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Summary

This report* considers the general problem of detection and MMSE estimation of nonlinear memoryless functionals of random processes. In all cases considered, the observation process is assumed to be contaminated by additive Gaussian white noise.

A Volterra functional expansion is derived for the likelihood ratio used in the detection of a nonlinear memoryless functional of a random process. This expansion is reduced to well known results for the special case of detection of a Gaussian process. For the case of detection of a nonlinear memoryless functional of a stationary Gaussian random process, it is shown that the likelihood ratio has an asymptotic form for which performance can be obtained provided the nonlinearities and processes satisfy Sun's theorem.

A Volterra functional expansion for MMSE estimation of a nonlinear memoryless functional of a random process using nonlinear observations is also derived. It is shown that, using linear observations, the Volterra expansion reduces to well known results for the case of MMSE estimation of a zero mean Gaussian process. A stochastic differential equation for the logarithm of the likelihood ratio is also derived to demonstrate agreement with known results.

*This report originally printed as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering in the Graduate School of Syracuse University, May 1979.

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CHAPTER I

DETECTION AND MMSE ESTIMATION OF MEMORYLESS NONLINEAR FUNCTIONALS OF RANDOM PROCESSES IN GAUSSIAN WHITE NOISE

1.1 Introduction and Dissertation Outline

The theory of detection and minimum mean square error (MMSE) estimation for known signals in Gaussian noise and for Gaussian processes in Gaussian noise are widely available [1,2] and well known. This is due to the successful application of the theory in solving many problems of practical interest. The nonlinear theory is not as widely known due both to a lack of solutions to problems of general interest, and to the mathematical background required to follow many of the proofs (which rely on measure theory and the stochastic calculus). Recently, several texts [3,4] have appeared to make the nonlinear theory more available to the average engineer.

In this dissertation, the Karhuenen-Loève expansion and the Volterra functional expansion are used to derive results for detection and MMSE estimation for nonlinear memoryless functionals of random processes. In all cases considered, the observation process is assumed to contain an additive Gaussian white noise component. Since the results are shown using simple concepts, they can be easily followed by most graduate students with no background in measure theory or the stochastic calculus.

In Chapter II, a Volterra functional expansion for the likelihood ratio is derived for detection of a nonlinear memoryless functional of a random process. For the special case of detection of a zero-mean Gaussian process, this Volterra functional expansion is reduced to well known results. Two examples of calculating Volterra kernels for detection of nonlinear memoryless functionals of zero-mean Gaussian processes are presented. Detection of nonlinear memoryless functionals of dc processes is considered, and performance of the truncated Volterra expansion compared with the optimum performance for a phase-modulated sinusoid. For long observation times, a class of nonlinearities and stationary Gaussian processes is considered for which it is possible to reduce the Volterra functional expansion. As an example, it is shown that first order Butterworth phase modulation of a sinusoid is a member of this class. Performance is derived for this example.

In Chapter III, MMSE estimation of a nonlinear memoryless functional of a random process using nonlinear observations is considered. Using Bayes' law and the Karhunen-Loève expansion, an expression for the aposteriori probability density function (pdf) of the coefficients of the Karhunen-Loève expansion of the random process is obtained. Using this, a Volterra functional expansion is given for the MMSE estimate of a nonlinear memoryless functional of the random process. For the special case of MMSE estimation of a zero-mean Gaussian process using linear observations, the Volterra expansion is reduced to well known results. Also, for the case of MMSE estimation of a nonlinear memoryless functional of a zero-mean Gaussian process using linear observations, the Volterra functional expansion is shown to reduce to results obtained by Olsen [5]. The connection between MMSE estimation and detection theory is shown by using the Volterra functional expansion to derive a stochastic differential equation for the logarithm of the likelihood ratio. This result was obtained by Kailath[6] using the Itô stochastic calculus. Systems of differential equations describing the MMSE estimate of a Gaussian first order Butterworth process with linear observations and the square of a Gaussian first order Butterworth process with squared observations are obtained. MMSE estimation of dc processes is considered, and as an example, MMSE estimation of the dc phase of a sinusoid is presented along with performance.

Chapter IV is a summary of the principal results of the dissertation and suggestions for future work.

1.2 Previous Results in Detection and MMSE Estimation for Nonlinear Problems

Systems of coupled nonlinear differential equations for the conditional moments of a random process were first rigorously derived by Kushner[7]. Development of these equations was preceded by the formulation of a partial differential equation for the a posteriori pdf of the state of a random process (known as Kushner's equation), also derived by Kushner[8] and later by Bucy[9]. An excellent presentation of this theory is given in Jazwinski[3]. Kushner's equation must be interpreted formally, since there is no theory for stochastic partial differential equations. The equations for the moments, however, have been rigorously established by Kushner. Unfortunately, this system of equations is infinite, and there does not appear to be any convenient way of solving them. Jazwinski has suggested several approximation techniques, such as truncating the system, or assuming a recursion between the moments (such as a Gaussian recursion) so that the system may be closed[3]. Some simulation results are available in the literature. Kushner[10], for example, successfully simulated a Gaussian type filter on a nonlinear second order system with linear measurements. These filters are not used as extensively as the linearized or extended Kalman filter[3], and so there are not many simulation results available.

Another approach that appears promising for the discrete filtering of Markov processes is based upon Bayes' law. Let Z^{K-1} denote the sequence of data $Z_0, \ldots, Z_{K-1}$. Using Bayes' law[11], the a posteriori pdf of the state of a random process is given by

$$P(X_K | Z^K) = \frac{P\left(X_K | Z^{K-1}\right) P(Z_K | X_K)}{P\left(Z_K | Z^{K-1}\right)} \tag{1-1}$$

$$P\left(X_K | Z^{K-1}\right) = \int P\left(X_{K-1} | Z^{K-1}\right) P(X_K | X_{K-1})\, dX_{K-1} \tag{1-2}$$

$$P\left(Z_K | Z^{K-1}\right) = \int P\left(X_K | Z^{K-1}\right) P(Z_K | X_K)\, dX_K \tag{1-3}$$

where the state X_K evolves according to a nonlinear difference equation

$$X_K = F_K(X_{K-1}, W_{K-1}), \tag{1-4}$$

and the measurement data is given by

$$Z_K = h_K(X_K, V_K), \tag{1-5}$$

and where W_{K-1} and V_K are uncorrelated Gaussian white sequences. In Sorenson[11], the a posteriori pdf in Equation (1-1) is approximated as a multivariate Gaussian density. In Hecht[12], multidimensional Gaussian quadrature is used to perform the integrations in Equations (1-2) and (1-3) and provide a very close approximation to the evolution of the a posteriori pdf. These ideas are extended in Bucy, et al.[13] and simulation results reported. The technique appears to be very promising even though the computational load is heavy, even for problems of low dimension.

Other techniques for MMSE estimation include the partitioning theorem of Lainiotis[14] and the Volterra approximation of Katzenelson and Gould[15]. Lainiotis obtains a Baysian relation for the a posteriori pdf of the state via his partitioning theorem. Katzenelson and Gould derive systems of integral equations to be solved to obtain the N^{th} order Volterra filter which minimizes the mean-squared-estimation error. To the best of the author's knowledge, neither technique has resulted in MMSE estimators for problems of general interest.

More recently, Olsen[5] has obtained exact results for MMSE estimation of a nonlinear memoryless functional of a Gaussian process in Gaussian white noise using linear observations.

The connection between detection theory and MMSE estimation was first discovered by Schweppe[16] for Gaussian processes in Gaussian white noise and Sosulin and Stratonovich[17] for detection of a diffusion process in Gaussian white noise. Duncan[18] later rigorously rederived the work of Sosulin and Stratonovich using the Itô interpretation of the stochastic integral. Kailath[6] later generalized these results to a larger class of random processes using the Itô stochastic calculus.

CHAPTER II

DETECTION OF NONLINEAR MEMORYLESS FUNCTIONALS OF RANDOM PROCESSES IN GAUSSIAN WHITE NOISE

2.1 Derivation of the Volterra Series

This paragraph considers the problem of detecting a nonlinear memoryless functional, $A\,S(t,\cdot)$, of a random process, $m(t)$, in additive Gaussian white noise $n(t)$. Using the Karhunen-Loève expansion, a Volterra functional expansion* is obtained for the likelihood ratio.

Consider the detection problem

$$H_1:\ r(t) = A\,S[t,\ m(t)] + n(t) \qquad 0 \le t \le T \tag{2-1}$$

$$H_0:\ r(t) = n(t) \qquad 0 \le t \le T \tag{2-2}$$

where $n(t)$ is a zero mean Gaussian process and $m(t)$ and $n(t)$ are independent processes with covariance

$$K_m(t_1,\ t_2) = E\left(\{m(t_1) - E[m(t_1)]\}\ \{m(t_2) - E[m(t_2)]\}\right) \tag{2-3}$$

$$K_n(t_1,\ t_2) = E[n(t_1)\,n(t_2)] = \frac{N_o}{2}\ \delta(t_1 - t_2) \tag{2-4}$$

Suppose that $m(t)$ has the Karhunen-Loève expansion

$$m(t) = \underset{N\to\infty}{\mathrm{L.I.M.}} \sum_{i=1}^{N} m_i\,\phi_i(t) \qquad 0 \le t \le T \tag{2-5}$$

where the eigenfunctions, $\phi_i(t)$, and eigenvalues, λ_i, are solutions of

$$\lambda_i\,\phi_i(t) = \int_o^T K_m(t,\ \tau)\,\phi_i(\tau)\,d\tau \tag{2-6}$$

and where

$$m_i = \int_o^T m(t)\,\phi_i(t)\,dt\ . \tag{2-7}$$

---

*A functional is a mapping from the observation space $\{r(t);\ 0 \le t \le T\}$ to the real line. The Volterra functional expansion is the functional analog of the Taylor series.

Consider the truncated representation of $m(t)$

$$m_N(t) = \sum_{i=1}^{N} m_i \, \phi_i(t) \tag{2-8}$$

and consider the hypothesis testing problem

$$H_1: \; r_N(t) = A\,S[t, m_N(t)] + n(t) \qquad 0 \le t \le T \tag{2-9}$$

$$H_0: \; r_N(t) = n(t) \quad . \tag{2-10}$$

Note that the m_i, $i = 1, 2, \ldots, N$ may be thought of as unwanted parameters with probability density function $p(m_1, \ldots, m_N)$.

To obtain the likelihood ratio for the hypothesis testing problem of Equations (2-1) and (2-2), the likelihood ratio for the hypothesis testing problem of Equations (2-9) and (2-10) will first be obtained. Then the likelihood ratio for Equations (2-1) and (2-2) will be determined by taking the limit as $N \to \infty$. Note that Equations (2-9) and (2-10) represent a composite hypothesis testing problem where the probability density function (pdf) of the unwanted parameters is $p(m_1, \ldots, m_N)$. For this type of problem [1, p. 87], the likelihood ratio is

$$\Lambda(\overline{R}) = \frac{\int p(\overline{R} \mid \overline{\theta}, H_1)\, p(\overline{\theta} \mid H_1)\, d\overline{\theta}}{\int p(\overline{R} \mid \overline{\theta}, H_0)\, p(\overline{\theta} \mid H_0)\, d\overline{\theta}} \tag{2-11}$$

where $\overline{R}$ is a random observation vector and $\overline{\theta}$ is a vector of unwanted random parameters. To pose the problem in a form where Equation (2-11) can be used, $r_N(t)$ is expanded in a Karhunen-Loève expansion (under the assumption $m(t)$ is given) in the set of orthonormal basis functions $\phi_i(t)$, $i = 1, \ldots$

where

$$\psi_1(t) = \frac{S\left[t, \sum_{i=1}^{N} m_i \phi_i(t)\right]}{\left\{\int_0^T S^2\left[t, \sum_{i=1}^{N} m_i \phi_i(t)\right] dt\right\}^{\frac{1}{2}}} \qquad 0 \le t \le T \tag{2-12}$$

and $\psi_i(t)$, $i = 2, \ldots$ are chosen arbitrarily to complete the set of basis functions.

The expansion is

$$r_N(t) = \underset{M \to \infty}{\text{L.I.M.}} \sum_{j=1}^{M} r_j \psi_j(t) \tag{2-13}$$

where

$$r_j = \int_0^T r_N(t) \psi_j(t) \, dt \, . \tag{2-14}$$

Now define

$$r_{N,M}(t) = \sum_{j=1}^{M} r_j \psi_j(t) \tag{2-15}$$

and form the hypothesis testing problem (given $m_1, \ldots, m_N$)

$$H_1 : \overline{R} = \begin{bmatrix} r_1 \\ \vdots \\ r_M \end{bmatrix} = \begin{bmatrix} \int_0^T \{A\,S[t, m_N(t)] + n(t)\} \psi_1(t) \, dt \\ \vdots \\ \int_0^T \{A\,S[t, m_N(t)] + n(t)\} \psi_M(t) \, dt \end{bmatrix} \tag{2-16}$$

$$H_0:\ \overline{R} = \begin{bmatrix} r_1 \\ \vdots \\ r_M \end{bmatrix} = \begin{bmatrix} \int_0^T n(t)\ \psi_1(t)\ dt \\ \vdots \\ \int_0^T n(t)\ \psi_M(t) dt \end{bmatrix} \ . \tag{2-17}$$

The expected value of r_1 conditioned on hypothesis H_1 and on $m_1, \ldots, m_N$ is given by

$$E(r_1 \mid H_1, m_1, \ldots, m_N) = E\left(A \int_0^T \frac{S^2[t,\ m_N(t)]}{\left[\int_0^T S^2[\tau,\ m_N(\tau)]\ d\tau\right]^{\frac{1}{2}}}\ dt + \int_0^T n(t)\ \psi_1(t)\ dt \right)$$

$$= A \left(\int_0^T S^2[t,\ m_N(t)]\ dt \right)^{\frac{1}{2}} \tag{2-18}$$

and, for $i > 1$

$$E(r_i \mid H_1, m_1, \ldots, m_N) = E\left(A \int_0^T S[t,\ m_N(t)]\ \psi_i(t)\ dt + \int_0^T n(t)\ \psi_i(t)\ dt \right) \ . \tag{2-19}$$

However, since

$$S[t,\ m_N(t)] = \left[\int_0^T S^2[t,\ m_N(t)]\ dt \right]^{\frac{1}{2}} \psi_1(t) \tag{2-20}$$

and $\{\psi_i(t)\}$ forms an orthonormal set, it follows from Equation (2-19) that

$$E(r_i \mid H_1, m_1, \ldots, m_N) = 0 \qquad i \geq 2 \ . \tag{2-21}$$

Similarly

$$E(r_i \mid H_0, m_1, \ldots, m_N) = 0 \qquad i \geq 1 \ . \tag{2-22}$$

Also, r_i under either hypothesis has variance $\frac{N_o}{2}$ and the r_i are uncorrelated and Gaussian (hence, statistically independent). From the above discussion it follows that

$$p(\overline{R} \mid \overline{\theta}, H_1) = \frac{1}{(\pi N_o)^{M/2}} \exp\left\{ -\frac{1}{N_o}\left[\left(r_1 - A\left\{ \int_o^T s^2[t, m_N(t)]\, dt \right\}^{\frac{1}{2}} \right)^2 + \sum_{i=2}^{M} r_i^2 \right] \right\} \tag{2-23}$$

and

$$p(\overline{R} \mid \overline{\theta}, H_0) = \frac{1}{(\pi N_o)^{M/2}} \exp\left[-\frac{1}{N_o} \sum_{i=1}^{M} r_i^2 \right] \tag{2-24}$$

where $\overline{\theta} = (m_1, \ldots, m_N)$ are the unwanted parameters and where $\overline{R} = (r_1, \ldots, r_M)$. From Equation (2-23), (2-24) and (2-11) and taking the limit as $N, M \to \infty$, the likelihood ratio for the hypothesis testing problem of Equations (2-1) and (2-2) is obtained as

$$\Lambda[r(t)] = \lim_{N\to\infty} \lim_{M\to\infty} \left\{ \frac{\underbrace{\int_{-\infty}^{\infty}\cdots\int}_{N} \frac{1}{(\pi N_o)^{M/2}} \exp\left[-\frac{1}{N_o}\left(\sum_{i=1}^{M} r_i^2 - 2A\left\{ \int_o^T s^2[t, m_N(t)]\, dt \right\}^{\frac{1}{2}} r_1 + A^2\left\{ \int_o^T s^2[t, m_N(t)]\, dt \right\} \right) \right] p(m_1, \ldots, m_N)\, dm_1 \ldots dm_N}{\underbrace{\int_{-\infty}^{\infty}\cdots\int}_{N} \frac{1}{(\pi N_o)^{M/2}} \exp\left[-\frac{1}{N_o}\left(\sum_{i=1}^{M} r_i^2 \right) \right] \delta(m_1) \ldots \delta(m_N)\, dm_1 \ldots dm_N} \right\} \tag{2-25}$$

Cancelling $(\pi N_o)^{-M/2} \exp\left(-\frac{1}{N_o}\sum_{i=1}^{M} r_i^2\right)$ out of the numerator and denominator in Equation (2-25), taking the limit $M \to \infty$ and integrating the denominator results in

$$\Lambda[r(t)] = \lim_{N\to\infty} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int \exp\left(\frac{2A}{N_o}\int_0^T r(t)\, S[t, m_N(t)]\, dt\right)$$

$$\exp\left(-\frac{A^2}{N_o}\int_0^T S^2[t, m_N(t)]\, dt\right) p(m_1 \ldots m_N)\, dm_1 \ldots dm_N \qquad (2\text{-}26)$$

which can be rewritten in the form

$$\Lambda[r(t)] = \lim_{N\to\infty} \left\{ \int_{-\infty}^{\infty} \overset{N}{\cdots} \int \exp\left[\frac{2A}{N_o}\int_0^T S[t, m_N(t)]\left(r(t) - \frac{A}{2} S[t, m_N(t)]\right) dt\right]\right.$$

$$\left. p(m_1, \ldots, m_N)\, dm_1 \ldots dm_N \right\} . \qquad (2\text{-}27)$$

Expanding the first exponential in Equation (2-26) yields

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$(2\text{-}28)$$

where the integrand contains the Volterra kernel $f_i\,(t_1, \ldots, t_i)$

$$f_i(t_1, \ldots, t_i) = \lim_{N\to\infty} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int S[t_1, m_N(t_1)] \ldots S[t_i, m_N(t_i)]$$

$$\exp\left(-\frac{A^2}{N_o}\int_0^T S^2[\tau, m_N(\tau)]\, d\tau\right)$$

$$p(m_1, \ldots, m_N)\, dm_1 \ldots dm_N \ . \qquad (2\text{-}29)$$

Letting $N \to \infty$ in Equation (2-29) $f_i(t_1, \ldots, t_i)$ becomes

$$f_i(t_1, \ldots, t_i) = E\left(S[t_1, m(t_1)] \ldots S[t_i, m(t_i)] \exp\left(-\frac{A^2}{N_o} \int_o^T S^2[\tau, m(\tau)]\, d\tau\right)\right) \tag{2-30}$$

where the expectation is carried out with respect to the random variables $m(t_1) \ldots, m(t_i)$ and $\exp\left(-\frac{A^2}{N_o} \int_o^T S^2[\tau, m(\tau)]\, d\tau\right)$. Expanding the exponential in Equation (2-30), $f_i(t_1, \ldots, t_i)$ can also be written as

$$f_i(t_1, \ldots, t_i) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\frac{A^2 T}{N_o}\right)^j \frac{1}{T^j} \int_o^T \overset{j}{\ldots} \int_o^T E\left(S_1, \ldots, S_i, S_{i+1}^2 \ldots S_{i+j}^2\right) dt_{i+1} \ldots dt_{i+j} \tag{2-31}$$

where

$$S_K = S[t_K, m(t_K)] \quad .$$

Summarizing, the likelihood ratio of the hypothesis testing problem of Equations (2-1) and (2-2) is given by the Volterra functional expansion of Equation (2-28) with the Volterra kernels defined by Equation (2-30).

Note that Equation (2-28) is a physically realizable nonlinear operation with memory on $r(t)$. It is easily put in the form

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \overset{i}{\ldots} \int_{-\infty}^{\infty} r(T - t_1) \ldots r(T - t_i)\, \overline{f}_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

where

$$\bar{f}_i(t_1, \ldots, t_i) = \frac{1}{i!}\left(\frac{2A}{N_o}\right)^i f_i(t_1, \ldots, t_i) \quad U(t_1) \ldots U(t_i) \qquad (2\text{-}32)$$

and U(t) is the unit step. Also, each kernel $f_i(t_1, \ldots, t_i)$ is symmetric in its arguments.

2.2 Rederivation of the Linear Result for Zero Mean Gaussian Processes

Consider

$$A\,S[t, m(t)] = m(t) \qquad (2\text{-}33)$$

where m(t) is a zero-mean Gaussian process with autocovariance

$$K_m(t_1, t_2) = E[m(t_1)\, m(t_2)] \quad . \qquad (2\text{-}34)$$

For this case, the hypothesis testing problem of Equations (2-1) and (2-2) becomes

$$H_1:\ r(t) = m(t) + n(t) \qquad 0 \leq t \leq T \qquad (2\text{-}35)$$

$$H_0:\ r(t) = n(t) \qquad 0 \leq t \leq T \qquad (2\text{-}36)$$

where m(t) and n(t) are independent zero-mean Gaussian processes. From Equations (2-28) and (2-30) the likelihood ratio is

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \qquad (2\text{-}37)$$

where

$$f_i(t_1, \ldots, t_i) = E\left(m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_o^T m^2(\tau)\, d\tau\right]\right) \quad . \tag{2-38}$$

Equation (2-37) is now expressed in closed form by obtaining $f_i(t_1, \ldots, t_i)$ as a sum of products of $f_2(t_k, t_j)$. From Equation (2-38)

$$f_o = E\left(\exp\left[-\frac{1}{N_o}\int_o^T m^2(\tau)\, d\tau\right]\right) \quad . \tag{2-39}$$

Expanding $m(\tau)$ in a Karhunen-Loève expansion and using the orthogonality of the eigenfunctions, Equation (2-39) can be written as

$$f_0 = E\left[\exp\left(-\frac{1}{N_o}\sum_{j=1}^{\infty} m_j^2\right)\right] \tag{2-40}$$

where the pdf of $m_1, m_2, \ldots, m_N$ is given by

$$p(m_1, \ldots, m_N) = \frac{1}{(2\pi)^{N/2}\left[\prod_{j=1}^{N} \lambda_j\right]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{j=1}^{N} \frac{m_j^2}{\lambda_j}\right] \tag{2-41}$$

and where the eigenvalues, λ_i, are solution of Equation (2-6). From Equations (2-40) and (2-41)

$$f_o = \lim_{N\to\infty} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}\left[\prod_{j=1}^{N} \lambda_j\right]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{\lambda_j} + \frac{2}{N_o}\right) m_j^2\right] dm_1 \ldots dm_N \quad . \tag{2-42}$$

Equation (2-42) may be rewritten as

$$f_o = \lim_{N \to \infty} \left[\prod_{i=1}^{N} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2} \left[\prod_{j=1}^{N} \frac{N_o/2\ \lambda_j}{\frac{N_o}{2} + \lambda_j} \right]^{\frac{1}{2}}}$$

$$\exp\left(- \frac{1}{2} \sum_{j=1}^{N} \frac{m_j^2}{\frac{(N_o/2)\lambda_j}{\frac{N_o}{2} + \lambda_j}} \right) dm_1 \ldots dm_N \quad . \qquad (2-43)$$

The N-fold integration is recognized to be unity since it is the volume of an N dimensional density function. Taking the limit in Equation (2-43) it follows that

$$f_o = \left[\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right]^{\frac{1}{2}} \quad . \qquad (2-44)$$

In the ensuing development it is shown that $f_i(t_1, \ldots, t_i) = 0$ for i odd. Proceeding with $f_2(t_1, t_2)$,

$$f_2(t_1, t_2) = E\left(m(t_1)\, m(t_2) \exp\left[- \frac{1}{N_o} \int_{o}^{T} m^2(\tau)\, d\tau \right] \right) . \qquad (2-45)$$

Expanding $m(t_1)$, $m(t_2)$ and $m(\tau)$ in Karhunen-Loéve expansions, there results

$$f_2(t_1, t_2) = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} E\left[m_{i_1} m_{i_2} \exp\left(- \frac{1}{N_o} \sum_{j=1}^{\infty} m_j^2 \right) \right]$$

$$\phi_{i_1}(t_1)\, \phi_{i_2}(t_2) \quad . \qquad (2-46)$$

In Equation (2-46) note that

$$E\left[m_{i_1} m_{i_2} \exp\left(-\frac{1}{N_o}\sum_{j=1}^{\infty} m_j^2\right)\right]$$

$$= \left[\lim_{N\to\infty}\prod_{i=1}^{N}\frac{N_o/2}{\frac{N_o}{2}+\lambda_i}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty}\overset{N}{\cdots}\int_{-\infty}^{\infty}\frac{m_{i_1} m_{i_2}}{(2\pi)^{N/2}\left[\prod_{j=1}^{N}\frac{(N_o/2)\lambda_j}{\frac{N_o}{2}+\lambda_j}\right]^{\frac{1}{2}}}$$

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\frac{m_j^2}{\frac{(N_o/2)\lambda_j}{\frac{N_o}{2}+\lambda_j}}\right) dm_1 \ldots dm_N \quad . \tag{2-47}$$

From Equation (2-47)

$$E\left(m_{i_1} m_{i_2} \exp\left[-\frac{1}{N_o}\sum_{j=1}^{\infty} m_j^2\right]\right)$$

$$= \left[\prod_{i=1}^{\infty}\frac{N_o/2}{\frac{N_o}{2}+\lambda_i}\right]^{\frac{1}{2}} \frac{(N_o/2)\lambda_{i_1}}{\frac{N_o}{2}+\lambda_{i_1}}\,\delta_{i_1 i_2} \tag{2-48}$$

where $\delta_{i_1 i_2}$ is the Kroenecker delta. From Equations (2-48) and (2-46)

$$f_2(t_1, t_2) = \left[\prod_{i=1}^{\infty}\frac{N_o/2}{\frac{N_o}{2}+\lambda_i}\right]^{\frac{1}{2}} \frac{N_o}{2}\sum_{i=1}^{\infty}\frac{\lambda_i}{\frac{N_o}{2}+\lambda_i}\,\phi_i(t_1)\,\phi_i(t_2) \quad . \tag{2-49}$$

Similarly,

$$f_4(t_1, t_2, t_3, t_4) = \sum_{i_1}\sum_{i_2}\sum_{i_3}\sum_{i_4} E\left[m_{i_1} m_{i_2} m_{i_3} m_{i_4} \exp\left(-\frac{1}{N_o}\sum_j m_j^2\right)\right]$$

$$\phi_{i_1}(t_1)\,\phi_{i_2}(t_2)\,\phi_{i_3}(t_3)\,\phi_{i_4}(t_4) \qquad (2\text{-}50)$$

where

$$E\left[m_{i_1} m_{i_2} m_{i_3} m_{i_4} \exp\left(-\frac{1}{N_o}\sum_j m_j^2\right)\right]$$

$$= \left[\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2}+\lambda_i}\right]^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^2 \left[\left(\frac{\lambda_{i_1}}{\frac{N_o}{2}+\lambda_{i_1}}\right)\left(\frac{\lambda_{i_3}}{\frac{N_o}{2}+\lambda_{i_3}}\right)\delta_{i_1 i_2}\,\delta_{i_3 i_4}\right.$$

$$+\left(\frac{\lambda_{i_1}}{\frac{N_o}{2}+\lambda_{i_1}}\right)\left(\frac{\lambda_{i_2}}{\frac{N_o}{2}+\lambda_{i_2}}\right)\delta_{i_1 i_3}\,\delta_{i_2 i_4}$$

$$\left.+\left(\frac{\lambda_{i_1}}{\frac{N_o}{2}+\lambda_{i_1}}\right)\left(\frac{\lambda_{i_2}}{\frac{N_o}{2}+\lambda_{i_2}}\right)\delta_{i_1 i_4}\,\delta_{i_2 i_3}\right] . \qquad (2\text{-}51)$$

From Equations (2-51) and (2-50)

$$f_4(t_1, t_2, t_3, t_4) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^2$$

$$\left[\sum_{i_1=1}^{\infty} \frac{\lambda_{i_1}}{\frac{N_o}{2} + \lambda_{i_1}} \phi_{i_1}(t_1) \phi_{i_1}(t_2) \sum_{i_2=1}^{\infty} \frac{\lambda_{i_2}}{\frac{N_o}{2} + \lambda_{i_2}} \right.$$

$$\phi_{i_2}(t_3) \phi_{i_2}(t_4)$$

$$+ \sum_{i_1=1}^{\infty} \frac{\lambda_{i_1}}{\frac{N_o}{2} + \lambda_{i_1}} \phi_{i_1}(t_1) \phi_{i_1}(t_3) \sum_{i_2=1}^{\infty} \frac{\lambda_{i_2}}{\frac{N_o}{2} + \lambda_{i_2}}$$

$$\phi_{i_2}(t_2) \phi_{i_2}(t_4)$$

$$+ \sum_{i_1=1}^{\infty} \frac{\lambda_{i_1}}{\frac{N_o}{2} + \lambda_{i_1}} \phi_{i_1}(t_1) \phi_{i_1}(t_4) \sum_{i_2=1}^{\infty} \frac{\lambda_{i_2}}{\frac{N_o}{2} + \lambda_{i_2}}$$

$$\left. \phi_{i_2}(t_2) \phi_{i_2}(t_3) \right] . \tag{2-52}$$

From Equation (2-52) note that

$$\int_o^T \int_o^T \int_o^T \int_o^T r(t_1)\, r(t_2)\, r(t_3)\, r(t_4)\; f_4(t_1, t_2, t_3, t_4)\, dt_1\, dt_2\, dt_3\, dt_4$$

$$= f_o \left(\frac{N_o}{2}\right)^2 3 \left[\int_o^T \int_o^T r(t_1)\, r(t_2) \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{N_o}{2} + \lambda_j} \phi_j(t_1)\, \phi_j(t_2)\, dt_1\, dt_2 \right]^2 . \tag{2-53}$$

Generalizing* from Equation (2-53) obtain

$$\int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= f_o \left(\frac{N_o}{2}\right)^2 \frac{i!}{2^{i/2} \frac{i}{2}!} \left[\int_0^T \int_0^T r(t_1)\, r(t_2) \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{N_o}{2} + \lambda_j} \phi_j(t_1)\, \phi_j(t_2)\, dt_1\, dt_2 \right]^{i/2} \quad \text{for } i \text{ even}$$

$$= 0 \qquad \text{for } i \text{ odd}.$$

(2-54)

From Equations (2-54), (2-44) and (2-37) the likelihood ratio is found to be

$$\Lambda[r(t)] = \left[\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right]^{\frac{1}{2}} \sum_{\substack{i=o \\ i \text{ even}}}^{\infty} \left(\frac{1}{N_o}\right)^{i/2} \frac{1}{(i/2)!}$$

$$\left[\int_o^T \int_o^T r(t_1)\, r(t_2) \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{N_o}{2} + \lambda_j} \phi_j(t_1)\, \phi_j(t_2)\, dt_1\, dt_2 \right]^{i/2} . \qquad (2\text{-}55)$$

---

*This follows from

$$E(m_1, \ldots, m_N) \begin{matrix} = \sum E(m_{i_1} m_{i_2}) \ldots E(m_{i_{N-1}} m_{i_N}) & i \text{ even} \\ = 0 \qquad i \text{ odd} & \end{matrix}$$

where the sum is over all $\frac{N!}{2^{N/2}\left(\frac{N}{2}\right)!}$ ways of dividing the N random variables into $\frac{N}{2}$ combinations of pairs and where $E(m_i) = 0$ for $i = 1, \ldots, N$. See [1, problem 3.3.12].

Redefining the index of the summation on i in Equation (2-55) the expression for $\Lambda[r(t)]$ becomes

$$\Lambda[r(t)] = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right|^{\frac{1}{2}} \exp\left[\frac{1}{N_o} \int_0^T \int_0^T r(t_1)\, r(t_2) \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{N_o}{2} + \lambda_j} \phi_j(t_1)\, \phi_j(t_2)\, dt_1\, dt_2 \right] . \tag{2-56}$$

The series in Equation (2-56) is recognized as the solution to the integral equation [1]

$$\frac{N_o}{2} h_*(t_1, t_2) + \int_0^T h_*(t_1, \tau)\, K_m(\tau, t_2)\, d\tau = K_m(t_1, t_2) . \tag{2-57}$$

Therefore,

$$h_*(t_1, t_2) = \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{N_o}{2} + \lambda_j} \phi_j(t_1)\, \phi_j(t_2) . \tag{2-58}$$

From Equation (2-56) and (2-58), the logarithm of the likelihood ratio is given by

$$\ln \Lambda[r(t)] = \frac{1}{N_o} \int_0^T \int_0^T r(t_1)\, r(t_2)\, h_*(t_1, t_2)\, dt_1\, dt_2 - \frac{1}{2} \sum_{i=1}^{\infty} \ln\left(1 + \frac{2\lambda_i}{N_o}\right) . \tag{2-59}$$

This is in agreement with known results [1, p. 12, Equation (2-26)] .

2.3 Calculation of Volterra Kernels - Some Nonlinear Examples

In this paragraph, three examples of computing Volterra kernels are presented. The first two examples, a hard-limited Gaussian process and the absolute value of a Gaussian process, are the only nonlinearities of Gaussian processes discovered to date for which exact calculation of the first three kernels is possible. The last example compares the performance of a receiver based on the likelihood ratio with the performance of a receiver based on the truncated Volterra expansion of the likelihood ratio for a nonlinear memoryless functional of a D.C. process.

2.3.1 Hard-Limited Gaussian Process in Gaussian White Noise

Consider

$$S[t, m(t)] = \mathrm{sgn}[m(t)] = 1 \qquad m(t) > 0$$
$$= 0 \qquad m(t) = 0$$
$$= -1 \qquad m(t) < 0 \tag{2-60}$$

where m(t) is a zero-mean Gaussian process. From Equations (2-28) and (2-30)

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{2-61}$$

where

$$f_i(t_1, \ldots, t_i) = E\left(\mathrm{sgn}[m(t_1)] \ldots \mathrm{sgn}[m(t_i)] \exp\left\{-\frac{A^2}{N_o}\int_0^T \mathrm{sgn}^2[m(\tau)]\, d\tau\right\}\right). \tag{2-62}$$

From Equations (2-62) and (2-60)

$$f_o = E\left\{\exp\left[-\frac{A^2}{N_o}\int_0^T \mathrm{sgn}^2[m(\tau)]\, d\tau\right]\right\} = \exp\left(-\frac{A^2 T}{N_o}\right). \tag{2-63}$$

Also, since sgn[m(t)] is an odd function of m(t)

$$f_1(t_1) = f_0\, E\{\mathrm{sgn}[m(t)]\} = 0. \tag{2-64}$$

Similarly,

$$f_2(t_1, t_2) = f_0\, E\{\mathrm{sgn}[m(t_1)]\, \mathrm{sgn}[m(t_2)]\}. \tag{2-65}$$

The expectation in Equation (2-65) is given in [19, p. 198, Equation 7-29] to obtain

$$f_2(t_1, t_2) = f_0 \frac{2}{\pi} \sin^{-1}\left[\frac{K_m(t_1, t_2)}{\sqrt{K_m(t_1, t_1)\, K_m(t_2, t_2)}}\right] \tag{2-66}$$

where $K_m(t_1, t_2)$ is the autocovariance of function of $m(t)$. Also

$$f_3(t_1, t_2, t_3) = f_0\, E\left\{\mathrm{sgn}[m(t_1)]\ \mathrm{sgn}[m(t_2)]\ \mathrm{sgn}[m(t_3)]\right\} = 0 \tag{2-67}$$

and $f_i(t_1, \ldots, t_i) = 0$ for i odd. The general expression for $f_i(t_1, \ldots, t_i)$ for i even appears to be intractable. An interesting integral expression is available in [20]

$$E[\mathrm{sgn}(m_1 \ldots m_N)] = \frac{1}{\pi^N} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \Phi_m(\omega_1, \ldots, \omega_N)\, \frac{d\omega_1 \ldots d\omega_N}{\omega_1 \ldots \omega_N} \tag{2-68}$$

where $\Phi_m(\omega_1, \ldots, \omega_N)$ is the joint characteristic function of m_i, $i = 1, \ldots, N$. This expression is not available in reduced form for $N \geq 4$.

From Equations (2-61), (2-63), (2-64), and (2-66), it follows that

$$\Lambda[r(t)] = \exp -\left(\frac{A^2 T}{N_o}\right) \left[1 + \left(\frac{2A}{N_o}\right)^2 \frac{1}{\pi} \int_o^T \int_o^T r(t_1)\, r(t_2) \sin^{-1}\left[\frac{K_m(t_1, t_2)}{\sqrt{K_m(t_1, t_1)\, K_m(t_2, t_2)}}\right] dt_1\, dt_2 + \ldots\right]. \tag{2-69}$$

2.3.2 Absolute Value of a Gaussian Process in Gaussian White Noise

Consider

$$S[t, m(t)] = |m(t)| \tag{2-70}$$

where m(t) is a zero-mean Gaussian process. Also, assume A = 1 to obtain from Equation (2-28) and (2-30)

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{2-71}$$

where

$$f_i(t_1, \ldots, t_i) = E\left\{|m(t_1) \ldots m(t_i)| \exp\left[-\frac{1}{N_o}\int_0^T |m(\tau)|^2 d\tau\right]\right\} . \tag{2-72}$$

Note that

$$\int_0^T |m(\tau)|^2 d\tau = \int_0^T m^2(\tau)\, d\tau \quad . \tag{2-73}$$

Hence, expanding $m(\tau)$ in a Karhunen-Loéve expansion where the pdf of the expansion coefficients is given by Equation (2-41), f_o becomes

$$f_o = E\left(\exp\left[-\frac{1}{N_o}\int_0^T m^2(\tau)\, d\tau\right]\right)$$

$$= \lim_{N\to\infty} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}\left(\prod_{i=1}^{N}\lambda_i\right)^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{1}{\lambda_i} + \frac{2}{N_o}\right) m_i^2\right] dm_1 \ldots dm_N \, . \tag{2-74}$$

With reference to the development of Equation (2-44), Equation (2-74) can be written as

$$f_o = \left(\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} . \tag{2-75}$$

Similarly,

$$f_1(t_1) = \lim_{N \to \infty} \left(\prod_{i=1}^{N} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \frac{\left| \sum_{i=1}^{N} m_i \, \phi_i(t_1) \right|}{(2\pi)^{N/2} \left(\prod_{i=1}^{N} \frac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}}}$$

$$\exp\left(-\frac{1}{2} \sum_{i=1}^{N} \frac{m_i^2}{\frac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i}} \right) dm_1 \ldots dm_N . \tag{2-76}$$

To proceed it is convenient to define the random variable

$$Z_{1N} = \sum_{i=1}^{N} m_i \, \phi_i(t_1) \tag{2-77}$$

where now, from Equation (2-76), m_i, $i = 1, \ldots, N$ are zero mean independent Gaussian random variables of variance $\dfrac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i}$. Thus, Z_{1N} is zero-mean Gaussian with variance

$$\sigma^2_{Z_{1N}} = \sum_{i=1}^{N} \frac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i} \phi_i^2(t_1) . \tag{2-78}$$

From Equation (2-76) through Equation (2-78)

$$f_1(t_1) = \lim_{N \to \infty} \left(\prod_{i=1}^{N} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} E(|Z_{1N}|) \quad . \tag{2-79}$$

From Papoulis [19, Equation (5-48)]

$$E(|Z_{1N}|) = \sqrt{\frac{2}{\pi}\, \sigma^2_{Z_{1N}}} \quad . \tag{2-80}$$

Utilizing Equation (2-58) in Equation (2-78), $f_1(t_1)$ may be expressed as

$$f_1(t_1) = \left(\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} \sqrt{\frac{N_o}{\pi}\, h_*(t_1, t_1)} \quad . \tag{2-81}$$

Similarly, note that

$$f_2(t_1, t_2) = \lim_{N \to \infty} \left(\prod_{i=1}^{N} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \overset{N}{\cdots} \int_{-\infty}^{\infty} \frac{\left| \sum_{i=1}^{N} m_i \, \phi_i(t_1) \right| \left| \sum_{i=1}^{N} m_i \phi_i(t_2) \right|}{(2\pi)^{N/2} \left(\prod_{i=1}^{N} \frac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}}}$$

$$\exp\left(-\frac{1}{2} \sum_{i=1}^{N} \frac{m_i^2}{\frac{\frac{N_o}{2} \lambda_i}{\frac{N_o}{2} + \lambda_i}} \right) dm_1 \ldots dm_N \quad . \tag{2-82}$$

As with Equation (2-77) define the zero mean jointly Gaussian random variables

$$Z_{1N} = \sum_{i=1}^{N} m_i \, \phi_i(t_1) \tag{2-83}$$

$$Z_{2N} = \sum_{i=1}^{N} m_i \, \phi_i(t_2) \quad . \tag{2-84}$$

Z_{1N} and Z_{2N} have the property that

$$\lim_{N \to \infty} \sigma_{Z_{1N}}^{2} = \frac{N_o}{2} \, h_*(t_1, t_1) \tag{2-85}$$

$$\lim_{N \to \infty} \sigma_{Z_{2N}}^{2} = \frac{N_o}{2} \, h_*(t_2, t_2) \tag{2-86}$$

$$\lim_{N \to \infty} E(Z_{1N} \, Z_{2N}) = \frac{N_o}{2} \, h_*(t_1, t_2) \quad . \tag{2-87}$$

In terms of Z_{1N} and Z_{2N},

$$f_2(t_1, t_2) = \lim_{N \to \infty} \left(\prod_{i=1}^{N} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} E(|Z_{1N} \, Z_{2N}|) \quad . \tag{2-88}$$

From Papoulis [19, p. 221, Equation (7-109) and Equation (7-110)]

$$E(|Z_{1N} \, Z_{2N}|) = \frac{2}{\pi} \, \sigma_{Z_{1N}} \, \sigma_{Z_{2N}} \left\{ \sqrt{1 - \left[\frac{E(Z_{1N} \, Z_{2N})}{\sigma_{Z_{1N}} \, \sigma_{Z_{2N}}} \right]^2} + \frac{E(Z_{1N} \, Z_{2N})}{\sigma_{Z_{1N}} \, \sigma_{Z_{2N}}} \sin^{-1} \left[\frac{E(Z_{1N} \, Z_{2N})}{\sigma_{Z_{1N}} \, \sigma_{Z_{2N}}} \right] \right\} \quad . \tag{2-89}$$

From Equation (2-85) through Equation (2-89)

$$f_2(t_1, t_2) = \left(\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{\frac{1}{2}} \frac{N_o}{\pi} \sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}$$

$$\left\{ \left[1 - \frac{h_*^2(t_1, t_2)}{h_*(t_1, t_1)\, h_*(t_2, t_2)} \right]^{\frac{1}{2}} + \frac{h_*(t_1, t_2)}{\sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}} \right.$$

$$\left. \sin^{-1} \left[\frac{h_*(t_1, t_2)}{\sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}} \right] \right\} . \tag{2-90}$$

Similarly, it can be shown that [21]

$$
\begin{aligned}
f_3(t_1, t_2, t_3) = {} & \left(\prod_{i=1}^{\infty} \frac{N_o/2}{\frac{N_o}{2} + \lambda_i} \right)^{1/2} \left(\frac{N_o}{\pi}\right)^{3/2} \sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)\, h_*(t_3, t_3)} \\
& \Bigg\{ \left[1 + 2\, \frac{h_*(t_1, t_2)\, h_*(t_2, t_3)\, h_*(t_1, t_3)}{h_*(t_1, t_1)\, h_*(t_2, t_2)\, h_*(t_3, t_3)} - \frac{h_*^2(t_1, t_3)}{h_*(t_1, t_1)\, h_*(t_3, t_3)} \right. \\
& \left. - \frac{h_*^2(t_2, t_3)}{h_*(t_2, t_2)\, h_*(t_3, t_3)} - \frac{h_*^2(t_1, t_2)}{h_*(t_1, t_1)\, h_*(t_2, t_2)} \right]^{1/2} \\
& + \left[\frac{h_*(t_1, t_2)}{\sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}} + \frac{h_*(t_1, t_3)\, h_*(t_2, t_3)}{h_*(t_3, t_3) \sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}} \right] \\
& \sin^{-1} \frac{\dfrac{h_*(t_1, t_2)}{\sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}} - \dfrac{h_*(t_1, t_3)\, h_*(t_2, t_3)}{h_*(t_3, t_3) \sqrt{h_*(t_1, t_1)\, h_*(t_2, t_2)}}}{\sqrt{\left[1 - \dfrac{h_*^2(t_1, t_3)}{h_*(t_1, t_1)\, h_*(t_3, t_3)} \right] \left[1 - \dfrac{h_*^2(t_2, t_3)}{h_*(t_2, t_2)\, h_*(t_3, t_3)} \right]}} \\
& + \left[\frac{h_*(t_1, t_3)}{\sqrt{h_*(t_1, t_1)\, h_*(t_3, t_3)}} + \frac{h_*(t_1, t_2)\, h_*(t_2, t_3)}{h_*(t_2, t_2) \sqrt{h_*(t_1, t_1)\, h_*(t_3, t_3)}} \right] \\
& \sin^{-1} \frac{\dfrac{h_*(t_1, t_3)}{\sqrt{h_*(t_1, t_1)\, h_*(t_3, t_3)}} - \dfrac{h_*(t_1, t_2)\, h_*(t_2, t_3)}{h_*(t_2, t_2) \sqrt{h_*(t_1, t_1)\, h_*(t_3, t_3)}}}{\sqrt{\left[- \dfrac{h_*^2(t_1, t_2)}{h_*(t_1, t_1)\, h_*(t_2, t_2)} \right] \left[- \dfrac{h_*^2(t_2, t_3)}{h_*(t_2, t_2)\, h_*(t_3, t_3)} \right]}} \\
& + \left[\frac{h_*(t_2, t_3)}{\sqrt{h_*(t_2, t_2)\, h_*(t_3, t_3)}} + \frac{h_*(t_1, t_2)\, h_*(t_1, t_3)}{h_*(t_1, t_1) \sqrt{h_*(t_2, t_2)\, h_*(t_3, t_3)}} \right] \\
& \sin^{-1} \left(\frac{\dfrac{h_*(t_2, t_3)}{\sqrt{h_*(t_2, t_2)\, h_*(t_3, t_3)}} - \dfrac{h_*(t_1, t_2)\, h_*(t_1, t_3)}{h_*(t_1, t_1) \sqrt{h_*(t_2, t_2)\, h_*(t_3, t_3)}}}{\sqrt{\left[1 - \dfrac{h_*^2(t_1, t_2)}{h_*(t_1, t_1)\, h_*(t_2, t_2)} \right] \left[1 - \dfrac{h_*^2(t_1, t_3)}{h_*(t_1, t_1)\, h_*(t_3, t_3)} \right]}} \right) \Bigg\}
\end{aligned}
\tag{2-91}
$$

A general expression for $f_i(t_1, \ldots, t_i)$ has not been obtained.

