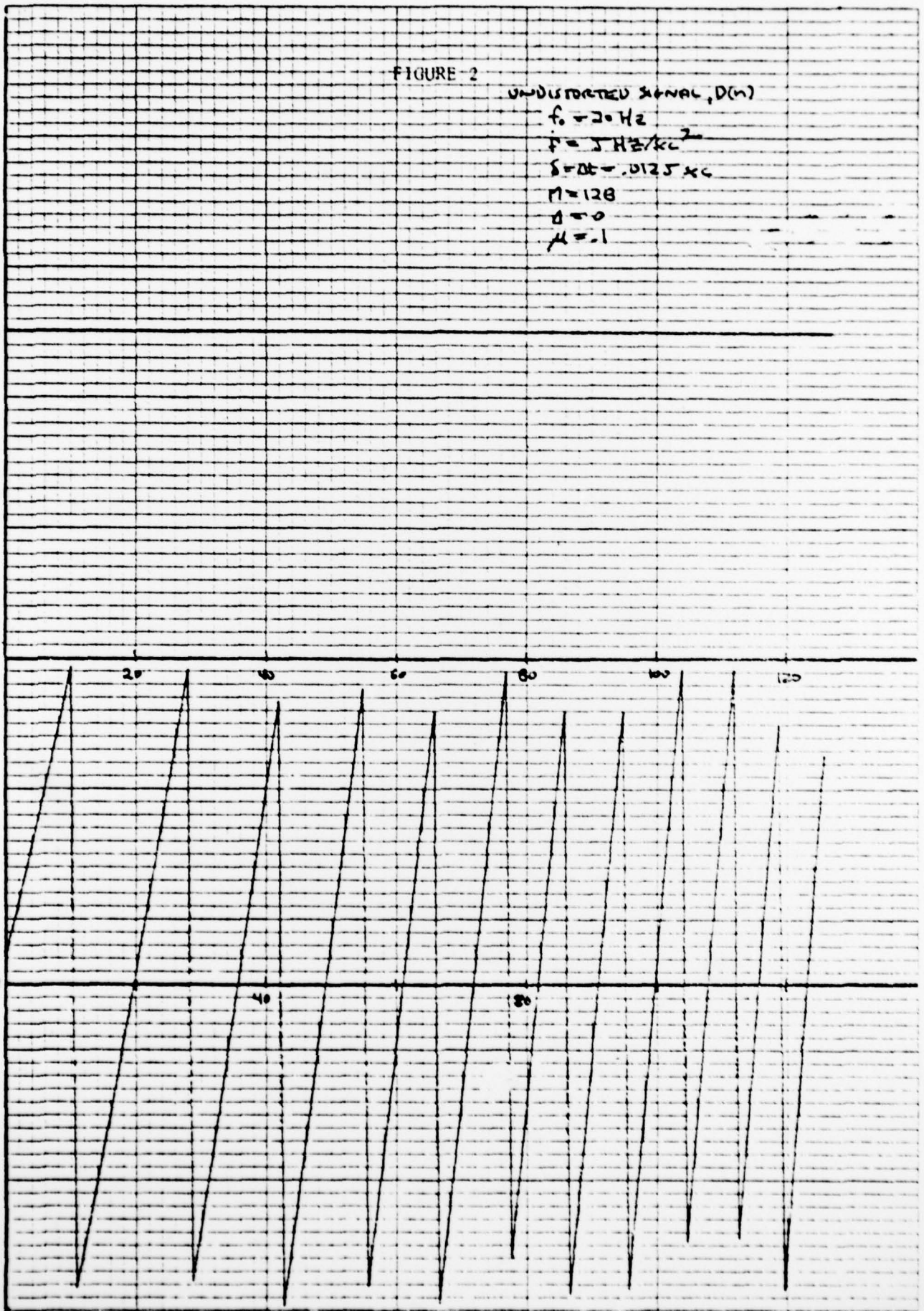


FIGURE 3

STEADY STATE WEIGHTS

$$f_s = 30 \text{ Hz}$$

$$\dot{f} = 5 \text{ Hz/sec}^2$$

$$\delta = \Delta t = .0125 \text{ sec}$$

$$n = 128$$

$$\Delta = 0$$

$$P_1 = 1$$

$$P_2 = 1$$

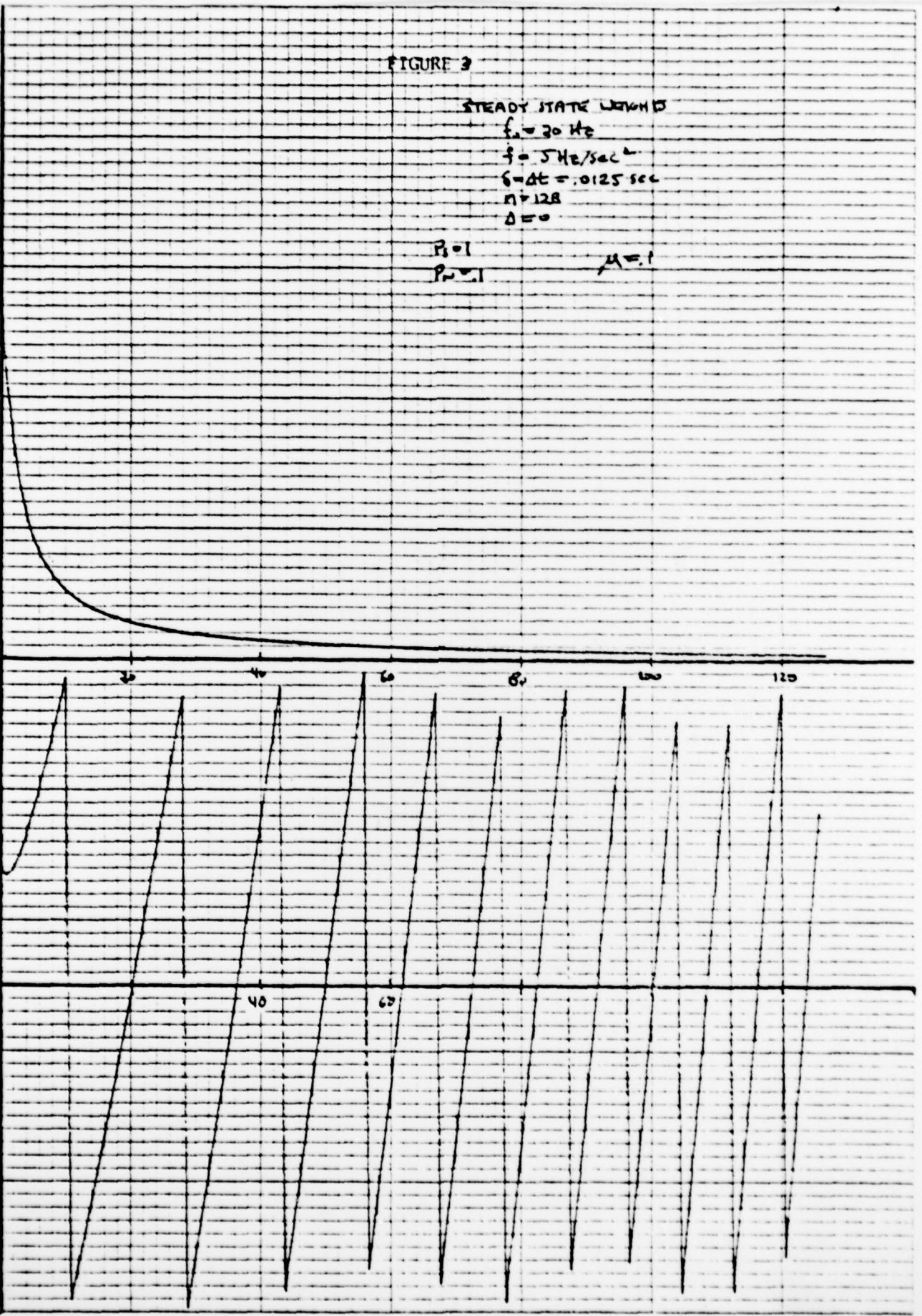
$$M = 1$$

MAGNITUDE

(5) 0

PHASE (0)

(11)



TAP NO.

FIGURE 4

STEADY STATE WEIGHTS

$$\rho_0 = 3.0172$$

$$F = 1.5 \text{ Hz/sec}^2$$

$$S = \Delta t = .0125$$

$$M = 128$$

$$\Delta = 0$$

$$P_1 = 1$$

$$P_2 = 1$$

$$\mu = .1$$

MAGNITUDE

0.05

0.01

0

120

PHASE (°)

0

60

120

180

240

300

360

420

480

540

600

660

720

780

840

900

960

1020

1080

1140

1200

TAP NO.

FIGURE 5

STEADY STATE WEIGHTS

$$E_0 = 20 \text{ Hz}$$

$$\dot{P} = 5 \text{ Hz/sec}^2$$

$$\Delta t = 5 = .0125$$

$$M = 120$$

$$\Delta = 0$$

$$P_1 = 1$$

$$P_2 = 10$$

$$M = 1$$

MAGNITUDE

.025

PHASE

(0)

(π)

TAP NO.

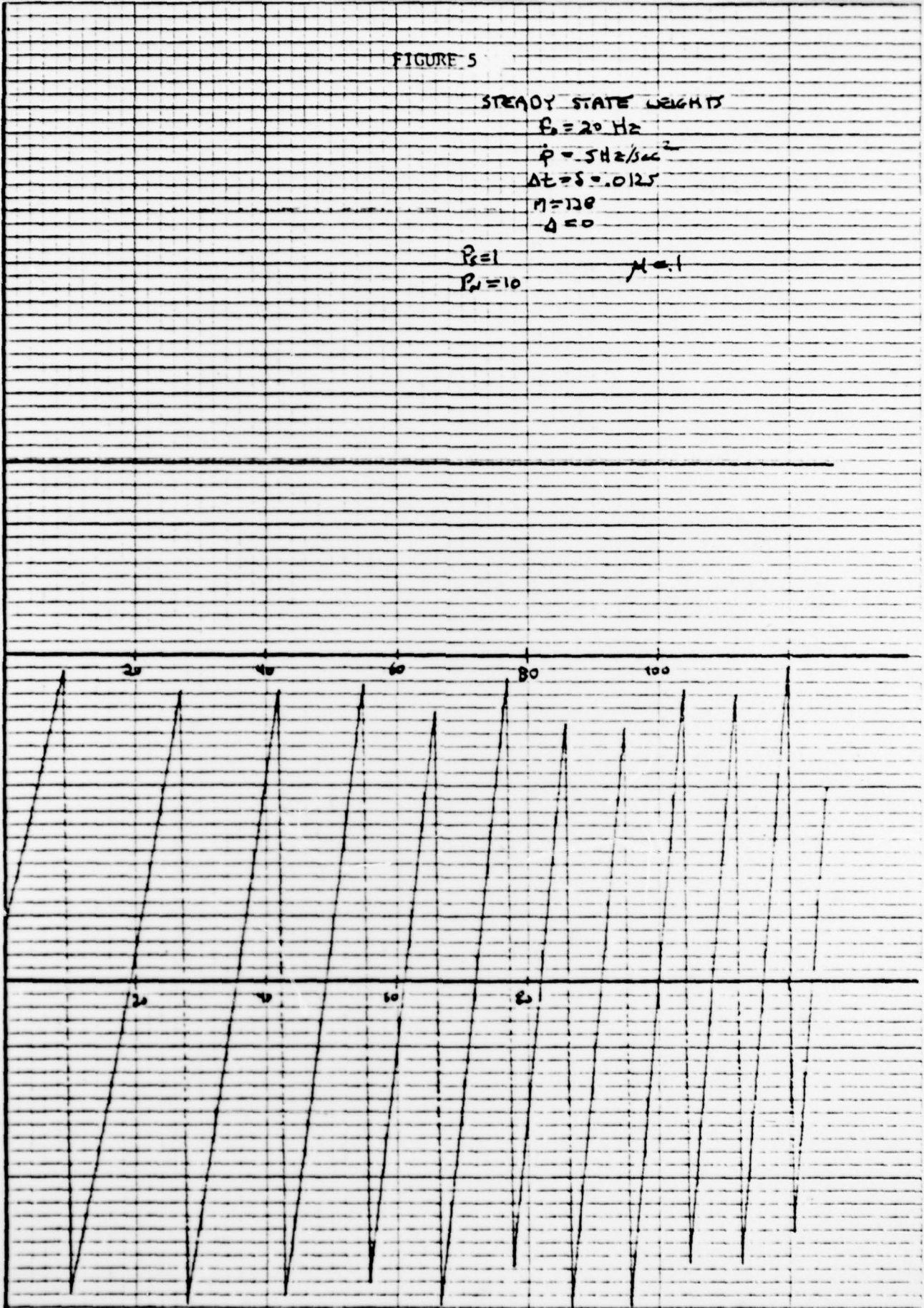


FIGURE 6

UNDISTORTED SIGNAL, $D(\omega)$

$$\omega_0 = 20 \text{ Hz}$$

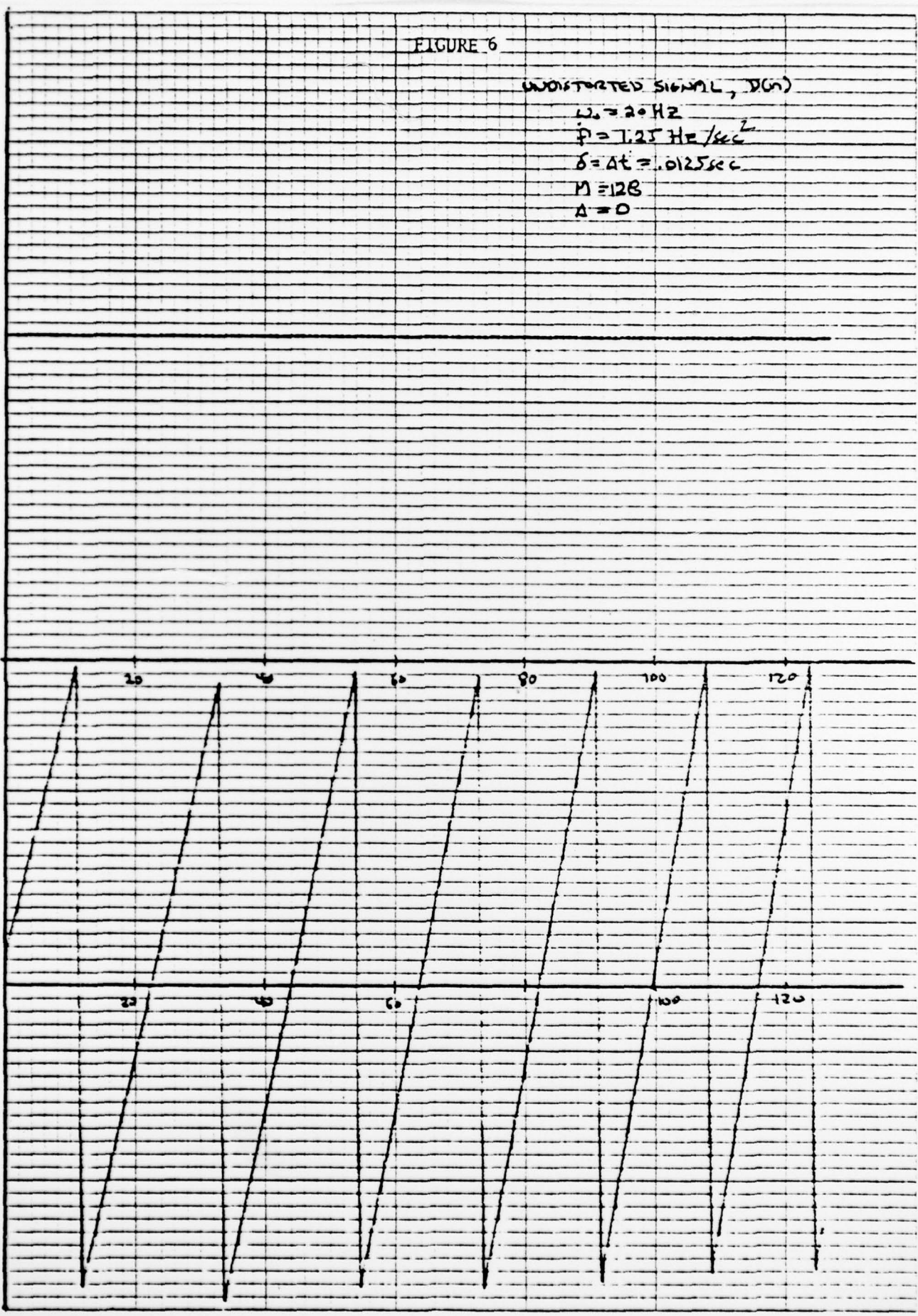
$$\dot{\omega} = 1.25 \text{ Hz/sec}^2$$