2.3.3 A Dc Example - Phase Modulated Sinusoid

For the dc case, the sample functions are random variables. Hence,

$$m(t) = m \qquad 0 \le t \le T \quad . \tag{2-92}$$

From Equation (2-28) and Equation (2-30), the likelihood ratio is given by

$$\Lambda[r(t)] = \sum_{i=o}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{2-93}$$

where

$$f_i(t_1, \ldots, t_i) = E\left[S(t_1, m) \ldots S(t_i, m) \exp\left(-\frac{A^2}{N_o} \int_o^T S^2(\tau, m)\, d\tau\right)\right] \quad . \tag{2-94}$$

Observe that the Karhunen-Loève expansion of m(t) consists of a single term. Hence another expression for the likelihood ratio is obtained from Equation (2-26) as

$$\Lambda[r(t)] = \int_{-\infty}^{\infty} \exp\left[\frac{2A}{N_o} \int_o^T r(t)\, S(t, m)\, dt - \frac{A^2}{N_o} \int_o^T S^2(t, m)\, dt\right] p(m)\, dm \tag{2-95}$$

where p(m) is the a priori pdf of m.

Let

$$S(t, m) = \cos(\omega_o t + m) \tag{2-96}$$

where

$$p(m) = \frac{1}{2\pi} \qquad 0 \le m \le 2\pi$$
$$= 0 \qquad \text{elsewhere} \quad . \tag{2-97}$$

From Equations (2-95) through (2-97), the likelihood ratio for this problem is given by

$$\Lambda[r(t)] = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{2A}{N_o}\int_0^T r(t)\cos(\omega_o t + m)\,dt - \frac{A^2}{N_o}\int_0^T \cos^2(\omega_o t + m)\,dt\right) dm. \tag{2-98}$$

Let

$$I = \int_0^T r(t)\cos\omega_o t\,dt \tag{2-99}$$

$$Q = \int_0^T r(t)\sin\omega_o t\,dt \quad . \tag{2-100}$$

Then Equation (2-98) can be written as

$$\Lambda[r(t)] = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[\frac{2A}{N_o}(I\cos m - Q\sin m) - \frac{A^2 T}{2N_o}\left(1 + \frac{\sin 2\omega_o T}{2\omega_o T}\cos 2m + \frac{\cos 2\omega_o T - 1}{2\omega_o T}\sin 2m\right)\right] dm \quad . \tag{2-101}$$

For $\omega_o T >> 1$ the second term in the argument of the exponential in Equation (2-101) reduces to a constant independent of m and may be lumped with the threshold. When this approximation is not made, note that Equation (2-101) can be rewritten as

$$\Lambda[r(t)] = \frac{e^{-\frac{A^2 T}{2N_o}}}{2\pi}\int_0^{2\pi} \exp\left[\frac{2A}{N_o}(I\cos m - Q\sin m) - \frac{A^2 T}{2N_o}\,\frac{\sin\omega_o T}{\omega_o T}\cos(2m + \omega_o T)\right] dm \quad . \tag{2-102}$$

From [22, Equation (9.6.34)]

$$e^{Z\cos\theta} = I_o(Z) + 2\sum_{k=1}^{\infty} I_k(Z)\cos(k\,\theta) \quad . \tag{2-103}$$

It follows that

$$\Lambda[r(t)] = \left\{ \frac{e^{-\frac{A^2T}{2N_o}}}{2\pi} \int_o^{2\pi} I_o\left(-\frac{A^2T}{2N_o}\,\frac{\sin\omega_oT}{\omega_oT}\right)\exp\left[\frac{2A}{N_o}\,(I\cos m - Q\sin m)\right] dm \right.$$

$$+ 2\sum_{k=1}^{\infty} I_k\left(-\frac{A^2T}{2N_o}\,\frac{\sin\omega_oT}{\omega_oT}\right)\int_o^{2\pi}$$

$$\left. \exp\left[\frac{2A}{N_o}\,(I\cos m - Q\sin m\right]\cos[k(2m + \omega_oT)]\,dm \right\} . \tag{2-104}$$

Note that

$$\int_o^{2\pi} \exp\left[\frac{2A}{N_o}\,(I\cos m - Q\sin m)\right]\cos[k(2m + \omega_oT)]\,dm$$

$$= \cos k\omega_oT \int_o^{2\pi} \exp\left[\frac{2A}{N_o}\,(I\cos m - Q\sin m)\right]\cos 2km\,dm$$

$$- \sin k\omega_oT \int_o^{2\pi} \exp\left[\frac{2A}{N_o}\,(I\cos m - Q\sin m)\right]\sin 2km\,dm\,. \tag{2-105}$$

From [23, Equations (3. 937-1) and (3. 937-2)]

$$\int_0^{2\pi} \exp\left[\frac{2A}{N_o} (I \cos m - Q \sin m)\right] \cos 2km \; dm$$

$$= \pi \quad (I^2 + Q^2)^{-k} \quad [(I - j\,Q)^{2k} + (I + j\,Q)^{2k}] \; I_{2k}\left[\frac{2A}{N_o} \; (I^2 + Q^2)^{\frac{1}{2}}\right] \tag{2-106}$$

and

$$\int_0^{2\pi} \exp\left[\frac{2A}{N_o} (I \cos m - Q \sin m)\right] \sin 2km \; dm$$

$$= j\pi \; (I^2 + Q^2)^{-k} \quad [(I - j\,Q)^{2k} - (I + j\,Q)^{2k}] \; I_{2k}\left[\frac{2A}{N_o} \; (I^2 + Q^2)^{\frac{1}{2}}\right]. \tag{2-107}$$

where $j = \sqrt{-1}$ and $k = 0, 1, \ldots$.

Consequently, Equation (2-105) is evaluated as

$$\int_0^{2\pi} \exp\left[\frac{2A}{N_o} (I \cos m - Q \sin m)\right] \cos [k(2m + \omega_o T)] \; dm$$

$$= \pi\,(I^2 + Q^2)^{-k} \quad I_{2k}\left[\frac{2A}{N_o} \; (I^2 + Q^2)^{\frac{1}{2}}\right]$$

$$\Big\{ \cos k\omega_o T \; [(I - jQ)^{2k} + (I + jQ)^{2k}]$$

$$- j \sin k\omega_o T \; [(I - jQ)^{2k} - (I + jQ)^{2k}] \Big\} \; . \tag{2-108}$$

In polar form

$$(I - jQ)^{2k} = (I^2 + Q^2)^k \, e^{-j2k \tan^{-1}(Q/I)} \tag{2-109}$$

$$(I - jQ)^{2k} = (I^2 + Q^2)^k \, e^{+j2k \tan^{-1}(Q/I)} \quad . \tag{2-110}$$

Expressing $\cos k\omega_o T$ and $\sin k\omega_o T$ as sums of exponentials and making use of Equations (2-109), (2-110) it follows that

$$(I - jQ)^{2k} (\cos k\omega_o T - j \sin k\omega_o T) + (I + jQ)^{2k} (\cos k\omega_o T + j \sin k\omega_o T)$$

$$= 2(I^2 + Q^2)^k \cos \left(2k \tan^{-1} \frac{Q}{I} + k\omega_o T\right) \quad . \tag{2-111}$$

From Equations (2-111) and (2-108)

$$\int_o^{2\pi} \exp\left[\frac{2A}{N_o} (I_o \cos m - Q \sin m)\right] \cos [k(2m + \omega_o T)] \, dm$$

$$= 2\pi \, I_{2k}\left[\frac{2A}{N_o} (I^2 + Q^2)^{\frac{1}{2}}\right] \cos\left(2k \tan^{-1} \frac{Q}{I} + k\omega_o T\right) \tag{2-112}$$

for $k = 0, 1, 2, \ldots$.

Substituting Equation (2-112) into (2-104) the expression for the likelihood ratio becomes

$$\Lambda[r(t)] = e^{-\frac{A^2 T}{2N_o}} \Bigg\{ I_o\left(-\frac{A^2 T}{2N_o} \frac{\sin \omega_o T}{\omega_o T}\right) I_o\left[\frac{2A}{N_o} (I^2 + Q^2)^{\frac{1}{2}}\right]$$

$$+ 2 \sum_{k=1}^{\infty} I_k\left(-\frac{A^2 T}{2N_o} \frac{\sin \omega_o T}{\omega_o T}\right) I_{2k}\left[\frac{2A}{N_o} (I^2 + Q^2)^{\frac{1}{2}}\right]$$

$$\cos\left[k(\omega_o T + 2 \tan^{-1} \frac{Q}{I})\right] \Bigg\} \quad . \tag{2-113}$$

Note that for $\omega_o T = n\pi$, $n \neq 0$ or for $\omega_o T >> 1$, Equation (2-113) simplifies to

$$\Lambda[r(t)] = e^{-\frac{A^2 T}{2N_o}} I_o\left[\frac{2A}{N_o} (I^2 + Q^2)^{\frac{1}{2}}\right] . \tag{2-114}$$

This is in agreement with a previously published result [1, Equation (367)]. The performance using Equation (2-114) is also documented [1, p. 346].

Now consider Equations (2-93) and (2-94)

$$f_o = E\left\{\exp\left[-\frac{A^2}{N_o}\int_o^T \cos^2(\omega_o \tau + m)\, d\tau\right]\right\}$$

$$= E\left(\exp\left\{-\frac{A^2 T}{2N_o}\left[1 + \frac{\sin\omega_o T}{\omega_o T}\cos(2m + \omega_o T)\right]\right\}\right) . \tag{2-115}$$

From Equations (2-115) and (2-97)

$$f_o = e^{-\frac{A^2 T}{2N_o}} \frac{1}{2\pi}\int_o^{2\pi} \exp\left[-\frac{A^2 T}{2N_o}\frac{\sin\omega_o T}{\omega_o T}\cos(2m + \omega_o T)\right] dm .$$

This involves a well known integral [22]. Consequently,

$$f_o = e^{-\frac{A^2 T}{2N_o}} I_o\left(-\frac{A^2 T}{2N_o}\frac{\sin\omega_o T}{\omega_o T}\right) . \tag{2-116}$$

Using $\omega_o T = n\pi$ in Equation (2-116), f_o reduces to

$$f_o = e^{-\frac{A^2 T}{2N_o}} . \tag{2-117}$$

From Equation (2-94), $f_1(t_1)$ is given by

$$f_1(t_1) = E\left\{\cos(\omega_o t_1 + m) \exp\left[-\frac{A^2}{N_o}\int_o^T \cos^2(\omega_o \tau + m)\, d\tau\right]\right\}. \quad (2\text{-}118)$$

Performing the integration in Equation (2-118) there results

$$f_1(t_1) = E\left(\cos(\omega_o t_1 + m) \exp\left\{-\frac{A^2 T}{2N_o}\left[1 + \frac{\sin \omega_o T}{\omega_o T} \cos(2m + \omega_o T)\right]\right\}\right)$$

(2-119)

For $\omega_o T >> 1$ or $\omega_o T = n\pi$, $n \neq 0$ Equation (2-119) simplifies to

$$f_1(t_1) = E[\cos(\omega_o t_1 + m)]\; e^{-\frac{A^2 T}{2N_o}} \quad (2\text{-}120)$$

Expanding the sinusoid in Equation (2-120) using trigonometric identities there results

$$f_1(t_1) = \left[\cos \omega_o t_1 \frac{1}{2\pi}\int_o^{2\pi} \cos m dm - \sin \omega_o t_1 \frac{1}{2\pi}\int_o^{2\pi} \sin m dm\right] e^{-\frac{A^2 T}{2N_o}}$$

$$= 0 \quad . \quad (2\text{-}121)$$

From Equation (2-94), $f_2(t_1, t_2)$ is given by

$$f_2(t_1, t_2) = E\left(\cos(\omega_o t_1 + m)\cos(\omega_o t_2 + m)\right.$$

$$\left.\exp\left[-\frac{A^2T}{2N_o}\left[1 + \frac{\sin \omega_o T}{\omega_o T}\cos(2m + \omega_o T)\right]\right]\right) \qquad (2-122)$$

For $\omega_o T >> 1$ or $\omega_o T = n\pi$, $n \neq 0$ $f_2(t_1, t_2)$ becomes

$$f_2(t_1, t_2) = E[\cos(\omega_o t_1 + m)\cos(\omega_o t_2 + m)]\ e^{-\frac{A^2T}{2N_o}} \quad . \qquad (2-123)$$

Expanding the sinusoids in Equation (2-123) and taking the expectation there results

$$f_2(t_1, t_2) = \frac{1}{2}[\cos \omega_o t_1 \cos \omega_o t_2 + \sin \omega_o t_1 \sin \omega_o t_2]$$

$$e^{-\frac{A^2T}{2N_o}} \quad . \qquad (2-124)$$

Using Equations (2-117), (2-121) and (2-124) in (2-93), an approximation to the likelihood ratio, using only the first three terms, can be written as

$$\Lambda_2[r(t)] = e^{-\frac{A^2T}{2N_o}}\left[1 + \frac{1}{4}\left(\frac{2A}{N_o}\right)^2 (I^2 + Q^2)\right] \qquad (2-125)$$

where I and Q are defined in Equations (2-99) and (2-100), respectively and the subscript refers to the exponent of $\frac{2A}{N_o}$.

Note that both Equations (2-125) and (2-114) are monotonic functions of $(I^2 + Q^2)^{\frac{1}{2}}$. It follows that for $\omega_o T = n\pi$, $n \neq 0$, the performance of a receiver based upon $\Lambda[r(t)]$ and $\Lambda_2[r(t)]$ are identical even though $\Lambda[r(t)]$ and $\Lambda_2[r(t)]$ are very different functions of $(I^2 + Q^2)^{\frac{1}{2}}$. Including the first five terms of the likelihood ratio, it can be shown that

$$\Lambda_4[r(t)] = e^{-\frac{A^2T}{2N_o}} \left[1 + \frac{1}{4}\left(\frac{2A}{N_o}\right)^2 (I^2 + Q^2) + \frac{1}{64}\left(\frac{2A}{N_o}\right)^4 (I^2 + Q^2)^2\right] \tag{2-126}$$

where the subscript refers to the highest exponent of $\frac{2A}{N_o}$. Equation (2-126) is also a monotonic function of $(I^2 + Q^2)^{\frac{1}{2}}$. Note that expanding $I_o(\cdot)$ in Equation (2-114) in a Taylor series results in

$$\Lambda[r(t)] = e^{-\frac{A^2T}{2N_o}} \left[1 + \frac{1}{4}\left(\frac{2A}{N_o}\right)^2 (I^2 + Q^2) + \frac{1}{64}\left(\frac{2A}{N_o}\right)^4 (I^2 + Q^2)^2 + \dots\right]. \tag{2-127}$$

Hence, $\Lambda_{2N}[r(t)]$ is obtained by truncating the expansion of $\Lambda[r(t)]$ after $2N + 1$ terms. Since all the coefficients in the expansion of $I_o(\cdot)$ are positive and since $(I^2 + Q^2)^{\frac{1}{2}}$ is positive it follows that both $\Lambda_{2N}[r(t)]$ and $\Lambda[r(t)]$ are monotonic functions of $(I^2 + Q^2)^{\frac{1}{2}}$. Generalizing, it follows that receivers based upon $\Lambda[r(t)]$ and $\Lambda_{2N}[r(t)]$ have identical performance for $\omega_o T = n\pi$. It follows that it is not always necessary to take many terms in the Volterra functional expansion of $\Lambda[r(t)]$ in order to obtain acceptable performance. Of course, for this example, $\Lambda_{2N}[r(t)]$ N = 1, 2, ... all turned out to be monotonic functions of the same statistic $(I^2 + Q^2)^{\frac{1}{2}}$. For most nonlinear problems the sufficient statistic for $\Lambda_{2N}[r(t)]$ will not be the same as the sufficient statistic for $\Lambda[r(t)]$. This is apparent from Sections 2.3.1 and 2.3.2 where the formulation of a simple sufficient statistic appears unlikely due to the form of the Volterra kernels (i.e., higher order kernels cannot be expressed as combinations of lower order kernels). Then, the closer $\Lambda_{2N}[r(t)]$ approximates $\Lambda[r(t)]$, the closer the receiver performance approaches the optimum. In general, it is not clear how large N must be to obtain acceptable performance.

2.4 Sun's Theorem and Asymptotic Receivers

In this section, it is shown that for the class of nonlinearities, $S[t, m(t)]$, and Gaussian processes, $m(t)$, satisfying Sun's Theorem* [24] the higher order Volterra kernels are representable in terms of lower order kernels as the length of the observation interval goes to infinity ($T \to \infty$). For this case, asymptotic sufficient statistics can be obtained for which performance can be determined.

2.4.1 Asymptotic Receiver Derivation

From par. 2.1, the likelihood ratio for the detection problem of Equations (2-1) and (2-2) is

$$\Lambda[r(t)] = \sum_{i=o}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{2-128}$$

*Sun's Theorem is discussed in Appendix A.

where

$$f_i(t_1, \ldots t_i) = E\left(S[t_1, m(t_1)] \ldots S[t_i, m(t_i)] \right.$$

$$\left. \exp\left\{ -\frac{A^2}{N_o} \int_o^T S^2[\tau, m(\tau)]\, d\tau \right\} \right) \quad . \tag{2-129}$$

In this section, asymptotically optimal receivers are obtained for the special case

$$f_i(t_1, \ldots, t_i) \simeq K\, E\{S[t_1, m(t_1)] \ldots S[t_i, m(t_i)]\} \tag{2-130}$$

where K is some proportionality constant independent of i.*

At this point it is convenient to define

$$d_i(t_1, \ldots, t_i) = E\{S[t_1, m(t_1)] \ldots S[t_i, m(t_i)]\} \quad . \tag{2-131}$$

From Equations (2-130) and (2-131)

$$f_i(t_1, \ldots, t_i) = K\, d_i(t_1, \ldots, t_i) \tag{2-132}$$

Suppose that $d_i(t_1, \ldots, t_i)$, where $0 \le t_j \le T$ for $j = 1, \ldots, i$, is expanded in a multidimensional Fourier series. For i = 1, 2, and 3 the expansions are

$$d_1(t_1) = \sum_{i_1 = o}^{\infty} d_{i_1}^{11} \cos \frac{2\pi i_1}{T} t_1 + d_{i_1}^{12} \sin \frac{2\pi i_1}{T} t_1 \tag{2-133}$$

*For many angle modulated sinusoids, Equation (2-130) applies. An angle modulation example is treated in Section 2.4.2.

$$
\begin{aligned}
d_2(t_1, t_2) = \sum_{i_1=o}^{\infty} \sum_{i_2=o}^{\infty} & d_{i_1 i_2}^{21} \cos \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \\
& + d_{i_1 i_2}^{22} \sin \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \\
& + d_{i_1 i_2}^{23} \cos \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2 \\
& + d_{i_1 i_2}^{24} \sin \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2
\end{aligned} \tag{2-134}
$$

$$
\begin{aligned}
d_3(t_1, t_2, t_3) = \sum_{i_1=o}^{\infty} \sum_{i_2=o}^{\infty} \sum_{i_3=o}^{\infty} & d_{i_1 i_2 i_3}^{31} \cos \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \cos \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{32} \sin \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \cos \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{33} \cos \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2 \cos \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{34} \cos \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \sin \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{35} \sin \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2 \cos \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{36} \sin \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \sin \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{37} \cos \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2 \sin \frac{2\pi i_3}{T} t_3 \\
& + d_{i_1 i_2 i_3}^{38} \sin \frac{2\pi i_1}{T} t_1 \sin \frac{2\pi i_2}{T} t_2 \sin \frac{2\pi i_3}{T} t_3 \quad .
\end{aligned} \tag{2-135}
$$

The Fourier coefficients are determined in the usual manner. The first few are

$$d_{i_1}^{11} = \frac{2}{T} \int_o^T d_1(t_1) \cos \frac{2\pi i_1}{T} t_1 \, dt_1$$

$$d_{i_1}^{12} = \frac{2}{T} \int_o^T d_1(t_1) \sin \frac{2\pi i_1}{T} t_1 \, dt_1$$

$$d_{i_1 i_2}^{21} = \left(\frac{2}{T}\right)^2 \int_o^T \int_o^T d_2(t_1, t_2) \cos \frac{2\pi i_1}{T} t_1 \cos \frac{2\pi i_2}{T} t_2 \, dt_1 \, dt_2 \quad .$$

At this point it is convenient to define

$$x_i = \frac{2A}{N_o} \int_o^T r(t) \cos \frac{2\pi i}{T} t \, dt \tag{2-136}$$

$$y_i = \frac{2A}{N_o} \int_o^T r(t) \sin \frac{2\pi i}{T} t \, dt \quad . \tag{2-137}$$

Under H_o, x_i and y_i, $i = 0, 1, \ldots,$ are jointly Gaussian and statistically independent with

$$E(x_i | H_o) = E(y_i | H_o) = 0 \tag{2-138}$$

and

$$E(x_i \, x_j | H_o) = \frac{1}{2} \frac{2A^2 T}{N_o} \delta_{ij}$$

$$E(y_i \, y_j | H_o) = \frac{1}{2} \frac{2A^2 T}{H_o} \delta_{ij} \tag{2-139}$$

$$E(x_i \, y_i | H_o) = 0 \quad .$$

Under H_1, x_i and y_i, $i = 0, 1 \ldots$, may be asymptotically jointly Gaussian. (In Appendix A, this is shown for the case $S(t, m(t)) = \cos(\omega_o t + m(t))$. The proof makes use of Sun's Theorem* and the Cramer-Wold Theorem.) The expected value of x_i conditioned on H_1 is

$$E(x_i | H_1) = \frac{2A}{N_o} \left\{ \int_0^T A\, E\{S[t, m(t)]\} \cos \frac{2\pi i}{T} t\, dt + \int_0^T E[n(t)] \cos \frac{2\pi i}{T} t\, dt \right\}$$

From Equations (2-133) and (2-131) it follows that

$$E(x_i | H_1) = \frac{1}{2}\left(\frac{2A^2 T}{N_o}\right) d_i^{11} \quad . \tag{2-140}$$

Similarly

$$E(y_i | H_1) = \frac{1}{2}\left(\frac{2A^2 T}{N_o}\right) d_i^{12} \tag{2-141}$$

In like manner

$$E(x_i\, x_j | H_1) = \frac{1}{4} \left(\frac{2A^2 T}{N_o}\right)^2 d_{ij}^{21} + \frac{1}{2} \frac{2A^2 T}{N_o} \delta_{ij} \tag{2-142}$$

$$E(x_i\, y_j | H_1) = \frac{1}{4} \left(\frac{2A^2 T}{N_o}\right)^2 d_{ij}^{23} \tag{2-143}$$

$$E(y_i\, x_j | H_1) = \frac{1}{4} \left(\frac{2A^2 T}{N_o}\right)^2 d_{ij}^{22} \tag{2-144}$$

$$E(y_i\, y_j | H_1) = \frac{1}{4} \left(\frac{2A^2 T}{N_o}\right)^2 d_{ij}^{24} + \frac{1}{2} \left(\frac{2A^2 T}{N_o}\right) \delta_{ij} \quad . \tag{2-145}$$

---

*Sun's theorem requires that $m(t)$ be a zero mean stationary Gaussian process. Other restrictions on $m(t)$ and $S[t, m(t)]$ are discussed in Appendix A. An example which satisfies Sun's theorem is presented in Section 2.4.3.

Utilizing the asymptotic statistics of Equations (2-136) and (2-137) it is possible to obtain the Fourier coefficients of Equation (2-135) in terms of the Fourier coefficients in Equation (2-133) and (2-134). To see this, recall that if z_i, z_j and z_k are jointly Gaussian, it follows that [25, p. 71]

$$
\begin{aligned}
E(z_i\, z_j\, z_k) = & \left[E(z_i\, z_j) - E(z_i)\, E(z_j)\right] E(z_k) \\
& + \left[E(z_i\, z_k) - E(z_i)\, E(z_k)\right] E(z_j) \\
& + \left[E(z_j\, z_k) - E(z_j)\, E(z_k)\right] E(z_i) \\
& + E(z_i)\, E(z_j)\, E(z_k) \quad . \qquad (2\text{-}146)
\end{aligned}
$$

Consider

$$
\begin{aligned}
E(x_i\, x_j\, x_k \mid H_1) = & \left(\frac{2A}{N_o}\right)^3 \int_0^T \int_0^T \int_0^T \Big[A^3\, d_3\,(t_1, t_2, t_3) \\
& + A\, \frac{N_o}{2}\, d_1(t_1)\, \delta(t_2 - t_3) \\
& + A\, \frac{N_o}{2}\, d_1(t_2)\, \delta(t_1 - t_3) \\
& + A\, \frac{N_o}{2}\, d_1(t_3)\, \delta(t_1 - t_2)\Big] \\
& \cos \frac{2\pi i}{T}\, t_1 \cos \frac{2\pi j}{T}\, t_2 \cos \frac{2\pi k}{T}\, t_3\, dt_1\, dt_2\, dt_3 \quad . \qquad (2\text{-}147)
\end{aligned}
$$

From Equations (2-147), (2-135) and (2-133)

$$
\begin{aligned}
E(x_i\, x_j\, x_k \mid H_1) = & \frac{1}{8}\left(\frac{2A^2T}{N_o}\right)^3 d^{31}_{ijk} + \frac{1}{4}\left(\frac{2A^2T}{N_o}\right)\Big(d^{11}_i\, \delta_{jk} \\
& + d^{11}_j\, \delta_{ik} + d^{11}_k\, \delta_{ij}\Big) \quad . \qquad (2\text{-}148)
\end{aligned}
$$

On the other hand, substituting Equations (2-140) and (2-142) into (2-146) yields

$$E(x_i\, x_j\, x_k \mid H_1) = \frac{1}{8}\left(\frac{2A^2T}{N_o}\right)^3 \left(d_{ij}^{21}\, d_k^{11} + d_{ik}^{21}\, d_j^{11} + d_{jk}^{21}\, d_i^{11} - 2\, d_i^{11}\, d_j^{11}\, d_k^{11}\right)$$

$$+ \frac{1}{4}\left(\frac{2A^2T}{N_o}\right)^2 \left(d_i^{11}\, \delta_{jk} + d_j^{11}\, \delta_{ik} + d_k^{11}\, \delta_{ij}\right) . \qquad (2\text{-}149)$$

Comparing Equations (2-148) with (2-149), it follows that

$$d_{ijk}^{31} = d_{ij}^{21}\, d_k^{11} + d_{ik}^{21}\, d_j^{11} + d_{jk}^{21}\, d_i^{11} - 2d_i^{11}\, d_j^{11}\, d_k^{11} \quad . \qquad (2\text{-}150)$$

In a manner similar to that used in the development of Equation (2-148),

$$E(y_i\, x_j\, x_k \mid H_1) = \frac{1}{8}\left(\frac{2A^2T}{N_o}\right)^3 d_{ijk}^{32} + \frac{1}{4}\left(\frac{2A^2T}{N_o}\right) \delta_{jk}\, d_i^{12} \qquad (2\text{-}151)$$

Substituting Equation (2-140) to (2-145) into (2-146) results in

$$E(y_i\, x_j\, x_k \mid H_1) = \frac{1}{8}\left(\frac{2A^2T}{N_o}\right)^3 \left(d_{ij}^{22}\, d_k^{11} + d_{ik}^{22}\, d_j^{11} + d_{jk}^{21}\, d_i^{12} - 2d_i^{12}\, d_j^{11}\, d_k^{11}\right)$$

$$+ \frac{1}{4}\left(\frac{2A^2T}{N_o}\right)^2 \delta_{jk}\, d_i^{12} \qquad (2\text{-}152)$$

Comparing Equations (2-151) with (2-152), observe that

$$d_{ijk}^{32} = d_{ij}^{22}\, d_k^{11} + d_{ik}^{22}\, d_j^{11} + d_{jk}^{21}\, d_i^{12} - 2d_i^{12}\, d_j^{11}\, d_k^{11} \quad . \qquad (2\text{-}153)$$

Similarly, it is readily shown that

$$d_{ijk}^{33} = d_{ij}^{23}\, d_k^{11} + d_{ik}^{21}\, d_j^{12} + d_{jk}^{22}\, d_i^{11} - 2d_i^{11}\, d_j^{12}\, d_k^{11} \qquad (2\text{-}154)$$

$$d_{ijk}^{34} = d_{ij}^{21}\, d_k^{12} + d_{ik}^{23}\, d_j^{11} + d_{jk}^{22}\, d_i^{11} - 2d_i^{11}\, d_j^{11}\, d_k^{12} \qquad (2\text{-}155)$$

$$d_{ijk}^{35} = d_{ij}^{24} d_k^{11} + d_{ik}^{22} d_j^{12} + d_{jk}^{22} d_i^{12} - 2d_i^{12} d_j^{12} d_k^{11} \qquad (2\text{-}156)$$

$$d_{ijk}^{36} = d_{ij}^{22} d_k^{12} + d_{ik}^{24} d_j^{11} + d_{jk}^{23} d_i^{12} - 2d_i^{12} d_j^{11} d_k^{12} \qquad (2\text{-}157)$$

$$d_{ijk}^{37} = d_{ij}^{23} d_k^{12} + d_{ik}^{23} d_j^{12} + d_{jk}^{24} d_i^{11} - 2d_i^{11} d_j^{12} d_k^{12} \qquad (2\text{-}158)$$

$$d_{ijk}^{38} = d_{ij}^{24} d_k^{12} + d_{ik}^{24} d_j^{12} + d_{jk}^{24} d_i^{12} - 2d_i^{12} d_j^{12} d_k^{12} \quad . \qquad (2\text{-}159)$$

With reference to Equations (2-133) and (2-134), now consider

$$d_1(t_1)\, d_2(t_2, t_3) + d_1(t_2)\, d_2(t_1, t_3) + d_1(t_3)\, d_2(t_1, t_2)$$

$$\begin{aligned}
= \sum_i \sum_j \sum_k & \left(d_i^{11} d_{jk}^{21} + d_j^{11} d_{ik}^{21} + d_k^{11} d_{ij}^{21}\right) \cos \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{12} d_{jk}^{21} + d_j^{11} d_{ik}^{22} + d_k^{11} d_{ij}^{22}\right) \sin \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{11} d_{jk}^{22} + d_j^{12} d_{ik}^{21} + d_k^{11} d_{ij}^{23}\right) \cos \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{11} d_{jk}^{23} + d_j^{11} d_{ik}^{23} + d_k^{12} d_{ij}^{21}\right) \cos \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{12} d_{jk}^{22} + d_j^{12} d_{ik}^{22} + d_k^{11} d_{ij}^{24}\right) \sin \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{12} d_{jk}^{23} + d_j^{11} d_{ik}^{24} + d_k^{12} d_{ij}^{22}\right) \sin \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{11} d_{jk}^{24} + d_j^{12} d_{ik}^{23} + d_k^{12} d_{ij}^{23}\right) \cos \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3 \\
& + \left(d_i^{12} d_{jk}^{24} + d_j^{12} d_{ik}^{24} + d_k^{12} d_{ij}^{24}\right) \sin \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3 \quad .
\end{aligned}$$

(2-160)

Also,

$$d_1(t_1)\, d_1(t_2)\, d_1(t_3)$$

$$= \sum_i \sum_j \sum_k d_i^{11}\, d_j^{11}\, d_k^{11} \cos \frac{2\pi i}{T} t_1 \cos \frac{2\pi i}{T} t_2 \cos \frac{2\pi k}{T} t_3$$

$$+ d_i^{12}\, d_j^{11}\, d_k^{11} \sin \frac{2\pi i}{T} t_1 \cos \frac{2\pi i}{T} t_2 \cos \frac{2\pi k}{T} t_3$$

$$+ d_i^{11}\, d_j^{12}\, d_k^{11} \cos \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3$$

$$+ d_i^{11}\, d_j^{11}\, d_k^{12} \cos \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3$$

$$+ d_i^{12}\, d_j^{12}\, d_k^{11} \sin \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \cos \frac{2\pi k}{T} t_3$$

$$+ d_i^{12}\, d_j^{11}\, d_k^{12} \sin \frac{2\pi i}{T} t_1 \cos \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3$$

$$+ d_i^{11}\, d_j^{12}\, d_k^{12} \cos \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3$$

$$+ d_i^{12}\, d_j^{12}\, d_k^{12} \sin \frac{2\pi i}{T} t_1 \sin \frac{2\pi j}{T} t_2 \sin \frac{2\pi k}{T} t_3 \,. \qquad (2\text{-}161)$$

Substituting Equations (2-150) and (2-153) through (2-159) into Equation (2-135) and making use of Equations (2-160) and (2-161) it follows that

$$d_3(t_1, t_2, t_3) \rightarrow d_1(t_1) d_2(t_2, t_3) + d_1(t_2)\, d_2(t_1, t_3) + d_1(t_3)\, d_2(t_1, t_2)$$

$$- 2d_1(t_1)\, d_1(t_2)\, d_1(t_3) \qquad (2\text{-}162)$$

where $\rightarrow$ denotes "is asymptotically equivalent to".

Note that Equation (2-162) can be written in the form

$$d_3(t_1, t_2, t_3) \rightarrow [d_2(t_2, t_3) - d_1(t_2)\, d_1(t_3)]\, d_1(t_1)$$

$$+ [d_2(t_1, t_3) - d_1(t_1)\, d_1(t_3)]\, d_1(t_2)$$

$$+ [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\, d_1(t_3)$$

$$+ d_1(t_1)\, d_1(t_2)\, d_1(t_3) \quad . \tag{2-163}$$

Also, note that the asymptotic form of $d_3(t_1, t_2, t_3)$ is symmetric in its arguments. Following the same procedure that led to Equation (2-162) it can be shown that

$$d_4(t_1, t_2, t_3, t_4) \rightarrow d_2(t_1, t_2)\, d_2(t_3, t_4) + d_2(t_1, t_3)\, d_2(t_2, t_4)$$

$$+ d_2(t_1, t_4)\, d_2(t_2, t_3) - 2d_1(t_1)\, d_1(t_2)\, d_1(t_3)\, d_1(t_4) \; . \tag{2-164}$$

Equation (2-164) can be rewritten in the form

$$d_4(t_1, t_2, t_3, t_4) \rightarrow [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\,[d_2(t_3, t_4) - d_1(t_3)\, d_1(t_4)]$$

$$+ [d_2(t_1, t_3) - d_1(t_1)\, d_1(t_3)]\,[d_2(t_2, t_4) - d_1(t_2)\, d_1(t_4)]$$

$$+ [d_2(t_1, t_4) - d_1(t_1)\, d_1(t_4)]\,[d_2(t_2, t_3) - d_1(t_2)\, d_1(t_3)]$$

$$+ [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\, d_1(t_3)\, d_1(t_4)$$

$$+ [d_2(t_1, t_3) - d_1(t_1)\, d_1(t_3)]\, d_1(t_2)\, d_1(t_4)$$

$$+ [d_2(t_1, t_4) - d_1(t_1)\, d_1(t_4)]\, d_1(t_2)\, d_1(t_3)$$

$$+ [d_2(t_2, t_3) - d_1(t_2)\, d_1(t_3)]\, d_1(t_1)\, d_1(t_4)$$

$$+ [d_2(t_2, t_4) - d_1(t_2)\, d_1(t_4)]\, d_1(t_1)\, d_1(t_3)$$

$$+ [d_2(t_3, t_4) - d_1(t_3)\, d_1(t_4)]\, d_1(t_1)\, d_1(t_2)$$

$$+ d_1(t_1)\, d_1(t_2)\, d_1(t_3)\, d_1(t_4) \quad . \tag{2-165}$$

At this point it is convenient to let

$$n_i = \int_0^T \overset{i}{\cdots} \int d_i(t_1, \ldots, t_i)\, r(t_1) \ldots r(t_i)\, dt_1 \ldots dt_i \tag{2-166}$$

$$n_1 = \eta = \int_0^T d_1(t_1)\, r(t_1)\, dt_1 \tag{2-167}$$

$$\nu = \int_0^T \int_0^T [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\, r(t_1)\, r(t_2)\, dt_1\, dt_2 \ , \tag{2-168}$$

Utilizing Equation (2-163), it follows that

$$n_3 \to 3\eta\, \nu + \eta^3 \ . \tag{2-169}$$

Similarly, making use of Equation (2-165), there results

$$n_4 \to 3\, \nu^2 + 6\, \eta^2\, \nu + \eta^4 \ . \tag{2-170}$$

Substituting Equation (2-166) into Equation (2-128) and making use of Equation (2-132) the likelihood ratio may be expressed as

$$\Lambda\,[r(t)] \to K \sum_{i=o}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i n_i \ . \tag{2-171}$$

However, for some cases n_i can be interpreted as the i^{th} moment of a Gaussian random variable z with mean η and variance ν. The characteristic function of z is given by

$$E\,[\exp(\omega z)] = \exp\left(\frac{1}{2}\, \omega^2 \nu + \omega\, \eta\right)$$

$$= \sum_{i=o}^{\infty} \frac{1}{i!}\, \omega^i\, n_i \ . \tag{2-172}$$

Comparing Equations (2-171) and (2-172), it follows that

$$\Lambda[r(t)] \to K \exp\left[\frac{1}{2}\left(\frac{2A}{N_o}\right)^2 \nu + \left(\frac{2A}{N_o}\right)\eta\right]^* \quad . \qquad (2\text{-}173)$$

*Justification for the generalization to η_i at this point, though convincing in the light of Equations (2-169) and (2-170), has not been given. Equation (2-173) can be proven by another method which relies on an observation due to A. Nuttall suggested to this author in a private communication. From Equations (2-128) and (2-130) it follows that

$$\Lambda[r(t)] = K\, E\left(\exp\left\{\frac{2A}{N_o}\int_o^T r(t)\, S[t, m(t)]\, dt\right\}\right) .$$

Nuttall recognized that $\Lambda[r(t)] = K\,\phi_z\left(\frac{2A}{N_o}\right)$ where $\phi_Z(\omega)$ is the moment generating function of

$$Z = \int_o^T r(t)\, S[t, m(t)]\, dt$$

with r(t) fixed. Note that Z has mean η and variance ν. If Z is asymptotically Gaussian, then Equation (2-173) follows. To see if Z is asymptotically Gaussian, expand r(t) in a Fourier series to obtain

$$Z = \sum_{i=o}^{\infty} r_i^{11}\int_o^T \cos\frac{2\pi i}{T}t\; S[t, m(t)]\; dt$$

$$+ r_i^{12}\int_o^T \sin\frac{2\pi i}{T}t\; S[t, m(t)]\; dt$$

where

$$r_i^{11} = \frac{2}{T}\int_o^T r(t)\cos\frac{2\pi i}{T}t\; dt$$

$$r_i^{12} = \frac{2}{T}\int_o^T r(t)\sin\frac{2\pi i}{T}t\; dt .$$

In general, Z is not asymptotically Gaussian. However, in Appendix A it is shown that for the special case $s(t,.) = \cos(\omega_o t + .)$

$$\int_o^T \cos\frac{2\pi i}{T}t\; S[t, m(t)]\, dt$$

and

$$\int_o^T \sin\frac{2\pi i}{T}t\; S[t, m(t)]\, dt$$

are asymptotically jointly Gaussian for i = o, . . . , N, N arbitrarily large. Hence, for this case, Z is asymptotically Gaussian and so (2-173) applies.

Substituting Equations (2-167) and (2-168) into (2-173) the logarithm of the likelihood ratio is asymptotically represented as

$$\ln \Lambda[r(t)] = \ell[r(t)] \rightarrow$$

$$\frac{1}{2}\left(\frac{2A}{N_o}\right)^2 \int_o^T \int_o^T [\, d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\, r(t_1)\, r(t_2)\, dt_1\, dt_2$$

$$+ \frac{2A}{N_o} \int_o^T d_1(t_1)\, r(t_1)\, dt_1 + \ln K \quad . \qquad (2\text{-}174)$$

Recall from Equation (2-131) that

$$d_2(t_1, t_2) = E\{S[t_1, m(t_1)]\ S[t_2, m(t_2)]\} \qquad (2\text{-}175)$$

$$d_1(t_1) = E\{S[t_1, m(t_1)]\} \quad . \qquad (2\text{-}176)$$

Equation (2-174) yields the asymptotic receiver. However, conditions have not been found under which T is large enough for (2-174) to apply.

2.4.2 Asymptotic Receiver Performance

From Equation (2-174), a receiver based on the logarithm of the likelihood ratio for long observation times is

$$\ell'_\infty[r(t)] = \ell_\infty[r(t)] - \ell n K = \frac{1}{2}\left(\frac{2A}{N_o}\right)^2 \int_o^T \int_o^T [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)]\, r(t_1)\, r(t_2)$$

$$+ \left(\frac{2A}{N_o}\right) \int_o^T d_1(t_1)\, r(t_1)\, dt_1 \qquad (2\text{-}177)$$

where

$$d_i\,(t_1, \ldots, t_i) = E\,\{S\,[\,t_1, m(t_1)]\ldots S(t_i, m(t_i)]\,\}\ . \tag{2-178}$$

Recall that $r(t) = AS\,[\,t, m(t)] + n(t)$ under hypothesis H_1. In this section, it is shown that the conditional moments of $\ell'_\infty\,[\,r(t)]$ given H_1 and the conditional moments of $\ell'_\infty\,[\,r(t)]$ where $r(t) = AS(t) + n(t)$ are asymptotically equal provided that $S(t)$ is a Gaussian processes with the same mean and autocovariance as $S\,[t, m(t)]$ and provided that $m(t)$ and $S(t, .)$ satisfy Sun's theorem. It then follows that the asymptotic performance of the asymptotic receiver may be found by considering the Gaussian process $S(t)$. Conditions on the length of the observation interval, T, under which the asymptotic performance of the asymptotic receiver closely approximate the performance of the optimum receiver, have not been obtained.

Consider the random variable

$$Z_1 = \int_0^T r(t)\, S[\,t, m(t)]\;dt \tag{2-179}$$

with moments

$$E(Z_1^i) = \eta_{1,i} = \int_0^{T_i} \cdots \int_0^T r(t_1)\ldots r(t_i) E\,\{S[\,t_1, m(t_1)]\ldots S[\,t_i, m(t_i]\,\}$$

$$dt_1 \ldots dt_i \tag{2-180}$$

and the Gaussian random variable

$$Z_2 = \int_0^T r(t)\, S(t)\;dt \tag{2-181}$$

with moments

$$E(Z_2^i) = \eta_{2,i} = \int_0^{T_i} \cdots \int_0^{T} r(t_1)\ldots r(t_i) E[S(t_1)\ldots S(t_i)]\, dt_1 \ldots dt_i . \tag{2-182}$$

If $m(t)$ and $S(t,.)$ satisfy Sun's theorem and if $\eta_{2,1} = \eta_{1,1}$ and $\eta_{2,2} = \eta_{1,2}$ then it follows from par. 2.4.1 (see footnote pg. 49) for $T \to \infty$ that

$$\eta_{1,i} \to \eta_{2,i} \tag{2-183}$$

for $i > 2$.