$$\delta = \Delta t = .0125 \text{ sec}$$

$$M = 128$$

$$A = 0$$

MAGNITUDE
PHASE
(E)
(S)
(F)



TAP NO.

FIGURE 7

STEADY STATE WEIGHTS

$\omega_s = 20 \text{ Hz}$
 $\dot{\phi} = 1.25 \text{ Hz/sec}^2$
 $\delta = \Delta t = .0125 \text{ sec}$
 $M = 128$
 $\Delta = 0$

$P_s = 1$
 $P_w = .1$

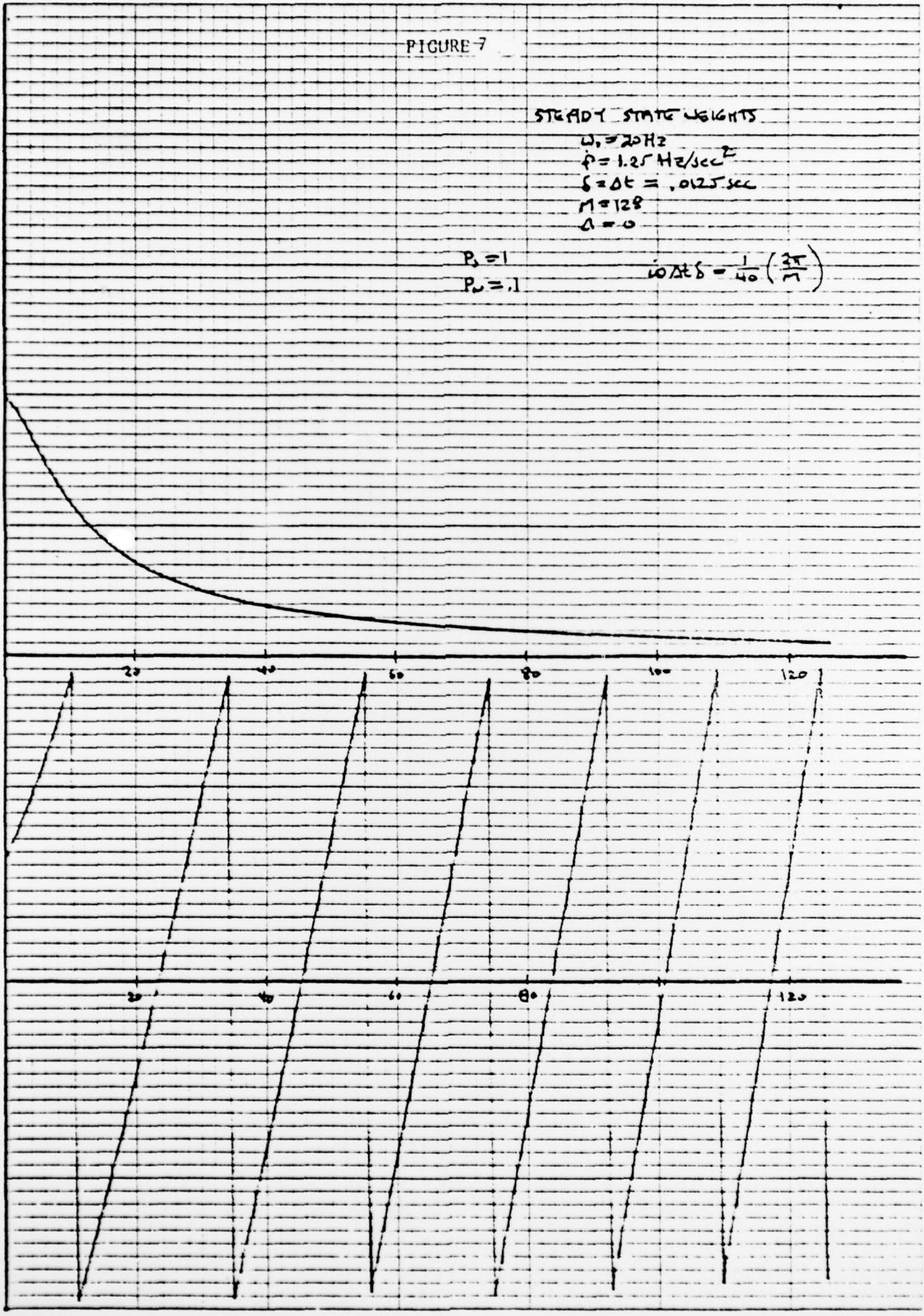
$$\omega \Delta t \delta = \frac{1}{40} \left(\frac{3\pi}{1} \right)$$

MAGNITUDE
46 0700

(E) 0

PHASE (0)

(F)



TAP NO.