Substituting Equations (2-180) and (2-182) into (2-183) there results

$$E\{S[t_1, m(t_1)]\ldots S[t_i, m(t_i)]\} \to E[S(t_1)\ldots S(t_i)] . \tag{2-184}$$

Now consider the moments of $\ell'_\infty[r(t)]$ under H_1. From Equation (2-177) it follows that

$$E\left(\left\{\ell'_\infty[r(t)]\right\}^K \mid H_1\right)$$

$$= \sum_{i=0}^{K} \binom{K}{i} \left(\frac{1}{2}\right)^i \left(\frac{2A}{N_0}\right)^{K+i}$$

$$E\left(\left\{\int_0^T \int_0^T r(t_1) r(t_2)[d_2(t_1, t_2) - d_1(t_1) d_1(t_2)]\, dt_1 dt_2\right\}^i\right.$$

$$\left.\left[\int_0^T r(t) d_1(t) dt\right]^{K-i} \mid H_1\right) . \tag{2-185}$$

Expanding the argument of the expectation in Equation (2-185) there results

$$E\left\{\left[\ell'_{\infty}(r(t))\right]^{K}\Big| H_1\right\} = \left(\frac{2A}{N_o}\right)^{K} \int_0^T \overset{K}{\cdots} \int_0^T E[r(t)\ldots r(t_K)|H_1]$$

$$d_1(t_1)\ldots d_1(t_K)\, dt_1\ldots dt_k$$

$$+ \sum_{i=1}^{K} \binom{K}{i}\left(\frac{1}{2}\right)^{i}\left(\frac{2A}{N_o}\right)^{K+i} \int_0^T \overset{K+i}{\cdots} \int_0^T E[r(t_1)\ldots r(t_{K+i})\,|\,H_1]$$

$$[d_2(t_1, t_2)-d_1(t_1)d_1(t_2)]\ldots[d_2(t_{2i-1}, t_{2i})-d_1(t_{2i-1})d_1(t_{2i})]$$

$$d_1(t_{2i+1})\ldots d_1(t_{K+i})\, dt_1\ldots dt_{K+i} \quad . \qquad (2\text{-}186)$$

Focusing attention on the expectation in Equation (2-186), note that

$$E[r(t_1)\ldots r(t_{K+i})\,|\,H_1]$$

$$= E\left(\{AS[t_1, m(t_1)] + n(t_1)\}\ldots\{AS[t_{K+i}, m(t_{K+i})] + n(t_{K+i})\}\,|\,H_1\right).$$

$$(2\text{-}187)$$

Consider first the case for which K+i is an even integer. Multiplying out the right-hand side of Equation (2-187) and simplifying those terms involving n(t) as a factor, it follows that

$$E[r(t_1)\ldots r(t_{K+i})\,|\,H_1]$$

$$= A^{K+i}\, E\{S[t_1, m(t_1)]\ldots S[t_{K+i}, m(t_{K+i})]\}$$

$$+ A^{K+i-2}\,\frac{N_o}{2} \sum E[n(t_{i_1})n(t_{i_2})]$$

$$E\left\{\prod_{j=1}^{i+K} S[t_j, m(t_j)]\right\} \quad j\neq i_1, i_2$$

$$+ \ldots$$

$$+ \left(\frac{N_o}{2}\right)^{\frac{K+i}{2}} E[n(t_1)\ldots n(t_{K+i})] \qquad (2\text{-}188)$$

where the first sum is over all $\frac{K+i!}{2!\,(K+i-2)!}$ ways of choosing two arguments from K+i. Evaluation of the expectations involving the noise term, although known, are not required for the argument to follow. Similarly, for (i+K) odd

$$
\begin{aligned}
&E[\, r(t_1)\ldots r(t_{i+K}) \mid H_1] \\
&= A^{K+i}\; E\{\, S[\, t_1, m(t_1)] \ldots S[\, t_{i+K}, m(t_{i+K})]\} \\
&+ A^{K+i-2}\, \frac{N_o}{2} \sum E[\, n(t_{i_1}) n(t_{i_2})] \\
&\quad E\left\{ \prod_{j=1}^{i+K} S[\, t_j, m(t_j)] \right\} \quad j \neq i_1, i_2 \\
&+ \ldots \\
&+ A \left(\frac{N_o}{2} \right)^{\frac{K+i-1}{2}} \sum E\{\, S[\, t_{i_1}, m(t_{i_1})] \\
&\qquad E[\, n(t_1)\ldots n(t_{i_1-1}) n(t_{i_1+1})\ldots n(t_{K+i})]\}
\end{aligned}
\tag{2-189}
$$

Substituting Equation (2-184) into both Equation (2-188) and (2-189) results in

$$E[\, r(t_1)\ldots r(t_{K+i}) \mid H_1] \rightarrow E[\, r(t_1)\ldots r(t_{K+i}) \mid r(t) = AS(t)+n(t)] \tag{2-190}$$

Use of (2-190) in (2-186) leads to the conclusion that

$$E\left(\{ \ell'_\infty [\, r(t)] \}^K \mid H_1 \right) \rightarrow E\left(\{ \ell'_\infty [\, r(t)] \}^K \mid r(t) = AS(t) + n(t) \right) \tag{2-191}$$

Since the probability density function of $\ell'_\infty [\, r(t)]$ given H_1 is determined by the moments of $\ell'_\infty [\, r(t)]$ given H_1, it follows that

$$P_{\ell'_\infty \mid H_1}(X \mid H_1) \rightarrow P_{\ell'_\infty \mid r(t)=AS(t)+n(t)} [\, X \mid r(t)=AS(t)+n(t)] \tag{2-192}$$

and that the detection probability

$$P_D = \int_{\gamma}^{\infty} P_{\ell_\infty' | H_1}(X | H_1)\, dX \qquad (2\text{-}193)$$

may be evaluated by considering

$$P_D \rightarrow \int_{\gamma}^{\infty} P_{\ell_\infty' | r(t)=AS(t)+n(t)}[X | r(t)=AS(t)+n(t)]\, dX \qquad (2\text{-}194)$$

for $T \rightarrow \infty$. Therefore, the asymptotic receiver operating characteristic (R.O.C) of the asymptotic receiver is evaluated considering an equivalent problem where, under H_1, $r(t) = AS(t)+n(t)$, $S(t)$ is a Guassian process with

$$E[S(t)] = E\{S[t, m(t)]\}, \qquad (2\text{-}195)$$

and

$$E[S(t_1)S(t_2)] = E\{S[t_1, m(t_1)]\, S[t_2, m(t_2)]\}. \qquad (2\text{-}196)$$

When Equation (2-177) is the low energy coherence (L. E. C.) receiver (see Appendix C) for $S(t)$, the asymptotic performance is easily obtained using the Chernoff approximation of Appendix C.

Assuming that T is large enough so that the asymptotic receiver is a good approximation to the optimum Neyman-Pearson receiver, performance of the optimum receiver will be closely approximated by the asymptotic performance of the asymptotic receiver. It has not been possible to obtain general conditions which assure that T is sufficiently large.

2.4.3 An Example - First Order Butterworth Phase Modulation of a Sinusoid

For this example, let the hypotheses be given by

$$H_1: \quad r(t) = A \cos [\omega_o t + m(t)] + n(t), \quad 0 \le t \le T \tag{2-197}$$

$$H_0: \quad r(t) = n(t) \qquad 0 \le t \le T \tag{2-198}$$

where m(t) and n(t) are statistically independent Gaussian random processes with the properties

$$E[m(t)] = E[n(t)] = 0 \tag{2-199}$$

$$E[m(t_1)m(t_2)] = Pe^{-\alpha |t_1 - t_2|} \tag{2-200}$$

$$E[n(t_1)n(t_2)] = \frac{N_o}{2} \delta(t_1 - t_2) \quad . \tag{2-201}$$

From the correlation function assumed in Equation (2-200), the spectrum of m(t) is

$$S_m(\omega) = \frac{2\alpha P}{\alpha^2 + \omega^2} \quad . \tag{2-202}$$

From par. 2.1, the likelihood ratio is

$$\Lambda [r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) f_i(t_1, \ldots, t_i) \, dt_1 \ldots \, dt_i \tag{2-203}$$

where

$$f_i(t_1, \ldots, t_i) = E \Bigg(\cos [\omega_o t_1 + m(t_1)] \ldots \cos [\omega_o t_i + m(t_i)]$$

$$\exp \left\{ - \frac{A^2}{N_o} \int_o^T \cos^2 [\omega_o t + m(t)] \, dt \right\} \Bigg) . \tag{2-204}$$

Expanding the exponential in Equation (2-204), $f_i(t_1, \ldots, t_i)$ becomes

$$f_i(t_1, \ldots, t_i) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{A^2}{N_o}\right)^j \int_o^T \overset{j}{\cdots} \int_o^T E\left\{\cos[\omega_o t_1 + m(t_1)] \ldots \cos[\omega_o t_i + m(t_i)]\right.$$

$$\left. \cos^2[\omega_o t_{i+1} + m(t_{i+1})] \ldots \cos^2[\omega_o t_{i+j} + m(t_{i+j})]\right\}$$

$$dt_{i+1} \ldots dt_{i+j} \ . \tag{2-205}$$

Recalling that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ it follows that

$$E\left\{\cos[\omega_o t_1 + m(t_1)] \ldots \cos[\omega_o t_i + m(t_i)] \cos^2[\omega_o t_{i+1} + m(t_{i+1})] \ldots\right.$$

$$\left.\cos^2[\omega_o t_{i+j} + m(t_{i+j})]\right\}$$

$$= \frac{1}{2^j} E\left\{\cos[\omega_o t_1 + m(t_1)] \ldots \cos[\omega_o t_i + m(t_i)]\right\}$$

$$+ \text{ Terms involving } 2\omega_o t_K ; K = i+1, \ldots \ . \tag{2-206}$$

Hence, the integrand in Equation (2-205) involves a constant with respect to the variables of integration plus rapidly oscillating terms which contribute negligibly to the integration. Consequently, Equation (2-205) can be written as

$$f_i(t_1, \ldots, t_i) \simeq e^{-\frac{A^2 T}{2N_o}} E\left\{\cos[\omega_o t_1 + m(t_1)] \ldots \cos[\omega_o t_i + m(t_i)]\right\} \ . \tag{2-207}$$

Comparison of Equation (2-207) with Equation (2-130) shows that $K = e^{-A^2T/2N_o}$. In Appendix A it is shown that x_i and y_i defined by Equations (2-136) and (2-137) are asymptotically Gaussian when $S(t, m(t)) = \cos(\omega_o t + m(t))$. Hence from Equation (2-177)

$$\ell'_\infty(r(t)) = \frac{1}{2}\left(\frac{2A}{N_o}\right)^2 \int_o^T\int_o^T [d_2(t_1,t_2) - d_1(t_1)d_1(t_2)]\, r(t_1)r(t_2)\, dt_1\, dt_2$$

$$+\left(\frac{2A}{N_o}\right)\int_o^T d_1(t_1)r(t_1)\, dt_1 \qquad (2\text{-}208)$$

where, from Appendix B,

$$d_2(t_1,t_2) = E\left\{\cos[\omega_o t_1 + m(t_1)]\cos[\omega_o t_2 + m(t_2)]\right\}$$

$$= e^{-P}\Big[\cosh(Pe^{-\alpha|t_1-t_2|})\cos\omega_o t_1 \cos\omega_o t_2$$

$$+ \sinh(Pe^{-\alpha|t_1-t_2|})\sin\omega_o t_1 \sin\omega_o t_2\Big] \qquad (2\text{-}209)$$

and

$$d_1(t_1) = E\left\{\cos[\omega_o t_1 + m(t_1)]\right\} = e^{-P/2}\cos\omega_o t_1 \quad . \qquad (2\text{-}210)$$

In Appendix C par. C.2 it is shown that Equation (2-208) can be interpreted as the LEC receiver for a Gaussian process with mean and autocorrelation given by Equations (2-210) and (2-209) respectively. In Appendix D par. D-2 it is shown that the LEC condition applies for this process over a wide range of values for P and αT. As discussed in par. 2.4.2, the asymptotic performance of the asymptotic receiver is evaluated by considering an equivalent problem where, under H_1, $r(t) = AS(t)+n(t)$ where $S(t)$ is a Gaussian process with mean and auto-covariance given by Equations (2-210) and (2-209) respectively. As pointed out in Appendix C (see C-23), the LEC receiver for the equivalent problem is determined from Equation (2-208). Using only values of P and αT such that the LEC condition applies, performance is obtained by employing the Chernoff approximation discussed in Appendix C. From Appendix C

$$P_{FA} \simeq e^{\mu(S)-S\dot{\mu}(S)} \left[e^{S^2\ddot{\mu}(S/2)} \operatorname{erfc}_*\left(S\sqrt{\ddot{\mu}(S)}\right) \left(1 - \frac{\dddot{\mu}(S)S^3}{6}\right) - \frac{\dddot{\mu}(S)\left[1-S^2\ddot{\mu}(S)\right]}{6[\ddot{\mu}(S)]^{3/2}\sqrt{2\pi}} \right] \tag{2-211}$$

$$P_D \simeq 1 - e^{\mu(S)+(1-S)\dot{\mu}(S)} \left\{ e^{(1-S)^2\ddot{\mu}(S)/2} \operatorname{erfc}_*\left[(1-S)\sqrt{\ddot{\mu}(S)}\right]\left[1 - \frac{\dddot{\mu}(S)(1-S)^3}{6}\right] - \frac{\dddot{\mu}(S)\left[1-\ddot{\mu}(S)(1-S)^2\right]}{6\sqrt{2\pi}\,(\ddot{\mu}(S))^{3/2}} \right\} \tag{2-212}$$

where S is chosen so that $\dot{\mu}(S) = \gamma$, $0 \le S \le 1$, and γ is the threshold. Also, from Appendix C

$$\mu(S) = \frac{-S(1-S)}{2} \left(\left(\frac{A^2}{N_o\alpha}\right)^2 e^{-2P} \left\{ \frac{1}{2}\sum_{i=1}^{\infty} \frac{(2P)^{2i}}{2i!}\left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2}\right] - (2-S)\sum_{i=1}^{\infty}\frac{P^{2i}}{2i!}\left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2}\right] \right\} + \left(\frac{A^2}{N_o\alpha}\right) e^{-P}\alpha T \right) . \tag{2-213}$$

Note that $\left(\frac{A^2}{N_o \alpha}\right)$ can be interpreted as a signal-to-noise ratio (SNR). Also, $\sqrt{P}$ can be interpreted as the modulation index. Figures 2-1 thru 2-11 show the receiver operating characteristics (ROC) for a wide range of SNR, $\sqrt{P}$ and αT. Note that decreasing SNR by 10 dB, say from -10 dB to -20 dB, and increasing αT by a factor of 100, say from 10^5 to 10^7 results in identical ROC's at the higher SNR's. At lower SNR's, decreasing SNR by 10 dB, say from -80 to -90 dB, and increasing αT by a factor of 10, say from 10^{18} to 10^{19}, also results in ROC's which are close. This suggests a threshold effect in receiver performance. Also, note that increasing T for α, A, $\frac{N_o}{2}$, P, and P_{FA} given, results in a monotone decreasing miss probability, P_M. In addition, increasing P with A, $\frac{N_o}{2}$, α, T, and P_{FA} given, results in an increasing P_M.

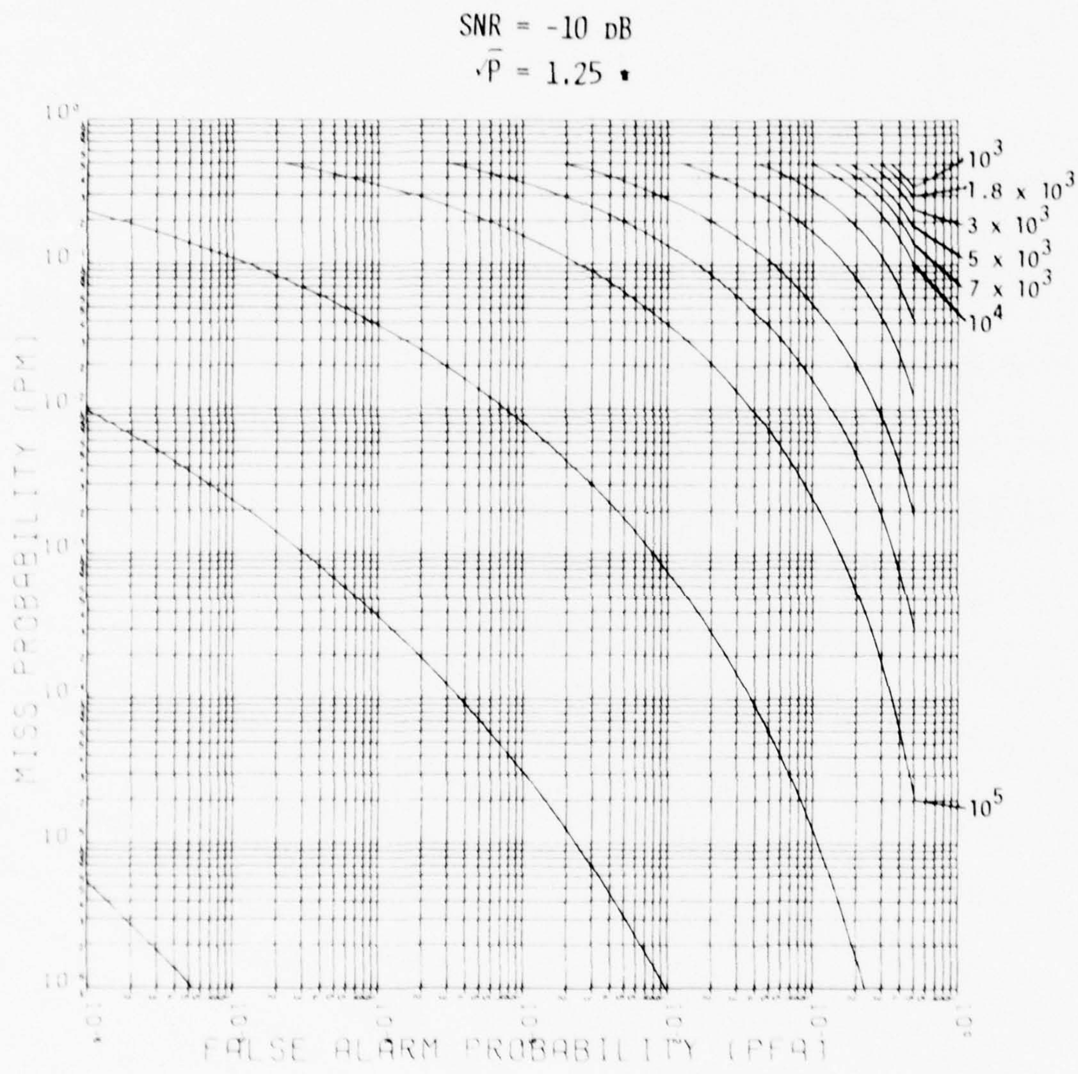

Figure 2-1. Miss Probability ($P_M = 1 - P_D$) vs False Alarm Probability (P_{FA}) Parametric in Time-Bandwidth Product (αT) for a Signal-to Noise Ratio ($SNR = \frac{A^2}{N_o \alpha}$) of -10 dB Modulation Index ($\sqrt{P}$) of 1.25 π.

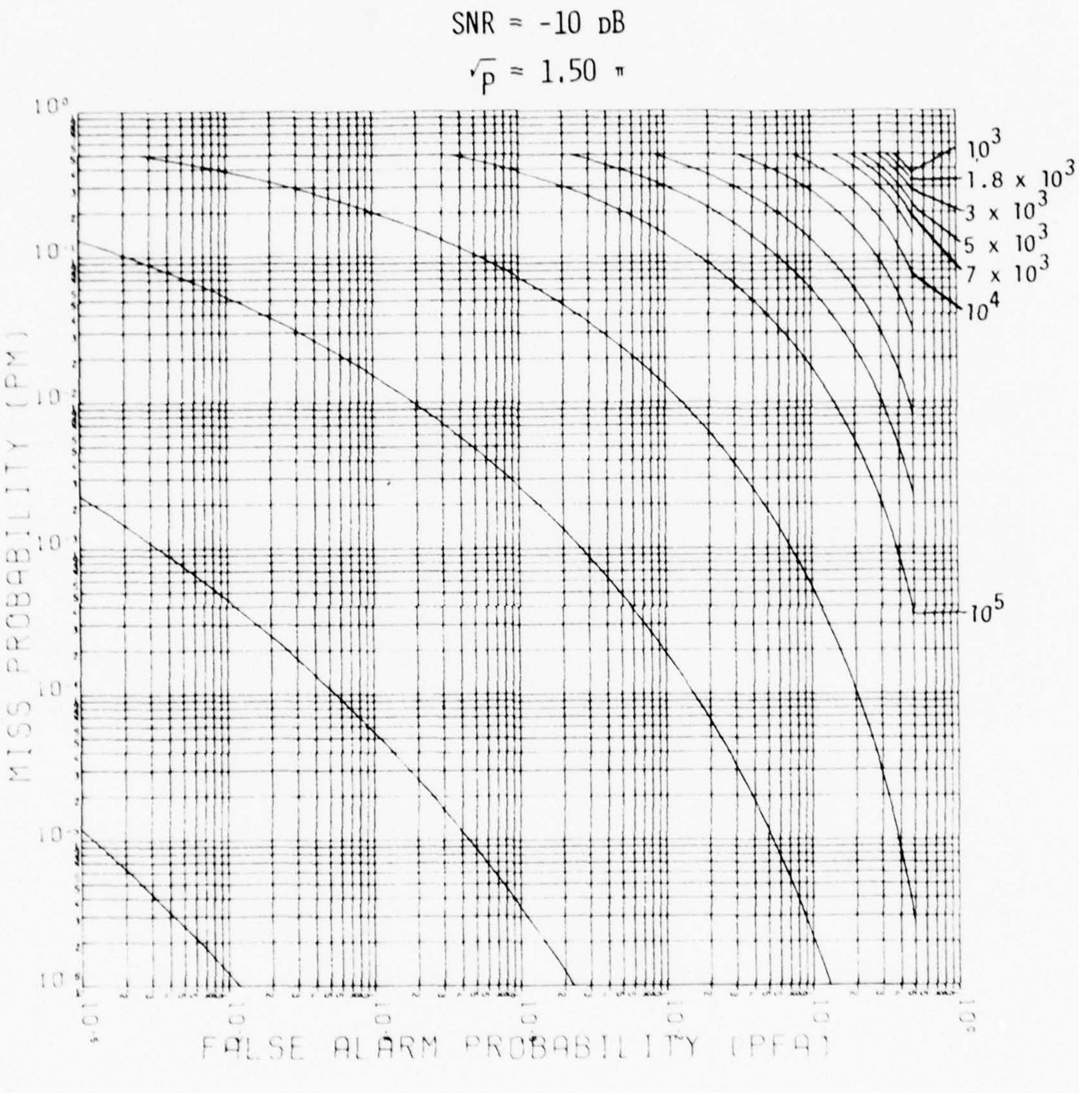

Figure 2-2. P_M vs P_{FA} Parametric in αT for SNR = -10dB and $\sqrt{P}$=1.50 π.

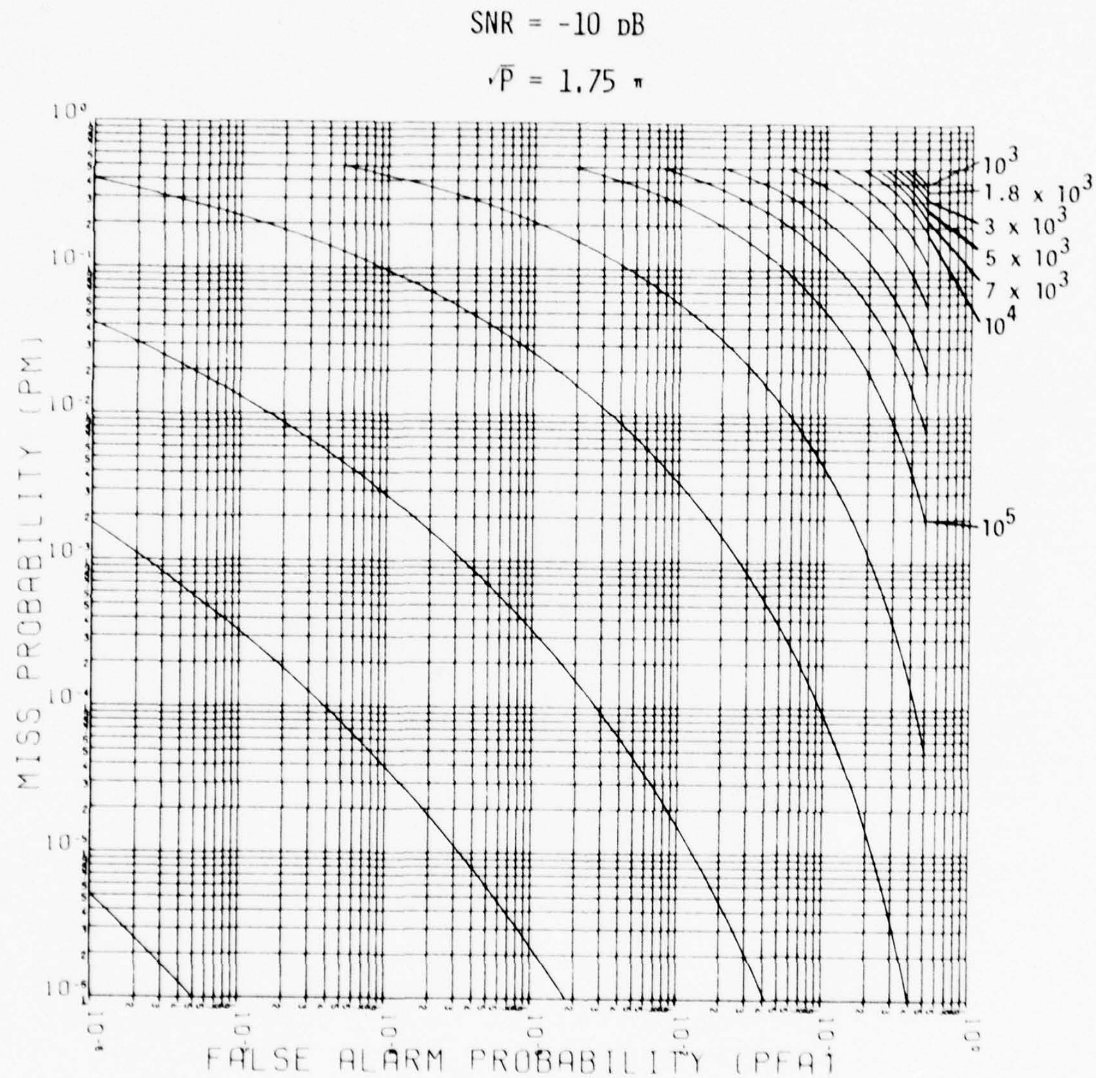

Figure 2-3. P_M vs P_{FA} Parametric in αT for SNR = -10dB and $\sqrt{P}$ = 1.75π.

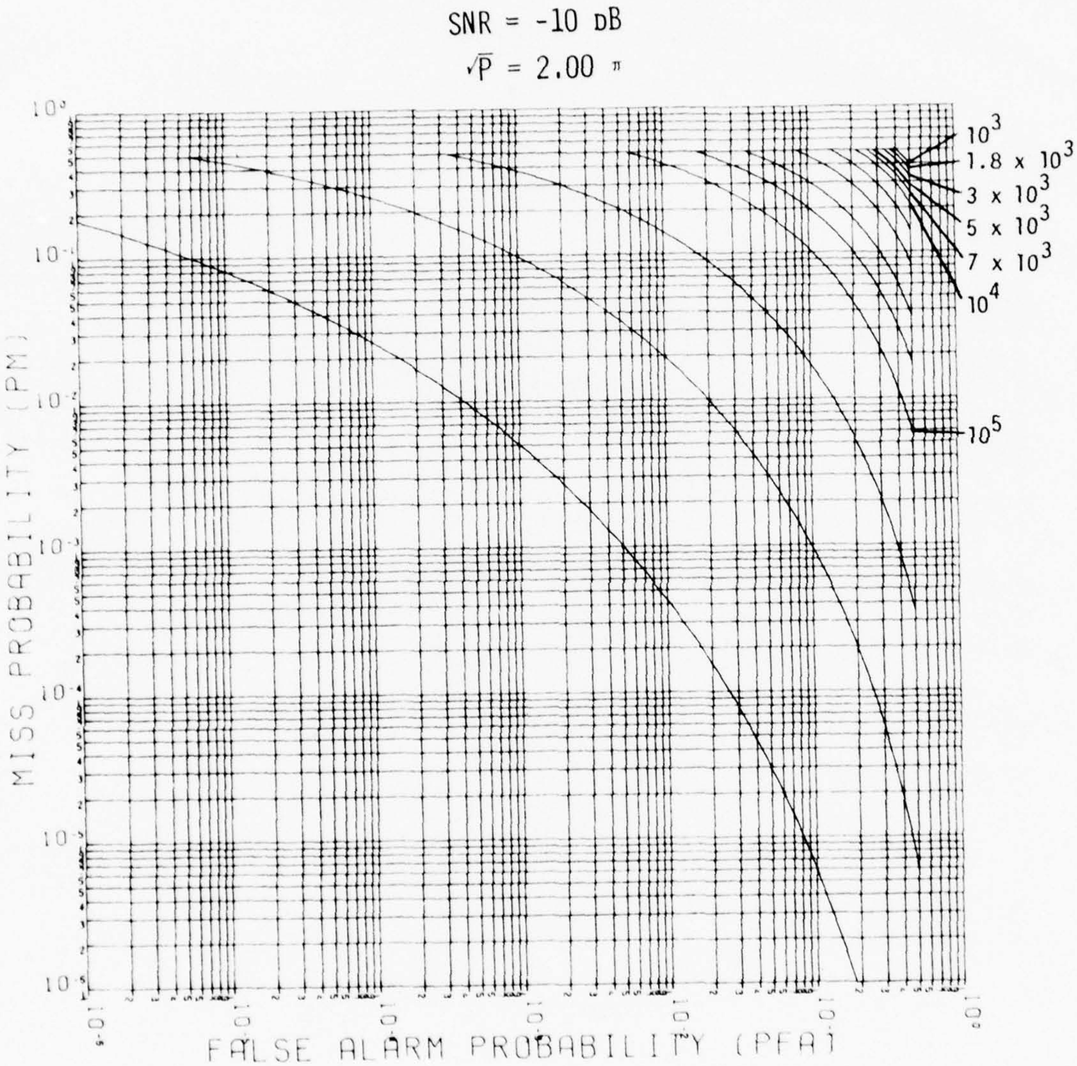

Figure 2-4. P_M vs P_{FA} Parametric in αT for SNR = -10dB and $\sqrt{P}$ = 2.00 π.

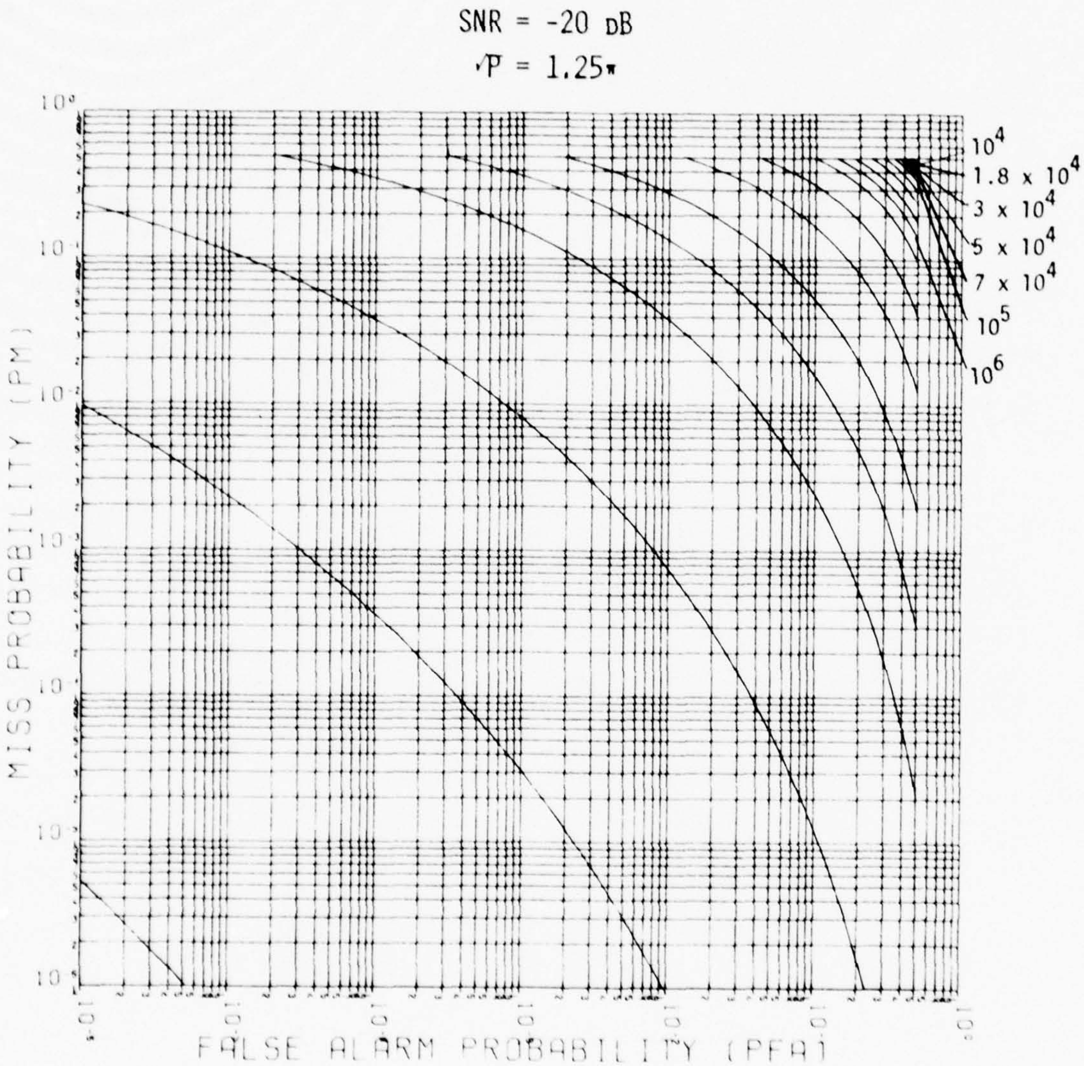

Figure 2-5. P_M vs P_{FA} Parametric in αT for SNR = -20dB and $\sqrt{P}$ = 1.25 π.

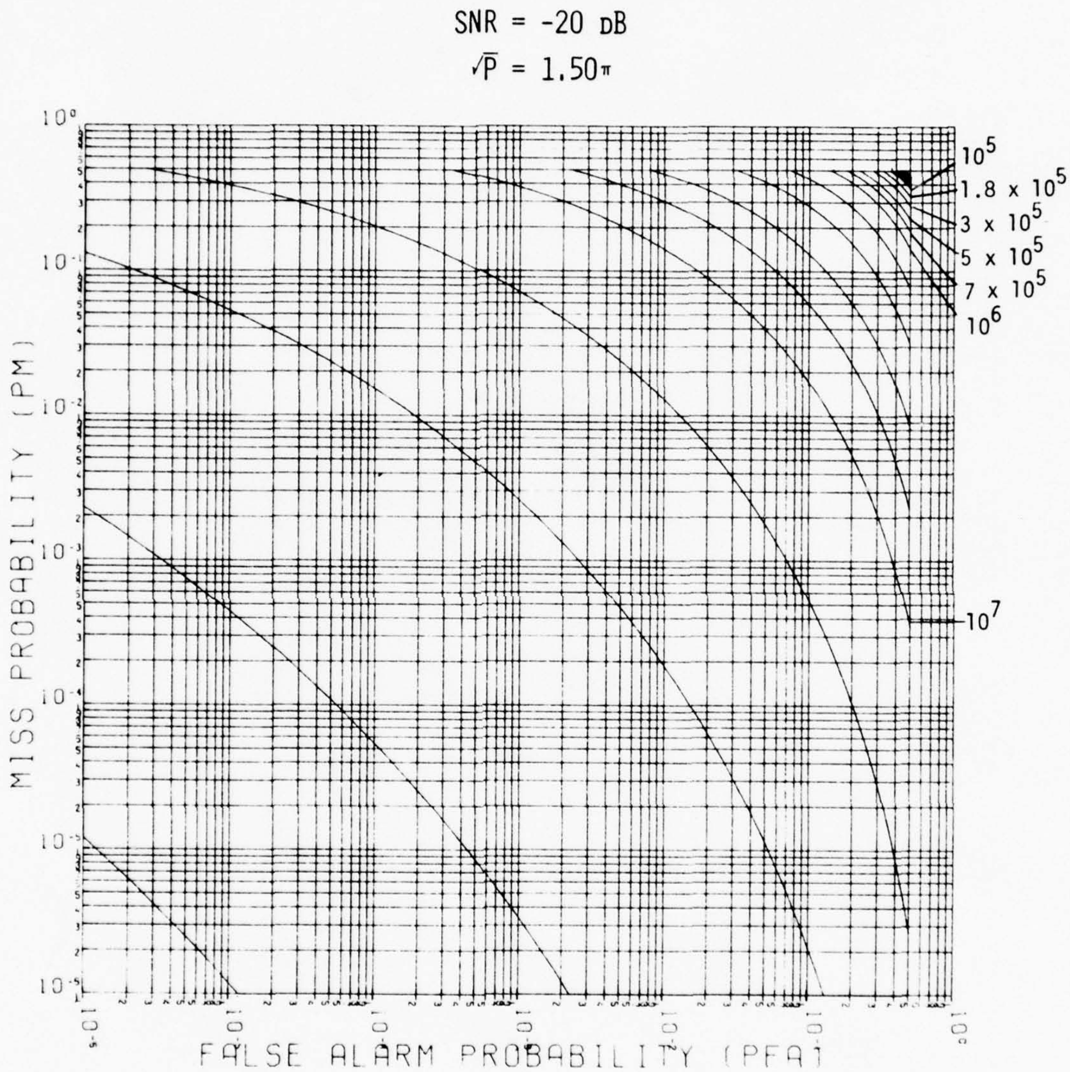

Figure 2-6. P_M vs P_{FA} Parametric in αT for SNR = -20dB and $\sqrt{P}$=1.5π.

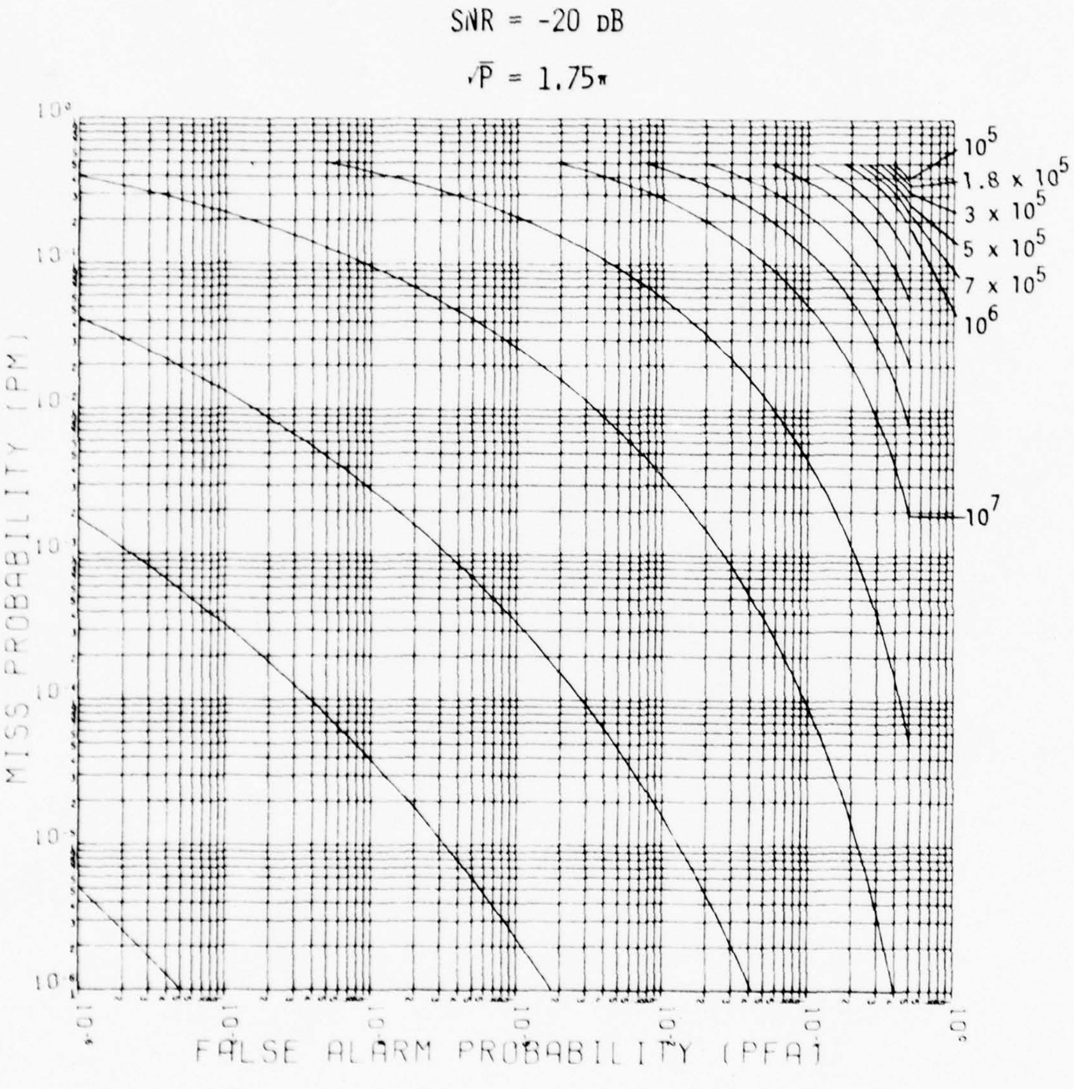

Figure 2-7. P_M vs P_{FA} Parametric in αT for SNR = -20dB and $\sqrt{P}$=1.75π.

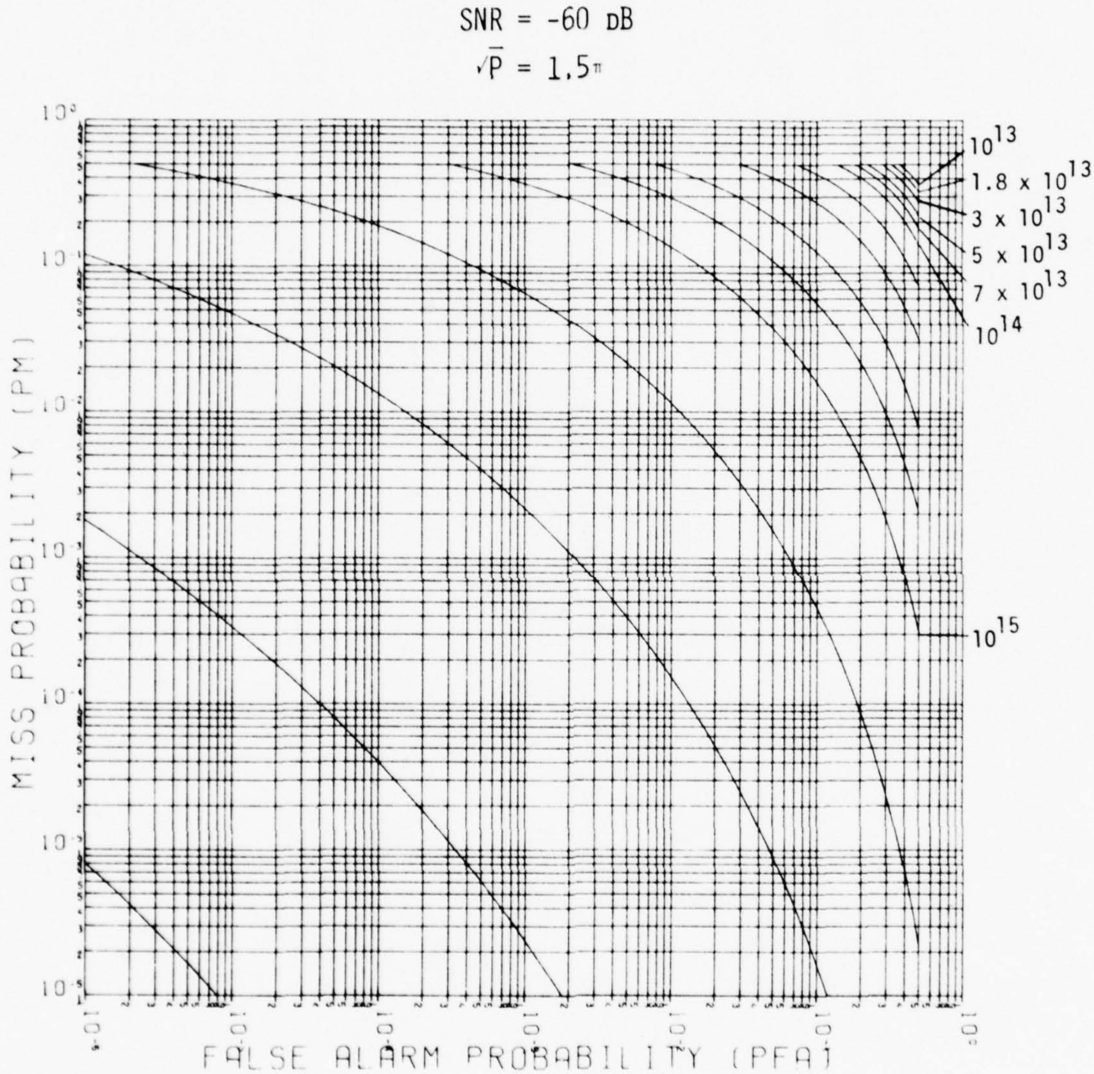

Figure 2-8. P_M vs P_{FA} Parametric in αT for SNR = -60dB and $\sqrt{P}$=1.5 π.

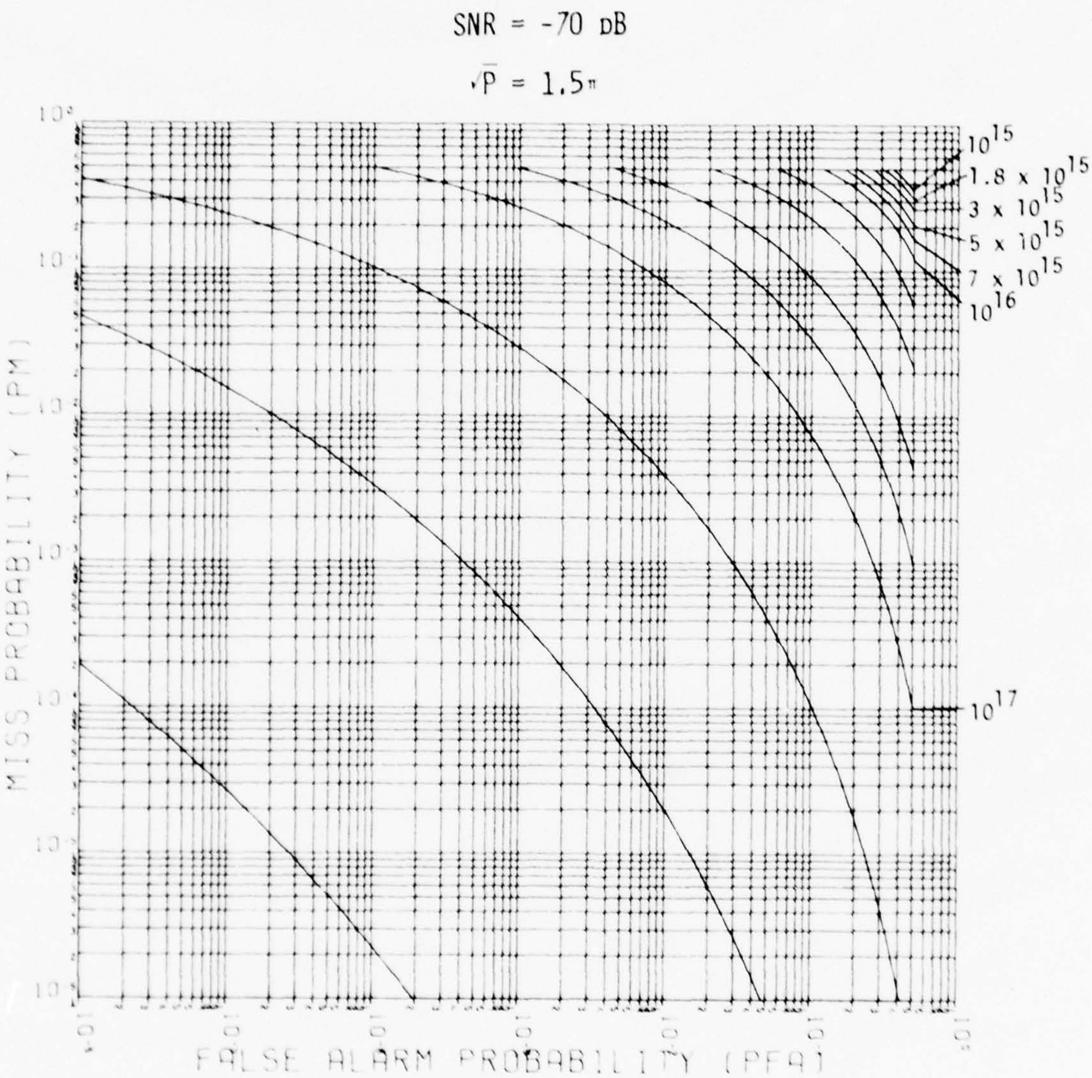

Figure 2-9. P_M vs P_{FA} Parametric in αT for SNR = -70dB and $\sqrt{P}$ = 1.5 π.

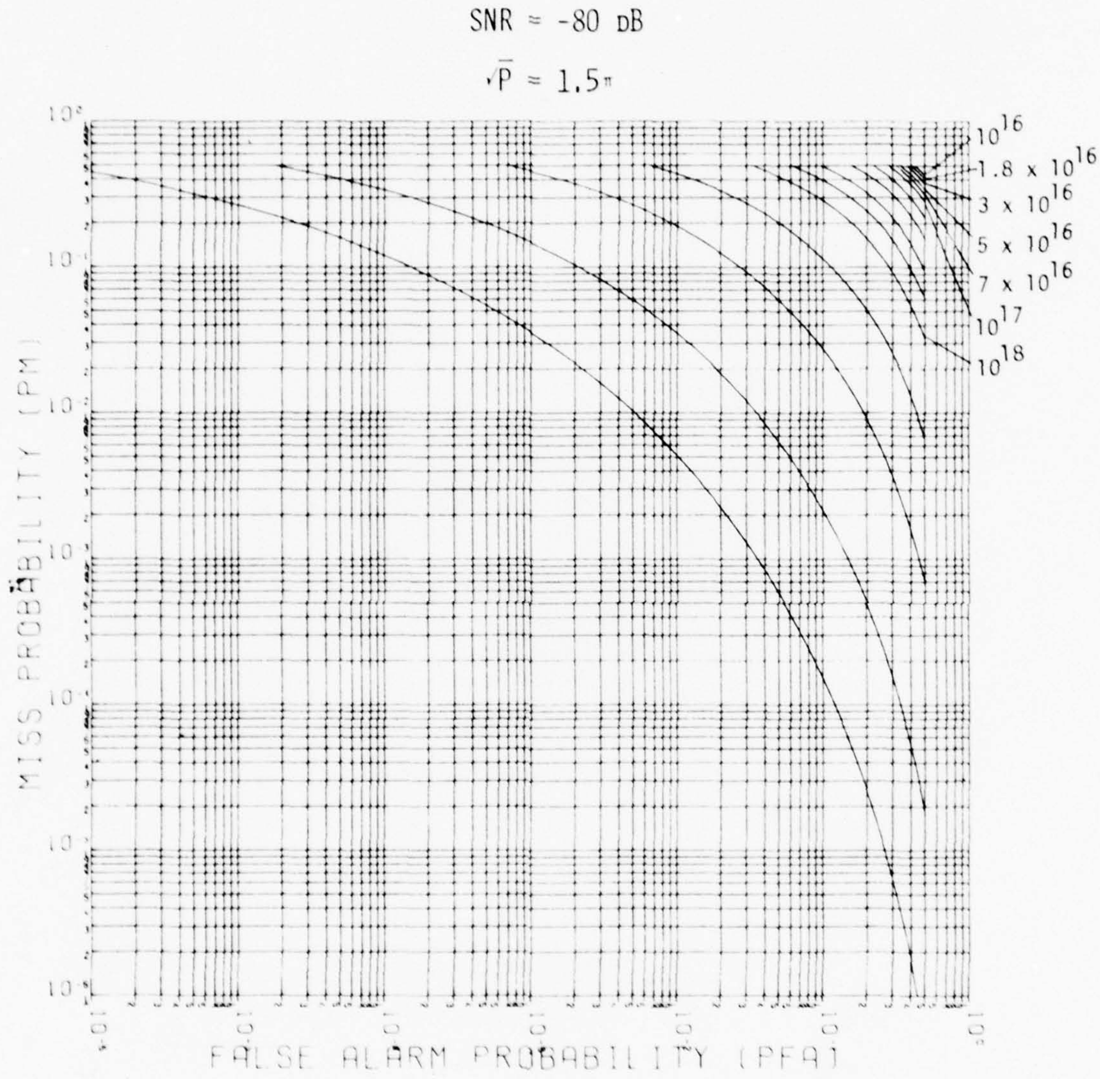

Figure 2-10. P_M vs P_{FA} Parametric in αT for SNR= -80dB and $\sqrt{P}$=1.5 π.

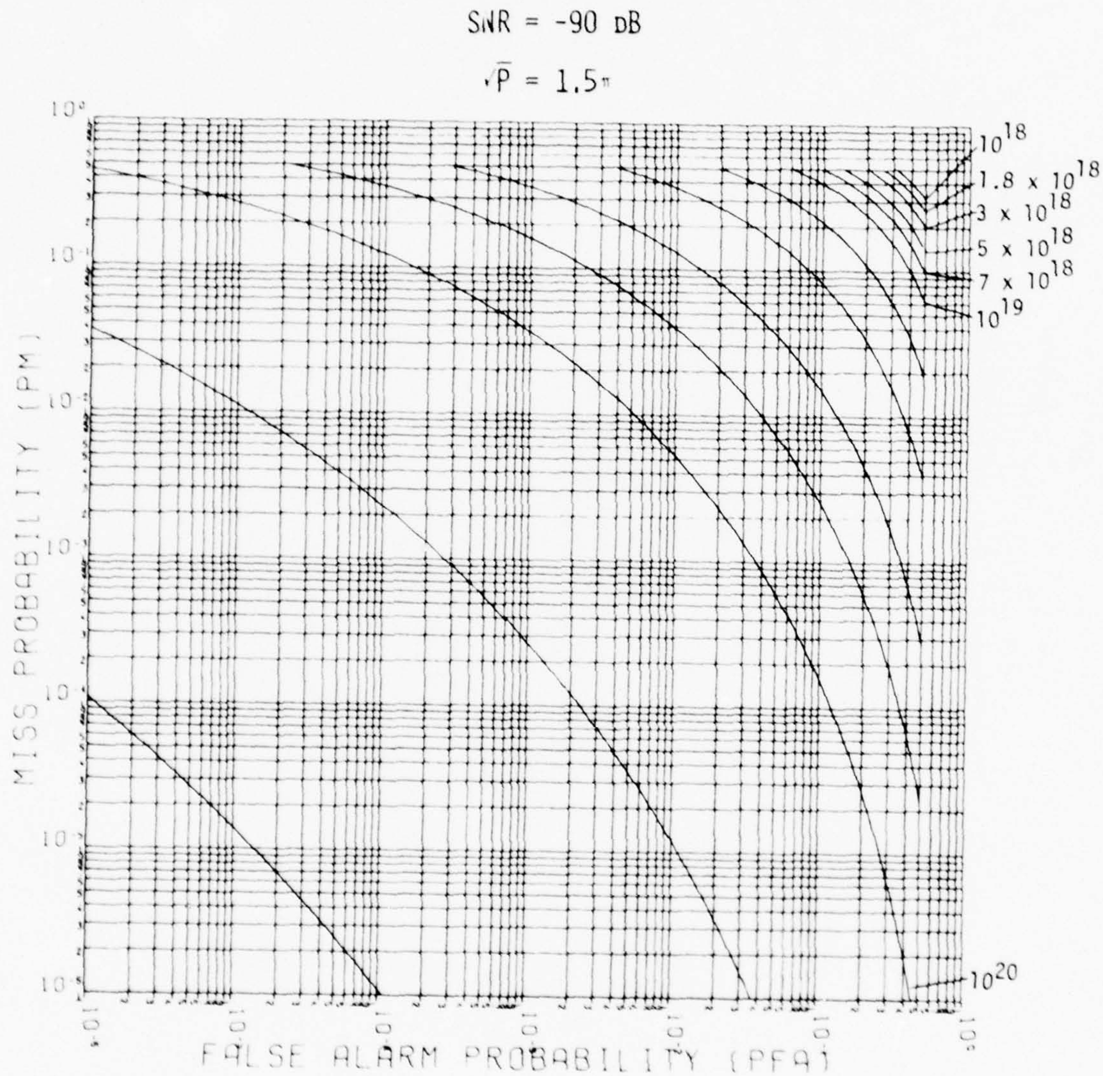

Figure 2-11. P_M vs P_{FA} Parametric in αT for SNR = -90dB and $\sqrt{P}$=1.5 π.

2.4.4 Suboptimum Detection of a First Order Butterworth Phase Modulated Sinusoid

In the hypothesis testing problem of par. 2.4.3, the hypotheses are

$$H_1: \; r(t) = A \cos [\omega_o t + m(t)] + n(t) , \qquad 0 \le t \le T \tag{2-214}$$

$$H_0: \; r(t) = n(t) , \qquad 0 \le t \le T \tag{2-215}$$

where m(t) and n(t) are zero mean independent Gaussian processes with autocorrelation functions

$$E[m(t_1)\, m(t_2)] = R_m(t_1, t_2) = P\, e^{-\alpha (t_1 - t_2)} \tag{2-216}$$

$$E[n(t_1)\, n(t_2)] = \frac{N_o}{2} \delta (t_1 - t_2) \quad . \tag{2-217}$$

Consider the detector sketched in Figure 2-12 which is optimum for detection of a sinusoid having a constant random phase angle uniformly distributed on $[0, 2\pi]$. In this section, the performance of this detector is determined for the hypotheses of Equations (2-214) and (2-215). As expected, the optimum detector is shown to outperform the detector of Figure 2-12. However, for signal to noise ratios (SNR's) below the threshold of the optimum detector, suboptimum detector performance is close to the asymptotic performance of the asymptotic receiver.

It is convenient to define

$$I = \frac{2A}{N_o} e^{-P/2} \int_0^T r(t) \cos \omega_o t \, dt \tag{2-218}$$

and

$$Q = \frac{2A}{N_o} e^{-P/2} \int_0^T r(t) \sin \omega_o t \, dt \quad . \tag{2-219}$$

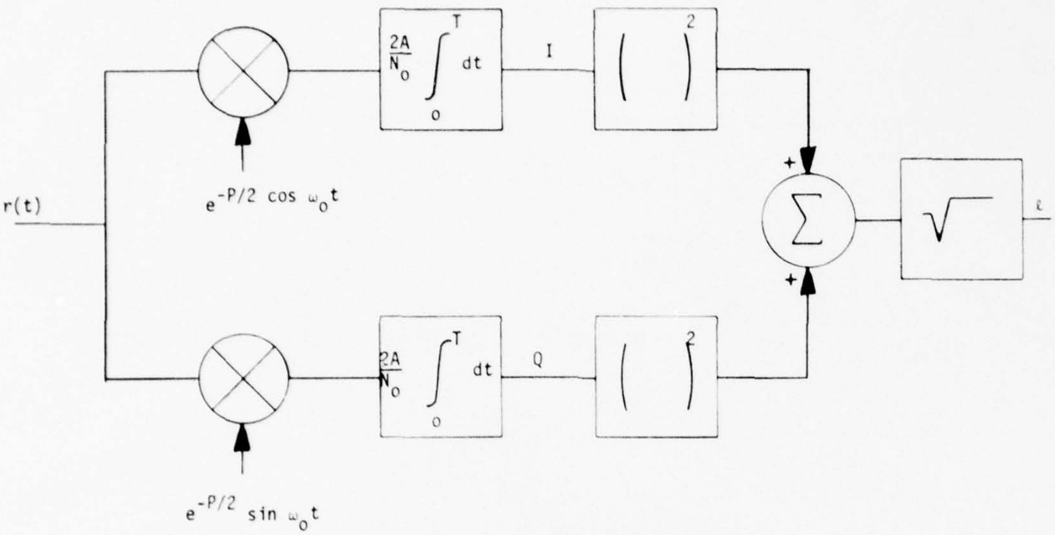

Figure 2-12. A Suboptimal Detector

Note that under hypothesis H_0, I and Q are zero mean statistically independent Gaussian random variables of variance

$$\sigma^2_{I|H_0} = \sigma^2_{Q|H_0} = \frac{2A^2}{N_o^{\alpha}} \cdot \frac{e^{-P}}{2} \cdot \alpha T \; . \tag{2-220}$$

From Appendix A, it follows that under hypothesis H_1, I and Q are asymptotically jointly Gaussian. From Equation (2-218)

$$E(I|H_1) = \frac{2A}{N_o} e^{-P/2} \int_0^T E\{A \cos [\omega_o t + m(t)]\} + n(t) \cos \omega_o t \, dt \; . \tag{2-221}$$

With reference to Equations (2-221) and (B-6), the mean of I under hypothesis H_1 is given by

$$\eta_{I|H_1} = E(I|H_1) \cong \frac{2A^2}{N_o^{\alpha}} \cdot \frac{e^{-P}}{2} \cdot \alpha T \tag{2-222}$$

where double frequency terms have been ignored. Similarly, the mean of Q under hypothesis H_1 is given by

$$\eta_{Q|H_1} = E(Q|H_1) \cong 0 \; . \qquad (2\text{-}223)$$

The second moment of I under hypothesis H_1 can be expressed as

$$E(I^2|H_1) = \left(\frac{2A^2}{N_o}\right)^2 e^{-P} \int_0^T \int_0^T \cos\omega_o t_1 \cos\omega_o t_2$$

$$E\Big(\{A\cos[\omega_o t_1 + m(t_1)] + n(t_1)\}\{A\cos[\omega_o t_2 + m(t_2)]$$

$$+ n(t_2)\}\Big)\, dt_1\, dt_2 \; . \qquad (2\text{-}224)$$

From Equations (B-23), (2-216), (2-217) and the statistical independence of m(t) and n(t), it follows that

$$E\Big(\{A\cos[\omega_o t_1 + m(t_1)] + n(t_1)\}\{A\cos[\omega_o t_2 + m(t_2)] + n(t_2)\}\Big)$$

$$= A^2 e^{-P}\Big[\cosh\left(P e^{-\alpha|t_1 - t_2|}\right)\cos\omega_o t_1 \cos\omega_o t_2$$

$$+ \sinh\left(P e^{-\alpha|t_1 - t_2|}\right)\sin\omega_o t_1 \sin\omega_o t_2\Big]$$

$$+ \frac{N_o}{2}\,\delta(t_1 - t_2) \; . \qquad (2\text{-}225)$$

Substituting Equation (2-225) into Equation (2-224) yields

$$E(I^2 | H_1) \cong \left(\frac{2A^2T}{N_o}\right)^2 \cdot \frac{e^{-2P}}{4} \cdot \frac{1}{T^2} \int_0^T \int_0^T \cosh\left(P\, e^{-\alpha|t_1 - t_2|}\right) dt_1\, dt_2$$

$$+ \frac{2A^2}{N_o \alpha} \cdot \frac{e^{-P}}{2} \cdot \alpha T \qquad (2\text{-}226)$$

where double frequency terms have been ignored. With the aid of Equations (C-69) and (2-226), it follows that the second moment of I given hypothesis H_1 can be written as

$$E(I^2 | H_1) \cong \left(\frac{A^2}{N_o \alpha}\right)^2 \cdot e^{-2P} \cdot \left\{ (\alpha T)^2 + \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1 - e^{-2i\alpha T})}{2i^2} \right] \right\}$$

$$+ \frac{A^2}{N_o \alpha} \cdot e^{-P} \cdot \alpha T \ . \qquad (2\text{-}227)$$

From Equations (2-227) and (2-222) it follows that the variance of I given H_1 is

$$\sigma^2_{I|H_1} \cong \left(\frac{A^2}{N_o \alpha}\right)^2 \cdot e^{-2P} \cdot \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1 - e^{-2i\alpha T})}{2i^2} \right]$$

$$+ \frac{A^2}{N_o \alpha} \cdot e^{-P} \cdot \alpha T \ . \qquad (2\text{-}228)$$

Similarly, it is readily shown that

$$\sigma^2_{Q|H_1} \cong \left(\frac{A^2}{N_o \alpha}\right)^2 \cdot e^{-2P} \cdot \sum_{i=0}^{\infty} \frac{P^{2i+1}}{(2i+1)!} \left[\frac{2\alpha T}{2i+1} - \frac{2(1 - e^{-(2i+1)\alpha T})}{(2i+1)^2} \right]$$

$$+ \frac{A^2}{N_o \alpha} \cdot e^{-P} \cdot \alpha T \qquad (2\text{-}229)$$

and

$$E(IQ | H_1) \cong 0 \ . \qquad (2\text{-}230)$$

The suboptimum receiver of Figure 2-12 forms the detection statistic

$$\ell = \sqrt{I^2 + Q^2} \qquad (2\text{-}231)$$

From Papoulis [3], the p.d.f. of ℓ given hypothesis H_0 is given by

$$P_{\ell | H_0}(L|H_0) = \frac{L}{\sigma_o^2} e^{-L^2/2\sigma_o^2}, \qquad L \geq 0$$

$$= 0 , \qquad L < 0 \qquad (2\text{-}232)$$

where

$$\sigma_o^2 = \sigma_{I|H_0}^2 = \sigma_{Q|H_0}^2 \quad . \qquad (2\text{-}233)$$

From Miller [25, p. 30], the asymptotic p.d.f. of ℓ given hypothesis H_1 follows as

$$P_{\ell | H_1}(L|H_1) = \frac{L}{\sigma_{I|H_1} \sigma_{Q|H_1}} \exp\left[- \frac{(L^2 + \eta_{I|H_1}^2)}{2\sigma_{I|H_1}} \right]$$

$$\cdot \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(1/2)\,\Gamma(m+1)}$$

$$\cdot \left[\frac{L(\sigma_{Q|H_1}^2 - \sigma_{I|H_1}^2)}{\eta_{I|H_1} \sigma_{Q|H_1}^2} \right]^m I_m \left(\frac{L\eta_{I|H_1}}{\sigma_{I|H_1}^2} \right)$$

$$, \quad L \geq 0$$

$$= 0 \qquad , \quad L < 0 \; . \qquad (2\text{-}234)$$

The false alarm probability, P_{FA}, is

$$P_{FA} = \int_{\gamma}^{\infty} P_{\ell | H_0}(L | H_0) \, dL \tag{2-235}$$

where γ is the threshold setting. Substituting Equation (2-232) into Equation (2-235) and performing the integration results in

$$P_{FA} = \exp\left(- \frac{\gamma^2}{2\sigma_o^2}\right) \quad . \tag{2-236}$$

The detection probability, P_D, is given by

$$P_D = \int_{\gamma}^{\infty} P_{\ell | H_1}(L | H_1) \, dL \quad . \tag{2-237}$$

It is convenient to define

$$a = \eta_{I | H_1} / \sigma_{I | H_1} \quad , \tag{2-238}$$

$$b = \gamma / \sigma_{I | H_1} \quad , \tag{2-239}$$

$$c = \sigma_{I | H_1} / \sigma_{Q | H_1} \quad , \tag{2-240}$$

$$d = (\sigma^2_{Q | H_1} - \sigma^2_{I | H_1}) / \sigma^2_{Q | H_1} \quad . \tag{2-241}$$

Substituting Equation (2-234) into Equation (2-237) and making use of Equations (2-238) - (2-241) there results

$$P_D = c \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(1/2)\, \Gamma(m+1)} d^m Q_{m+1}(a, b) \tag{2-242}$$

where $Q_K(a, b)$ is the generalized Q function

$$Q_K(a, b) = \int_b^\infty X \left(\frac{X}{a}\right)^{K-1} e^{-(X^2+a^2)/2} I_{K-1}(aX)\, dX \ . \tag{2-243}$$

From Equations (2-236) and (2-239), the parameter b can be written as

$$b = \frac{\left[-2\sigma_o^2 \ln P_{FA}\right]^{1/2}}{\sigma_{I|H_1}} \ . \tag{2-244}$$

Consequently, Equation (2-242) expresses P_D as a function of P_{FA}, $(A^2/N_o\alpha)$, P, and αT. Making use of Equations (2-220), (2-222), (2-228), (2-229) and (2-238) - (2-244), miss probability, $(P_M = 1 - P_D)$, is plotted in Figures 2-13 through 2-17 as a function of the time-bandwidth product, αT, for various signal to noise ratios, $A^2/N_o\alpha$, and the fixed false alarm probability, $P_{FA} = 10^{-3}$. Suboptimal performance is compared with the asymptotic performance of the asymptotic receiver in Figures 2-18 through 2-21 for parameters of interest. Note that suboptimum performance approaches the asymptotic receiver performance as the signal to noise ratio, $A^2/N_o\alpha$, decreases.

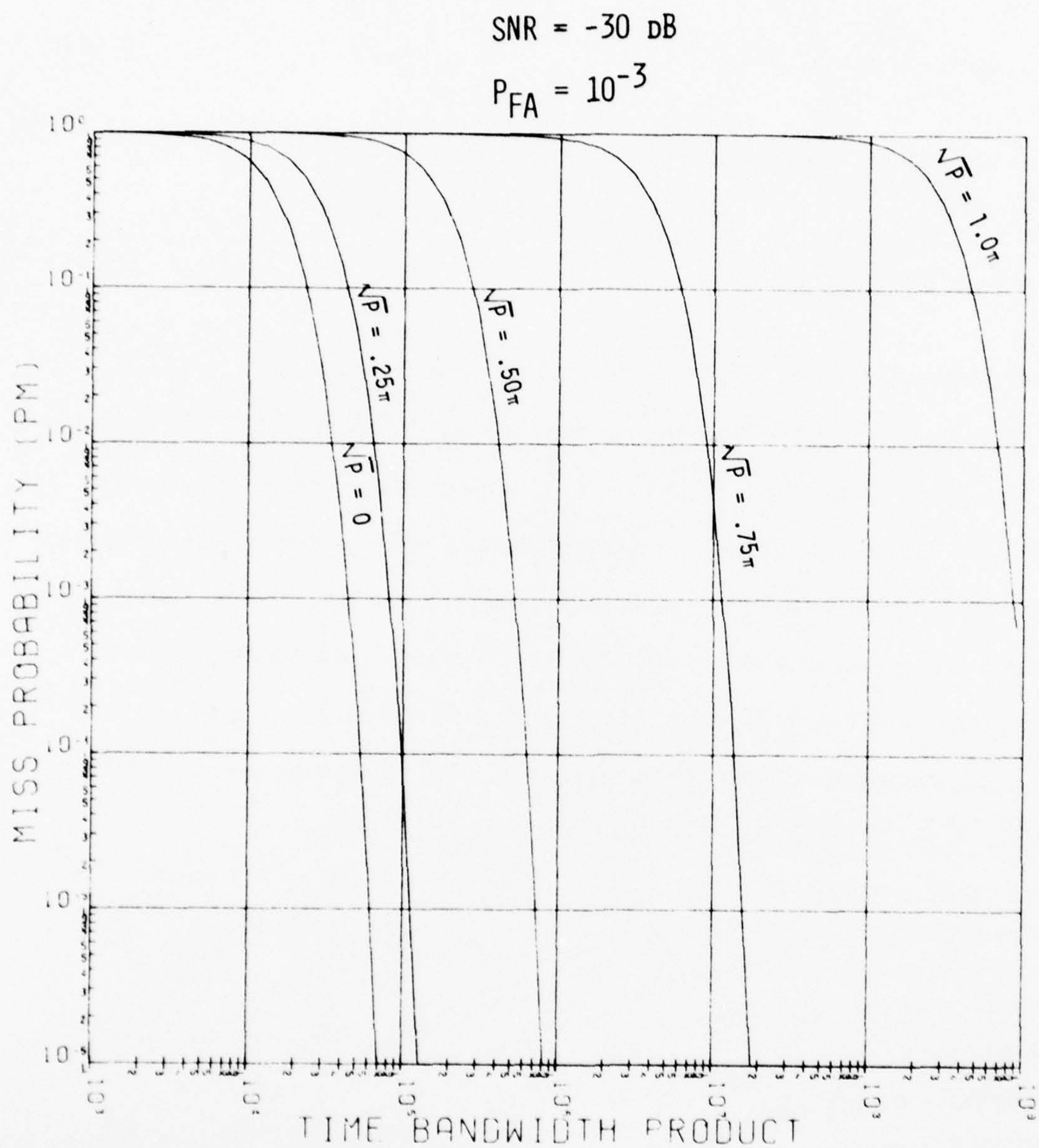

Figure 2-13. Miss Probability ($P_M = 1 - P_D$) vs Time-Bandwidth Product (αT) Parametric in Modulation Index ($\sqrt{P}$) for a Signal to Noise Ratio ($SNR = \frac{A^2}{N_o \alpha}$) of -30 dB and False Alarm Probability (P_{FA}) of 10^{-3}

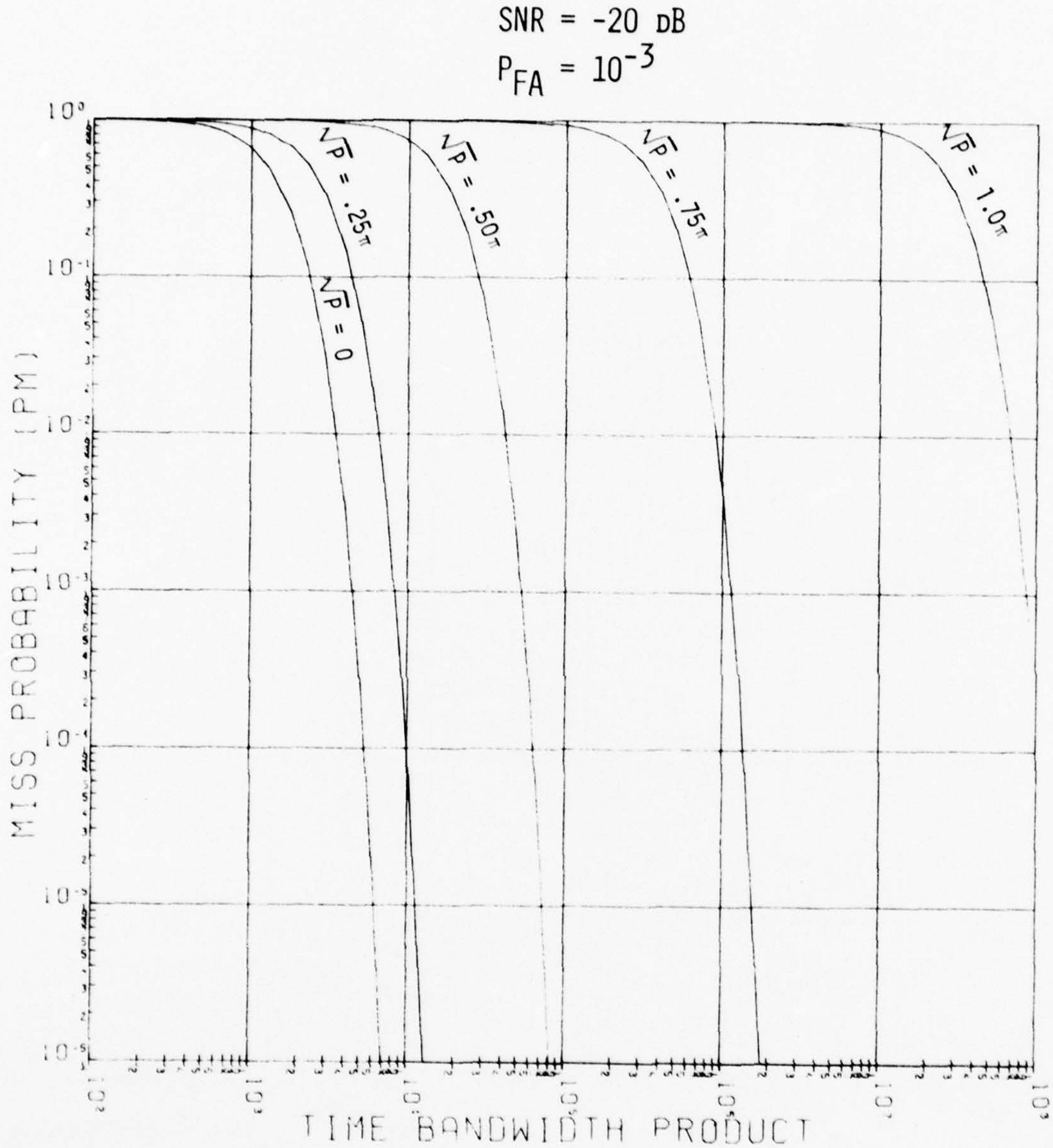

Figure 2-14. P_M vs αT Parametric in $\sqrt{P}$ for SNR = -20 dB and $P_{FA} = 10^{-3}$

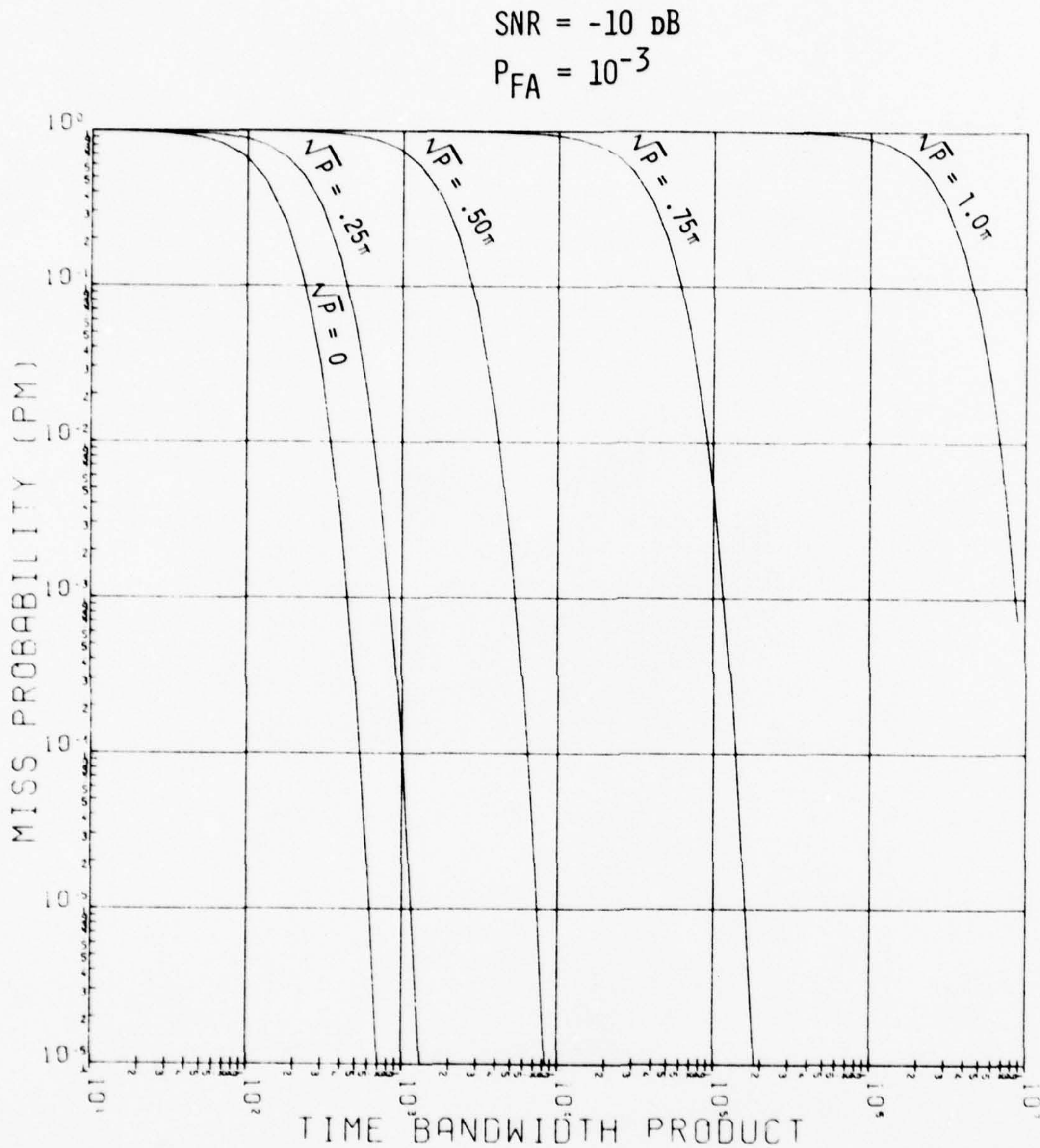

Figure 2-15. P_M vs αT Parametric in $\sqrt{P}$ for SNR = -10 dB and $P_{FA} = 10^{-3}$

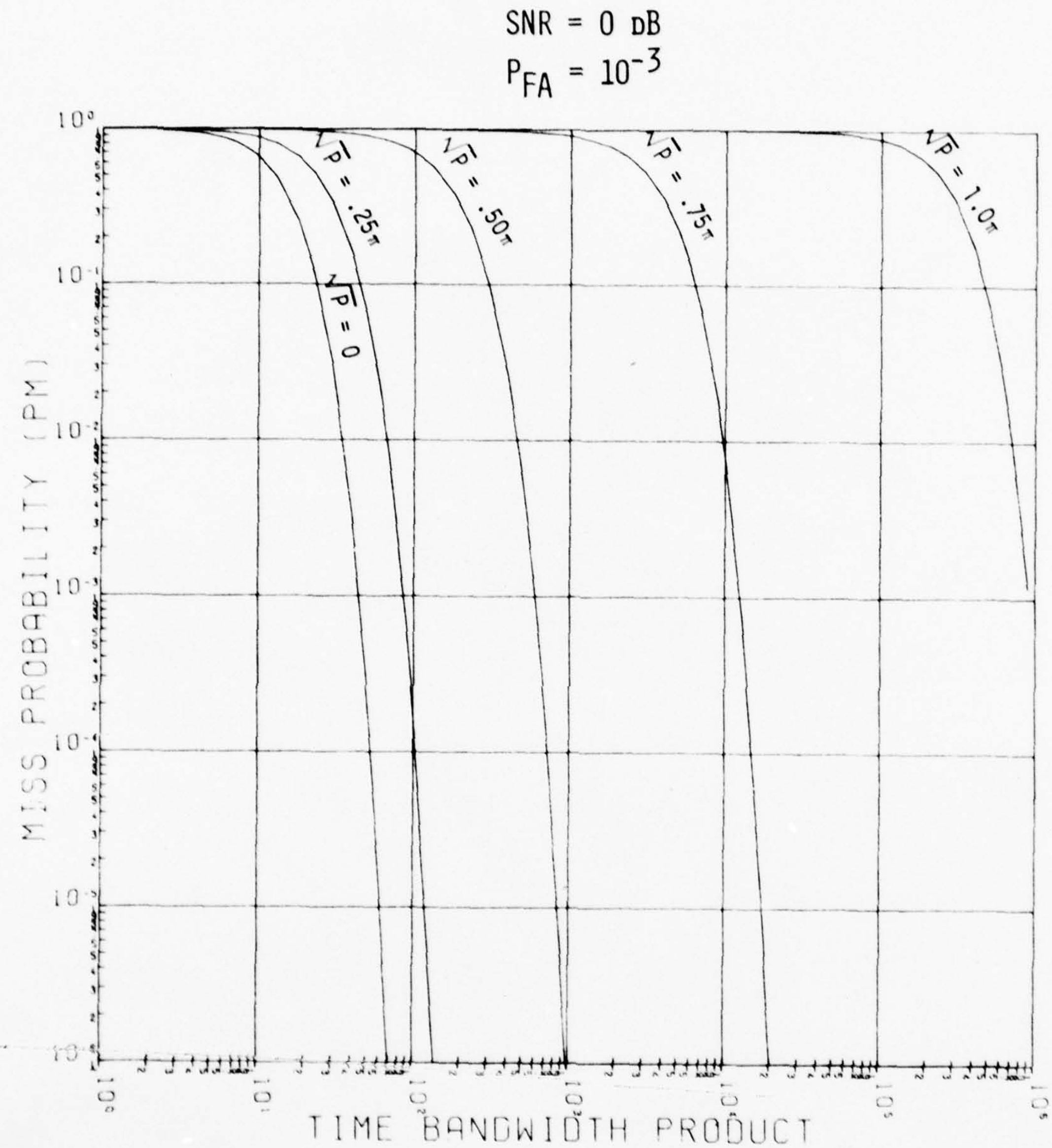

Figure 2-16. P_M vs αT Parametric in $\sqrt{P}$ for SNR = 0 dB and $P_{FA} = 10^{-3}$

SNR = -10 dB

$P_{FA} = 10^{-3}$

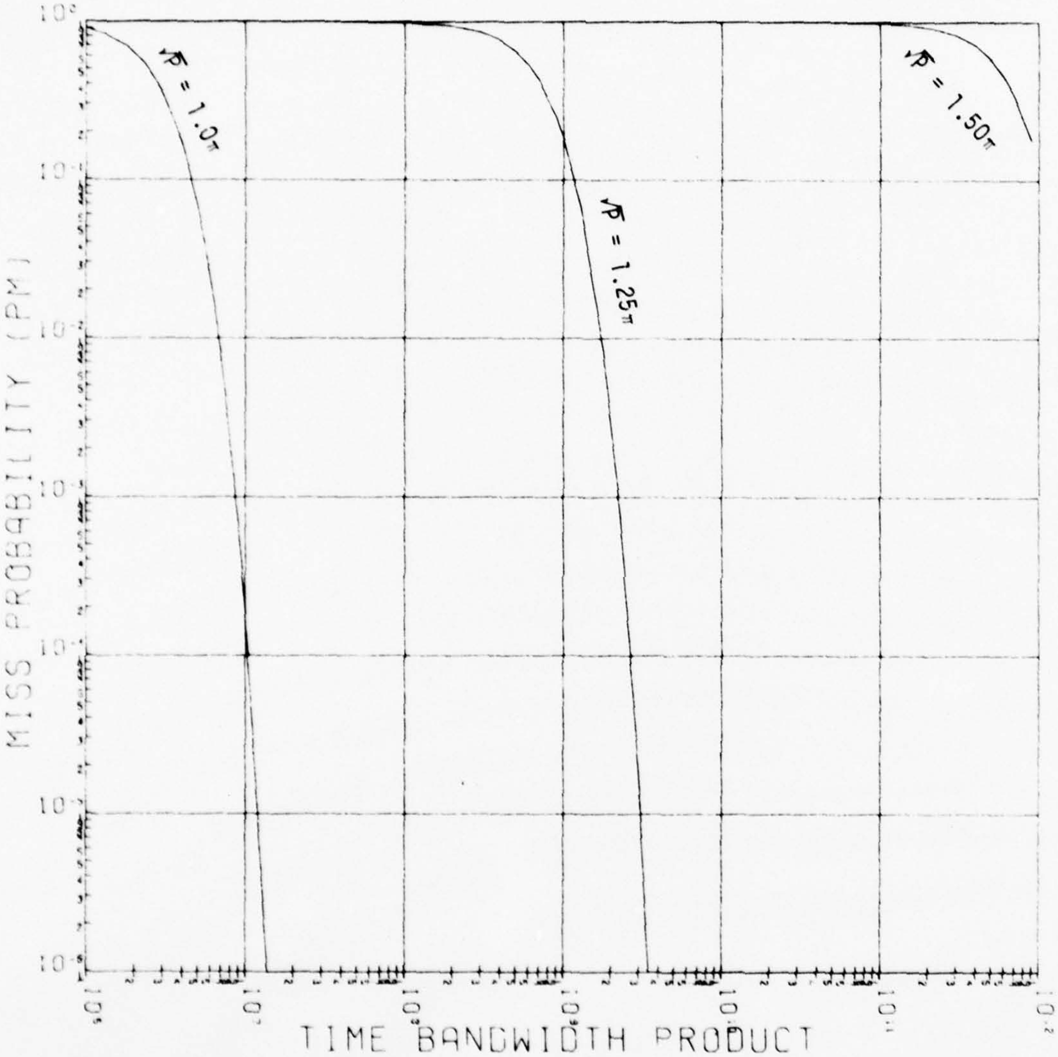

Figure 2-17. P_M vs αT Parametric in $\sqrt{P}$ for SNR = -10 dB and $P_{FA} = 10^{-3}$