FIGURE 8

STEADY STATE RESULTS

$\omega_0 = 20 \text{ Hz}$

$\rho = 1.25 \text{ Hz/sec}^2$

$S = \Delta t = .0125$

$M = 128$

$D = 0$

$P_0 = 1$

$P_w = 1$

$\omega_{ATS} = \frac{1}{60} \left(\frac{M}{T} \right)$

PHASE (RADIANS)

.02

.01

0

PHASE (DEGREES)

0

(-180)

20

40

60

80

100

120

TAP NO.

FIGURE 9

STEADY STATE WEIGHTS

$$\omega_0 = 20 \text{ Hz}$$

$$P = 1.25 \text{ Hz/sec}^2$$

$$\delta = \Delta t = .0125 \text{ sec}^2$$

$$M = 128$$

$$\Delta = 0$$

$$P_0 = 10$$

$$P_1 = 1$$

$$\omega \Delta t \delta = \frac{1}{40} \left(\frac{2\pi}{M} \right)$$

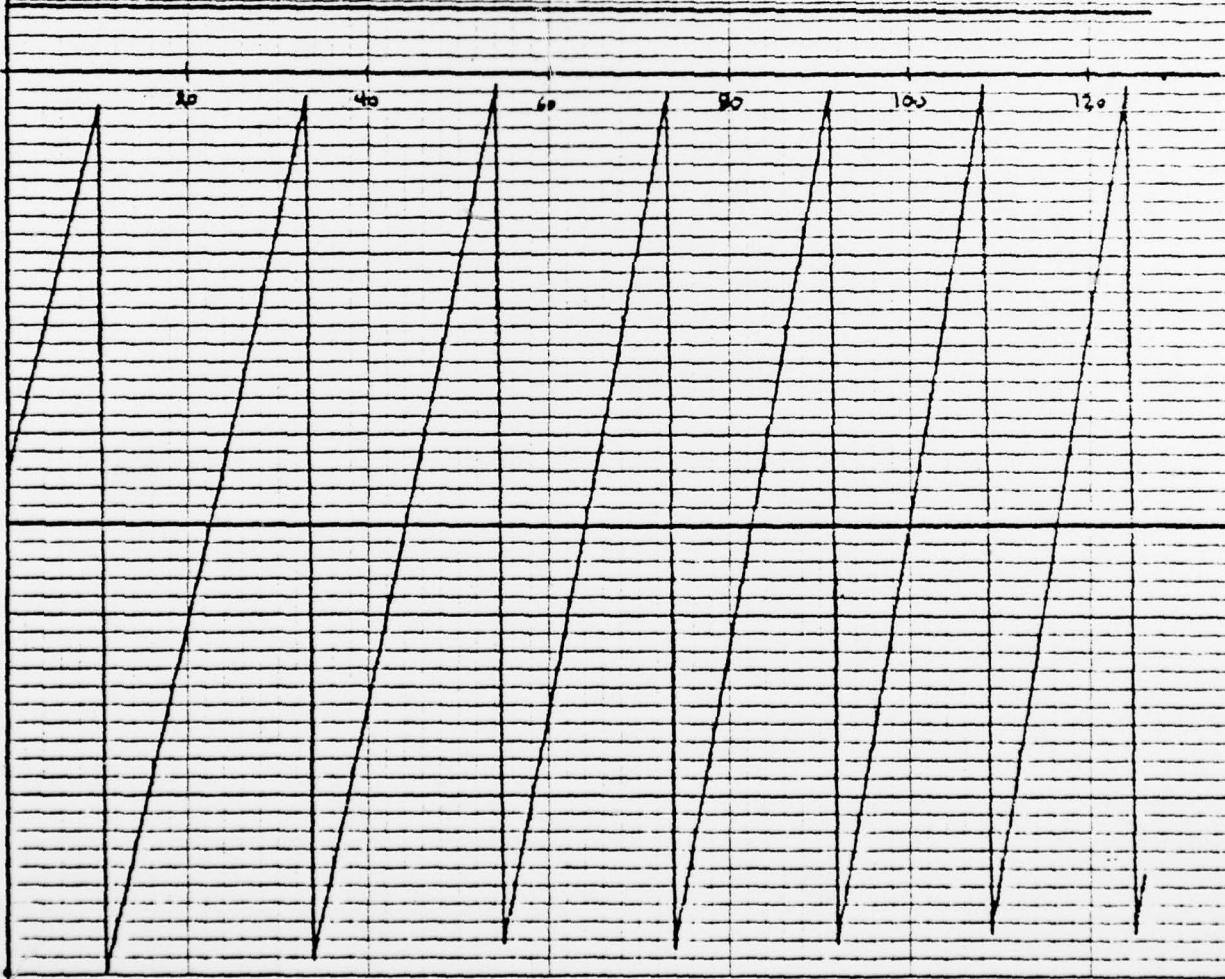
46 0700

MAGNITUDE

(π)₀

PHASE (ϕ)

($-\pi$)



TAP NO.