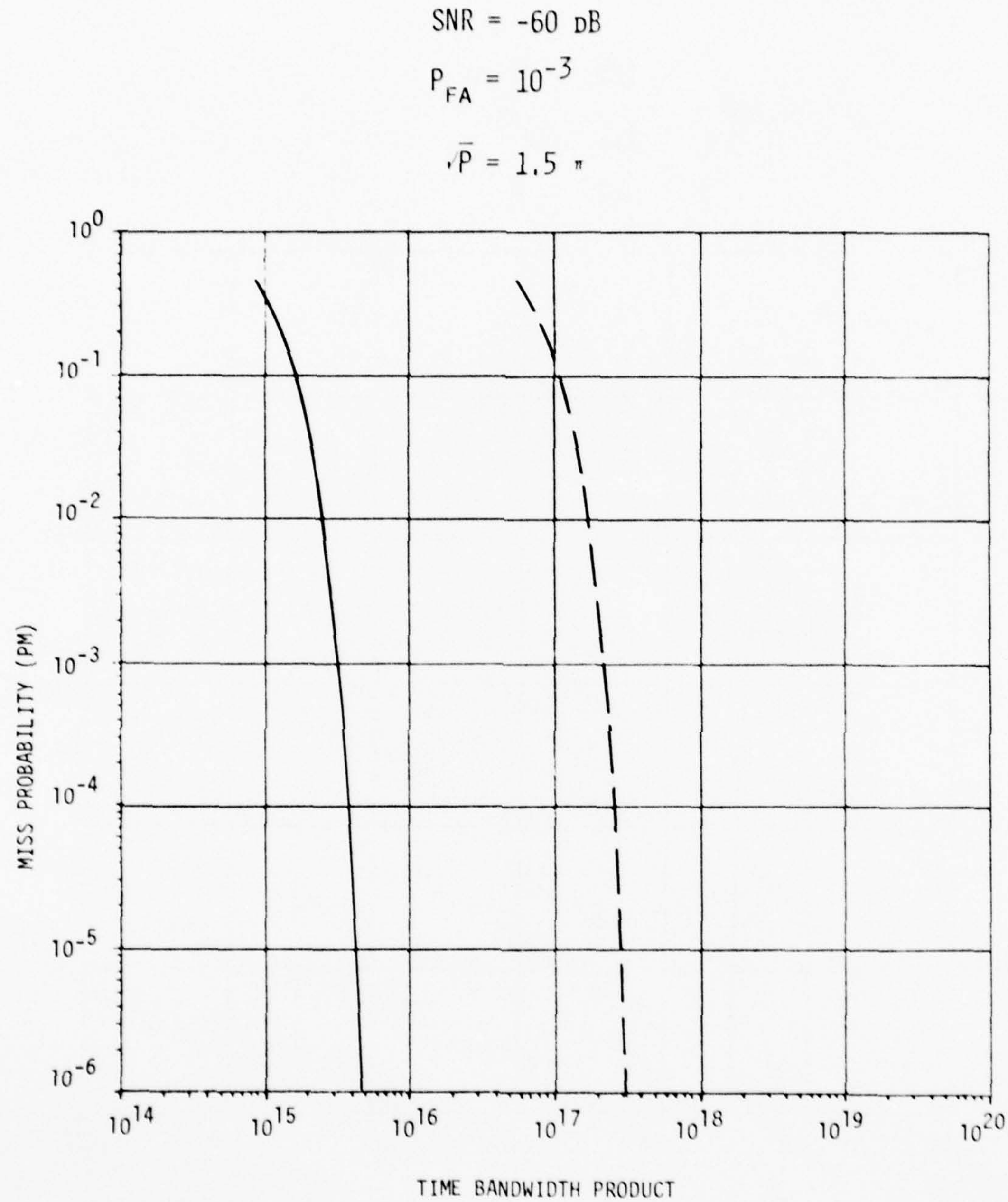

Figure 2-18. Comparison of Optimum Detector (Solid Line) and Suboptimum Detector (Broken Line) Performance for SNR = -60 dB, $P_{FA} = 10^{-3}$, and $\sqrt{P} = 1.5\pi$

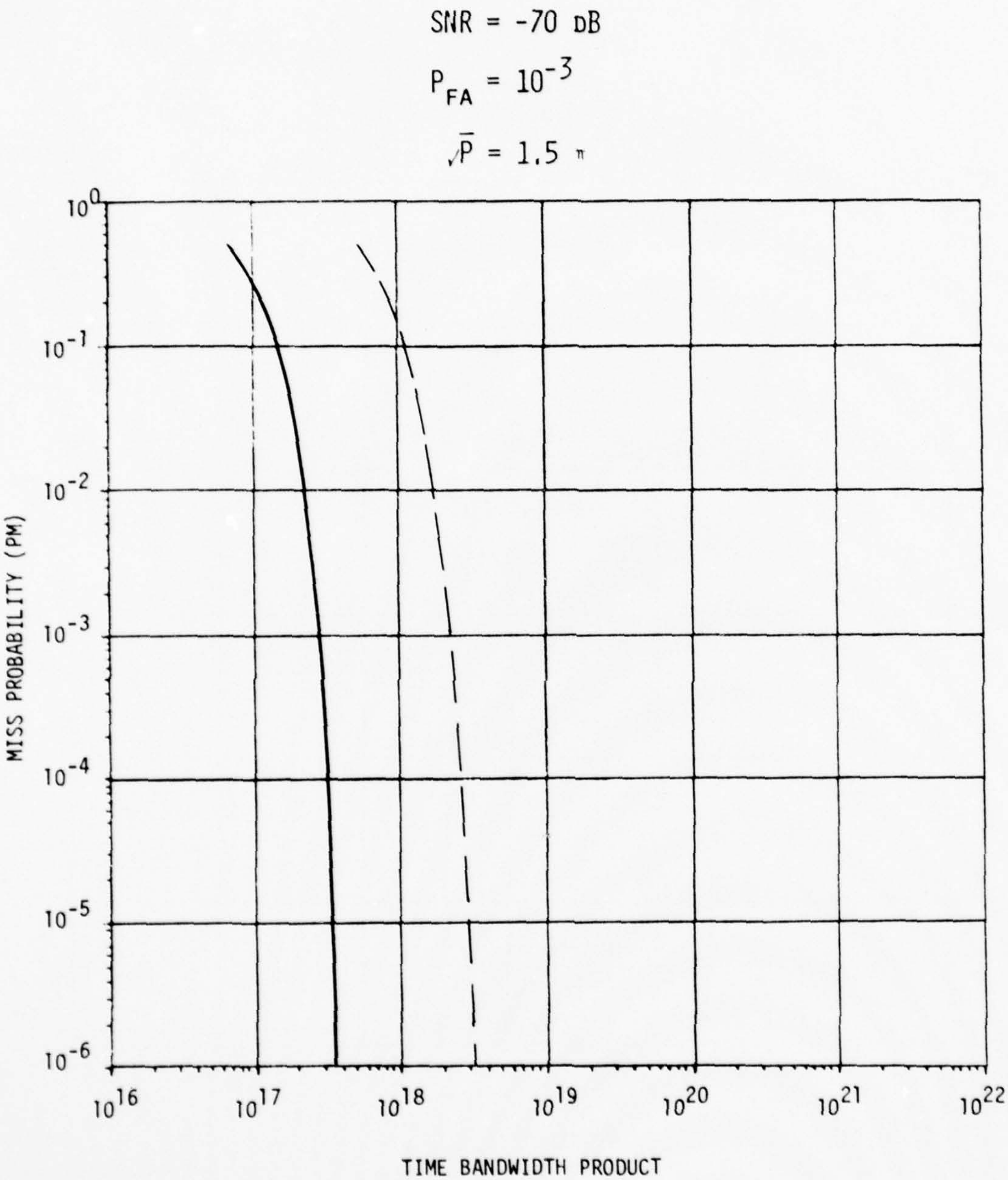

Figure 2-19. Comparison of Optimum Detector and Suboptimum Detector Performance for SNR = -70 dB, $P_{FA} = 10^{-3}$, and $\sqrt{P} = 1.5\pi$

SNR = -80 dB

$P_{FA} = 10^{-3}$

$\sqrt{P} = 1.5\ \pi$

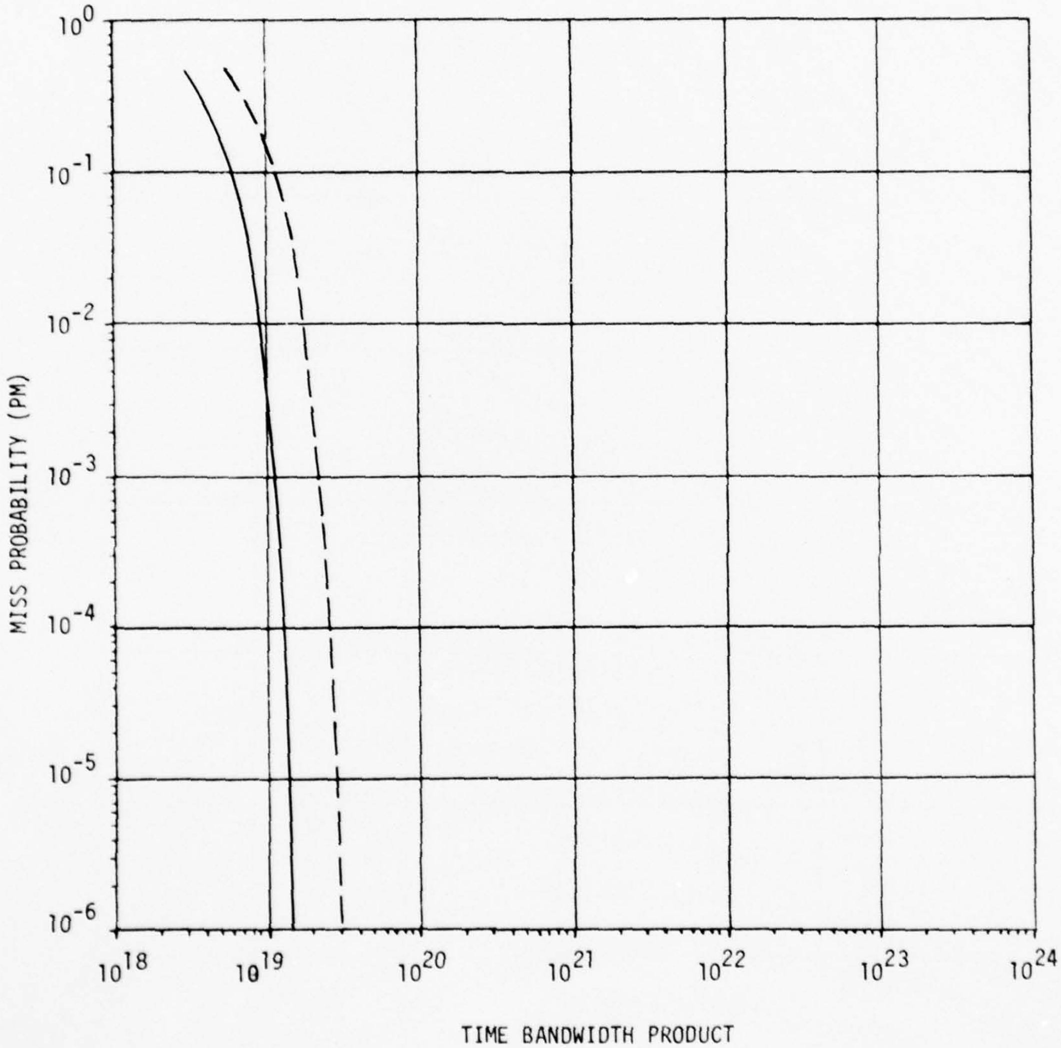

Figure 2-20. Comparison of Optimum Detector and Suboptimum Detector Performance for SNR = -80 dB, $P_{FA} = 10^{-3}$ and $\sqrt{P} = 1.5\pi$

$SNR = -90$ dB

$P_{FA} = 10^{-3}$

$\sqrt{P} = 1.5\pi$

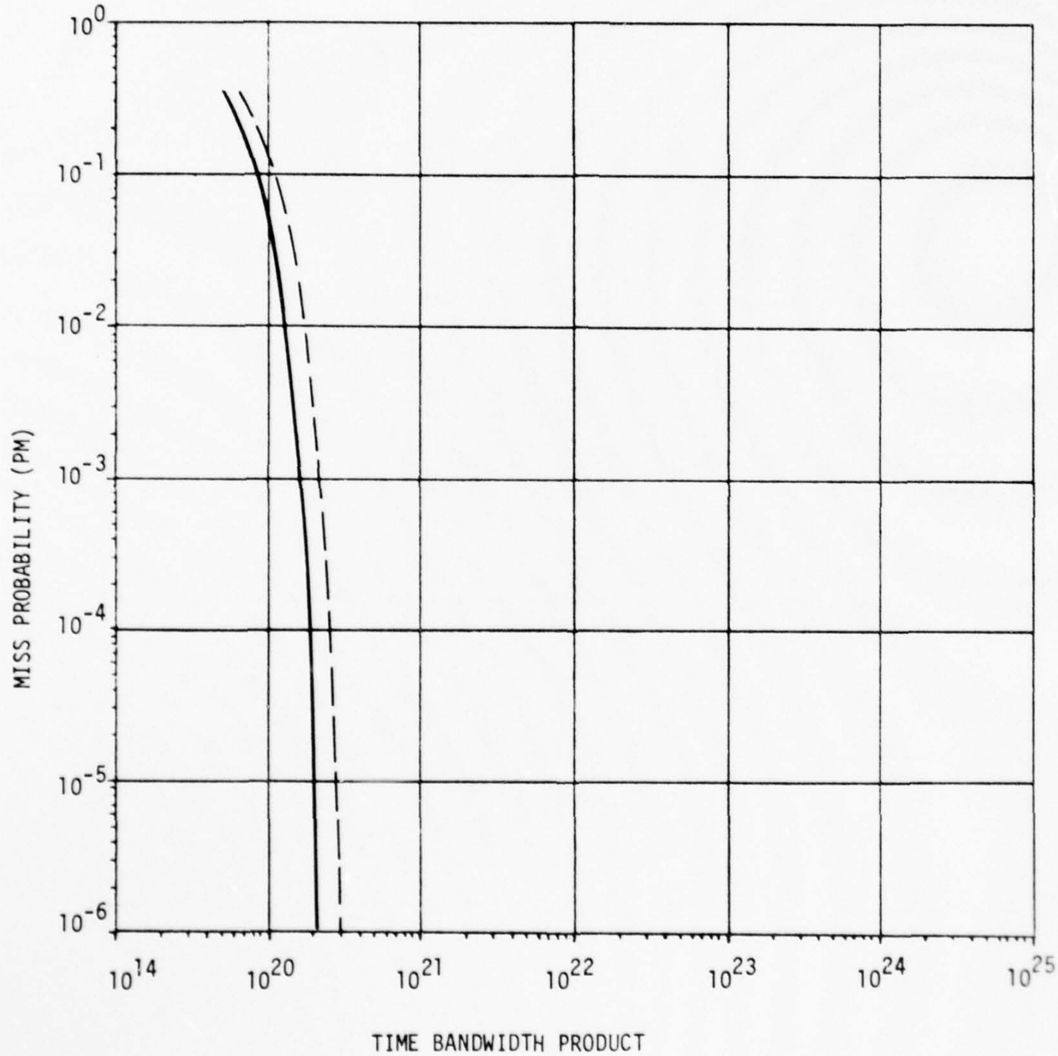

Figure 2-21. Comparison of Optimum Detector and Suboptimum Detector Performance for SNR = -90 dB, $P_{FA} = 10^{-3}$ and $\sqrt{P} = 1.5\pi$

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CHAPTER III

MMSE ESTIMATION IN GAUSSIAN WHITE NOISE OF A NONLINEAR MEMORYLESS FUNCTIONAL OF A RANDOM PROCESS USING NONLINEAR OBSERVATIONS

3.1 Derivation of the Volterra Series

Consider the problem of finding the minimum-mean squared error (MMSE) estimate of a nonlinear memoryless functional of a random process when the observations have the form

$$r(t) = AS\,[t,\ m(t)] + n(t) \qquad 0 \le t \le T \tag{3-1}$$

and the MMSE estimate desired is

$$\hat{g}\,[T,\ m(T)] = E\{g\,[T,\ m(T)]\ |\ r(t);\ 0 \le t \le T\} \tag{3-2}$$

where n(t) is zero-mean Gaussian white noise and

$$E\,[n(t_1)\,n(t_2)] = N_o/2\ \delta\,(t_1 - t_2). \tag{3-3}$$

To find the MMSE estimate m(t) is expanded in a Karhunen-Loève expansion on [0, T]. This yields

$$m(t) = \underset{N\to\infty}{\text{L.I.M.}} \sum_{i=1}^{N} m_i\,\phi_i\,(t) \qquad 0 \le t \le T \tag{3-4}$$

where

$$\lambda_i\,\phi_i\,(t_1) = \int_o^T K_m(t_1, t_2)\,\phi_i(t_2)dt_2 \tag{3-5}$$

$$m_i = \int_o^T m(t)\,\phi_i(t)dt \tag{3-6}$$

$$E\,[m(t)] = 0 \tag{3-7}$$

$$K_m(t_1, t_2) = E\,[m(t_1)\,m(t_2)] \quad . \tag{3-8}$$

The joint probability density function (p. d. f.) of $m_1, \ldots, m_N$ is denoted by $p(m_1, \ldots, m_N)$. Let

$$m_N(t) = \sum_{i=1}^{N} m_i \, \phi_i(t) \tag{3-9}$$

and consider

$$r_N(t) = AS\,[\,t,\, m_N(t)\,] + n(t) \quad . \tag{3-10}$$

Assuming m_i, $i=1, \ldots, N$, are given, $r_N(t)$ may be expanded in a Karhunen-Loève expansion to obtain

$$r_N(t) = \lim_{M \to \infty} \sum_{j=1}^{M} r_j \, \psi_j(t) \qquad 0 \le t \le T \tag{3-11}$$

where the first eigenfunction is chosen to be

$$\psi_1(t) = \frac{S\left[t,\, \sum_{i=1}^{N} m_i \, \phi_i(t)\right]}{\left\{\int_o^T S^2\left[t,\, \sum_{i=1}^{N} m_i \, \phi_i(t)\right] dt\right\}^{\frac{1}{2}}} \tag{3-12}$$

and $\psi_j(t)$, $j \ge 2$ are chosen arbitrarily to complete the set. Clearly

$$E\,(r_1 \mid m_1, \ldots, m_N) = E\left(\int_o^T \frac{r_N(t)\, S\left[t,\, \sum_{i=1}^{N} m_i \, \phi_i(t)\right]}{\left\{\int_o^T S^2\left[t,\, \sum_{i=1}^{N} m_i \, \phi_i(t)\right] dt\right\}^{\frac{1}{2}}} dt\right) . \tag{3-13}$$

From Equations (3-13) and (3-10)

$$E(r_1 \mid m_1, \ldots, m_N) = A\left\{\int_0^T S^2\left[t, \sum_{i=1}^{N} m_i \phi_i(t)\right] dt\right\}^{\frac{1}{2}} . \qquad (3\text{-}14)$$

Also, from the orthogonality of $\psi_j(t)$ and Equation (3-11)

$$E(r_j \mid m_1, \ldots, m_N) = 0 \quad , \; j \geq 2 \quad . \qquad (3\text{-}15)$$

In addition, the conditional variance of r_j is given by

$$\sigma^2_{r_j \mid m_1, \ldots, m_N} = \frac{N_o}{2} \quad , \; j \geq 1 \, . \qquad (3\text{-}16)$$

It follows from Equations (3-14) through (3-16) and the joint normality and statistical independence of the r_j when $\{ m_i \}$ are given that

$$p(r_1, \ldots, r_M \mid m_1, \ldots, m_N) = \frac{\exp\left(-\dfrac{\left\{r_1 - A\left(\int_0^T S^2\,[t, m_N(t)]\,dt\right)^{\frac{1}{2}}\right\}^2 + \sum_{j=2}^{M} r_j^{\,2}}{N_o}\right)}{\left(\pi N_o\right)^{M/2}} . \qquad (3\text{-}17)$$

Applying Bayes' Law to Equation (3-17) results in

$$p(m_1, \ldots, m_N \mid r_1, \ldots, r_M)$$

$$= \frac{\dfrac{\exp\left\{-\frac{1}{N_o}\left[\left(r_1 - A\left\{\int_0^T S^2\left[t, m_N(t)\right] dt\right\}^{\frac{1}{2}}\right)^2 + \sum_{j=2}^{M} r_j^2\right]\right\} p(m_1, \ldots, m_N)}{\left(\pi N_o\right)^{M/2}}}{\dfrac{\int_{-\infty}^{\infty} \cdots \int \exp\left\{-\frac{1}{N_o}\left[\left(r_1 - A\left\{\int_0^T S^2\left[t, m_N(t)\right]\right\}^{\frac{1}{2}} dt\right)^2 + \sum_{j=2}^{M} r_j^2\right]\right\} p(m_1, \ldots, m_N)\, dm_1 \ldots dm_N}{\left(\pi N_o\right)^{M/2}}} . \qquad (3\text{-}18)$$

Let

$$g[t, m_N(t)] = g[t, \sum_{i=1}^{N} m_i \phi_i(t)] \quad . \tag{3-19}$$

The expectation of $g(T, m_N(T))$, given $r_1, \ldots, r_M$ may then be expressed as

$$E\left\{ g[T, m_N(T)] \mid r_1, \ldots, r_M \right\}$$

$$= \lim_{M \to \infty} \frac{\int_{-\infty}^{\infty} \overset{N}{\cdots} \int g[T, m_N(T)] \dfrac{\exp\left\{ -\frac{1}{N_o} \left[\left(r_1 - A \left\{ \int_0^T S^2[t, m_N(t)]\, dt \right\}^{\frac{1}{2}} \right)^2 + \sum_{j=2}^{M} r_j^2 \right] \right\}}{\left(\pi N_o\right)^{M/2}} p(m_1, \ldots, m_N) dm_1 \ldots dm_N}{\int_{-\infty}^{\infty} \overset{N}{\cdots} \int \dfrac{\exp\left\{ -\frac{1}{N_o} \left[\left(r_1 - A \left\{ \int_0^T S^2[t, m_N(t)]\, dt \right\}^{\frac{1}{2}} \right)^2 + \sum_{j=2}^{M} r_j^2 \right] \right\}}{\left(\pi N_o\right)^{M/2}} p(m_1, \ldots, m_N) dm_1, \ldots, dm_N} \; . \tag{3-20}$$

Cancelling common factors in the numerator and denominator of Equation (3-20) and taking the limit as $M \to \infty$, the conditional expectation becomes

$$E\left\{ g[T, m_N(T)] \mid r(t);\ 0 \le t \le T \right\}$$

$$= \frac{\int_{-\infty}^{\infty} \overset{N}{\cdots} \int g_N[T, m_N(T)] \exp\left(\frac{2A}{N_o} \int_0^T S[t, m_N(t)] \left\{ r(t) - \frac{A}{2} S[t, m_N(t)] \right\} dt \right) p(m_1, \ldots, m_N)\, dm_1, \ldots, dm_N}{\int_{-\infty}^{\infty} \overset{N}{\cdots} \int \exp\left(\frac{2A}{N_o} \int_0^T S[t, m_N(t)] \left\{ r(t) - \frac{A}{2} S[t, m_N(t)] \right\} dt \right) p(m_1, \ldots, m_N)\, dm_1, \ldots, dm_N} \tag{3-21}$$

Taking $\lim N \to \infty$, the denominator of Equation (3-21) is identical to Equation (2-27) while the numerator differs only by the factor $g(T, m_N(T))$. Following the same procedure used in developing Equation (2-28), it follows that

$$E\ \{ g[T, m(T)] \mid r(t);\ 0 \le t \le T \}$$

$$= \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1, \ldots dt_i}{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) f_i(t_1, \ldots, t_i)\, dt_1, \ldots, dt_i} \tag{3-22}$$

where

$$f_i(t_1, \ldots, t_i) = E\left(S[t_1, m(t_1)] \ldots S[t_i, m(t_i)] \exp\left\{ -\frac{A^2}{N_o} \int_o^T S^2[\tau, m(\tau)]\, d\tau \right\} \right), \tag{3-23}$$

$$h_i\ (T, t_1, \ldots, t_i) = E\Bigg(g[T, m(T)]\ S[t_1, m(t_1)] \ldots$$

$$S[t_i, m(t_i)] \exp\left\{ -\frac{A^2}{N_o} \int_o^T S^2\, [\tau, m(\tau)]\, d\tau \right\} \Bigg). \tag{3-24}$$

Equation (3-22) - (3-24) are used later in this chapter to demonstrate that previously known results can be obtained by this alternate approach. To reduce Equations (3-22) - (3-24) to a form which is more suitable for implementation of the estimator, it is assumed that the MMSE estimate of $g(T, m(T))$ is expressed in terms of a Volterra functional expansion according to the relation

$$\hat{g}\ [T, m(T)] = \sum_{i_2=0}^{\infty} \frac{1}{i_2!} \left(\frac{2A}{N_o}\right)^{i_2} \int_o^T \overset{i_2}{\cdots} \int_o^T r(\tau_1) \ldots r(\tau_{i_2})$$

$$e_{i_2}\ (\tau_1, \ldots, \tau_{i_2})\, d\tau_1 \ldots d\tau_{i_2} \tag{3-25}$$

where $e_i(\tau_1, \ldots, \tau_i)$ is the i^{th} Volterra kernel of the estimate. Equating Equations (3-25) and (3-22), there results

$$\sum_{i_2=0}^{\infty} \frac{1}{i_2!} \left(\frac{2A}{N_o}\right)^{i_2} \int_0^T \overset{i_2}{\cdots} \int_0^T r(\tau_1)\ldots r(\tau_{i_2}) e_{i_2}(\tau_1, \ldots, \tau_{i_2})\, d\tau_1 \ldots d\tau_{i_2}$$

$$= \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^{i} \int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1, \ldots, dt_i}{\displaystyle\sum_{i_1=0}^{\infty} \frac{1}{i_1!} \left(\frac{2A}{N_o}\right)^{i_1} \int_0^T \overset{i_1}{\cdots} \int_0^T r(t_1), \ldots, r(t_{i_1}) f_{i_1}(t_1, \ldots, t_{i_1}) dt_1, \ldots, dt_{i_1}} . \tag{3-26}$$

Multiplying both sides by the likelihood ratio, which is the denominator of Equation (3-22), (3-26) becomes

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{1}{i_1! i_2!} \left(\frac{2A}{N_o}\right)^{i_1+i_2} \int_0^T \overset{i_1+i_2}{\cdots} \int_0^T r(t_1)\ldots r(t_{i_1}) f_{i_1}(t_1, \ldots, t_{i_1})$$

$$r(\tau_1)\ldots r(\tau_{i_2}) e_{i_2}(\tau_1, \ldots, \tau_{i_2})$$

$$dt_1 \ldots dt_{i_1}\, d\tau_1 \ldots d\tau_{i_2}$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^{i} \int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i . \tag{3-27}$$

Equating terms of like power in $\left(\frac{2A}{N_o}\right)$, it follows that

$$e_0 = \frac{h_0(T)}{f_0} , \tag{3-28}$$

$$e_1(t_1) = \frac{h_1(T, t_1) - f_1(t_1) e_0}{f_0} , \tag{3-29}$$

$$e_2(t_1,t_2) = \frac{h_2(T,t_1,t_2) - 2\,f_1(t_1)e_1(t_2) - f_2(t_1,t_2)e_0}{f_0} \quad . \tag{3-30}$$

$$\vdots$$

In general, $e_i(t_1,\ldots,t_i)$ is given by

$$e_i(t_1,\ldots,t_i) = \frac{h_i(T,t_1,\ldots,t_i)}{f_o} - \sum_{k=1}^{i-1} \binom{i}{k} \frac{f_k(t_1,\ldots,t_k)e_{i-k}(t_{k+1},\ldots,t_i)}{f_o}$$

$$- \frac{f_k(t_1,\ldots,t_i)e_0}{f_o} \quad . \tag{3-31}$$

Observe that $e_i(t_1,\ldots,t_i)$ can be obtained explicitly in terms of $h_j(T,t_1,\ldots,t_j)$ and $f_k(t_1,\ldots,t_k)$; $j,k=0,1,\ldots,i$. Consequently, the assumed form of the MMSE estimate, given in Equation (3-25), is valid.

3.2 Rederivation of the Linear Result for Zero Mean Gaussian Processes

In this section it is assumed that

$$r(t) = m(t) + n(t) \;; \quad 0 \le t \le T \tag{3-32}$$

where m(t) is a zero-mean Gaussian process and the desired MMSE estimate is

$$\hat{g}\,(T, m(T)) = \hat{m}\,(T) = E(m(T) \mid r(t);\ 0 \le t \le T) \quad . \tag{3-33}$$

From Equation (3-22) - (3-24), $\hat{m}(T)$ is given by

$$\hat{m}(T) = \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i)h_i(T,t_1,\ldots,t_i)\, dt_1,\ldots,dt_i}{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i)f_i(t_1,\ldots,t_i)\, dt_1\ldots dt_i} \tag{3-34}$$

where

$$f_i(t_1, \ldots, t_i) = E\left\{ m(t_1)\ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)d\tau \right] \right\} \tag{3-35}$$

and

$$h_i(T, t_1, \ldots, t_i) = E\left\{ m(T)m(t_1)\ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)d\tau \right] \right\} \tag{3-36}$$

.

The denominator of Equation (3-34) is the likelihood ratio of Chapter II par. 2.2. Hence, from Equations (2-56) - (2-58)

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\ldots} \int_o^T r(t_1)\ldots r(t_i) f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \exp\left(\frac{1}{N_o} \int_o^T \int_o^T r(t_1)r(t_2)h_*(t_1, t_2)\, dt_1 dt_2 \right) \tag{3-37}$$

where $h_*(t_1, t_2)$ is the solution to the integral equation

$$\frac{N_o}{2} h_*(t_1, t_2) + \int_o^T h_*(t_1, \tau) K_m(\tau, t_2)\, d\tau = K_m(t_1, t_2) \tag{3-38}$$

and where

$$K_m(t_1, t_2) = E(m(t_1)m(t_2)) \quad . \tag{3-39}$$

To develop the numerator in closed form, note that

$$h_0(T) = E\left(m(T) \exp\left(-\frac{1}{N_o} \int_o^T m^2(\tau)d\tau \right) \right) \quad . \tag{3-40}$$

Following the procedure previously used in Chapter II par. 2.2, m(t) is expanded in a Karhunen-Loève expansion to obtain

$$h_0(T) = \sum_{i=1}^{\infty} E\left(m_i \exp\left(-\frac{1}{N_o}\sum_{j=1}^{\infty} m_j^2\right)\right)\phi_i(T) . \tag{3-41}$$

However,

$$E\left(m_i \exp\left(-\frac{1}{N_o}\sum_{j=1}^{\infty} m_j^2\right)\right) = \lim_{N\to\infty} \int_{-\infty}^{\infty}\overset{N}{\cdots}\int m_i \frac{\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{\lambda_j}+\frac{2}{N_o}\right)m_j^2\right)}{(2\pi)^{N/2}\left|\prod_{j=1}^{N}\lambda_j\right|^{\frac{1}{2}}}$$

$$dm_1 \ldots dm_N$$

$$= 0 \quad . \tag{3-42}$$

It follows that

$$h_0(T) = 0 . \tag{3-43}$$

Similarly,

$$h_1(T, t_1) = E\left(m(T)m(t_1)\exp\left(-\frac{1}{N_o}\int_0^T m^2(\tau)\,d\tau\right)\right) \tag{3-44}$$

With reference to Equation (2-45), $h_1(T, t_1)$ is identical to $f_2(T, t_1)$. From Equation (2-49) it is concluded that

$$h_1(T, t_1) = \left|\prod_{i=1}^{\infty}\frac{N_o/2}{N_o/2+\lambda_i}\right|^{\frac{1}{2}} \frac{N_o}{2}\sum_{i=1}^{\infty}\frac{\lambda_i}{N_o/2+\lambda_i}\phi_i(T)\,\phi_i(t_1) . \tag{3-45}$$

Use of Equation (2-58) in Equation (3-45) results in

$$h_1(T, t_1) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \frac{N_o}{2} \; h_*(T, t_1) \quad . \tag{3-46}$$

It is also readily shown that

$$h_2(T, t_1, t_2) = 0 \tag{3-47}$$

$$h_3(T, t_1, t_2, t_3) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^2$$

$$[\,h_*(T, t_1)h_*(t_2, t_3) + h_*(T, t_2)h_*(t_1, t_3) + h_*(T, t_3)h_*(t_1, t_2)] \; . \tag{3-48}$$

In general, $h_i(T, t_1, \ldots, t_i)$ is given by

$$h_i(T, t_1, \ldots, t_i) = \left| \prod_{j=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_j} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^{\frac{i+1}{2}}$$

$$\sum h_*(T, t_{k_1})h_*(t_{k_2}, t_{k_3}) \ldots h_*(t_{k_{i-1}}, t_{k_i}), \text{ i odd}$$

$$= 0 \;\; , \text{ i even} \tag{3-49}$$

where the sum is overall $i!/2^{\frac{i-1}{2}}\left(\frac{i-1}{2}\right)!$ ways of partitioning the set $[T, t_1, \ldots, t_i]$ into $\frac{i+1}{2}$ pairs. From Equation (3-49), the i^{th} i-fold integration in the numerator of Equation (3-34) can be expressed as

$$\int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i) h_i(T, t_1, \ldots t_i)\, dt_1 \ldots dt_i$$

$$= \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^{\frac{i+1}{2}} \frac{i!}{2^{\frac{i-1}{2}}\left(\frac{i-1}{2}\right)!}$$

$$\left[\int_0^T \int_0^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1\, dt_2\right]^{\frac{i-1}{2}}$$

$$\left[\int_0^T h_*(T, t_1) r(t_1)\, dt_1\right] \qquad i \text{ odd}$$

$$= 0 \qquad i \text{ even} \qquad . \qquad (3\text{-}50)$$

With the aid of Equation (3-50), the numerator of Equation (3-34) becomes

$$\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1)\ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \sum_{\substack{i=0 \\ i\ \mathrm{odd}}}^{\infty} \left(\frac{1}{N_o}\right)^{\frac{i-1}{2}} \frac{1}{\left(\frac{i-1}{2}\right)!} \left[\int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1\, dt_2\right]^{\frac{i-1}{2}}$$

$$\cdot \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \int_o^T h_*(T, t_1) r(t_1) dt_1$$

$$= \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \int_o^T h_*(T, t_1) r(t_1) dt_1$$

$$\exp\left(\frac{1}{N_o} \int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2) dt_1 dt_2\right) \quad . \qquad (3\text{-}51)$$

Substituting Equations (3-37) and (3-51) into Equation (3-34), the expression for $\hat{m}(T)$ reduces to

$$\hat{m}(T) = \int_o^T h_*(T, t_1) r(t_1) dt_1 \qquad (3\text{-}52)$$

where $h_*(T, t_1)$ is the solution to the integral equation

$$\frac{N_o}{2} h_*(T, t_1) + \int_o^T h_*(T, \tau) K_m(\tau, t_1)\, d\tau = K_m(T, t_1) \quad . \qquad (3\text{-}53)$$

This agrees with a well known result [1, Chapter 6 Equations (23) and (17)] .

3.3 A Stochastic Differential Equation for the Logarithm of the Likelihood Ratio

In this paragraph, a stochastic differential equation for the logarithm of the likelihood ratio is developed.

From Chapter II par. 2.1, the likehood ratio is

$$\Lambda\,(r(t)) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1)\ldots r(t_i) f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i \qquad (3\text{-}54)$$

where

$$f_i(t_1, \ldots, t_i) = E(S(t_1, m(t_1)) \ldots S(t_i, m(t_i)) \exp\left(-\frac{A^2}{N_o} \int_o^T S^2(\tau, m(\tau)) d\tau\right) . \qquad (3\text{-}55)$$

Observe that the likelihood ratio is a function of T. Consequently, we consider the derivative of $\Lambda(r(t))$ with respect to T. First consider

$$\frac{d}{dT} \int_o^T f_1(t_1) r(t_1)\, dt_1 = \int_o^T \left[\frac{d}{dT} f_1(t_1)\right] r(t_1)\, dt_1 + f_1(T) r(T) . \qquad (3\text{-}56)$$

Similarly,

$$\frac{d}{dT} \int_o^T \int_o^T f_2(t_1, t_2)\, r(t_1) r(t_2) dt_1 dt_2$$

$$= \int_o^T \left[\frac{d}{dT} \int_o^T f_2(t_1, t_2) r(t_2) dt_2\right] r(t_1) dt_1$$

$$+ r(T) \int_o^T f_2(T, t_2) r(t_2) dt_2 . \qquad (3\text{-}57)$$

Since $f_2(t_1, t_2)$ is symmetric in its arguments, Equation (3-57) simplifies to

$$\frac{d}{dT} \int_0^T \int_0^T f_2(t_1, t_2) r(t_1) r(t_2)\, dt_1\, dt_2$$

$$= \int_0^T \int_0^T \left[\frac{d}{dT} f_2(t_1, t_2) \right] r(t_1) r(t_2)\, dt_1\, dt_2$$

$$+ 2r(T) \int_0^T f_2(T, t_2) r(t_2)\, dt_2 \quad . \tag{3-58}$$

More generally, it can be shown that

$$\frac{d}{dT} \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i) f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \int_0^T \overset{i}{\cdots} \int_0^T \left[\frac{d}{dT} f_i(t_1, \ldots, t_i) \right] r(t_1) \ldots r(t_i)\, dt_1 \ldots dt_i$$

$$+ i\, r(T) \int_0^T \overset{i-1}{\cdots} \int_0^T f_i(T, t_1, \ldots, t_{i-1}) r(t_1) \ldots r(t_{i-1})\, dt_1 \ldots dt_{i-1} \quad . \tag{3-59}$$

Therefore, the derivative of Λ (r(t)) with respect to T may be expressed as

$$\frac{d}{dT}\,\Lambda(r(t)) = \sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{2A}{N_o}\right)^{i}\int_0^T\overset{i}{\cdots}\int_0^T\left[\frac{d}{dT}\,f_i(t_1,\ldots,t_i)\right] r(t_1)\ldots r(t_i)\,dt_1\ldots dt_i$$

$$+\left(\frac{2A}{N_o}\right) r(T)\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{2A}{N_o}\right)^{i}\int_0^T\overset{i}{\cdots}\int_0^T f_{i+1}(T,t_1,\ldots,t_i)r(t_1)\ldots r(t_i)dt_1\ldots dt_i\ . \tag{3-60}$$

Equation (3-60) is further simplified using the estimation results from par. 3.1. Consider the MMSE estimate of S(T, m(T)), then

$$g(t, m(t)) = \hat{S}(t, m(t))\ . \tag{3-61}$$

From Equation (3-24), $h_i(T, t_1, \ldots, t_i)$ becomes

$$h_i(T,t_1,\ldots,t_i) = E\left\{S[T,m(T)]\,S[t_1,m(t_1)]\ldots\right.$$

$$\left.S[t_i,m(t_i)]\,\exp\left[-\frac{A^2}{N_o}\int_0^T S^2(\tau,m(\tau))\,d\tau\right]\right\} \tag{3-62}$$

which, from Equation (3-23) may also be expressed as

$$h_i(T,t_1,\ldots,t_i) = f_{i+1}(T,t_1,\ldots,t_i)\ . \tag{3-63}$$

Substituting Equation (3-63) into the numerator of Equation (3-22), it is concluded that the second term of Equation (3-60) is given by

$$\frac{2A}{N_o}\,r(T)\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{2A}{N_o}\right)^{i}\int_0^T\overset{i}{\cdots}\int_0^T f_{i+1}(T,t_1,\ldots,t_i)\,dt_1\ldots dt_i$$

$$= \frac{2A}{N_o}\,r(T)\,\hat{S}\,[T,m(T)]\,\Lambda\,[r(t)]\ . \tag{3-64}$$

To simplify the first term of Equation (3-60), consider the MMSE estimate of $S^2(T, m(T))$. Now

$$g(T, m(T)) = S^2(T, m(T)). \tag{3-65}$$

Observe that

$$\frac{d}{dT} f_i(t_1, \ldots, t_i) = -\frac{A^2}{N_o} E \left\{ S^2 [T, m(T)] S[t_1, m(t_1)] \ldots \right.$$

$$\left. S[t_i, m(t_i)] \exp\left(-\frac{A^2}{N_o} \int_0^T S^2[\tau, m(\tau)] d\tau \right) \right\}$$

$$= -\frac{A^2}{N_o} h_i (T, t_1, \ldots, t_i) \quad . \tag{3-66}$$

Substituting Equation (3-66) into the numerator of Equation (3-22), the first term in Equation (3-60) is expressed as

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2A}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T \left[\frac{d}{dT} f_i(t_1, \ldots, t_i)\right] r(t_1) \ldots r(t_i) \, dt_1 \ldots dt_i$$

$$= -\frac{A^2}{N_o} \widehat{S^2} [T, m(T)] \Lambda [r(t)] \quad . \tag{3-67}$$

Combining the above results Equation (3-60) becomes

$$\frac{d}{dT} \Lambda [r(t)] = -\frac{A^2}{N_o} \widehat{S^2}[T, m(T)] \Lambda [r(t)]$$

$$+ \frac{2A}{N_o} r(T) \hat{S} [T, m(T)] \Lambda [r(t)] \quad . \tag{3-68}$$

Dividing both sides of Equation (3-68) by $\Lambda(r(t))$ it is recognized that

$$\frac{d}{dT} \ell n \, (\Lambda(r(t)) = \frac{2}{N_o} \left\{ r(t) \, A\hat{S}\,(T, m(T)) - \frac{1}{2} A^2 \, \widehat{S^2}[T, m(T)] \right\} . \qquad (3\text{-}69)$$

The result in Equation (3-69) is consistent with the use of the Stratonovich stochastic integral implied in pars. 2.1 and 3.1 where integrals were manipulated using the ordinary integral calculus. Equation (3-69) is easily modified to be consistent with the use of the Itô stochastic calculus via addition of the correction term

$$\frac{1}{N_o} \left\{ A^2 \widehat{S^2}[T, m(T)] - A^2 \, \hat{S}^2[T, m(T)] \right\}$$

as pointed out by Duncan [18] to obtain

$$\frac{d}{dT} \ell n \left\{ \Lambda[r(t)] \right\} = \frac{2}{N_o} \; A\hat{S}[T, m(T)] \left\{ r(T) - \frac{1}{2} \, A\hat{S}[T, m(T)] \right\} . \qquad (3\text{-}70)$$

This latter result was obtained by Kailath [6] using the Itô stochastic calculus.