Mean Square Error
vs. $\mu\sigma_n^2$

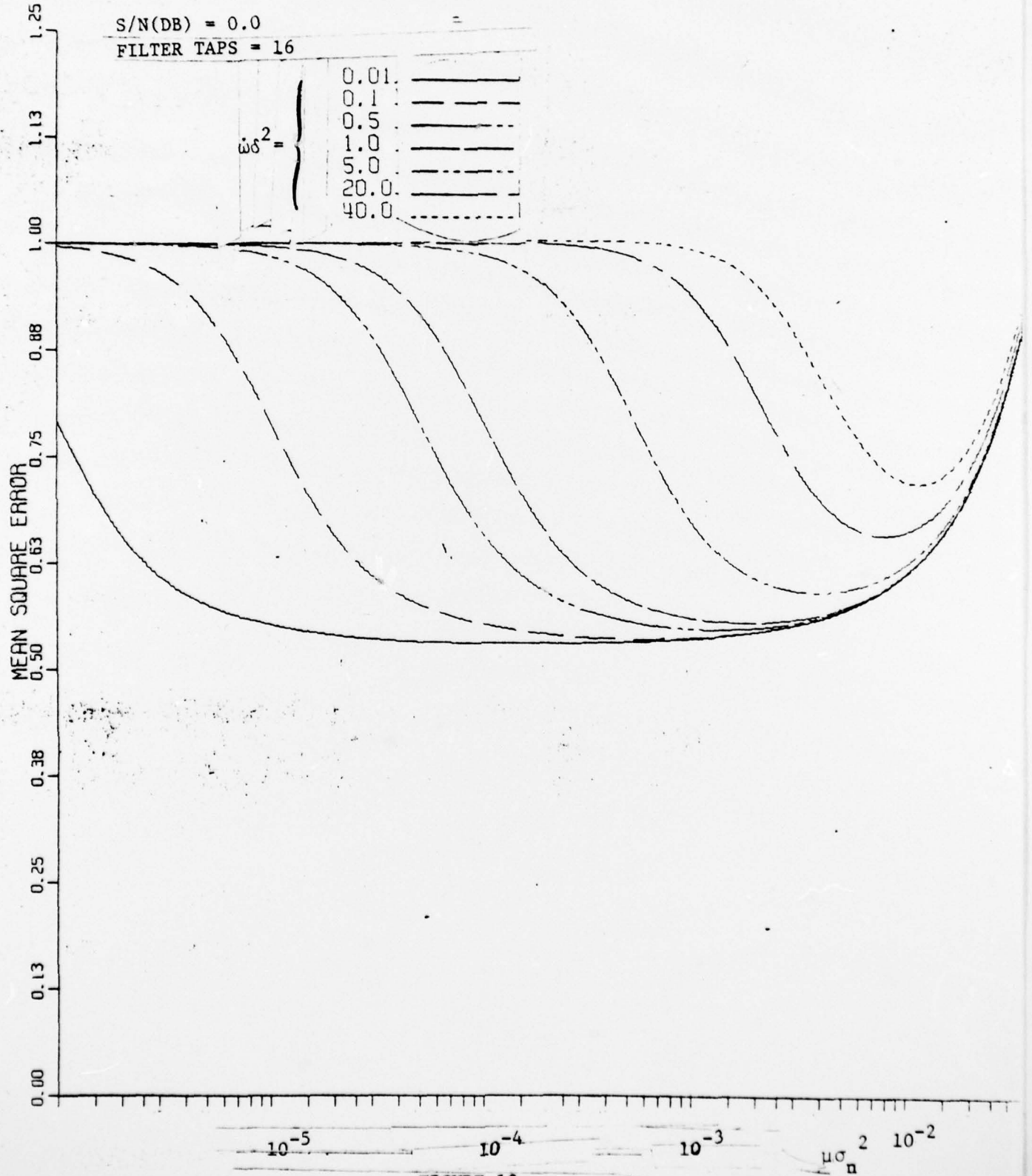


FIGURE 10

Mean Square Error
vs. $\mu\sigma_n^2$

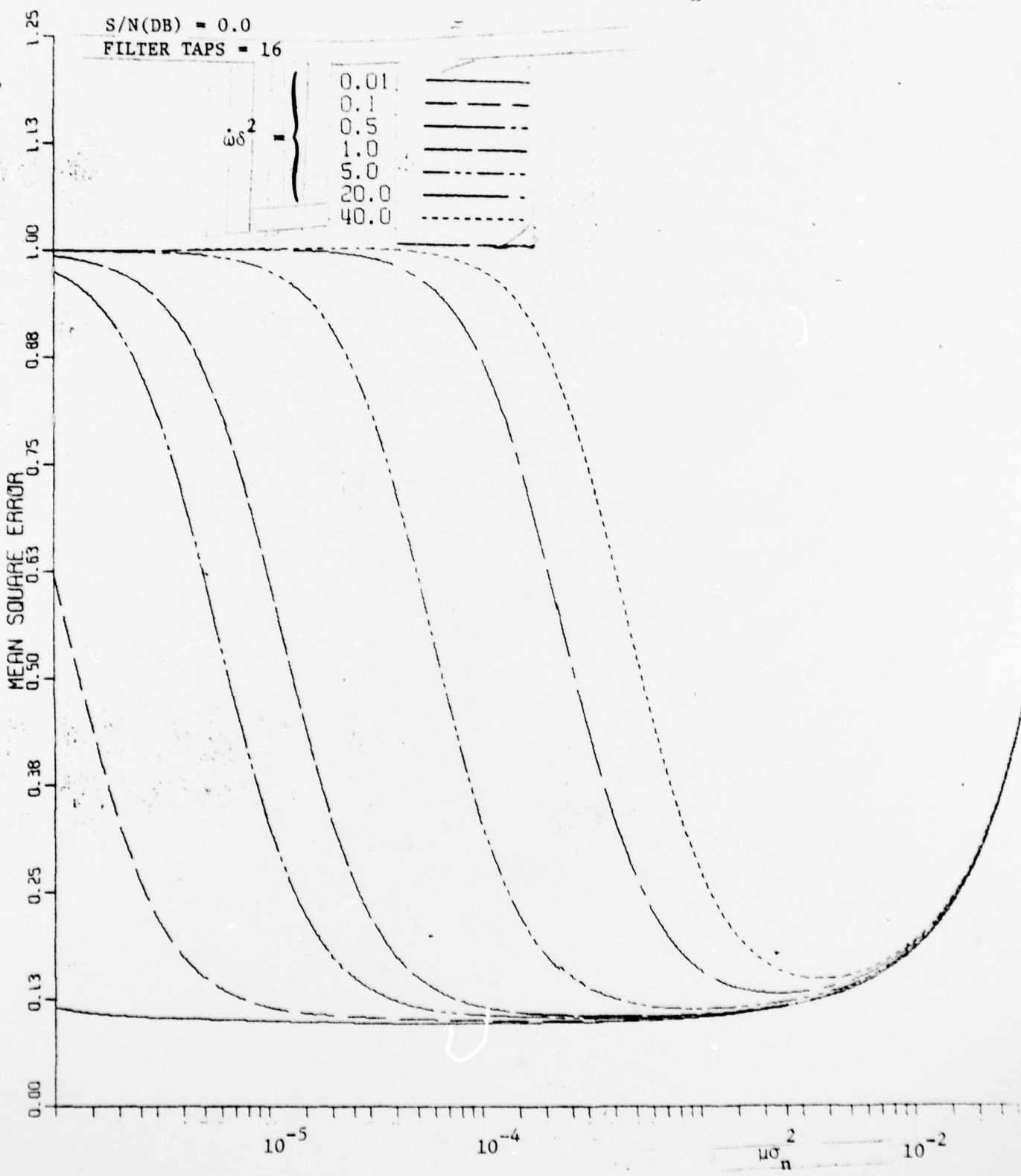


FIGURE 11

Mean Square Error
vs. $\mu\sigma_n^2$

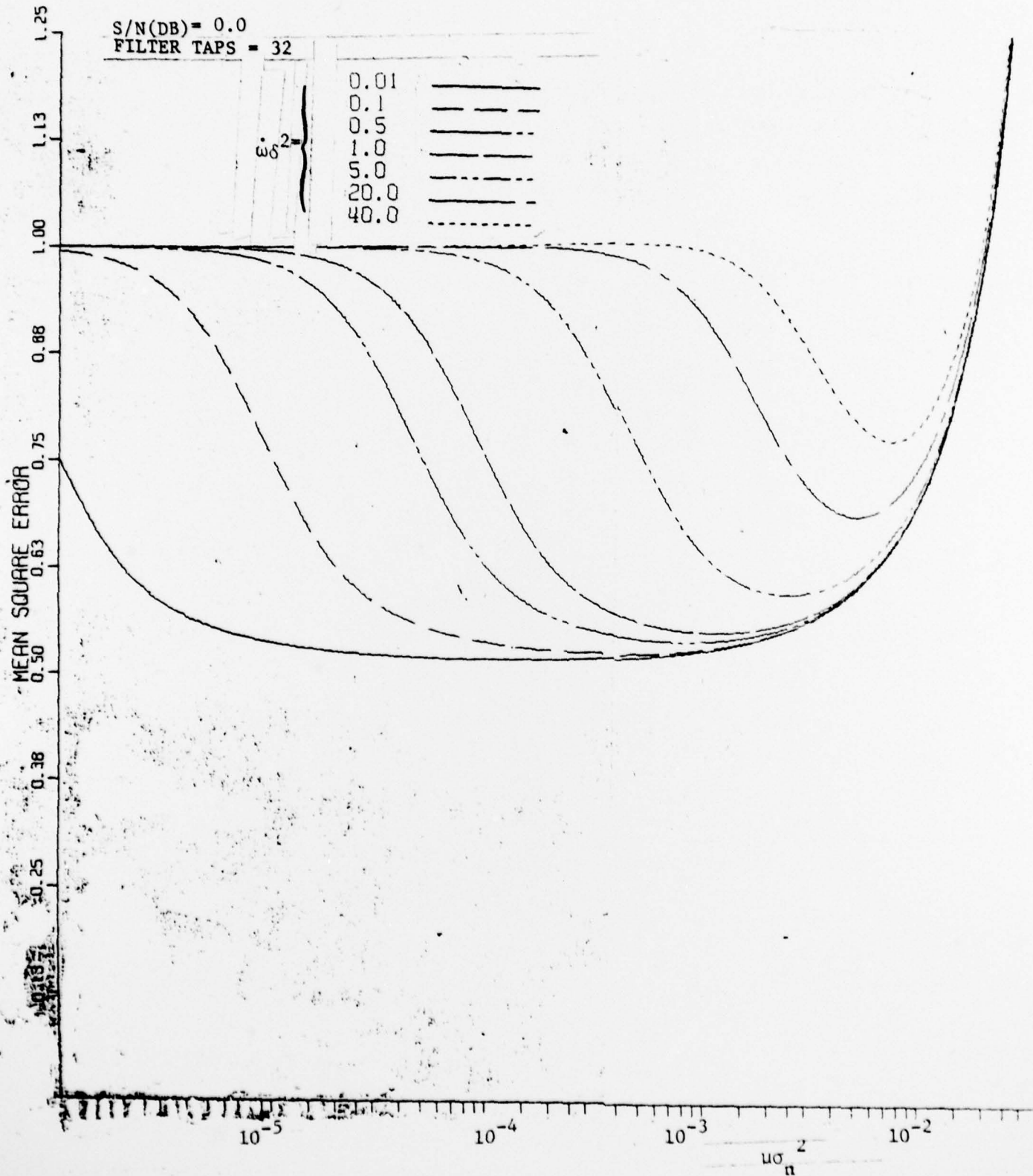


FIGURE 12

Mean Square Error