3.4 MMSE Estimation in Gaussian White Noise of a Nonlinear Functional of a Zero Mean Gaussian Random Process Using Linear Observations

In this section

$$r(t) = m(t) + n(t) \qquad 0 \le t \le T \tag{3-71}$$

where $m(t)$ is a zero-mean Gaussian process with autocovariance

$$K_m(t_1, t_2) = E(m(t_1)m(t_2)) \tag{3-72}$$

and where $n(t)$ is zero-mean Gaussian white noise with

$$E(n(t_1)n(t_2)) = \frac{N_o}{2}\,\delta(t_1 - t_2) \quad . \tag{3-73}$$

The MMSE estimate desired is

$$\hat{g}(T, m(T)) = E(g(T, m(T)) \mid r(t);\ 0 \le t \le T) \quad . \tag{3-74}$$

From Equations (3-22) - (3-24)

$$\hat{g}(T, m(T)) = \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1)\ldots r(t_i)h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i}{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1)\ldots r(t_i)f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i} \tag{3-75}$$

where

$$f_i(t_1, \ldots, t_i) = E\left(m(t_1)\ldots m(t_i)\, \exp\left(-\frac{1}{N_o}\int_o^T m^2(\tau)\, d\tau\right)\right) \tag{3-76}$$

and

$$h_i(T, t_1, \ldots, t_i) = E\left(g(T, m(T))m(t_1)\ldots m(t_i)\, \exp\left(-\frac{1}{N_o}\int_o^T m^2(\tau)d\tau\right)\right) \quad . \tag{3-77}$$

The denominator of Equation (3-75) is recognized as the likelihood ratio. From Equations (2-56) and (2-58) the likelihood ratio is expressed as

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_o^T \!\!.\overset{i}{.}.\!\! \int_o^T r(t_1) \ldots r(t_i) f_i(t_1, \ldots t_i) dt_1, \ldots dt_i$$

$$= \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \exp\left[\frac{1}{N_o} \int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1 dt_2 \right] \qquad (3\text{-}78)$$

where $h_*(t_1, t_2)$ is given by Equation (2-57). The numerator of Equation (3-75) must be treated separately for each choice of g(T, m(T)). As an example, let

$$g(T, m(T)) = m^2(T) \qquad (3\text{-}79)$$

For this case

$$h_0(T) = E\left(m^2(T) \exp\left(-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right)\right) \qquad (3\text{-}80)$$

With reference to Equations (2-45), (2-49) and (2-58) it follows that

$$h_0(T) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \frac{N_o}{2}\, h_*(T, T)\, . \qquad (3\text{-}81)$$

Similarly,

$$h_1(T, t_1) = 0\, . \qquad (3\text{-}82)$$

Also, from Equations (2-52) and (2-58),

$$h_2(T, t_1, t_2) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^2 [h_*(T, T)h_*(t_1, t_2)$$

$$+ 2\, h_*(T, t_1)h_*(T, t_2)] \ . \tag{3-83}$$

To proceed, it is readily shown that

$$h_3(T, t_1, t_2, t_3) = 0 \tag{3-84}$$

$$h_4(T, t_1, t_2, t_3, t_4) = \left| \prod_{i=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_i} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^3$$

$$[h_*(T, T)h_*(t_1, t_2)h_*(t_3, t_4)$$

$$+ h_*(T, T)h_*(t_1, t_3)h_*(t_2, t_4)$$

$$+ h_*(T, T)h_*(t_1, t_4)h_*(t_2, t_3)$$

$$+ 2h_*(T, t_1)h_*(T, t_2)h_*(t_3, t_4)$$

$$+ 2h_*(T, t_1)h_*(T, t_3)h_*(t_2, t_4)$$

$$+ 2h_*(T, t_2)h_*(T, t_3)h_*(t_1, t_4)$$

$$+ 2h_*(T, t_1)h_*(T, t_4)h_*(t_2, t_3)$$

$$+ 2h_*(T, t_2)h_*(T, t_4)h_*(t_1, t_3)$$

$$+ 2h_*(T, t_3)h_*(T, t_4)h_*(t_1, t_2)] \tag{3-85}$$

and so on. In general, $h_i(T, t_1, \ldots, t_i)$ is given by

$$h_i(T, t_1, \ldots, t_i) = \left| \prod_{j=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_j} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^{\frac{i+2}{2}}$$

$$\left[\sum h_*(T, T) h_*(t_{j_1}, t_{j_2}) \ldots h_*(t_{j_{i-1}}, t_{j_i}) \right.$$

$$\left. + \sum h_*(T, t_{j_1}) h_*(T, t_{j_2}) \ldots h_*(t_{j_{i-1}}, t_{j_i}) \right] \quad , \text{ i even}$$

$$= 0 \quad , \text{ i odd} \qquad (3\text{-}86)$$

where the first sum is over all $i!/2^{\frac{i}{2}} \left(\frac{i}{2}\right)!$ ways of partitioning $[t_1, \ldots, t_i]$ into $\frac{i}{2}$ pairs and where the second sum is over all $i \cdot i!/2^{i/2} \left(\frac{i}{2}\right)!$ ways of partitioning $[T, T, t_1, \ldots, t_i]$ into $\frac{i}{2} + 1$ pairs not included in the first sum. From Equation (3-86), the i^{th} i-fold integration in the numerator of Equation (3-75) can be expressed as

$$\int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) h_i(T, t_1, \ldots t_i) dt_1 \ldots dt_i = \left| \prod_{j=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_j} \right|^{\frac{1}{2}} \left(\frac{N_o}{2}\right)^{\frac{i+2}{2}}$$

$$\left\{ \frac{i!}{2^{\frac{i}{2}} \left(\frac{i}{2}\right)!} h_*(T, T) \left[\int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2) dt_1 dt_2 \right]^{i/2} \right.$$

$$\left. + \frac{i\, i!}{2^{\frac{i}{2}} \left(\frac{i}{2}\right)!} \left[\int_o^T r(t_1) h_*(T, t_1) dt_1 \right]^2 \left[\int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2) dt_1 dt_2 \right]^{\frac{i}{2} - 1} \right\}$$

$$, \text{ i even}$$

$$= 0 \, , \text{ i odd} \qquad (3\text{-}87)$$

From Equation (3-87), it follows that the numerator of Equation (3-75) is

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^{i} \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \left| \prod_{j=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_j} \right|^{\frac{1}{2}} \left[h_*(T,T) \sum_{\substack{i=0 \\ i \text{ even}}}^{\infty} \frac{1}{2^{i/2}\left(\frac{i}{2}\right)!} \left(\frac{2}{N_o}\right)^{\frac{i}{2}-1} \right.$$

$$\left(\int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1 dt_2 \right)^{i/2}$$

$$+ \left(\int_o^T r(t_1) h_*(T, t_1) dt_1 \right)^2 \sum_{\substack{i=0 \\ i \text{ even}}}^{\infty} \frac{i}{2^{i/2}\left(\frac{i}{2}\right)!} \left(\frac{2}{N_o}\right)^{\frac{i}{2}-1}$$

$$\left. \left(\int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1\, dt_2 \right)^{\frac{i}{2}-1} \right] . \qquad (3\text{-}88)$$

Reducing the summations in Equation (3-88) to closed form results in

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^{i} \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i) h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \left| \prod_{j=1}^{\infty} \frac{N_o/2}{N_o/2 + \lambda_j} \right|^{\frac{1}{2}} \exp\left[\frac{1}{N_o} \int_o^T \int_o^T r(t_1) r(t_2) h_*(t_1, t_2)\, dt_1 dt_2 \right]$$

$$\left\{ \frac{N_o}{2} h_*(T,T) + \left[\int_o^T r(t_1) h_*(T, t_1) dt_1 \right]^2 \right\} . \qquad (3\text{-}89)$$

From Equations (3-75), (3-78), and (3-89) the MMSE estimate of $m^2(T)$ is given by

$$\widehat{m^2}(T) = \frac{N_o}{2} h_*(T, T) + \left[\int_o^T h_*(T, t_1) r(t_1) dt_1 \right]^2 . \tag{3-90}$$

However, from Equation (3-52)

$$\hat{m}(T) = \int_o^T h_*(T, t_1) r(t_1) \, dt_1 . \tag{3-91}$$

Furthermore, it is known that [1]

$$\begin{aligned} \xi(T) &= E\left\{ [m(T) - \hat{m}(T)]^2 \mid r(t);\, 0 \le t \le T \right\} \\ &= \frac{N_o}{2} h^*(T, T) . \end{aligned} \tag{3-92}$$

Hence, Equation (3-90) may be expressed as

$$\widehat{m^2}(T) = \xi(T) + \hat{m}^2(T) \tag{3-93}$$

which is in agreement with Olsen [5] . MMSE estimation given by Equation (3-74) using the observation Equation (3-71) has been studied by Olsen [5] using a technique much more efficient than the one presented here. For this reason, no further consideration will be given to this class of MMSE estimation problems.

3.5 Systems of Differential Equations Describing the MMSE Estimate

In this section, Equations (3-22) - (3-24) are used to obtain differential equations for $\hat{g}(T, m(T))$. Two examples are considered. In the first, the method is illustrated by considering a linear example for which $g(T, m(T)) = AS(T, m(T)) = m(T)$ where m(t) is a zero-mean Gaussian process with a first-order Butterworth spectrum. Using results derived by Olsen [5], the Kalman filter equations are obtained. In the second, a nonlinear example is considered for which $g(T, m(T)) = AS(T, m(T)) = m^2(T)$ where m(t) is a zero-mean Gaussian process with a first-order Butterworth spectrum. It is shown that, since results comparable to those obtained by Olsen for the case $AS(T, m(T)) = m(T)$ are not available in the more general nonlinear case, the system of differential equations contains an infinite number of equations.

3.5.1 A Linear Example: First-Order Butterworth Process

In this section, the Kalman filter equations are derived for the MMSE estimate of a zero-mean Gaussian first-order Butterworth process in Gaussian white noise. The observation equation is

$$r(t) = m(t) + n(t) \quad , \quad 0 \le t \le T \tag{3-94}$$

where

$$K_m(t_1, t_2) = E\left[m(t_1)\, m(t_2)\right] = Pe^{-\alpha |t_1 - t_2|} \tag{3-95}$$

and

$$E\left[n(t_1)\, n(t_2)\right] = \frac{N_o}{2}\, \delta(t_1 - t_2) \ . \tag{3-96}$$

The MMSE estimate desired is

$$\hat{g}(T, m(T)) = \hat{m}(T) \ . \tag{3-97}$$

From Van Trees [1, p. 547], m(t) can be realized by the differential equation

$$\dot{m}(t) = -\alpha\, m(t) + w(t) \tag{3-98}$$

where w(t) is a zero-mean Gaussian white noise process with autocorrelation

$$E(w(t_1)\, w(t_2)) = 2\alpha\, P\, \delta(t_1 - t_2) \quad . \tag{3-99}$$

From Equations (3-22) - (3-24), the MMSE estimate of m(T) is

$$\hat{m}(T) = \frac{\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i}{\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i} \tag{3-100}$$

where

$$h_i(T, t_1, \ldots, t_i) = E\left\{ m(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^2(\tau)\, d\tau\right]\right\} \tag{3-101}$$

and

$$f_i(t_1, \ldots, t_i) = E\left\{ m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^2(\tau)\, d\tau\right]\right\} . \tag{3-102}$$

Recognizing the denominator of Equation (3-100) as the likelihood ratio, $\hat{m}(T)$ can be written as

$$\hat{m}\,\Lambda = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{3-103}$$

where the argument of $\hat{m}(T)$ and $\Lambda\,(r(t))$ have been suppressed for notational economy. Taking the derivative of Equation (3-103) with respect to T and making use of Equation (3-59), there results

$$\begin{aligned} \dot{\hat{m}}\,\Lambda + \hat{m}\,\dot{\Lambda} = {} & \frac{2}{N_o}\, r(T) \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, h_{i+1}(T, T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \\ & + \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, \dot{h}_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \end{aligned} \tag{3-104}$$

where the "dot" denotes $\frac{d}{dT}$.

From Equation (3-101) it follows that

$$\dot{h}_i(T, t_1, \ldots, t_i) = \frac{1}{N_o} E\left\{ m^3(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}$$

$$+ E\left\{ \left[-\alpha\, m(T) + w(T) \right] m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}.$$

(3-105)

Since $m(t)$ and $w(T)$ are statistically independent for $t < T$ it follows that

$$\dot{h}_i(T, t_1, \ldots, t_i) = -\frac{1}{N_o} E\left\{ m^3(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}$$

$$- \alpha E\left\{ m(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\} .$$

(3-106)

Consider the second series in Equation (3-104). Substituting Equation (3-106) back into this series and making use of Equations (3-22) - (3-24) there results

$$\sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\frac{2}{N_o} \right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, \dot{h}_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= -\frac{1}{N_o} \widehat{m^3} \Lambda - \alpha \hat{m} \Lambda . \qquad (3\text{-}107)$$

Similarly, $h_{i+1}(T, T, t_1, \ldots, t_i)$ can be written as

$$h_{i+1}(T, T, t_1, \ldots, t_i) = E\left[m^2(T)\, m(t_1) \ldots m(t_i) \exp\left(-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right) \right] .$$

(3-108)

From Equation (3-108) and Equations (3-22) - (3-24), the first series in Equation (3-104) is given by

$$\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, h_{i+1}(T, T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$= \widehat{m^2} \Lambda \ . \qquad (3\text{-}109)$$

From Equation (3-68)

$$\dot{\Lambda} = \frac{1}{N_o} \widehat{m^2} \Lambda + \frac{2}{N_o} r(T)\, \hat{m}\, \Lambda \ . \qquad (3\text{-}110)$$

Substituting Equations (3-107), (3-109), and (3-110) into Equation (3-104) there results

$$\dot{\hat{m}} = -\alpha\, \hat{m} + \frac{2}{N_o} \left[\left(\widehat{m^2} - \hat{m}^2 \right) r(T) - \frac{1}{2} \widehat{m^3} + \frac{1}{2} \widehat{m^2}\, \hat{m} \right] . \qquad (3\text{-}111)$$

From Olsen [5, Table 2.2]

$$\widehat{m^3} = \hat{m} \left(3\, \widehat{m^2} - 2\, \hat{m}^2 \right) . \qquad (3\text{-}112)$$

Substituting Equation (3-112) into Equation (3-111) yields

$$\hat{m} = -\alpha\, \hat{m} + \frac{2}{N_o} \left(\widehat{m^2} - \hat{m}^2 \right) \left(r(T) - \hat{m} \right) . \qquad (3\text{-}113)$$

To proceed, it is necessary to derive a differential equation for $\widehat{m^2} - \hat{m}^2$. To accomplish this, a differential equation is first derived for $\widehat{m^2}$. Let $g(T, m(T)) = m^2(T)$, then from Equations (3-22) - (3-24) it follows that

$$\widehat{m^2} \Lambda = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$(3\text{-}114)$$

where

$$h_i(T, t_1, \ldots, t_i) = E\left\{ m^2(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\} . \tag{3-115}$$

Taking the derivative of Equation (3-114) there results

$$\dot{\widehat{m^2}}\,\Lambda + \widehat{m^2}\,\dot{\Lambda} = \frac{2}{N_o}\, r(T)\, \widehat{m^3}\,\Lambda$$

$$+ \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_o^T \overset{i}{\cdots} \int_o^T r(t_1) \ldots r(t_i)\, \dot{h}_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i . \tag{3-116}$$

From Equation (3-115)

$$\dot{h}_i(T, t_1, \ldots, t_i) = -\frac{1}{N_o}\, E\left\{ m^4(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}$$

$$+ 2\, E\left\{ m(T) \left[-\alpha m(T) + w(T) \right] m(t_1) \ldots m(t_i) \left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\} . \tag{3-117}$$

Note that both $m(T)$ and $w(T)$ appear in the second expectation in Equation (3-117). Equation (3-117) can be written as

$$\dot{h}_i(T, t_1, \ldots, t_i) = -\frac{1}{N_o} E\left\{m^4(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^2(\tau)\, d\tau\right]\right\}$$

$$-2\alpha E\left\{m^2(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^2(\tau)\, d\tau\right]\right\}$$

$$+2\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{1}{N_o}\right)^j \overbrace{\int_0^T \cdots \int_0^T}^{j} E\left[w(T)\, m(T)\, m(t_1) \ldots m(t_i) m^2(t_{i+1}) \ldots m^2(t_{i+j})\right]$$

$$dt_{i+1} \ldots dt_{i+j} \, . \qquad (3\text{-}118)$$

Note that

$$E\left[w(T)\, m(T)\, m(t_1) \ldots m(t_i)\, m^2(t_{i+1}) \ldots m^2(t_{i+j})\right]$$

$$= E\left[w(T)\, m(T)\right] E\left[m(t_1) \ldots m(t_i)\, m^2(t_{i+1}) \ldots m^2(t_{i+j})\right]$$

$$+ \text{ terms involving } E\left[w(T)\, m(t_k)\right] \; ; \; k=1, \ldots, i+j \quad . \qquad (3\text{-}119)$$

Since $w(T)$ and $m(t_k)$; $k=1, \ldots, i+j$; are statistically independent, only the first term in Equation (3-119) is nonzero. Substituting Equation (3-119) into Equation (3-118) and recalling from Van Trees [1, p. 532] that

$$E\left[w(T)\, m(T)\right] = \alpha P, \qquad (3\text{-}120)$$

it follows that

$$\dot{h}_i(T, t_1, \ldots t_i) = -\frac{1}{N_o} E\left\{ m^4(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}$$

$$-2\alpha\, E\left\{ m^2(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o} \int_o^T m^2(\tau)\, d\tau \right] \right\}$$

$$+ 2\alpha P f_i(t_1, \ldots, t_i) . \qquad (3\text{-}121)$$

Substituting Equation (3-121) into Equation (3-116) and making use of Equations (3-22) - (3-24) and Equation (3-68) there results

$$\dot{\widehat{m^2}} = -2\alpha \widehat{m^2} - \frac{1}{N_o} \widehat{m^4} + 2\alpha P + \frac{2}{N_o} r(T)\, \widehat{m^3}$$

$$+ \frac{1}{N_o} \widehat{m^2}^{\,2} - \frac{2}{N_o} r(T)\, \hat{m}\, \widehat{m^2} . \qquad (3\text{-}122)$$

From Olsen [5, Table 2.2]

$$\widehat{m^4} = 3\, \widehat{m^2}^{\,2} - 2\, \hat{m}^4 . \qquad (3\text{-}123)$$

Substituting Equations (3-123) and (3-112) into Equation (3-122), $\dot{\widehat{m^2}}$ is given by

$$\dot{\widehat{m^2}} = -2\alpha \widehat{m^2} + \frac{2}{N_o} \left[\left(3\, \hat{m}\, \widehat{m^2} - 2\, \hat{m}^3 - \hat{m}\, \widehat{m^2} \right) r(T) \right.$$

$$\left. + \frac{1}{2} \left(\widehat{m^2}^{\,2} - 3\, \widehat{m^2}^{\,2} + 2\hat{m}^4 \right) \right] + 2\alpha P . \qquad (3\text{-}124)$$

Let

$$\zeta = \widehat{m^2} - \hat{m}^2 . \qquad (3\text{-}125)$$

It follows that

$$\dot{\zeta} = \dot{\widehat{m^2}} - 2\, \hat{m}\, \dot{\hat{m}} . \qquad (3\text{-}126)$$

Substituting Equation (3-124) and (3-113) into Equation (3-126) yields

$$\xi = -2\,\alpha\,\widehat{m^2} + 2\,\alpha\,\hat{m}^2 + \frac{1}{N_o}\left[-2\,\widehat{m^2}^{\,2} + 2\,\hat{m}^4 + 4\,\widehat{m^2}\,\hat{m}^2 - 4\,\hat{m}^4\right] + 2\,\alpha\,P$$

$$= -2\,\alpha\,\xi - \frac{2}{N_o}\,\xi^2 + 2\,\alpha\,P\ . \qquad (3\text{-}127)$$

Summarizing,

$$\dot{\hat{m}} = -\alpha\,\hat{m} + \frac{2}{N_o}\,\xi\left[r(T) - \hat{m}\right] \qquad (3\text{-}128)$$

$$\dot{\xi} = -2\,\alpha\,\xi - \frac{2}{N_o}\,\xi^2 + 2\,\alpha\,P \qquad (3\text{-}129)$$

in agreement with well known results [1].

Relations like Equations (3-112) and (3-122) are not generally available and the system of differential equations cannot be reduced to a finite system of equations as in the linear case. A nonlinear example which leads to an infinite system of equations is considered in the next section.

3.5.2 A Nonlinear Example - Squared First-Order Butterworth Process

In this paragraph, a nonlinear observation equation is considered. In particular, the received signal is assumed to be

$$r(t) = m^2(t) + n(t)\qquad ,\qquad 0 \le t \le T \qquad (3\text{-}130)$$

where m(t) is a zero mean Gaussian process with first-order Butterworth autocorrelation

$$K_m(t_1,\ t_2) = P\,e^{-\alpha|t_1 - t_2|} \qquad (3\text{-}131)$$

and where n(t) is zero mean Gaussian white noise with autocorrelation

$$E[n(t_1)\, n(t_2)] = \frac{N_o}{2}\, \delta(t_1 - t_2) \quad . \tag{3-132}$$

The desired MMSE estimate is

$$\hat{g}[T, m(T)] = \widehat{m^2}(T) \quad . \tag{3-133}$$

The objective is to obtain a system of differential equations whose solution yields $\hat{g}(T, m(T))$. From Equation (3-22) - (3-24), the MMSE estimate of $m^2(T)$ is

$$\widehat{m^2}(T) = \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i}{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, f_i(t_1, \ldots, t_i)\, dt_1 \ldots dt_i} \tag{3-134}$$

where

$$h_i(T, t_1, \ldots, t_i) = E\left\{ m^2(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[- \frac{2}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\} \tag{3-135}$$

and

$$f_i(t_1, \ldots, t_i) = E\left\{ m^2(t_1) \ldots m^2(t_i) \exp\left[- \frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\} \quad . \tag{3-136}$$

Recognizing the denominator of Equation (3-134) as the likelihood ratio, $m^2(T)$ can be written in the form

$$\widehat{m^2 \Lambda} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \quad . \tag{3-137}$$

Taking the derivative and making use of Equation (3-59) yields

$$\dot{\widehat{m^2 \Lambda}} + \widehat{m^2 \dot{\Lambda}} = \frac{2}{N_o} r(T) \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T$$

$$r(t_1) \ldots r(t_i)\, h_{i+1}(T, T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i$$

$$+ \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_1)$$

$$\dot{h}_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \quad . \tag{3-138}$$

From Equations (3-135), (3-98) and the argument leading to Equation (3-121) it follows that

$$\dot{h}_i(T, t_1, \ldots t_i) = -\frac{1}{N_o} E \left\{ m^6(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\}$$

$$- 2\alpha E \left\{ m^2(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\}$$

$$+ 2\alpha P E \left\{ m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\} \quad . \tag{3-139}$$

Also,

$$h_{i+1}(T, T, t_1, \ldots, t_i) = E\left\{m^4(T)\, m(t_1) \ldots m(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^4(\tau)\, d\tau\right]\right\} . \tag{3-140}$$

Substituting Equations (3-140) and (3-139) into Equation (3-138) and making use of Equation (3-22) - (3-24) results in

$$\dot{\widehat{m^2\Lambda}} + \widehat{m^2}\dot{\Lambda} = \frac{2}{N_o} r(T)\, \widehat{m^4\Lambda} - \frac{1}{N_o}\widehat{m^6\Lambda} - 2\alpha\, \widehat{m^2\Lambda} + 2\alpha P \Lambda . \tag{3-141}$$

From Equation (3-68)

$$\dot{\Lambda} = -\frac{1}{N_o}\widehat{m^4\Lambda} + \frac{2}{N_o} r(T)\, \widehat{m^2\Lambda} . \tag{3-142}$$

Substituting Equation (3-142) into Equation (3-141) and simplifying yields

$$\dot{\widehat{m^2}} = -2\alpha\, \widehat{m^2} + \frac{2}{N_o}\left(\widehat{m^4} - \widehat{m^2}^{\,2}\right) r(T) + \frac{1}{N_o}\left(\widehat{m^4}\,\widehat{m^2} - \widehat{m^6}\right)$$
$$+ 2\alpha P \quad . \tag{3-143}$$

Similar results can be derived for $\widehat{m^4}$ and $\widehat{m^6}$. For $\widehat{m^4}$, let $g(T, m(T)) = m^4(T)$. From Equations (3-22) - (3-24) it follows that

$$\widehat{m^4\Lambda} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\ldots} \int_0^T r(t_1) \ldots r(t_i)\, h_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \tag{3-144}$$

where

$$h_i(T, t_1, \ldots t_i) = E\left\{m^4(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o}\int_0^T m^4(\tau)\, d\tau\right]\right\} . \tag{3-145}$$

Taking the derivative of Equation (3-144) there results

$$\dot{\widehat{m^4 \Lambda}} + \widehat{m^4 \dot{\Lambda}} = \frac{2}{N_o} r(T) \widehat{m^6 \Lambda}$$

$$+ \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2}{N_o}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1) \ldots r(t_i)\, \dot{h}_i(T, t_1, \ldots, t_i)\, dt_1 \ldots dt_i \; . \tag{3-146}$$

From Equations (3-145) and (3-98), the derivative of $h_i(T, t_1, \ldots, t_i)$ is given by

$$\dot{h}_i(T, t_1, \ldots, t_i) = -\frac{1}{N_o} E\left\{ m^8(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\}$$

$$- 4\alpha E\left\{ m^4(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\}$$

$$+ 4 E\left\{ m^3(T)\, w(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\} . \tag{3-147}$$

To simplify the last term of Equation (3-147), first expand the exponential to obtain

$$E\left\{ m^3(T)\, w(T)\, m^2(t_1) \ldots m^2(t_i) \exp\left[-\frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right] \right\}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{1}{N_o}\right)^j \int_0^T \cdots \int_0^T E(m^3(T)\, w(T)\, m^2(t_1) \ldots m^2(t_i)\, m^4(t_{i+1}) \ldots$$

$$m^4(t_{i+j}))\, dt_{i+1} \ldots dt_{i+j} \; . \tag{3-148}$$

Since w(T) and m(t) are independent for t < T it follows that

$$E[\, m^3(T)\, w(T)\, m^2(t_1)\ldots m^2(t_i)\, m^4(t_{i+1})\ldots m^4(t_{i+j})]$$

$$= 3\, E[\, m(T)\, w(T)]\ E[\, m^2(T)\, m^2(t_1)\ldots m^2(t_i)\, m^4(t_{i+1})\ldots m^4(t_{i+j})] \quad . \tag{3-149}$$

Substituting Equation (3-149) into Equation (3-148) it follows that $\dot{h}_i(T,\, t_1, \ldots,\, t_i)$ can be written as

$$\begin{aligned}
\dot{h}_i(T, t_1, \ldots, t_i) = &- \frac{1}{N_o}\, E\left\{ m^8(T)\, m^2(t_1)\ldots m^2(t_i) \exp\left[- \frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right]\right\} \\
&- 4\alpha\, E\left\{ m^4(T)\, m^2(t_1)\ldots m^2(t_i) \exp\left[- \frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right]\right\} \\
&+ 12\alpha\, P\, E\left\{ m^2(T)\, m^2(t_1)\ldots m^2(t_i) \exp\left[- \frac{1}{N_o} \int_0^T m^4(\tau)\, d\tau \right]\right\} .
\end{aligned} \tag{3-150}$$

Substituting Equation (3-150) into Equation (3-146) and making use of Equations (3-22) - (3-24) and (3-142) results in

$$\begin{aligned}
\dot{\widehat{m^4}} = &-4\alpha\, \widehat{m^4} + \frac{2}{N_o}\left(\widehat{m^6} - \widehat{m^2}\, \widehat{m^4}\right) r(T) \\
&+ \frac{1}{N_o}\left(\widehat{m^4}^{\,2} - \widehat{m^8}\right) + 12\alpha\, P\, \widehat{m^2} \quad .
\end{aligned} \tag{3-151}$$

In general, $\dot{\widehat{m^{2n}}}$ can be written as

$$\dot{\widehat{m^{2n}}} = -2n\alpha\, \widehat{m^{2n}} + \frac{2}{N_o} (\widehat{m^{2(n+1)}} - \widehat{m^2}\, \widehat{m^{2n}})\, r(T)$$

$$+ \frac{1}{N_o} (\widehat{m^4}\, \widehat{m^{2n}} - \widehat{m^{2(n+2)}}) + (2n)(2n-1)\,\alpha P\, \widehat{m^{2(n-1)}} \quad . \quad (3\text{-}152)$$

for n = 1, 2. . . . Equation (3-152) represents an infinite system of coupled nonlinear differential equations for estimates of the moments of $m^2(T)$ conditioned on the observation record r(t); $0 \leq t \leq T$. Approximations to the solution of Equation (3-152) may be obtained by assuming a value for N such that $\widehat{m^{2n}} = 0$ for $n \geq N$ or by assuming a particular form for the conditional p.d.f. of $m^2(T)$ given r(t); $0 \leq t \leq T$, so that higher order moments can be expressed in terms of lower-order moments. These techniques have been applied [5] with some success.

3.6 MMSE Estimation for dc Processes - An Example: Estimating the Phase of a Sinusoid

Detection of nonlinear memoryless functionals of dc processes was considered in par. 2.3.3. In this paragraph, the estimation problem is considered. For the special case of dc processes

$$m(t) = m \quad , \qquad 0 \leq t \leq T \qquad (3\text{-}153)$$

where the probability density function (p.d.f.) of m is given by $p_m(M)$. Since there is only one term in the Karhunen-Loéve expansion of m(t), it follows from Equation (3-18) that the conditional p.d.f. of m conditioned on the measurement record $r(t) = A\,S(t, m) + n(t)$, $0 \leq t \leq T$, is given by

$$p_m(M \mid r(t);\ 0 \leq t \leq T)$$

$$= \frac{\exp\left[\frac{2A}{N_o}\int_0^T r(t)\,S(t, M)\,dt - \frac{A^2}{N_o}\int_0^T S^2(t, M)\,dt\right] p_m(M)}{\int_{-\infty}^{\infty} \exp\left[\frac{2A}{N_o}\int_0^T r(t)\,S(t, M)\,dt - \frac{A^2}{N_o}\int_0^T S^2(t, M)\,dt\right] p_m(M)\,dM} . \tag{3-154}$$

As an example, consider

$$S(t, m) = \cos(\omega_o t + m) \tag{3-155}$$

and

$$p_m(M) = \frac{1}{2\pi}, \qquad 0 \leq M \leq 2\pi$$

$$= 0 \qquad \text{elsewhere} . \tag{3-156}$$

The observation equation is then given by

$$r(t) = A\cos(\omega_o t + m) + n(t) . \tag{3-157}$$

Estimation of a constant random phase angle of a sinusoid embedded in additive white Gaussian noise has been considered by Abbate and Schilling [26] and also by Babcock [27] . In this section, after obtaining an expression for the MMSE estimate of m, performance of the estimator is derived using a new approach. The method used in this paragraph for deriving the estimator and its performance is more efficient than is the Volterra functional expansion.

Substituting Equations (3-155) and (3-156) into Equation (3-154) there results

$$p_m(M \mid r(t);\ 0 \le t \le T)$$

$$= \frac{\exp\left[\frac{2A}{N_o}\int_0^T r(t)\cos(\omega_o t + M)\,dt - \frac{A^2}{N_o}\int_0^T \cos^2(\omega_o t + M)\,dt\right]}{\int_0^{2\pi}\exp\left[r(t)\cos(\omega_o t + M)\,dt - \frac{A^2}{N_o}\int_0^T \cos^2(\omega_o t + M)\,dt\right]dM}\quad,\ 0 \le M \le 2\pi$$

$$= 0 \qquad\qquad ,\ \text{elsewhere}\ . \qquad (3\text{-}158)$$

From Equation (3-158) and the development leading to Equation (2-113) it follows that

$$p_m(M \mid r(t);\ 0 \le t \le T)$$

$$= \frac{\exp\left[\frac{2A}{N_o}(I\cos M - Q\sin M) - \frac{A^2 T}{2N_o}\cdot\frac{\sin\omega_o T}{\omega_o T}\cdot\cos(\omega_o T + 2M)\right]}{2\pi\left\{I_o\left(-\frac{A^2T}{2N_o}\cdot\frac{\sin\omega_oT}{\omega_oT}\right)I_o\left[\frac{2A}{N_o}(I^2+Q^2)^{1/2}\right] + 2\sum_{k=1}^{\infty} I_k\left(-\frac{A^2T}{2N_o}\cdot\frac{\sin\omega_oT}{\omega_oT}\right)I_{2k}\left[\frac{2A}{N_o}(I^2+Q^2)^{1/2}\right]\cdot\cos\left[k\left(2\tan^{-1}\frac{Q}{I} + \omega_oT\right)\right]\right\}}\quad,\ 0 \le M \le 2\pi$$

$$= 0\ , \qquad\qquad ,\ \text{elsewhere} \qquad (3\text{-}159)$$

where

$$I = \int_0^T r(t) \cos \omega_o t \, dt \tag{3-160}$$

$$Q = \int_0^T r(t) \sin \omega_o t \, dt \quad . \tag{3-161}$$

For $\omega_o T = n\pi$, $n \neq 0$, or $\omega_o T >> 1$, Equation (3-157) simplifies to

$$\begin{aligned} p_m(M \mid r(t);\ 0 \leq t \leq T) &= \frac{\exp\left[\frac{A^2 T}{N_o}(\overline{I} \cos M - \overline{Q} \sin M)\right]}{2\pi \ I_o\left[\frac{A^2 T}{N_o}\left(\overline{I}^2 + \overline{Q}^2\right)^{1/2}\right]} \quad , \qquad 0 \leq M \leq 2\pi \\ &= 0 \ , \qquad \text{elsewhere} \end{aligned} \tag{3-162}$$

where

$$\overline{I} = \frac{2}{AT} I \tag{3-163}$$

$$\overline{Q} = \frac{2}{AT} Q \quad . \tag{3-164}$$

From Equations (3-157), (3-160) and (3-163), the mean of $\overline{I}$ conditioned on m is given by

$$E(\overline{I} \mid m) = \frac{2}{AT} \int_0^T A \cos (\omega_o t + m) \cos \omega_o t \, dt \quad . \tag{3-165}$$

Carrying out the integration in Equation (3-165) there results

$$E(\overline{I}\,|\,m) = \cos m \,. \tag{3-166}$$

Similarly, the second moment of $\overline{I}$ conditioned on m follows as

$$E(\overline{I}^2\,|\,m) = \frac{4}{A^2T^2}\int_0^T\int_0^T\left[A^2\cos(\omega_o t_1 + m)\cos(\omega_o t_2 + m)\right.$$

$$\left. + \frac{N_o}{2}\,\delta(t_1 - t_2)\right]\cos\omega_o t_1 \cos\omega_o t_2 \; dt_1\, dt_2 \,. \tag{3-167}$$

Carrying out the integration in Equation (3-167) yields

$$E(\overline{I}^2\,|\,m) = E^2(\overline{I}\,|\,m) + \frac{N_o}{A^2T} \,. \tag{3-168}$$

From Equation (3-168) it follows that the variance of $\overline{I}$ conditioned on m is given by

$$\sigma^2_{\overline{I}|m} = \frac{N_o}{A^2T} \,. \tag{3-169}$$

Similarly, it can readily be shown that

$$E(\overline{Q}\,|\,m) = -\sin m \,, \tag{3-170}$$

$$\sigma^2_{\overline{Q}|m} = \frac{N_o}{A^2T} \tag{3-171}$$

$$E(\overline{I}\,\overline{Q}\,|\,m) = E(\overline{I}\,|\,m)\,E(\overline{Q}\,|\,m) \,. \tag{3-172}$$

Since, given m, $\overline{I}$ and $\overline{Q}$ are jointly Gaussian, Equations (3-166) and (3-169)-(3-172) completely describe the statistics of $\overline{I}$ and $\overline{Q}$.

The MMSE estimate of m is the conditional mean of m given r(t); $0 \le t \le T$. Therefore, from Equation (3-162),

$$\hat{m} = \frac{1}{2\pi} \int_0^{2\pi} M \frac{\exp\left[\frac{A^2T}{N_o} (\bar{I} \cos M - \bar{Q} \sin M)\right]}{I_o\left[\frac{A^2T}{N_o}\left(\bar{I}^2 + \bar{Q}^2\right)^{1/2}\right]} dM \; . \qquad (3\text{-}173)$$

Equation (3-173) is useful for obtaining the conditional bias and variance of $\hat{m}$ by numerical integration. The integration in Equation (3-173) can be carried out by first considering the generating function

$$\exp(Z \cos \theta) = I_o(Z) + 2 \sum_{k=1}^{\infty} I_k(Z) \cos k\theta \; . \qquad (3\text{-}174)$$

Substituting Equation (3-174) into Equation (3-173) there results

$$\hat{m} = \pi + 2 \sum_{k=1}^{\infty} \frac{I_k(R)}{I_o(R)} \frac{1}{2\pi} \int_0^{2\pi} M \cos\left[k(M + \theta)\right] dM \qquad (3\text{-}175)$$

where

$$R = \frac{A^2T}{N_o} \sqrt{\bar{I}^2 + \bar{Q}^2} \qquad (3\text{-}176)$$

$$\theta = \tan^{-1} \frac{\bar{Q}}{\bar{I}} \; . \qquad (3\text{-}177)$$

From the C.R.C. tables [28, Equations 389, 393]

$$\frac{1}{2\pi}\int_0^{2\pi} M \cos[k(M+\theta)]\, dM$$

$$= \cos k\theta \, \frac{1}{2\pi} \int_0^{2\pi} M \cos kM \, dM$$

$$- \sin k\theta \, \frac{1}{2\pi} \int_0^{2\pi} M \sin kM \, dM$$

$$= \frac{1}{k} \sin k\theta \, . \tag{3-178}$$

Substituting Equation (3-178) into Equation (3-175) yields

$$\hat{m} = \pi + 2 \sum_{k=1}^{\infty} \frac{I_k(R)}{k\, I_o(R)} \sin k\theta \, . \tag{3-179}$$

Note that for $A^2T/N_o = 0$ (no observations), $R = 0$ and $\hat{m} = \pi$, the a priori estimate. Also, for $A^2T/N_o = \infty$ (observations on $[0, \infty]$), it follows that the variances of $\overline{I}$ and $\overline{Q}$ are zero. This results in $\theta = -m$. In addition, $\lim_{R\to\infty} \frac{I_k(R)}{I_o(R)} = 1$. Consequently, for $A^2T/N_o \to \infty$ Equation (3-179) can be written as

$$\lim_{\frac{A^2T}{N_o}\to\infty} \hat{m} = \pi - 2 \sum_{k=1}^{\infty} \frac{\sin km}{k} \, . \tag{3-180}$$

From the CRC tables [28, p. 464] the right-hand side of Equation (3-180) is recognized as the Fourier series for m on $[0, 2\pi]$. Hence, $\hat{m} \to m$ as $\frac{A^2T}{N_o} \to \infty$. Equation (3-179) is plotted in Figure 3-1.

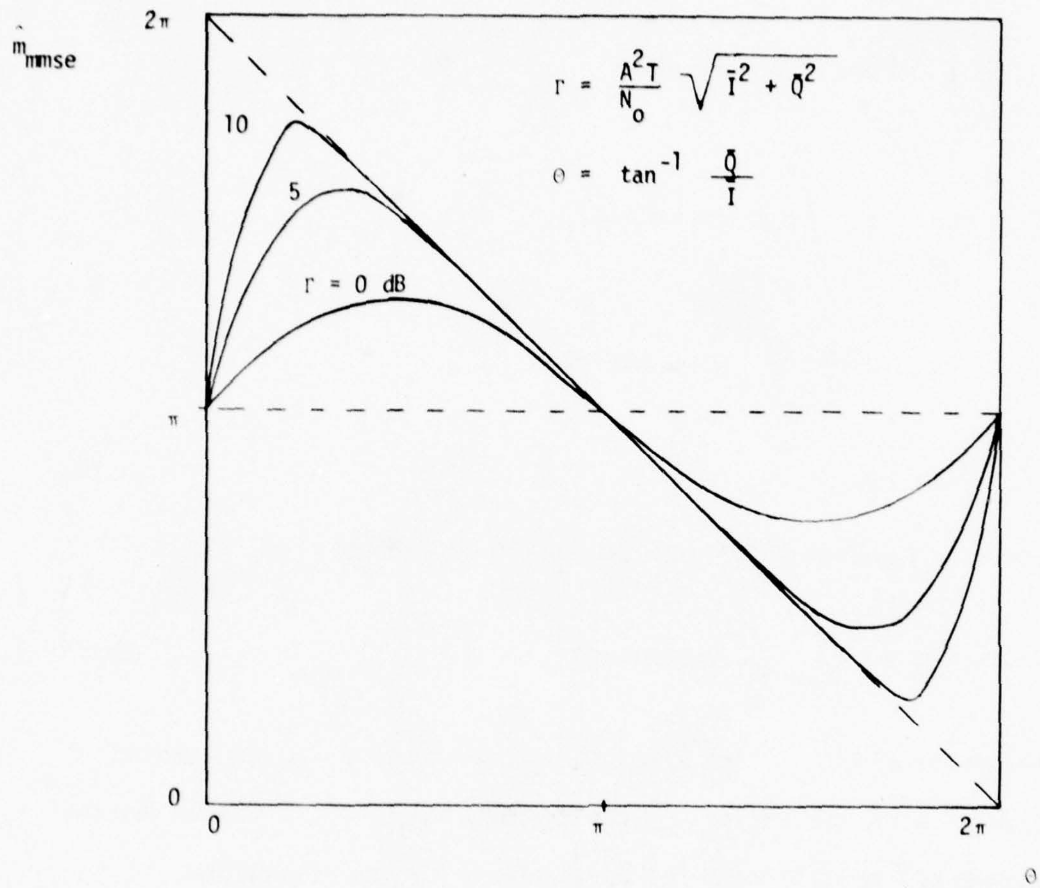

Figure 3-1. $\hat{m}_{mmse}$ vs θ Parametric in Signal-to-Noise Ratio (Γ)
$\omega_o T = n\pi$, $n \neq 0$ or $\omega_o T >> 1$

The bias and variance of $\hat{m}$ can be obtained by expanding the integrand of Equation (3-173) in a Taylor series about the conditional means of $\bar{I}$ and $\bar{Q}$ as suggested by Papoulis [19, p. 212]. Since the variance of $\bar{I}$ and $\bar{Q}$ approaches zero as A^2T/N_o approaches infinity, this approximation to the bias and variance of $\hat{m}$ is best for large SNR's, A^2T/N_o. From Papoulis [19] and the independence of $\bar{I}$ and $\bar{Q}$

$$\sigma^2_{\hat{m}|m} \simeq \left(\frac{\partial \hat{m}}{\partial \bar{I}} \Bigg|_{\substack{\bar{I} = E(\bar{I}|m) \\ \bar{Q} = E(\bar{Q}|m)}} \right)^2 \sigma^2_{\bar{I}|m} + \left(\frac{\partial \hat{m}}{\partial \bar{Q}} \Bigg|_{\substack{\bar{I} = E(\bar{I}|m) \\ \bar{Q} = E(\bar{Q}|m)}} \right)^2 \sigma^2_{\bar{Q}|m} \tag{3-181}$$

and

$$E(\hat{m}|m) \simeq \hat{m} \Bigg|_{\substack{\bar{I} = E(\bar{I}|m) \\ \bar{Q} = E(\bar{Q}|m)}} . \tag{3-182}$$

The mean squared estimation error is given by

$$\begin{aligned} \sigma^2_{\epsilon|m} &= E[(\hat{m} - m)^2 |m] \\ &= \sigma^2_{\hat{m}|m} + (E(\hat{m}|m) - m)^2 . \end{aligned} \tag{3-183}$$

Making use of Equations (3-166), (3-170) and (3-173) in Equation (3-182) there results

$$E(\hat{m}|m) = \frac{1}{2\pi} \int_0^{2\pi} X \frac{\exp\left(\frac{A^2T}{N_o} \cos(X - m)\right)}{I_o\left(\frac{A^2T}{N_o}\right)} dX . \tag{3-184}$$

Taking the partial derivative of m with respect to $\bar{I}$ in Equation (3-173) yields

$$\frac{\partial \hat{m}}{\partial \bar{I}} = \frac{1}{2\pi} \int_0^{2\pi} X \left\{ \frac{\left(\frac{A^2T}{N_o} \cos X\right)}{I_o\left(\frac{A^2T}{N_o}\sqrt{\bar{I}^2 + \bar{Q}^2}\right)} - \left(\frac{A^2T}{N_o} \frac{\bar{I}}{\sqrt{\bar{I}^2 + \bar{Q}^2}}\right) \left[\frac{I_1\left(\frac{A^2T}{N_o}\sqrt{\bar{I}^2 + \bar{Q}^2}\right)}{I_o^2\left(\frac{A^2T}{N_o}\sqrt{\bar{I}^2 + \bar{Q}^2}\right)}\right]\right\} \exp\left[\frac{A^2T}{N_o}(\bar{I}\cos X - \bar{Q}\sin X)\right] dX \,. \tag{3-185}$$

Substituting Equations (3-166) and (3-170) into Equation (3-185) it follows that

$$\left.\frac{\partial \hat{m}}{\partial \bar{I}}\right|_{\substack{\bar{I} = E(\bar{I} | m) \\ \bar{Q} = E(\bar{Q} | m)}} = \frac{\frac{A^2T}{N_o}}{I_o\left(\frac{A^2T}{N_o}\right)} \frac{1}{2\pi} \int_0^{2\pi} \left[X \cos X - \frac{I_1\left(\frac{A^2T}{N_o}\right)}{I_o\left(\frac{A^2T}{N_o}\right)} X \cos m\right] \exp\left[\frac{A^2T}{N_o}\cos(X - m)\right] dX \,. \tag{3-186}$$

Similarly, taking the partial of $\hat{m}$ with respect to $\bar{Q}$ there results

$$\frac{\partial \hat{m}}{\partial \bar{Q}} = \frac{1}{2\pi} \int_0^{2\pi} X \left\{ \frac{\left(- \frac{A^2 T}{N_o} \sin X \right)}{I_o \left(\frac{A^2 T}{N_o} \sqrt{\bar{I}^2 + \bar{Q}^2} \right)} - \left(\frac{A^2 T}{N_o} \frac{\bar{Q}}{\sqrt{\bar{I}^2 + \bar{Q}^2}} \right) \right.$$

$$\left. \left[\frac{I_1 \left(\frac{A^2 T}{N_o} \sqrt{\bar{I}^2 + \bar{Q}^2} \right)}{I_o^2 \left(\frac{A^2 T}{N_o} \sqrt{\bar{I}^2 + \bar{Q}^2} \right)} \right] \right\} \exp \left[\frac{A^2 T}{N_o} (\bar{I} \cos X - \bar{Q} \sin X) \right] dX \, . \tag{3-187}$$

Substituting Equations (3-166) and (3-170) into Equation (3-187) yields

$$\left. \frac{\partial \hat{m}}{\partial \bar{Q}} \right|_{\substack{\bar{I} = E(\bar{I} \mid m) \\ Q = E(\bar{Q} \mid m)}} = \frac{\frac{A^2 T}{N_o}}{I_o \left(\frac{A^2 T}{N_o} \right)} \frac{1}{2\pi} \int_0^{2\pi} \left[X \sin X - \frac{I_1 \left(\frac{A^2 T}{N_o} \right)}{I_o \left(\frac{A^2 T}{N_o} \right)} X \sin m \right]$$

$$\exp \left[\frac{A^2 T}{N_o} \cos (X - m) \right] dX \, . \tag{3-188}$$

Equation (3-184) was evaluated by numerical integration to provide the estimation bias plotted in Figure 3-2. Equations (3-186) and (3-188) were also evaluated by numerical integration and used in Equations (3-181) and (3-183) to provide the estimator variance and mean squared estimation error plotted in Figures 3-3 and 3-4, respectively. Note that the estimator bias, variance and mean squared estimation error are all conditioned on m and, therefore, are sample function dependent.