vs. $\mu\sigma_n^2$

S/N(DB) = 10.0
FILTER TAPS = 32

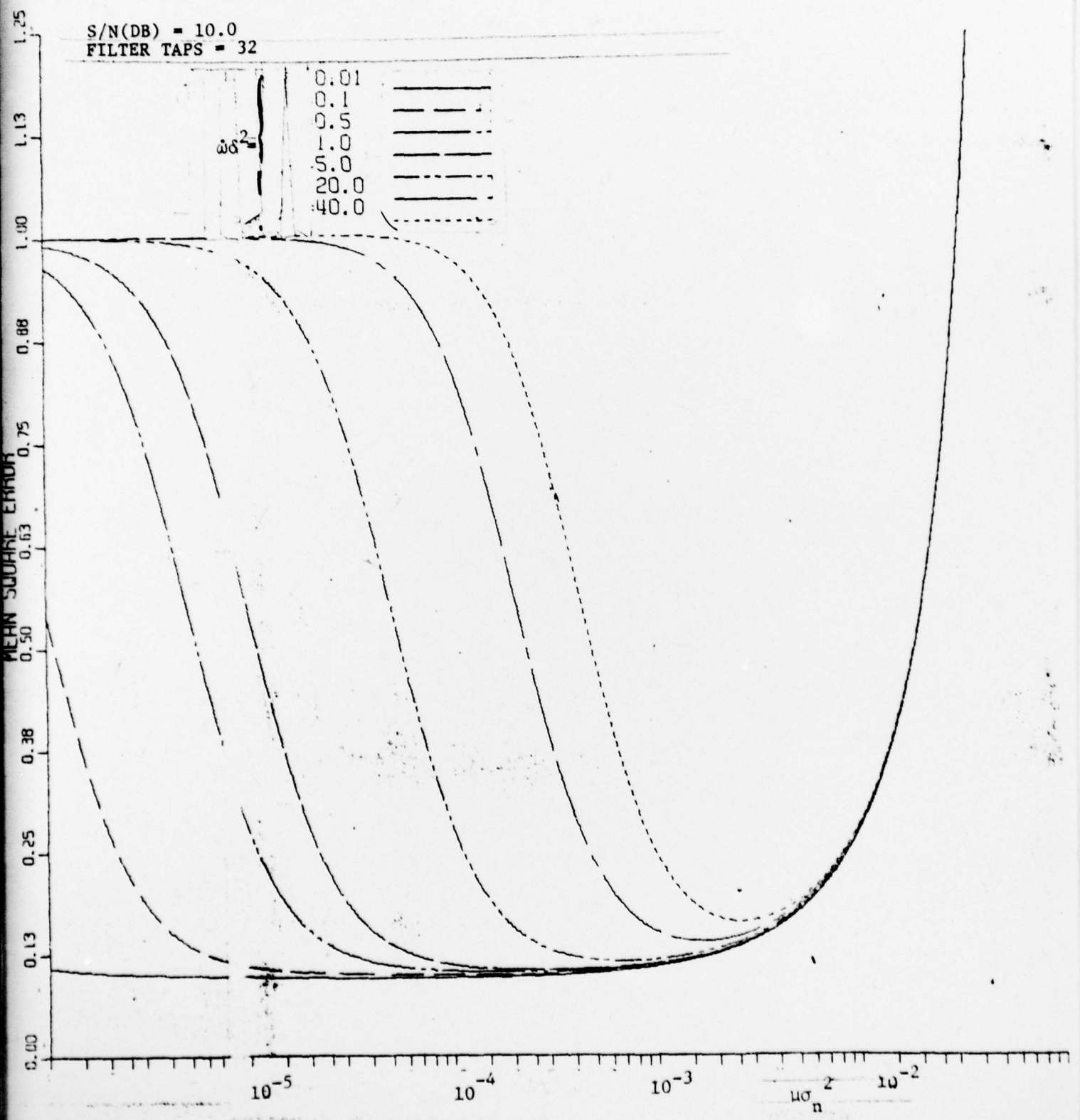
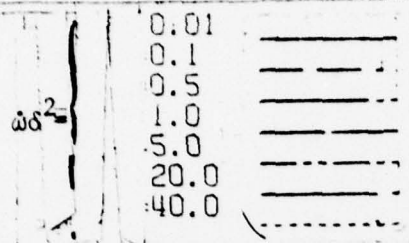


FIGURE 13