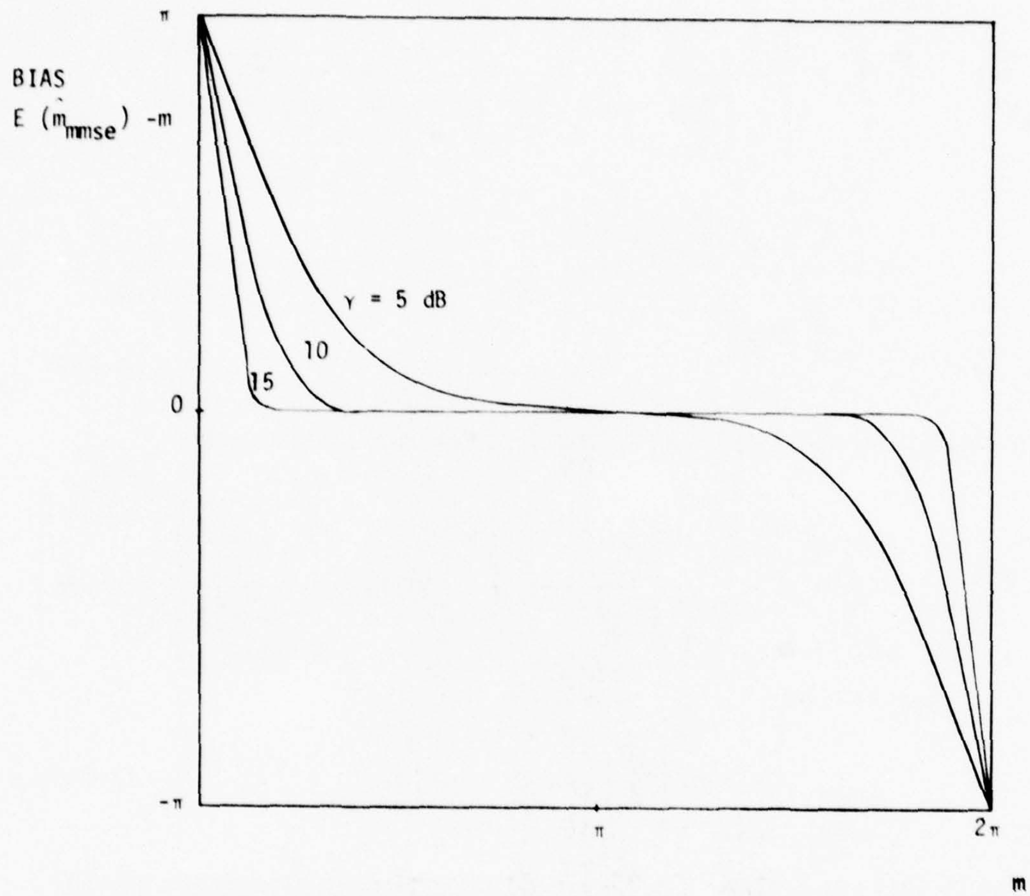

Figure 3-2. Estimation Bias vs m Parametric in SNR

$$\gamma = \frac{A^2T}{N_o} \qquad \omega_o T = n\pi,\ n \neq 0 \text{ or } \omega_o T >> 1$$

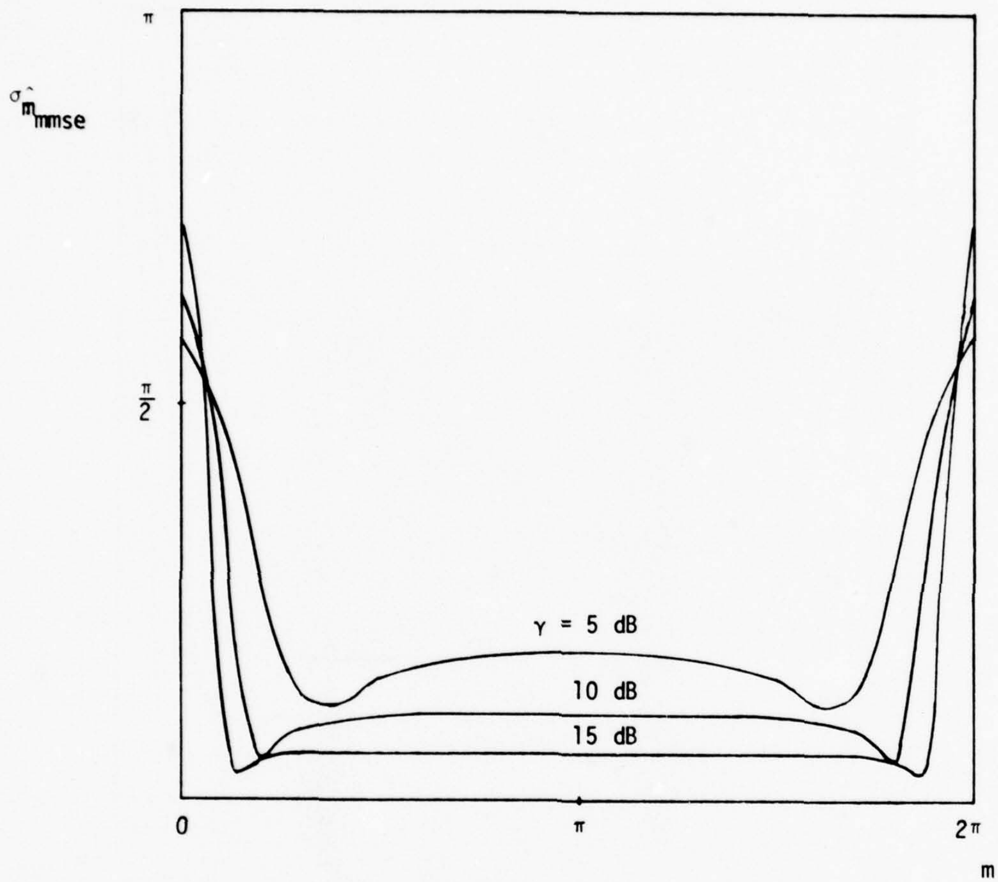

Figure 3-3. Standard Deviation of $\hat{m}_{mmse}$ vs m Parametric in SNR

$$\gamma = \frac{A^2 T}{N_o} \quad , \quad \omega_o T = n\pi,\ n \neq 0 \text{ or } \omega_o T >> 1$$

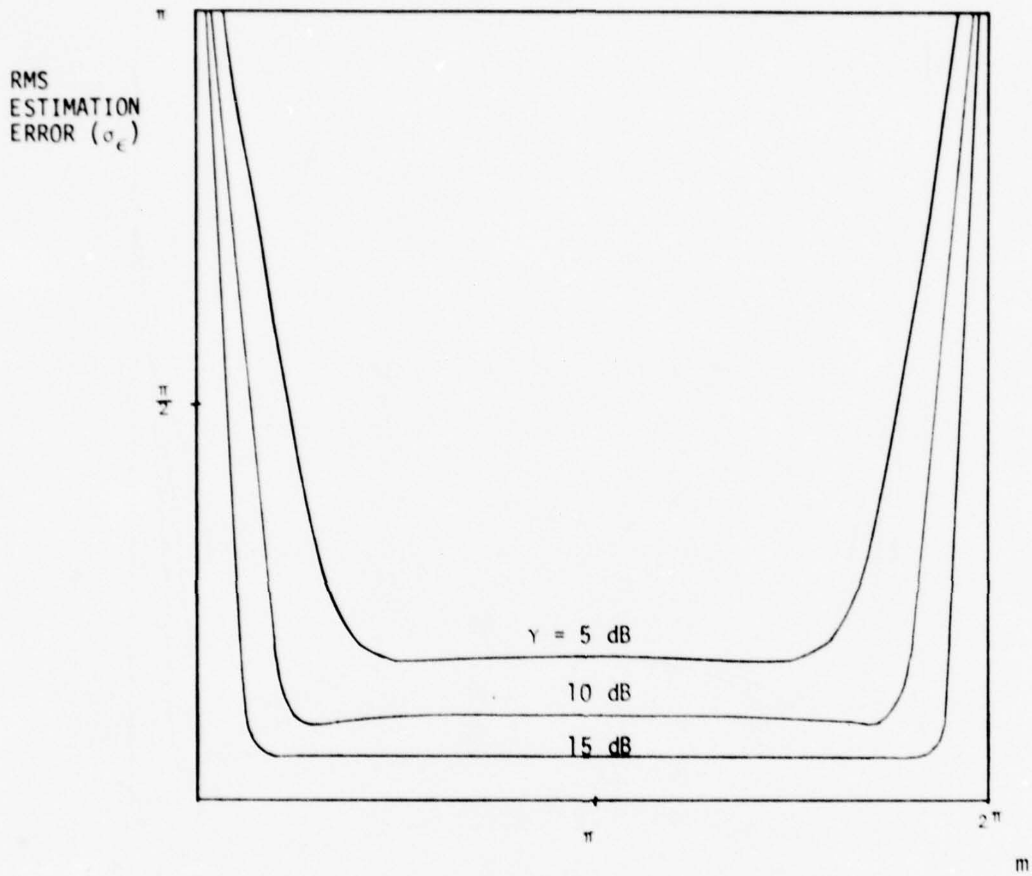

Figure 3-4. RMS Estimation Errors vs m Parametric in SNR

$$\gamma = \frac{A^2 T}{N_0} \quad \text{and} \quad \sigma_\epsilon = \sqrt{E\left((\hat{m}_{mmse} - m)^2 | m\right)}$$

$$\text{WHERE } \omega_0 T = n\pi,\ n \neq 0 \text{ or } \omega_0 T >> 1$$

CHAPTER IV

SUMMARY

4.1 Principal Results

In chapter II, par. 2.1, the hypothesis testing problem

$$H_1 : r(t) = A\,S\,[\,t,\ m(t)] + n(t),\ 0 \le t \le T \tag{4-1}$$

$$H_0 : r(t) = n(t),\ 0 \le t \le T \tag{4-2}$$

is considered where n(t) is a zero mean Gaussian process and m(t) and n(t) are independent processes with covariance

$$K_m(t_1, t_2) = E\,[\,m(t_1)\,m(t_2)\,] \tag{4-3}$$

$$K_n(t_1, t_2) = E\,[\,n(t_1)\,n(t_2)\,] = \frac{N_0}{2}\,\delta(t_1 - t_2)\,. \tag{4-4}$$

Using the Karhunen-Loève expansion, it is shown that the Volterra functional expansion for the likelihood ratio is given by

$$\Lambda[r(t)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_0}\right)^i \int_0^T \overset{i}{\cdots} \int_0^T r(t_1)\ldots r(t_i)\, f_i\,(t_1, \ldots, t_i)\; dt_1 \ldots dt_i \tag{4-5}$$

where the Volterra kernels, $f_i\,(t_1, \ldots, t_i)$ are expressed as

$$f_i\,(t_1, \ldots, t_i) = E\left(S[\,t_1,\ m(t_1)] \ldots S\,[\,t_i,\ m(t_i)]\ \exp\left\{-\frac{A^2}{N_0}\int_0^T S^2\,[\tau,\ m(\tau)]\,d\tau\right\}\right). \tag{4-6}$$

In par. 2.2, Equations (4-5) and (4-6) are reduced to the well known results for detection of a zero mean Gaussian process. In par. 2.3, three nonlinear examples are presented. In the first example, the first three Volterra kernels are obtained for detection for a hard-limited Gaussian process. In the second example, the first three Volterra kernels are obtained for detection of the absolute value of a Gaussian process. In the last example, a sinusoid phase modulated by a dc process uniform on $[0, 2\pi]$ is considered. The likelihood ratio is derived for this example. For the special case $\omega_o T = n\pi$, $n \neq 0$ or $\omega_o T >> 1$ it is shown that the performance of a receiver based on the likelihood ratio is identical to the performance of a receiver based on the trucated Volterra expansion of Equation (4-5) provided that at least two terms are used in the expansion.

In par. 2.4, Sun's theorem is used to sum the series in Equation (4-5) for large T. This is accomplished for the special case where the Volterra kernels have the form

$$f_i(t_1, \ldots, t_i) \cong K\,E\left\{S[t_1, m(t_1)] \ldots S[t_i, m(t_i)]\right\}, \tag{4-7}$$

where K is some constant independent of the index i, and for nonlinearities $S(t, .)$ and processes $m(t)$ which satisfy Sun's theorem. As an example, the asymptotic receiver is derived for a sinusoid phase modulated by a first-order Butterworth process. The asymptotic performance of this asymptotic receiver is also obtained and compared with the performance of a suboptimum detector.

In chapter III, par. 3.1, the problem of finding the minimum-mean squared error (MMSE) estimate

$$\hat{g}[T, m(T)] = E\left\{g[T, m(T)] \mid r(t) ; 0 \leq t \leq T\right\} \tag{4-8}$$

is considered where the observation equation is given by

$$r(t) = A\,S\,[t,\ m(t)] + n(t). \tag{4-9}$$

Using the Karhunen-Loève expansion and Bayes' law, it is shown that

$$\hat{g}[T,\ m(T)] = \frac{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_0}\right)^i \int_0^T \overset{i}{\ldots} \int_0^T r(t_1)\ldots r(t_i)\, h_i\,(T,\ t_1,\ \ldots,\ t_i)\, dt_1 \ldots dt_i}{\displaystyle\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_0}\right)^i \int_0^T \overset{i}{\ldots} \int_0^T r(t_1)\ldots r(t_i)\, f_i\,(t_1,\ \ldots,\ t_i)\, dt_1 \ldots dt_i} \tag{4-10}$$

where

$$h_i(T, t_1, \ldots, t_i) = E\left(g[T, m(T)]\, S\,[\,t_1, m(t_1)] \ldots S\,[t_i, m\,(t_i)] \exp\left\{ - \frac{A^2}{N_0} \int_0^T S^2\,[\tau,\ m(\tau)]\ d\tau \right\}\right) \tag{4-11}$$

and where $f_i\ (t_1,\ \ldots,\ t_i)$ is given by Equation (4-6). It is shown that Equation (4-10) can be reduced to

$$\hat{g}\,[T,\ m(T)] = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{2A}{N_0}\right)^i \int_0^T \overset{i}{\ldots} \int_0^T r(t_1)\ldots r(t_i)\ e_i\ (t_1, \ldots, t_i)\ dt_1 \ldots dt_i \tag{4-12}$$

where

$$e_0 = \frac{h_0(T)}{f_0} \tag{4-13}$$

$$e_1\,(t_1) = \frac{h_1\,(T, t_1) - f_1\,(t_1)\, e_0}{f_0} \tag{4-14}$$

and, in general,

$$e_i(t_1, \ldots, t_i) = \frac{h_i(T, t_1, \ldots t_i)}{f_0} - \sum_{k=1}^{i-1} \binom{i}{k} \frac{f_k(t_1, \ldots, t_k)\, e_{i-k}(t_{k+1}, \ldots, t_i)}{f_0} - \frac{f_i(t_1, \ldots, t_i)\, e_0}{f_0} \quad . \tag{4-15}$$

In par. 3.2, Equations (4-10), (4-11) and (4-6) are reduced to well known results for MMSE estimation of a Gaussian process using linear observations. In par. 3.3, Equation (4-10) is used to obtain a stochastic differential equation for the logarithm of the likelihood ratio. In par. 3.4, Equation (4-10) is used to determine the MMSE estimate of the square of a zero mean Gaussian process using linear observations. This result is shown to be in agreement with the work of Olsen [5] . In par. 3.5, Equation (4-10) is used to obtain systems of coupled nonlinear differential equations for the MMSE estimate. Two examples are considered. In the first, the Kalman filter equations are derived for the MMSE estimate of a first-order Butterworth process with a linear observation equation. In the second, an infinite system of differential equations is derived for the MMSE estimate of the square of a first-order Butterworth process with the observation equation $r(t) = m^2(t) + n(t)$. In par. 3.6, the MMSE estimate of the dc phase of a sinusoid is derived and the estimator bias, variance and mean squared error are obtained.

4.2 Suggestions for Future Work

The results of par. 2.4 apply only for large T. It has not been possible, however, to determine how large T must be for the asymptotic results to be good approximations. A useful extension of the results of this par. would be an estimate of the length of the observation interval required for the asymptotic results to apply. This might be accomplished for a particular problem by a Monte Carlo simulation of the asymptotic receiver to obtain performance. Another interesting extension of the results of par. 2.4 would consider frequency modulation of a sinusoid.

An interesting application of the results of par. 3.1 would be found in estimating the phase of a sinusoid. For large T, the Volterra kernels in Equation (4-12) can be approximated by the approach utilized in par. 2.4.3 leading to Equation (2-207). In principle, any number of Volterra kernels can be approximated using this method.

APPENDIX A

ASYMPTOTIC NORMALITY OF x_i AND y_i

A.1 Sun's Theorem

The asymptotic behavior of x_i and y_i are determined from Sun's theorem [24]. Sun's theorem is stated as follows.

Let m(t) be a real, stationary Gaussian process which is continuous in the second mean, i.e.,

1. $\lim_{t \to t_o} E\left\{ [m(t) - m(t_o)]^2 \right\} = 0 \qquad \forall t, t_o$

and

2. $E[m(t)] = 0 \qquad \forall t$

3. $R_m(\tau) = E[m(t)\, m(t+\tau)] = \int_{-\infty}^{\infty} e^{j\omega\tau} S_m(\omega)\, d\omega$

   $\forall t, \tau$

   with $\int_{-\infty}^{\infty} S_m^2(\omega)\, d\omega < \infty$

Let g(t, m(t)) be a time-dependent memoryless function that satisfies

4. $E\{g[t, m(t)]\} = 0 \qquad \forall t$

5. $E\{g^2[t, m(t)]\} < \infty \qquad \forall t$

6. $\lim_{t \to t_o} E\left\{ \left(g[t, m(t)] - g[t_o, m(t_o)] \right)^2 \right\} = 0 \qquad \forall t_o, t$

7. A $p > 0$ exists such that

   $g[t + np, m(t+np)] = g[t, m(t+np)]$

   for $n = 0, \pm 1, \pm 2, \ldots$

Let

$$S_m^*(\omega) = \sum_{n=-\infty}^{\infty} S_m(\omega + \frac{2n\pi}{p}) \quad . \tag{A-1}$$

If

8. $$\int_{-\pi/p}^{\pi/p} S_m^{*2}(\omega)\, d\omega < \infty$$

and

9. $$\lim_{N\to\infty} \frac{1}{2\pi N} \int_{-\pi/p}^{\pi/p} \left(\frac{\sin \frac{N}{2}\omega}{\sin \frac{1}{2}\omega} \right)^2 S_m^*(\omega)\, d\omega$$

exists and is finite

then as $T \to \infty$

$$Z_T = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} g[t, m(t)]\, dt \tag{A-2}$$

is asymptotically Gaussian with variance

$$\sigma_Z^2 = \lim_{T\to\infty} E(Z_T^2) < \infty \quad . \tag{A-3}$$

From Fitelson, [29] condition 1 implies that $R_m(\tau)$ is continuous. Also, from Fitelson, [30] condition 8 is related to the large frequency behavior of $S_m(\omega)$. For instance if

$$S_m(\omega) = \frac{C_1}{1 + C_2 |\omega|^{\alpha}} \tag{A-4}$$

where C_1, $C_2 > 0$ and $\alpha > 1$, then $S^*_m(\omega)$ would satisfy

$$S^*_m(\omega) \leq \sum_{n=-\infty}^{\infty} \frac{C_1}{1 + C_2 \left| \omega + \frac{2n\pi}{P} \right|^{\alpha}} < \infty \tag{A-5}$$

and condition 8 follows. In condition 9, note that

$$\lim_{N \to \infty} \frac{1}{2\pi N} \left(\frac{\sin \frac{N}{2} \omega}{\sin \frac{1}{2} \omega} \right)^2 = \delta(\omega) \quad . \tag{A-6}$$

Thus, condition 9 is also satisfied if Equation (A-4) applies.

A. 2 Joint Asymptotic Normality of x_i and y_i in par. 2.4.3 for $S[t, m(t)] = \cos[\omega_o t + m(t)]$

In this paragraph it is shown that

$$x_i = \frac{2A}{N_o} \int_0^T \{ A \cos[\omega_o t + m(t)] + n(t) \} \cos \frac{2\pi i}{T} t \, dt \tag{A-7}$$

$$y_i = \frac{2A}{N_o} \int_0^T \{ A \cos[\omega_o t + m(t)] + n(t) \} \sin \frac{2\pi i}{T} t \, dt \tag{A-8}$$

where $i = 0, 1, \ldots$ are asymptotically jointly Gaussian. It is assumed that $m(t)$ and $n(t)$ are independent zero mean Gaussian processes with

$$\begin{aligned} E[m(t)\, m(t+\tau)] &= R_m(\tau) \\ E[n(t)\, n(t+\tau)] &= \frac{N_o}{2} \delta(\tau) \end{aligned} \tag{A-9}$$

First note that x_i and y_i can be written as the sum of two random variables. Specifically

$$x_i = \frac{2A^2\sqrt{T}}{N_o} \frac{1}{\sqrt{T}} \int_0^T \cos[\omega_o t + m(t)] \cos \frac{2\pi i}{T} t \, dt$$

$$+ \frac{2A\sqrt{T}}{N_o} \frac{1}{\sqrt{T}} \int_0^T n(t) \cos \frac{2\pi i}{T} t \, dt \qquad \text{(A-10)}$$

and

$$y_i = \frac{2A^2\sqrt{T}}{N_o} \frac{1}{\sqrt{T}} \int_0^T \cos[\omega_o t + m(t)] \sin \frac{2\pi i}{T} t \, dt$$

$$+ \frac{2A\sqrt{T}}{N_o} \cdot \frac{1}{\sqrt{T}} \int_0^T n(t) \sin \frac{2\pi i}{T} t \, dt \quad . \qquad \text{(A-11)}$$

since the terms in Equations (A-10) and (A-11) involving n(t) are jointly Gaussian for $i = 0, 1, \ldots$, it remains to show that the first terms involving m(t) in Equations (A-10) and (A-11) are asymptotically jointly Gaussian in order to show that x_i and y_i are asymptotically jointly Gaussian for $i = 0, 1, \ldots$ By the Cramer-Wold Theorem* it is sufficient to show that

$$\overline{Z}_T = \frac{1}{\sqrt{T}} \int_0^T \overline{g}[t, m(t)] = \frac{2A^2\sqrt{T}}{N_o} \sum_{i=0}^{N} a_i \left\{ \frac{1}{\sqrt{T}} \int_0^T \cos[\omega_o t + m(t)] \right.$$

$$\left. \cos \frac{2\pi i}{T} t \, dt \right\} + b_i \left\{ \frac{1}{\sqrt{T}} \int_0^T \cos[\omega_o t + m(t)] \sin \frac{2\pi i}{T} t \, dt \right\} \qquad \text{(A-12)}$$

---

*In the Cramer-Wold Theorem, if $Z = \sum_{i=1}^{N} a_i x_i$ is Gaussian for any $a_1, \ldots, a_N$, then x_i, $i = 1, \ldots, N$ are jointly Gaussian. See, for example, [19, p. 231, problem 7-23.]

is asymptotically Gaussian for any arbitrary a_i, b_i, $i = 0, 1, \ldots, N$ to show that $\left\{ \frac{1}{\sqrt{T}} \int_0^T \cos [\omega_o t + m(t)] \cos \frac{2\pi i}{T} t \, dt \right\}$ and

$\left\{ \frac{1}{\sqrt{T}} \int_0^T \cos [\omega_o t + m(t)] \sin \frac{2\pi i}{T} t \, dt \right\}$ for $i = 0, 1, \ldots, N$ are asymptotically jointly Gaussian. Assume that the process $m(t)$ satisfies conditions (1) - (3) of Sun's theorem. Define $g(t, m(t))$ to be

$$g[t, m(t)] = \bar{g}[t, m(t)] - E\{\bar{g}[t, m(t)]\} \tag{A-13}$$

Observe that conditions (4) and (5) are satisfied [condition (5) is satisfied since $\bar{g}(t, m(t))$ is bounded]. Condition (6) is satisfied because the continuous and bounded behavior of $g(t, m(t))$ guarantees the continuity of $R_g(t, t_o)$ for all t and t_o. To see that condition (7) is satisfied, let $\omega_o = \frac{2\pi M}{KT}$ for some M, K and let $p = KT$.* Then, from Equations (A-12) and (A-13)

$$\begin{aligned}
& g[t + np, m(t + np)] \\
&= \frac{2A^2\sqrt{T}}{N_o} \sum_{i=0}^{N} a_i \left\{ \cos\left[\frac{2\pi M}{KT} t + \frac{2\pi M}{KT} \cdot nKT + m(t + np)\right] \right. \\
&\qquad \left. \cos\left(\frac{2\pi i}{T} t + \frac{2\pi i}{T} \cdot nKT\right) dt \right\} \\
&\quad + b_i \left\{ \sin\left[\frac{2\pi M}{KT} t + \frac{2\pi M}{KT} \cdot nKT + m(t + np)\right] \right. \\
&\qquad \left. \sin\left(\frac{2\pi i}{T} t + \frac{2\pi i}{T} \cdot nKT\right) dt \right\} \\
&\quad - E\{\bar{g}[t + np, m(t + np)]\} \\
&= g[t, m(t + np)]
\end{aligned} \tag{A-14}$$

---

* Fitelson has conjectured, [29] in a similar context, that Sun's Theorem applies also for $\omega_o \in R$ rather than just for the dense set used here. This dense set is not very restrictive since $2\pi M/KT$ can be made arbitrarily close to any $\omega_o \in R$ by a suitable choice of M and K.

Assume that the spectrum $S_m(\omega)$ satisfies conditions (8) and (9). (Any spectrum satisfying Equation (A-4) will also satisfy conditions (8) and (9).) One example is the first-order Butterworth process for which

$$S_m(\omega) = \frac{2\alpha P}{\alpha^2 + \omega^2} \tag{A-15}$$

which satisfies Equation (A-4) by inspection. Hence, it follows from Sun's theorem that $\overline{Z}_T$ is asymptotically Gaussian for all a_i, b_i, and it follows from the Cramer-Wold theorem that

$$\begin{aligned} &\frac{1}{\sqrt{T}} \int_0^T \cos\left[\omega_o t + m(t)\right] \cos \frac{2\pi i}{T} t \, dt \\ &\frac{1}{\sqrt{T}} \int_0^T \cos\left[\omega_o t + m(t)\right] \sin \frac{2\pi i}{T} t \, dt \end{aligned} \tag{A-16}$$

are asymptotically jointly Gaussian for $i = 0, 1, \ldots, N$ and N arbitrarily large, but finite, and for $\omega_o = \frac{2\pi M}{KT}$. This in turn implies that x_i and y_i, as given by Equations (A-7) and (A-8) are asymptotically jointly Gaussian.

APPENDIX B

VOLTERRA KERNELS FOR A PHASE-MODULATED SINUSOID UP TO THIRD ORDER

Assume

$$S[t, m(t)] = \cos[\omega_o t + m(t)] \tag{B-1}$$

where m(t) is a zero-mean Gaussian process and

$$E[m(t_1)\, m(t_2)] = R_m(t_1, t_2) \quad . \tag{B-2}$$

B.1 Evaluation of $d_1(t_1)$

By definition

$$d_1(t_1) = E\{\cos[\omega_o t_1 + m(t_1)]\}$$

$$= \cos \omega_o t_1 \; E\left[\frac{e^{jm(t_1)} + e^{-jm(t_1)}}{2}\right]$$

$$- \sin \omega_o t_1 \; E\left[\frac{e^{jm(t_1)} - e^{-jm(t_1)}}{2j}\right] \quad . \tag{B-3}$$

By assumption, $m(t_1) = m_1$ is a zero-mean Gaussian random variable with probability density function

$$p(m_1) = \frac{1}{\sqrt{2\pi R_m(t_1, t_1)}} \; e^{-\frac{m_1^2}{2R_m(t_1, t_1)}} \quad . \tag{B-4}$$

In addition, observe that

$$E\left(e^{jm_1}\right) = \phi_{m_1}(1) = e^{-\frac{1}{2} R_m(t_1, t_1)} \tag{B-5}$$

where $\phi_{m_1}(\omega)$ is the characteristic function associated with $p(m_1)$. Use of Equation (B-5) in Equation (B-3) yields

$$d_1(t_1) = e^{-\frac{1}{2} R_m(t_1, t_1)} \cos \omega_o t_1 \quad . \tag{B-6}$$

B.2 Evaluation of $d_2(t_1, t_2)$

By definition,

$$d_2(t_1, t_2) = E\{\cos[\omega_o t_1 + m(t_1)] \cos[\omega_o t_2 + m(t_2)]\} \quad . \tag{B-7}$$

Using trigonometric identities, Equation (B-7) becomes

$$\begin{aligned} d_2(t_1, t_2) = {} & \cos \omega_o t_1 \cos \omega_o t_2 \ E[\cos m(t_1) \cos m(t_2)] \\ & - \sin \omega_o t_1 \cos \omega_o t_2 \ E[\sin m(t_1) \cos m(t_2)] \\ & - \cos \omega_o t_1 \sin \omega_o t_2 \ E[\cos m(t_1) \sin m(t_2)] \\ & + \sin \omega_o t_1 \sin \omega_o t_2 \ E[\sin m(t_1) \sin m(t_2)] \quad . \end{aligned} \tag{B-8}$$

Let $m(t_1) = m_1$, $m(t_2) = m_2$. Then

$$\begin{aligned} E(\cos m_1 \cos m_2) &= \frac{1}{4} E\left[\left(e^{jm_1} + e^{-jm_1}\right)\left(e^{jm_2} + e^{-jm_2}\right)\right] \\ &= \frac{1}{4}\left[E\left(e^{j(m_1 + m_2)}\right) + E\left(e^{j(m_1 - m_2)}\right)\right. \\ &\quad + E\left(e^{j(-m_1 + m_2)}\right) \\ &\quad \left. + E\left(e^{j(-m_1 - m_2)}\right)\right] \quad . \end{aligned} \tag{B-9}$$

Since m(t) is a zero-mean Gaussian random process, m_1 and m_2 are zero-mean Gaussian random variables with probability density function

$$p(m_1, m_2) = \frac{1}{2\pi |K_m|^{\frac{1}{2}}} e^{-\frac{1}{2}(m_1, m_2) K_m^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}} \tag{B-10}$$

where

$$K_m = \begin{bmatrix} R_m(t_1, t_1) & R_m(t_1, t_2) \\ R_m(t_2, t_1) & R_m(t_2, t_2) \end{bmatrix} \tag{B-11}$$

and the corresponding characteristic function is

$$\begin{aligned} \phi(\omega_1, \omega_2) &= E\left(e^{j(\omega_1 m_1 + \omega_2 m_2)}\right) \\ &= e^{-\frac{1}{2}(\omega_1 \ \omega_2) K_m \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}} \quad . \end{aligned} \tag{B-12}$$

It follows that

$$\phi(1, 1) = E\left(e^{j(m_1 + m_2)}\right) = e^{-\frac{1}{2}\left[R_m(t_1, t_1) + R_m(t_2, t_2) + 2R_m(t_1, t_2)\right]} \tag{B-13}$$

$$\phi(1, -1) = E\left(e^{j(m_1 - m_2)}\right) = e^{-\frac{1}{2}\left[R_m(t_1, t_1) + R_m(t_2, t_2) - 2R_m(t_1, t_2)\right]} \tag{B-14}$$

$$\phi(-1, -1) = E\left(e^{-j(m_1 + m_2)}\right) = E^*\left(e^{j(m_1 + m_2)}\right) \tag{B-15}$$

$$\phi(-1, 1) = E\left(e^{-j(m_1 - m_2)}\right) = E^*\left(e^{j(m_1 - m_2)}\right) \tag{B-16}$$

where * denotes complex conjugation. Use of Equation (B-13) through (B-16) in Equation (B-9) results in

$$E(\cos m_1 \cos m_2) = e^{-\frac{1}{2}\left[R_m(t_1, t_1) + R_m(t_2, t_2)\right]} \cosh\left[R_m(t_1, t_2)\right] \quad . \tag{B-17}$$

Similarly,

$$E(\cos m_1 \sin m_2) = \frac{1}{4j} [\phi(1, 1) - \phi(1, -1) + \phi(-1, 1) - \phi(-1, -1)] \quad . \tag{B-18}$$

Use of Equations (B-13) through (B-16) in Equation (B-18) yields

$$E(\cos m_1 \sin m_2) = 0 \quad . \tag{B-19}$$

In a similar manner,

$$E(\sin m_1 \cos m_2) = 0 \quad . \tag{B-20}$$

Finally,

$$E(\sin m_1 \sin m_2) = -\frac{1}{4} [\phi(1, 1) - \phi(1, -1) - \phi(-1, 1) + \phi(-1, -1)] \quad . \tag{B-21}$$

Use of Equations (B-13) through (B-16) in Equation (B-21) results in

$$E(\sin m_1 \sin m_2) = e^{-\frac{1}{2}[R_m(t_1, t_1) + R_m(t_2, t_2)]} \sinh[R_m(t_1, t_2)] \quad . \tag{B-22}$$

Substituting the above results into Equation (B-8), it follows that

$$\begin{aligned} d_2(t_1, t_2) = {} & e^{-\frac{1}{2}[R_m(t_1, t_1) + R_m(t_2, t_2)]} \\ & \{\cosh[R_m(t_1, t_2)] \cos \omega_o t_1 \cos \omega_o t_2 \\ & + \sinh[R_m(t_1, t_2)] \sin \omega_o t_1 \sin \omega_o t_2\} \quad . \end{aligned} \tag{B-23}$$

B.3 <u>Evaluation of $d_3(t_1, t_2, t_3)$</u>

By definition,

$$d_3(t_1, t_2, t_3) = E\{\cos[\omega_o t_1 + m(t_1)] \cos[\omega_o t_2 + m(t_2)] \cos[\omega_o t_3 + m(t_3)]\}. \tag{B-24}$$

From trigonometric identities, Equation (B-24) becomes

$$d_3(t_1, t_2, t_3) = E[(\cos \omega_o t_1 \cos m_1 - \sin \omega_o t_1 \sin m_1)$$
$$(\cos \omega_o t_2 \cos m_2 - \sin \omega_o t_2 \sin m_2)$$
$$(\cos \omega_o t_3 \cos m_3 - \sin \omega_o t_3 \sin m_3)] \qquad \text{(B-25)}$$

where $m_i = m(t_i)$, $i = 1, 2, 3$. Carrying out the products in Equation (B-25) results in

$$\begin{aligned}
d_3(t_1, t_2, t_3) = {} & \cos \omega_o t_1 \cos \omega_o t_2 \cos \omega_o t_3 \; E(\cos m_1 \cos m_2 \cos m_3) \\
& - \sin \omega_o t_1 \cos \omega_o t_2 \cos \omega_o t_3 \; E(\sin m_1 \cos m_2 \cos m_3) \\
& - \cos \omega_o t_1 \sin \omega_o t_2 \cos \omega_o t_3 \; E(\cos m_1 \sin m_2 \cos m_3) \\
& - \cos \omega_o t_1 \cos \omega_o t_2 \sin \omega_o t_3 \; E(\cos m_1 \cos m_2 \sin m_3) \\
& + \sin \omega_o t_1 \sin \omega_o t_2 \cos \omega_o t_3 \; E(\sin m_1 \sin m_2 \cos m_3) \\
& + \sin \omega_o t_1 \cos \omega_o t_2 \sin \omega_o t_3 \; E(\sin m_1 \cos m_2 \sin m_3) \\
& + \cos \omega_o t_1 \sin \omega_o t_2 \sin \omega_o t_3 \; E(\cos m_1 \sin m_2 \sin m_3) \\
& - \sin \omega_o t_1 \sin \omega_o t_2 \sin \omega_o t_3 \; E(\sin m_1 \sin m_2 \sin m_3) \; .
\end{aligned} \qquad \text{(B-26)}$$

Note that the joint probability density function of m_1, m_2, and m_3 is

$$p(m_1, m_2, m_3) = \frac{1}{(2\pi)^{3/2} |K_m|^{1/2}} \, e^{-\frac{1}{2} (m_1 \; m_2 \; m_3) K_m^{-1} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}} \qquad \text{(B-27)}$$

where the covariance matrix is

$$K_m = \begin{bmatrix} R_m(t_1, t_1) & R_m(t_1, t_2) & R_m(t_1, t_3) \\ R_m(t_2, t_1) & R_m(t_2, t_2) & R_m(t_2, t_3) \\ R_m(t_3, t_1) & R_m(t_3, t_2) & R_m(t_3, t_3) \end{bmatrix} . \tag{B-28}$$

The corresponding characteristic function is

$$\phi(\omega_1, \omega_2, \omega_3) = e^{-\frac{1}{2}(\omega_1\ \omega_2\ \omega_3)\, K_m \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}} . \tag{B-29}$$

It follows that

$$\begin{aligned} E(\cos m_1 \cos m_2 \cos m_3) = \frac{1}{8} [& \phi(1, 1, 1) + \phi(1, 1, -1) + \phi(1, -1, 1) \\ & + \phi(-1, 1, 1) + \phi(1, -1, -1) + \phi(-1, 1, -1) \\ & + \phi(-1, -1, 1) + \phi(-1, -1, -1)] . \end{aligned} \tag{B-30}$$

As a result, (B-30) can be written in the form

$$
\begin{aligned}
E(\cos m_1 \cos m_2 \cos m_3) = \frac{1}{8} \Bigg\{ & e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{+ 2R(t_1, t_2) + 2R(t_1, t_3) + 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{+ 2R(t_1, t_2) - 2R(t_1, t_3) - 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{- 2R(t_1, t_2) + 2R(t_1, t_3) - 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{- 2R(t_1, t_2) - 2R(t_1, t_3) + 2R(t_2, t_3)} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{- 2R(t_1, t_2) - 2R(t_1, t_3) + 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{- 2R(t_1, t_2) + 2R(t_1, t_3) - 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{+ 2R(t_1, t_2) - 2R(t_1, t_3) - 2R(t_2, t_3)]} \\
& + e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)} \\
& {}^{+ 2R(t_1, t_2) + 2R(t_1, t_3) + 2R(t_2, t_3)]} \Bigg\} .
\end{aligned}
\tag{B-31}
$$

Upon simplification, (B-31) becomes

$$E(\cos m_1 \cos m_2 \cos m_3) = \frac{1}{4} e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)]}$$

$$\left\{ e^{-[R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)]} \right.$$

$$+ e^{-[R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)]}$$

$$+ e^{-[-R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)]}$$

$$\left. + e^{-[-R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)]} \right\} . \qquad \text{(B-32)}$$

In a similar manner, it is found that

$$E(\cos m_1 \cos m_2 \sin m_3) = \frac{1}{8j} [\phi(1, 1, 1) + \phi(1, -1, 1) + \phi(-1, 1, 1)$$

$$+ \phi(-1, -1, 1) - \phi(1, 1, -1) - \phi(1, -1, -1)$$

$$- \phi(-1, 1, -1) - \phi(-1, -1, -1)]$$

$$= 0 , \qquad \text{(B-33)}$$

$$E(\cos m_1 \sin m_2 \cos m_3) = 0 , \qquad \text{(B-34)}$$

$$E(\sin m_1 \cos m_2 \cos m_3) = 0 , \qquad \text{(B-35)}$$

$$E(\sin m_1 \sin m_2 \sin m_3) = 0 , \qquad \text{(B-36)}$$

$$
\begin{aligned}
E(\cos m_1 \sin m_2 \sin m_3) = -\frac{1}{4}\, e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)]} \\
\Big\{ e^{-[R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)]} \\
- e^{-[-R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)]} \\
+ e^{-[-R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)]} \\
- e^{-[R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)]} \Big\} ,
\end{aligned}
\tag{B-37}
$$

$$
\begin{aligned}
E(\sin m_1 \cos m_2 \sin m_3) = -\frac{1}{4}\, e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)]} \\
\Big\{ e^{-[R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)]} \\
- e^{-[-R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)]} \\
+ e^{-[-R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)]} \\
- e^{-[R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)]} \Big\} ,
\end{aligned}
\tag{B-38}
$$

and

$$
\begin{aligned}
E(\sin m_1 \sin m_3 \cos m_3) = -\frac{1}{4}\, e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)]} \\
\Big\{ e^{-[R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)]} \\
- e^{-[-R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)]} \\
+ e^{-[R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)]} \\
- e^{-[-R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)]} \Big\} .
\end{aligned}
\tag{B-39}
$$

Use of the above in (B-26) yields

$$
\begin{aligned}
d_3(t_1, t_2, t_3) = {} & e^{-\frac{1}{2}[R(t_1, t_1) + R(t_2, t_2) + R(t_3, t_3)]} \\
& \Big\{ \cos \omega_o t_1 \cos \omega_o t_2 \cos \omega_o t_3 \\
& \cdot \frac{1}{4} \Big[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} \\
& + e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)} \\
& + e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} \\
& + e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)} \Big] \Big\} \\
& - \Big\{ \sin \omega_o t_1 \sin \omega_o t_2 \cos \omega_o t_3 \\
& \cdot \frac{1}{4} \Big[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} \\
& + e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)} \\
& - e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} \\
& - e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)} \Big] \Big\} \\
& - \Big\{ \sin \omega_o t_1 \cos \omega_o t_2 \sin \omega_o t_3 \\
& \cdot \frac{1}{4} \Big[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} \\
& - e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)} \\
& + e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} \\
& - e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)} \Big] \Big\} \\
& - \Big\{ \cos \omega_o t_1 \sin \omega_o t_2 \sin \omega_o t_3 \\
& \cdot \frac{1}{4} \quad e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} \\
& - e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)} \\
& - e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} \\
& + e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)} \Big] \Big\} .
\end{aligned}
\tag{B-40}
$$

Equation (B-40) can be simplified by noting that

$$\cosh a \cosh b \cosh c - \sinh a \sinh b \sinh c$$

$$= \frac{1}{8} [(e^{a+b+c} + e^{a+b-c} + e^{a-b+c} + e^{a-b-c} + e^{-a+b+c}$$

$$+ e^{-a+b-c} + e^{-a-b+c} + e^{-a-b-c}) - (e^{a+b+c} - e^{a+b-c}$$

$$- e^{a-b+c} + e^{a-b-c} - e^{-a+b+c} + e^{-a+b-c} + e^{-a-b+c}$$

$$- e^{-a-b-c})] \ .$$

This simplifies to

$$\cosh a \cosh b \cosh c - \sinh a \sinh b \sinh c$$

$$= \frac{1}{4} [e^{a+b-c} + e^{a-b+c} + e^{-a+b+c} + e^{-a-b-c}] \ . \tag{B-41}$$

Comparing Equation (B-41) with the first group of terms in Equation (B-40), where

$$a = R(t_1, t_2)$$

$$b = R(t_1, t_3)$$

$$c = R(t_2, t_3) \ ,$$

it is observed that

$$\frac{1}{4} \left[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} + e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)} \right.$$

$$\left. + e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} + e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)} \right]$$

$$= \cosh [R(t_1, t_2)] \cosh [R(t_1, t_3)] \cosh [R(t_2, t_3)]$$

$$- \sinh [R(t_1, t_2)] \sinh [R(t_1, t_3)] \sinh [R(t_2, t_3)] \ . \tag{B-42}$$

Similarly,

$$\cosh a \cosh b \sinh c - \sinh a \sinh b \cosh c$$

$$= \frac{1}{8} [(e^{a+b+c} - e^{a+b-c} + e^{a-b+c} - e^{a-b-c} + e^{-a+b+c}$$

$$- e^{-a+b-c} + e^{-a-b+c} - e^{-a-b-c}) - (e^{a+b+c} + e^{a+b-c}$$

$$- e^{a-b+c} - e^{a-b-c} - e^{-a+b+c} - e^{-a+b-c}$$

$$+ e^{-a-b+c} + e^{-a-b-c})]$$

$$= -\frac{1}{4} [e^{a+b-c} - e^{a-b+c} - e^{-a+b+c} + e^{-a-b-c}] \quad . \tag{B-43}$$

Hence, the fourth group of terms in Equation (B-40) may be expressed as

$$-\frac{1}{4} \Big[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} - e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)}$$

$$- e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} + e^{+R(t_1, t_2) + R(t_2, t_3) - R(t_2, t_3)} \Big]$$

$$= \cosh [R(t_1, t_2)] \cosh [R(t_1, t_3)] \sinh [R(t_2, t_3)]$$

$$- \sinh [R(t_1, t_2)] \sinh [R(t_1, t_3)] \cosh [R(t_2, t_3)] \quad . \tag{B-44}$$

Redefining a, b, and c as

$$a = R(t_2, t_3)$$

$$b = R(t_1, t_2)$$

$$c = R(t_1, t_3) \quad ,$$

The third group of terms in Equation (B-40) become

$$-\frac{1}{4}\Big[e^{-R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} - e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)}$$

$$+ e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} - e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)}\Big]$$

$$= \cosh [R(t_1, t_2)] \sinh [R(t_1, t_3)] \cosh [R(t_2, t_3)]$$

$$- \sinh [R(t_1, t_2)] \cosh [R(t_1, t_3)] \sinh [R(t_2, t_3)] \quad . \qquad \text{(B-45)}$$

Similarly, the second group of terms in Equation (B-40) may be expressed as

$$-\frac{1}{4}\Big[e^{+R(t_1, t_2) - R(t_1, t_3) - R(t_2, t_3)} + e^{-R(t_1, t_2) + R(t_1, t_3) + R(t_2, t_3)}$$

$$- e^{+R(t_1, t_2) - R(t_1, t_3) + R(t_2, t_3)} - e^{+R(t_1, t_2) + R(t_1, t_3) - R(t_2, t_3)}\Big]$$

$$= \sinh [R(t_1, t_2)] \cosh [R(t_1, t_3)] \cosh [R(t_2, t_3)]$$

$$- \cosh [R(t_1, t_2)] \sinh [R(t_1, t_3)] \sinh [R(t_2, t_3)] \quad . \qquad \text{(B-46)}$$

From the above results it is concluded that

$$d_3(t_1, t_2, t_3) = e^{-\frac{1}{2}[R_m(t_1, t_1) + R_m(t_2, t_2) + R_m(t_3, t_3)]}$$

$$\Big(\cos \omega_o t_1 \cos \omega_o t_2 \cos \omega_o t_3 \{\cosh[R_m(t_1, t_2)] \cosh[R_m(t_1, t_3)] \cosh[R_m(t_2, t_3)]$$

$$- \sinh[R_m(t_1, t_2)] \sinh[R_m(t_1, t_3)] \sinh[R_m(t_2, t_3)]\}$$

$$+ \sin \omega_o t_1 \sin \omega_o t_2 \cos \omega_o t_3 \{\sinh[R_m(t_1, t_2)] \cosh[R_m(t_1, t_3)] \cosh[R(t_2, t_3)]$$

$$- \cosh[R_m(t_1, t_2)] \sinh[R_m(t_1, t_3)] \sinh[R_m(t_2, t_3)]\}$$

$$+ \sin \omega_o t_1 \cos \omega_o t_2 \sin \omega_o t_3 \{\cosh[R_m(t_1, t_2)] \sinh[R_m(t_1, t_3)] \cosh[R_m(t_2, t_3)]$$

$$- \sinh[R_m(t_1, t_2)] \cosh[R_m(t_1, t_3)] \sinh[R_m(t_2, t_3)]\}$$

$$+ \cos \omega_o t_1 \sin \omega_o t_2 \sin \omega_o t_3 \{\cosh[R_m(t_1, t_2)] \cosh[R_m(t_1, t_3)] \sinh[R_m(t_2, t_3)]$$

$$- \sinh[R_m(t_1, t_2)] \sinh[R_m(t_1, t_3)] \cosh[R_m(t_2, t_3)]\}\Big) \quad . \tag{B-47}$$

APPENDIX C

LEC RECEIVER FOR A NONZERO MEAN GAUSSIAN PROCESS

C.1 Receiver Derivation

Consider the hypothesis testing problem

$$H_1: \quad r(t) = s(t) + n(t); \quad 0 \leq t \leq T$$

$$H_0: \quad r(t) = n(t) \tag{C-1}$$

where s(t) and n(t) are independent Gaussian processes with

$$E[\, s(t_1),\ s(t_2)\,] = K_s(t_1, t_2)$$

$$E[\, n(t_1),\ n(t_2)\,] = \frac{N_o}{2}\, \delta(t_1 - t_2) \tag{C-2}$$

$$E[\, n(t)\,] = 0 \quad .$$

The low energy coherence (LEC) condition is [2, p. 31]

$$\lambda_i << \frac{N_o}{2} \tag{C-3}$$

where the eigenvalues, λ_i, and eigenfunctions, $\phi_i(t)$ are solutions of the integral equation

$$\lambda_i\, \phi_i(t_1) = \int_0^T K_s(t_1, t_2)\, \phi_i(t_2)\, dt_2 \quad . \tag{C-4}$$

From Van Trees [2, p. 11, Equation (19)], the logarithm of the likelihood ratio is

$$\ell n\, \Lambda\,(r(t)) = \frac{1}{N_o} \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + \frac{N_o}{2}}\, r_i^2 + \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + \frac{N_o}{2}}\, \eta_i\, r_i$$

$$- \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{\lambda_i + \frac{N_o}{2}}\, \eta_i^2 - \frac{1}{2} \sum_{i=1}^{\infty} \ell n \left(1 + \frac{2\lambda_i}{N_o} \right) \tag{C-5}$$

where

$$r_i = \int_0^T r(t)\ \phi_i(t)\ dt \tag{C-6}$$

$$\eta_i = A\int_0^T E[\,s(t)\,]\ \ \phi_i(t)\ dt \quad . \tag{C-7}$$

Equation (C-5) may be rewritten as

$$\begin{aligned} \ell n\ \Lambda[\,r(t)\,] &= \frac{1}{2}\left(\frac{2}{N_o}\right)^2 \sum_{i=1}^{\infty} \lambda_i \left(1 + \frac{2\lambda_i}{N_o}\right)^{-1} r_i^2 \\ &+ \left(\frac{2}{N_o}\right)\sum_{i=1}^{\infty} \eta_i \left(1 + \frac{2\lambda_i}{N_o}\right)^{-1} r_i \\ &- \frac{1}{2}\left(\frac{2}{N_o}\right)\sum_{i=1}^{\infty} \eta_i^2 \left(1 + \frac{2\lambda_i}{N_o}\right)^{-1} \\ &- \frac{1}{2}\sum_{i=1}^{\infty} \ell n \left(1 + \frac{2\lambda_i}{N_o}\right) \quad . \end{aligned} \tag{C-8}$$

Using Taylor series expansions for both $(1 + 2\lambda_i/N_o)^{-1}$ and $\ell n\ (1 + 2\lambda_i/N_o)$ and retaining terms up to order $2\lambda_i/N_o$ in Equation (C-8), the logarithm of the likelihood ratio simplifies to

$$\ell n\ \Lambda(r(t)) = \frac{1}{2}\left(\frac{2}{N_o}\right)^2 \sum_{i=1}^{\infty} \lambda_i r_i^2$$

$$+ \frac{2}{N_o} \sum_{i=1}^{\infty} \eta_i \left(1 - \frac{2\lambda_i}{N_o}\right) r_i$$

$$- \frac{1}{2}\ \frac{2}{N_o} \sum_{i=1}^{\infty} \eta_i^2 \left(1 - \frac{2\lambda_i}{N_o}\right)$$

$$- \frac{1}{2} \sum_{i=1}^{\infty} \frac{2\lambda_i}{N_o} \quad . \tag{C-9}$$

However,

$$K_s(t_1, t_2) = \sum_{i=1}^{\infty} \lambda_i\ \phi_i(t_1)\ \phi_i(t_2) \quad ,$$

$$E[\,s(t)\,] = \sum_{i=1}^{\infty} \eta_i\ \phi_i(t) \quad , \tag{C-10}$$

$$r(t) = \sum_{i=1}^{\infty} r_i\ \phi_i(t) \quad .$$

From Equation (C-10) and the o rthogonality of the eigenfunctions $\phi_i(t)$ on $[0, T]$ it follows that

$$\sum_{i=1}^{\infty} \lambda_i r_i^2 = \int_0^T \int_0^T r(t_1)\, r(t_2)\, K_s(t_1, t_2)\, dt_1\, dt_2 \qquad \text{(C-11)}$$

$$\sum_{i=1}^{\infty} \eta_i r_i = \int_0^T E[s(t)]\; r(t)\, dt \qquad \text{(C-12)}$$

$$\sum_{i=1}^{\infty} \eta_i \lambda_i r_i = \int_0^T \int_0^T E[s(t_1)]\; r(t_2) K_s(t_1, t_2)\, dt_1\, dt_2 \qquad \text{(C-13)}$$

$$\sum_{i=1}^{\infty} \eta_i^2 = \int_0^T \int_0^T E^2[s(t)]\; dt \qquad \text{(C-14)}$$

$$\sum_{i=1}^{\infty} \eta_i^2 \lambda_i = \int_0^T \int_0^T E[s(t_1)]\; E[s(t_2)]\; K_s(t_1, t_2)\, dt_1\, dt_2 \qquad \text{(C-15)}$$

$$\sum_{i=1}^{\infty} \lambda_i = \int_0^T K_s(t, t)\, dt \qquad . \qquad \text{(C-16)}$$

With the aid of Equations (C-11) through (C-16), Equation (C-9) can be expressed

$$\ell n\ \Lambda\,(r(t)) = \frac{1}{2}\left(\frac{2}{N_o}\right)^2 \int_0^T \int_0^T r(t_1)\,r(t_2)\ K_s\,(t_1,\ t_2)\ dt_1\ dt_2$$

$$+ \frac{2}{N_o} \int_0^T E\,[\,s\,(t_1)\,]\ \ r(t_1)\ dt_1$$

$$- \left(\frac{2}{N_o}\right)^2 \int_0^T \int_0^T E\,[\,s\,(t_1)\,]\ \ r(t_2)\ K_s\,(t_1,\ t_2)\ dt_1\ dt_2$$

$$- \frac{1}{2}\left(\frac{2}{N_o}\right) \int_0^T E^2\,[\,s\,(t)\,]\ dt$$

$$+ \frac{1}{2}\left(\frac{2}{N_o}\right)^2 \int_0^T \int_0^T E\,[\,s\,(t_1)\,]\ E\,[\,s\,(t_2)\,]\ \ K_s\,(t_1,\ t_2)\,dt_1\,dt_2$$

$$- \frac{1}{2}\left(\frac{2}{N_o}\right) \int_0^T K_s\,(t_1,\ t_1)\ dt_1 \qquad . \qquad \text{(C-17)}$$

Focusing attention on those terms in (C-17) involving the observation r(t), it is clear that the LEC receiver is given by

$$\frac{1}{2}\left(\frac{2}{N_o}\right)^2 \int_0^T \int_0^T r(t_1) r(t_2)\, K_s(t_1, t_2)\, dt_1\, dt_2$$

$$+\frac{2}{N_o}\int_0^T r(t_1)\left\{E[s(t_1)] - \frac{2}{N_o}\int_0^T E[s(t_2)]\, K_s(t_2, t_1)\, dt_2\right\} dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \qquad \text{(C-18)}$$

where γ is the threshold.

In Section C.2 a case of interest is considered for which $\frac{2}{N_o}\int_0^T E[s(t_2)]\, K_s(t_1, t_2)\, dt_2$ is negligible with respect to $E[s(t_1)]$. Let

$$y_1(t_1) = E[s(t_1)] \qquad \text{(C-19)}$$

$$y_2(t_1) = E[s(t_1)] - \frac{2}{N_o}\int_0^T E[s(t_2)]\, K_s(t_2, t_1)\, dt_2 . \qquad \text{(C-20)}$$

Define the norm of a signal y(t) to be

$$\| y \| = \sqrt{\int_0^T y^2(t) dt} . \qquad \text{(C-21)}$$

It follows that the squared relative error between $y_1(t_1)$ and $y_2(t_2)$ is given by

$$\epsilon^2 = \left(\frac{\| y_1(t_1) - y_2(t_1) \|}{\| y_1(t_1) \|}\right)^2 = \frac{\frac{2}{N_o}\int_0^T \left[\int_0^T E[s(t_2)]\, K_s(t_2, t_1) dt_2\right]^2 dt_1}{\int_0^T E^2[s(t_1)]\, dt_1} . \qquad \text{(C-22)}$$

$\epsilon^2 << 1$ implies that $y_1(t_1)$ is "close" to $y_2(t_1)$ in a squared error sense, or, that

$\frac{2}{N_o}\int_0^T E[s(t_2)]\, K_s\,(t_2, t_1)\, dt_1$ is negligible with respect to $E[s(t_1)]$. For the case

considered in paragraph C.2, it is shown that $\epsilon^2 << 1$.

The LEC receiver then becomes

$$\frac{1}{2}\left(\frac{2}{N_o}\right)^2 \int_0^T\int_0^T r(t_1) r(t_2)\, K_s\,(t_1, t_2)\, dt_1\, dt_2$$

$$+ \frac{2}{N_o}\int_0^T r(t_1)\, E[s(t_1)]\, dt_1 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma' \tag{C-23}$$

C.2 <u>Justification of LEC Receiver Interpretation in Par. 2.4.3</u>

In par. 2.4.3 $S(t, m(t)) = \cos(\omega_o t + m(t))$. From Equations (2-209) and (2-210), the mean and variance of the equivalent Gaussian random process, $s(t) = AS(t, m(t))$, used in determining the asymptotic performance of the asymptotic receiver are given by

$$E[s(t_1)] = A\, e^{-\frac{P}{2}} \cos \omega_o t_1$$

$$K_s(t_1, t_2) = A^2 e^{-P} \Big\{ [\cosh\,(Pe^{-\alpha|t_1 - t_2|}) - 1]\, \cos\omega_o t_1 \cos\omega_o t_2$$

$$+ \sinh\,(Pe^{-\alpha|t_1-t_2|})\, \sin\omega_o t_1 \sin\omega_o t_2 \Big\} .$$

With respect to (C-18)

$$\int_0^T E[s(t_2)] K_s(t_2, t_1) dt_2 \cong \frac{Ae}{2}^{3 - \frac{3P}{2}} \int_0^T [\cosh\,(Pe^{-\alpha|t_1-t_2|}) - 1]\, dt_2 \cos\omega_o t_1 \tag{C-24}$$

where the contribution from the double frequency terms is recognized as being negligible. Expanding cosh (·) in Equation (C-24) obtain

$$\int_0^T E[s(t_2)]\, K_s(t_2, t_1)\, dt_2$$

$$\simeq \frac{A^3 e^{-3P/2}}{2} \cos \omega_o t_1 \left[\frac{1}{2} \int_0^T \exp(Pe^{-\alpha|t_1-t_2|})\, dt_2 \right.$$

$$\left. + \frac{1}{2} \int_0^T \exp(-Pe^{-\alpha|t_1-t_2|})\, dt_2 - T \right] .$$

However,

$$\int_0^T \exp(Pe^{-\alpha|t_1-t_2|})\, dt_2 = T \left(1 + \sum_{i=1}^{\infty} \frac{P^i}{i!} \frac{1}{T} \int_0^T e^{-i\alpha|t_1-t_2|} dt_2 \right) .$$

Note that

$$\frac{1}{T} \int_0^T e^{-i\alpha|t_1-t_2|}\, dt_2 = \frac{1}{T} \int_0^{t_1} e^{-i\alpha|t_1-t_2|}\, dt_2 + \frac{1}{T} \int_{t_1}^T e^{i\alpha|t_1-t_2|}\, dt_2$$

$$= e^{-i\alpha t_1} \frac{(e^{i\alpha t_1} - 1)}{i\alpha T} + e^{i\alpha t_1} \frac{(e^{-i\alpha T} - e^{-i\alpha t_1})}{-i\alpha T} .$$

Similarly, $\int_0^T \exp(-Pe^{-\alpha|t_1-t_2|})\, dt_2$ is obtained by changing the sign of P in the above result. Consequently,

$$\int_0^T E[s(t_2)]\, K_s\,(t_2, t_1)\, dt_2$$

$$= \frac{A^3 e^{-3P/2}}{2}\, T \cos \omega_o t_1 \left\{ \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{1-e^{-2i\alpha t_1}}{2i\alpha T} + \frac{(1-e^{-2i\alpha T} e^{2i\alpha t_1})}{2i\alpha T} \right] \right\} . \tag{C-25}$$

Also, with respect to Equation (C-22)

$$\int_0^T E^2\,[\,s(t)]\, dt = \frac{A^2 e^{-P} T}{2} \quad . \tag{C-26}$$

Substitution of Equations (C-25) and (C-26) into Equation (C-22) results in

$$\epsilon^2 = \frac{\left(\frac{2}{N_o}\right)^2 \int_0^T \left\{ \int_0^T E[s(t_2)]\, K_s\,(t_2, t_1)\, dt_2 \right\}^2 dt_1}{\int_0^T E^2[s(t_1)]\, dt_1}$$

$$= \left(\frac{2}{N_o}\right)^2 \frac{A^4 e^{-2P} T^2}{2} \cdot \frac{1}{T} \int_0^T \cos^2 \omega_o t_1 \left[\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left(\frac{2-e^{-2i\alpha t_1} - e^{-2i\alpha T} e^{2i\alpha t_1}}{2i\alpha T} \right) \right]^2 dt_1$$

$$\leq \left(\frac{A^2}{N_o \alpha}\right)^2 e^{-2P} \cdot \frac{1}{T} \int_0^T \cos^2 \omega_o t_1 \left(\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!\,i} \right)^2 dt_1$$

$$= \frac{1}{2} \left(\frac{A^2}{N_o \alpha}\, e^{-P} \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!\,i} \right)^2 . \tag{C-27}$$

Observe in Equation (C-27) that

$$e^{-P} \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!\, i} \leq e^{-P} \sum_{i=1}^{\infty} \frac{P^{i}}{i!} = e^{-P} \left(e^{+P} - 1\right) < 1 . \tag{C-28}$$

It is concluded that the inequality in Equation (C-22) is easily satisfied provided $\frac{A^2}{N_o \alpha} << 1$. This is the case which is considered in par. 2.4.3. where use is made of the LEC receiver given by Equation (C-23).

C.3 Receiver Performance

The receiver performance presented here follows the development given by Van Trees [2] .

C.3.1 Chernoff Approximation

The Chernoff approximation to receiver performance was first introduced by Collins [31]. If ℓ is the logarithm of the likelihood ratio, the false alarm probability is given by

$$P_{FA} = \int_{\gamma}^{\infty} P_{\ell | H_0} (L | H_0)\, d\, L \tag{C-29}$$

while the detection probability is given by

$$P_D = \int_{\gamma}^{\infty} P_{\ell | H_1} (L | H_1)\, d\, L \tag{C-30}$$

where $P_{\ell | H_i} (L | H_i)$ is the conditional p.d.f. of ℓ conditioned on H_i. In Van Trees [1] it is shown that

$$P_{\ell | H_1} (L | H_1) = e^{L} P_{\ell | H_0} (L | H_0) . \tag{C-31}$$

The moment generating function of ℓ, conditioned on H_0, is

$$\phi_{\ell|H_0}(S) = \int_{-\infty}^{\infty} e^{SL} P_{\ell|H_0}(L|H_0)\, dL \ . \tag{C-32}$$

Now let

$$\mu(S) = \ell n\,(\phi_{\ell|H_0}(S)) \tag{C-33}$$

and next define a tilted random variable x_s with pdf

$$P_{x_s}(X) = \frac{e^{SX} P_{\ell|H_0}(X|H_0)}{\int_{-\infty}^{\infty} e^{SL} P_{\ell|H_0}(L|H_0)dL} = e^{SX-\mu(S)} P_{\ell|H_0}(X|H_0) \ . \tag{C-34}$$

From Equations (C-34) and (C-29)

$$P_{FA} = \int_{\gamma}^{\infty} e^{\mu(S) - SX} P_{x_s}(X)\, dX \ . \tag{C-35}$$

Equation (C-35) may be rewritten as

$$P_{FA} = e^{\mu(S)-S\gamma} \int_{\gamma}^{\infty} e^{S(\gamma-X)} P_{x_s}(X)dX \ . \tag{C-36}$$

Before proceeding, note that

$$E(x_s) = \frac{\int_{-\infty}^{\infty} X\, e^{SX} P_{\ell|H_0}(X|H_0)\, dX}{\int_{-\infty}^{\infty} e^{SL} P_{\ell|H_0}(L|H_0)dL} \ . \tag{C-37}$$

It follows from Equations (C-37), (C-32) and (C-33) that

$$E(x_s) = \frac{d}{ds} \mu(S) = \dot{\mu}(S) . \tag{C-38}$$

Similarly,

$$\sigma^2_{x_s} = \ddot{\mu}(S) . \tag{C-39}$$

A standardized random variable is now defined as

$$y = \frac{x_s - \dot{\mu}(S)}{\sqrt{\ddot{\mu}(S)}} . \tag{C-40}$$

In the Chernoff bound one chooses s so that

$$\dot{\mu}(S) = \gamma . \tag{C-41}$$

With the aid of Equations (C-41), (C-36) and (C-40), the false alarm probability is expressed as

$$P_{FA} = e^{\mu(S) - S\dot{\mu}(S)} \int_0^\infty e^{-S\sqrt{\ddot{\mu}(S)}\,Y} P_y(Y)\, dY . \tag{C-42}$$

Expand $P_y(Y)$ in an Edgeworth series to obtain

$$P_y(Y) = \phi(Y) - \frac{\alpha_3}{3!} \phi^{(3)}(Y) + \left[\frac{\alpha_4}{4!} \phi^{(4)}(Y) + \frac{10\alpha_3}{6!} \phi^{(6)}(Y) \right] + \ldots \tag{C-43}$$

where

$$\left.\begin{aligned} H_0(Y) &= 1 \\ H_1(Y) &= Y \\ H_2(Y) &= Y^2-1 \\ H_3(Y) &= Y^3-3Y \end{aligned}\right\} \tag{C-44}$$

$$\vdots$$

and

$$\alpha_n = \frac{\overset{(n)}{\mu}(S)}{(\ddot{\mu}(S))^{n/2}} \quad . \tag{C-45}$$

Writing out the first two terms of the Edgeworth series it follows that

$$\int_0^\infty e^{-S\sqrt{\ddot{\mu}(S)}\,Y} \quad P_y(Y)\, d\, Y$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-S\sqrt{\ddot{\mu}(S)}\,Y}\, e^{-Y^2/2}\, dY$$

$$+ \frac{\dddot{\mu}(S)}{6(\ddot{\mu}(S))^{3/2}} \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty Y^3 e^{-S\sqrt{\ddot{\mu}(S)}\,Y} e^{-Y^{2}/2} dY \right.$$

$$\left. - \frac{3}{\sqrt{2\pi}} \int_0^\infty Y e^{-S\sqrt{\ddot{\mu}(S)}\,Y}\, e^{-Y^{2}/2}\, dY \right] + \ldots \tag{C-46}$$

At this point it is convenient to define

$$I_n(\beta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^n\, e^{-\beta x}\, e^{-x^{2}/2}\, dx \quad . \tag{C-47}$$

Integrating Equation (C-47) by parts, there results

$$I_n(\beta) = (n-1)\, I_{n-2}(\beta) - \beta\, I_{n-1}(\beta) \quad . \tag{C-48}$$

Note, by direct integration, that

$$I_0(\beta) = e^{-\beta^2/2} \operatorname{erfc}^*(\beta) \tag{C-49}$$

$$I_1(\beta) = \frac{1}{\sqrt{2\pi}} - \beta e^{-\beta^2/2} \operatorname{erfc}^*(\beta) \tag{C-50}$$

where

$$\operatorname{erfc}^*(\beta) = \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} e^{-\frac{x^2}{2}} \, dx \, . \tag{C-51}$$

Using the recursive result of Equation (C-48)

$$I_3(\beta) = \frac{2+\beta^2}{\sqrt{2\pi}} - (\beta^3 + 3\beta)\, e^{\beta^2/2} \operatorname{erfc}^*(\beta) \quad . \tag{C-52}$$

Identifying β with S $\ddot{\mu}(S)$, it follows that

$$\int_0^{\infty} e^{-S\sqrt{\ddot{\mu}(S)}\,Y} \; P_y(Y)\, dY$$

$$= e^{\frac{S^2 \ddot{\mu}(S)}{2}} \operatorname{erfc}^*\left(S\sqrt{\ddot{\mu}(S)}\right)$$

$$+ \frac{\dddot{\mu}(S)}{6(\ddot{\mu}(S))^{3/2}} \left[\frac{S^2 \ddot{\mu}(S) - 1}{\sqrt{2\pi}} - S^3 (\ddot{\mu}(S))^{3/2} e^{\frac{S^2 \ddot{\mu}(S)}{2}} \operatorname{erfc}^*\left(S\sqrt{\ddot{\mu}(S)}\right) \right]$$

$$+ \ldots \quad . \tag{C-53}$$

From Equations (C-53) and (C-42), the false alarm probability is approximated as

$$
\begin{aligned}
P_{FA} \cong\ & e^{\mu(S) - S\dot{\mu}(S) + S^2\ddot{\mu}(S)/2} \ \mathrm{erfc}_* \left(S\sqrt{\ddot{\mu}(S)}\right) \\
& - \frac{\dddot{\mu}(S) S^3}{6} \ e^{\mu(S) - S\dot{\mu}(S) + S^2\ddot{\mu}(S)/2} \ \mathrm{erfc}_* \left(S\sqrt{\ddot{\mu}(S)}\right) \\
& - \frac{\dddot{\mu}(S)(1 - S^2\ddot{\mu}(S))}{6(\ddot{\mu}(S))^{3/2}\sqrt{2\pi}} \ e^{\mu(S) - S\dot{\mu}(S)}
\end{aligned}
\tag{C-54}
$$

where $0 \leq S \leq 1$ and is chosen so that $\dot{\mu}(S) = \gamma$.

From Equations (C-31) and (C-34), observe that

$$
P_{\ell | H_1}(X | H_1) = e^{-(S-1)X + \mu(S)} \ P_{x_s}(X) \quad . \tag{C-55}
$$

Use of Equations (C-55) and (C-30) result in the detection probability being expressed as

$$
P_D = 1 - e^{\mu(S) + (1-S)\dot{\mu}(S)} \int_{-\infty}^{0} e^{(1-S)\sqrt{\ddot{\mu}(S)}\ Y} \ P_y(Y)\, d\,Y \quad . \tag{C-56}
$$

In a manner similar to the development of Equation (C-54) from (C-42), it can be shown that Equation (C-56) can be written as

$$
\begin{aligned}
P_D \cong\ & 1 - e^{\mu(S) + (1-S)\dot{\mu}(S) + (1-S)^2\ddot{\mu}(S)/2} \ \mathrm{erfc}_* \left[(1-S)\sqrt{\ddot{\mu}(S)} \right] \\
& + \frac{\dddot{\mu}(S)(1-S)^3}{6} \ e^{\mu(S) + (1-S)\dot{\mu}(S) + (1-S)^2\ddot{\mu}(S)/2} \ \mathrm{erfc}_* \left[(1-S)\sqrt{\ddot{\mu}(S)} \right] \\
& + \frac{\dddot{\mu}(S)(1 - \ddot{\mu}(S)(1-S)^2)}{6\sqrt{2\pi}\ (\ddot{\mu}(S))^{3/2}} \ e^{\mu(S) + (1-S)\dot{\mu}(S)}
\end{aligned}
\tag{C-57}
$$

where, as before, $0 \leq S \leq 1$ and is chosen so that $\dot{\mu}(S) = \gamma$.

C.3.2 LEC Approximation for $\mu(S)$

From Van Trees [2, p.35] a simple expression for $\mu(S)$ can be obtained for the LEC case.

$$\mu(S) = \frac{1}{2} \sum_{i=1}^{\infty} \left[(1-S)\, \ell n \left(1 + \frac{2\lambda_i}{N_o}\right) - \ell n \left(1 + \frac{2(1-S)\lambda_i}{N_o}\right) \right]$$

$$- \frac{S}{2} \sum_{i=1}^{\infty} \frac{\eta_i^2}{N_o/2(1-S) + \lambda_i} \qquad 0 \le S \le 1 \quad . \tag{C-58}$$

Denoting the first term in Equation (C-58) as $\mu_R(S)$ and the second as $\mu_D(S)$, note that

$$\mu_R(S) \cong \frac{1}{2} \sum_{i=1}^{\infty} \left\{ (1-S) \left[\frac{2}{N_o} \lambda_i - \frac{1}{2} \left(\frac{2}{N_o}\right)^2 \lambda_i^2 + \ldots \right] \right.$$

$$\left. - \left[(1-S) \frac{2}{N_o} \lambda_i - \frac{(1-S)^2}{2} \left(\frac{2}{N_o}\right)^2 \lambda_i^2 + \ldots \right] \right\} . \tag{C-59}$$

Because of the LEC assumption it is necessary to retain terms only up to order $\left(\frac{2\lambda_i}{N_o}\right)^2$ (since the linear terms cancel). Hence, for the LEC case

$$\mu_R(S) \cong \frac{-S(1-S)}{2} \left\{ \frac{1}{2} \left(\frac{2}{N_o}\right)^2 \int_0^T \int_0^T K_s^2 (t_1, t_2)\, dt_1\, dt_2 \right\} . \tag{C-60}$$

Similarly,

$$\mu_D(S) = \frac{-S(1-S)}{2} \left\{ \left(\frac{2}{N_o}\right) \sum_{i=1}^{\infty} \eta_i^2 \left[1 - \frac{2\lambda_i (1-S)}{N_o} + \ldots \right] \right\} . \tag{C-61}$$

Retaining terms only of order $\frac{2\lambda_i}{N_o}$ in Equation (C-61), there results

$$\mu_D(S) \simeq \frac{-S(1-S)}{2} \left\{ \frac{2}{N_o} \int_0^T E^2 [s(t)]\, dt - \left(\frac{2}{N_o}\right)^2 (1-S) \int_0^T \int_0^T E[s(t_1)]\, E[s(t_2)]\, K_s(t_1, t_2) dt_1\, dt_2 \right\}. \tag{C-62}$$

It follows from Equations (C-58), (C-60) and (C-62) that

$$\mu(S) \simeq \mu_R(S) + \mu_D(S) \quad . \tag{C-63}$$

C.3.3 Evaluation of $\mu(S)$ as Required in Par. 2.4.3

From par. 2.4.3

$$E(s(t)) = Ae^{-\frac{P}{2}} \cos \omega_o t_1 \tag{C-64}$$

and

$$K_s(t_1, t_2) = A^2 e^{-P} \left\{ [\cosh (Pe^{-\alpha|t_1-t_2|})-1] \cos \omega_o t_1 \cos \omega_o t_2 + \sinh (Pe^{-\alpha|t_1-t_2|}) \sin \omega_o t_1 \sin \omega_o t_2 \right\} \quad . \tag{C-65}$$

Consequently

$$\int_0^T \int_0^T K_s^2 (t_1, t_2)\, dt_1\, dt_2$$

$$\simeq \frac{A^4 e^{-2P} T^2}{4} \cdot \frac{1}{T^2} \int_0^T \int_0^T \left[\cosh (2Pe^{-\alpha|t_1-t_2|}) - 2 \cosh(Pe^{-\alpha|t_1-t_2|}) + 1\right] dt_1\, dt_2 \tag{C-66}$$

where double frequency terms have been ignored.

To perform the integration in Equation (C-66) consider

$$\frac{1}{T^2} \int_0^T \int_0^T \exp\left(Pe^{-\alpha |t_1 - t_2|}\right) dt_1 \, dt_2$$

$$= 1 + \sum_{i=1}^{\infty} \frac{P^i}{i!} \frac{1}{T^2} \int_0^T \int_0^T e^{-i\alpha |t_1 - t_2|} \, dt_1 \, dt_2 \ . \qquad \text{(C-67)}$$

However,

$$\frac{1}{T^2} \int_0^T \int_0^T e^{-i\alpha |t_1 - t_2|} \, dt_1 \, dt_2$$

$$= \frac{1}{T} \int_0^T \left[\frac{1}{T} \int_0^{t_2} e^{+i\alpha (t_1 - t_2)} \, dt_1 + \frac{1}{T} \int_{t_2}^T e^{-\alpha (t_1 - t_2)} \, dt_1 \right] dt_2$$

$$= \frac{1}{T} \int_0^T \left[e^{-i\alpha t_2} \left(\frac{e^{+i\alpha t_2} - 1}{+i\alpha T} \right) + e^{+i\alpha t_2} \left(\frac{e^{-i\alpha T} - e^{-i\alpha t_2}}{-i\alpha T} \right) \right] dt_2$$

$$= \frac{1}{+i\alpha T} + \frac{(e^{-i\alpha T} - 1)}{(i\alpha T)^2} + \frac{e^{-i\alpha T}(e^{i\alpha T} - 1)}{-(i\alpha T)^2} + \frac{1}{i\alpha T}$$

$$= \frac{2}{i\alpha T} - \frac{2(1 - e^{-i\alpha T})}{(i\alpha T)^2} \ . \qquad \text{(C-68)}$$

Therefore, from Equations (C-67) and (C-68),

$$\frac{1}{T^2} \int_0^T \int_0^T \exp\left(Pe^{-\alpha |t_1 - t_2|}\right) dt_1 \, dt_2$$

$$= 1 + \sum_{i=1}^{\infty} \frac{P^i}{i!} \left[\frac{2}{i\alpha T} - \frac{2(1 - e^{-i\alpha T})}{(i\alpha T)^2} \right] \ . \qquad \text{(C-69)}$$

Use of Equations (C-69) in (C-66) results in

$$\int_0^T \int_0^T K_s^2(t_1, t_2)\, dt_1\, dt_2 =$$

$$\frac{A^4 e^{-P} T^2}{4} \left\{ \sum_{i=1}^{\infty} \frac{(2P)^{2i}}{2i!} \left[\frac{2}{2i\alpha T} - \frac{2(1-e^{-2i\alpha T})}{(2i\alpha T)^2} \right] \right.$$

$$\left. - 2\sum_{i=1}^{\infty} \frac{P^{2i}}{2i} \left[\frac{2}{2i\alpha T} - \frac{2(1-e^{-2i\alpha T})}{(2i\alpha T)^2} \right] \right\} \quad . \qquad \text{(C-70)}$$

Hence, with reference to Equation (C-70)

$$\mu_R(S) = \frac{-S(1-S)}{2} \left[\left(\frac{A^2}{N_o \alpha} \right)^2 \frac{e}{2}^{-2P} \left\{ \sum_{i=1}^{\infty} \frac{(2P)^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right] \right. \right.$$

$$\left. \left. - 2\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right] \right\} \right] \quad . \qquad \text{(C-71)}$$

Similarly,

$$\frac{2}{N_o} \int_0^T E^2 [s(t)]\, dt = \frac{2A^2 e^{-P} T}{N_o^2}$$

$$= \frac{A^2}{N_o \alpha} e^{-P} \alpha T \qquad \text{(C-72)}$$

and

$$\left(\frac{2}{N_o}\right)^2 \int_o^T \int_o^T E[s(t_1)]\, E[s(t_2)]\, K_s(t_1, t_2)\, dt_1\, dt_2$$

$$\simeq \left(\frac{2}{N_o}\right)^2 A^4 e^{-2P} \cdot \frac{1}{4} \int_o^T \int_o^T [\cosh (Pe^{-\alpha|t_1-t_2|}) - 1]\, dt_1\, dt_2 \qquad \text{(C-73)}$$

where double frequency terms have once again been ignored. Use of Equation (C-69) in Equation (C-73) results in

$$\left(\frac{2}{N_o}\right)^2 \int_o^T \int_o^T E[s(t_1)]\, E[s(t_2)]\, K_s(t_1, t_2)\, dt_1\, dt_2$$

$$= \left(\frac{A^2}{N_o\alpha}\right)^2 e^{-2P} \left[\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left(\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right) \right]. \qquad \text{(C-74)}$$

Hence, from Equations (C-74), (C-72) and (C-6?)

$$\mu_D(S) \cong \frac{-S(1-S)}{2} \left(\frac{A^2}{N_o\alpha} e^{-P}\alpha T - (1-S) \left(\frac{A^2}{N_o\alpha}\right)^2 e^{-2P} \left\{ \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right] \right\} \right) . \qquad \text{(C-75)}$$

Adding $\mu_R(S)$ and $\mu_D(S)$ it follows that

$$\mu(S) \cong \frac{-S(1-S)}{2} \left(\left(\frac{A^2}{N_o\alpha}\right)^2 e^{-2P} \left\{ \frac{1}{2} \sum_{i=1}^{\infty} \frac{(2P)^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right] - (2-S) \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left[\frac{\alpha T}{i} - \frac{(1-e^{-2i\alpha T})}{2i^2} \right] \right\} + \left(\frac{A^2}{N_o\alpha}\right) e^{-P}\alpha T \right) . \qquad \text{(C-76)}$$

APPENDIX D

THE LEC CONDITION FOR NONSTATIONARY PROCESSES

D.1 Derivation

Consider the integral equation

$$\lambda \phi(t_1) = \int_0^T K(t_1, t_2)\, \phi(t_2)\, dt_2 \tag{D-1}$$

where λ is an eigenvalue and $\phi_i(t)$ is an eigenfunction associated with λ_i. The low energy coherence (LEC) condition applies if [2]

$$|\lambda_{MAX}| << \frac{N_o}{2} \tag{D-2}$$

where $|\lambda_{MAX}|$ is the modulus of the largest eigenvalue and $\frac{N_o}{2}$ is a white noise level. If $K(t_1, t_2)$ is an autocovariance for a stationary process, the eigenvalues can be bounded by the maximum value of the spectrum [1]. It is then easy to see whether Equation (D-2) applies without actually solving Equation (D-1) for the eigenvalues. Unfortunately, no simple technique for determining a tight bound on λ_{MAX} is available when $K(t_1, t_2)$ is an autocovariance for a nonstationary process. The purpose of this appendix is to provide a technique for determining a tight bound on λ_{MAX} for the more general nonstationary case.

Let

$$K(t_1, t_2) - \overline{K}(t_1, t_2) = E(t_1, t_2) \tag{D-3}$$

where $\overline{K}(t_1, t_2)$ is an autocovariance for a stationary process. From Equations (D-1) and (D-3)

$$\lambda \phi(t_1) - \int_0^T E(t_1, t_2)\, \phi(t_2)\, dt_2 = \int_0^T \overline{K}(t_1, t_2)\, \phi(t_2)\, dt_2 \, . \tag{D-4}$$

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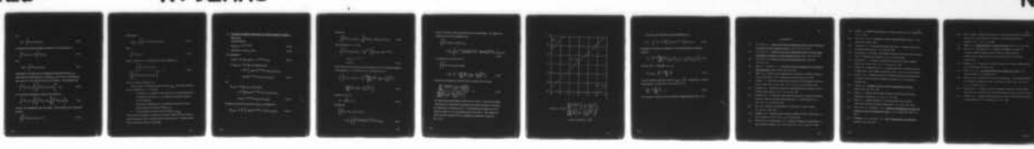

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Let

$$e(t_1) = \int_0^T E(t_1, t_2)\, \phi(t_2)\, dt_2 . \tag{D-5}$$

Following the development leading to Equation (C-22) it follows that if

$$\int_0^T e^2(t_1)\, dt_1 << \lambda^2 \int_0^T \phi^2(t_1)\, dt_1 \tag{D-6}$$

then

$$\lambda\, \phi(t_1) \simeq \int_0^T \overline{K}(t_1, t_2)\, \phi(t_2)\, dt_2 . \tag{D-7}$$

Consequently, the eigenvalues and eigenfunctions associated with $K(t_1, t_2)$ approximately satisfy Equation (D-7) and $\overline{K}(t_1, t_2)$ can be used to obtain approximate values for λ provided that Equation (D-6) applies. From Equation (D-6)

$$\int_0^T e^2(t_1)\, dt_1 = \int_0^T \left[\int_0^T E(t_1, t_2)\, \phi(t_2)\, dt_2 \right]^2 dt_1 . \tag{D-8}$$

Use of Schwartz's inequality in Equation (D-8) results in

$$\int_0^T e^2(t_1)\, dt_1 \leq \int_0^T \left[\int_0^T E^2(t_1, t_2)\, dt_2 \right] \left[\int_0^T \phi^2(t_2)\, dt_2 \right] dt_1 . \tag{D-9}$$

However, the eigenfunctions have unit energy. Hence Equation (D-6) is applicable provided

$$\int_0^T \int_0^T E^2(t_1, t_2)\, dt_1\, dt_2 << \lambda^2 . \tag{D-10}$$

Recall that [1]

$$\lambda_{MAX} \geq \int_0^T \int_0^T f(t_1)\, K(t_1, t_2)\, f(t_2)\, dt_1\, dt_2 \tag{D-11}$$

where

$$\int_0^T f^2(t_1)\, dt_1 = 1. \tag{D-12}$$

Hence, Equation (D-6) is satisfied for the largest eigenvalue if

$$\frac{\displaystyle\int_0^T \int_0^T E^2(t_1, t_2)\, dt_1\, dt_2}{\left[\displaystyle\int_0^T \int_0^T f(t_1) K(t_1, t_2) f(t_2)\, dt_1\, dt_2\right]^2} << 1 \tag{D-13}$$

where f(t) satisfies Equation (D-12).

The procedure for obtaining an upper bound on λ_{MAX}, the largest eigenvalue associated with $K(t_1, t_2)$, is as follows:

1. Given $K(t_1, t_2)$, formulate, by trial and error, a covariance function of a stationary process, $\overline{K}(t_1, t_2)$, for which the inequality in Equation (D-13) is satisfied.
2. Having found a suitable $\overline{K}(t_1, t_2)$, determine an upper bound on the largest eigenvalue associated with $\overline{K}(t_1, t_2)$.
3. This upper bound is used as an upper bound for λ_{MAX}.

Observe that the inequality in Equation (D-13) may be satisfied only for a limited range of parameter values associated with the random process. The upper bound is then valid only for this restricted range.

D.2 Calculation of Region for Which the LEC Condition Applies in Chapter II Par. 2.4.3.

In this paragraph

$$R_m(t_1, t_2) = Pe^{-\alpha |t_1 - t_2|} \tag{D-14}$$

$$AS[t, m(t)] = A \cos [\omega_o t + m(t)] \ . \tag{D-15}$$

From Appendix B

$$A\, d_1(t_1) = AE \{S[t_1, m(t_1)]\} = Ae^{-P/2} \cos \omega_o t_1 \tag{D-16}$$

$$\begin{aligned} A^2\, d_2(t_1, t_2) &= A^2 E \{S[t_1, m(t_1)]\, S[t_2, m(t_2)]\} \\ &= A^2 e^{-P} \Big[\cosh (Pe^{-\alpha|t_1 - t_2|}) \cos \omega_o t_1 \cos \omega_o t_2 \\ &\quad + \sinh (Pe^{-\alpha|t_1 - t_2|}) \sin \omega_o t_1 \sin \omega_o t_2 \Big] \ . \end{aligned} \tag{D-17}$$

Let

$$\begin{aligned} K(t_1, t_2) &= A^2 [d_2(t_1, t_2) - d_1(t_1)\, d_1(t_2)] \\ &= A^2 e^{-P} \Big[[\cosh (Pe^{-\alpha|t_1 - t_2|}) - 1] \cos \omega_o t_1 \cos \omega_o t_2 \\ &\quad + \sinh (Pe^{-\alpha|t_1 - t_2|}) \sin \omega_o t_1 \sin \omega_o t_2 \Big] \ . \end{aligned} \tag{D-18}$$

Through a process of trial and error, $\overline{K}(t_1, t_2)$ is chosen to be

$$\overline{K}(t_1, t_2) = A^2 \frac{e^{-P}}{2} \left[\exp (Pe^{-\alpha|t_1 - t_2|}) - 1 \right] \cos \omega_o (t_1 - t_2) \ . \tag{D-19}$$

Recall that

$$\int_0^T \int_0^T E^2(t_1, t_2)\, dt_1\, dt_2 = \int_0^T \int_0^T (K(t_1, t_2) - \overline{K}(t_1, t_2))^2\, dt_1\, dt_2 \ . \tag{D-20}$$

From Equation (D-18) - (D-20)

$$\int_0^T \int_0^T E^2(t_1, t_2)\, dt_1\, dt_2 \simeq A^4 \frac{e}{16}^{-2P} \int_0^T \int_0^T (2 \exp (-2Pe^{-\alpha|t_1-t_2|})$$

$$- 4 \exp (-Pe^{-\alpha|t_1-t_2|})$$

$$+ 2)\, dt_1\, dt_2 \tag{D-21}$$

where double frequency terms have been dropped since they contribute negligibly to the integration. From Equation (C-67) and Equation (D-21)

$$\int_0^T \int_0^T E^2(t_1, t_2)\, dt_1\, dt_2 \simeq A^4 \frac{e}{4}^{-2P} \left[\sum_{i=1}^{\infty} \frac{(-2P)^i}{i!} \left(\frac{1}{i\alpha T} - \frac{(1-e^{-i\alpha T})}{(i\alpha T)^2} \right) \right.$$

$$\left. -2 \sum_{i=1}^{\infty} \frac{(-P)^i}{i!} \left(\frac{1}{i\alpha T} - \frac{(1-e^{-i\alpha T})}{(i\alpha T)^2} \right) \right] \ . \tag{D-22}$$

In Equation (D-13) let

$$f(t) = \sqrt{\frac{2}{T}} \cos \omega_o t \ . \tag{D-23}$$

It follows that

$$\int_0^T \int_0^T f(t_1)\, K(t_1, t_2)\, f(t_2)\, dt_1\, dt_2$$

$$\simeq A^2 \frac{2}{T} \int_0^T \int_0^T \frac{e^{-P}(\cosh (Pe^{-\alpha|t_1-t_2|}) -1)}{4}\, dt_1\, dt_2 \tag{D-24}$$

where, once again, double frequency terms have been dropped. The right side of Equation (D-24) can be expanded so that

$$\int_0^T \int_0^T f(t_1)\, K(t_1, t_2)\, f(t_2)\, dt_1\, dt_2$$

$$\simeq A^2 \frac{2}{T} \int_0^T \int_0^T \frac{e^{-P}}{4} \left(\frac{\exp\left(Pe^{-\alpha|t_1-t_2|}\right)}{2} + \frac{\exp\left(-Pe^{-\alpha|t_1-t_2|}\right)}{2} - 1 \right) dt_1\, dt_2 \,. \tag{D-25}$$

From Equations (D-25) and (C-67)

$$\int_0^T \int_0^T f(t_1)\, K(t_1, t_2)\, f(t_2)\, dt_1\, dt_2$$

$$\simeq A^2 \frac{2}{T} \cdot \frac{e^{-P}}{4} \cdot 2 \sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left(\frac{1}{2i\alpha T} - \frac{\left(1-e^{-2i\alpha T}\right)}{(2i\alpha T)^2} \right) \,. \tag{D-26}$$

With the aid of Equations (D-26) and (D-22), Equation (D-13) becomes

$$\frac{\displaystyle\sum_{i=1}^{\infty} \frac{(-P)^i (2^i-2)}{i!} \left(\frac{1}{i\alpha T} - \frac{\left(1-e^{-i\alpha T}\right)}{(i\alpha T)^2} \right)}{\left(\displaystyle\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!} \left(\frac{1}{i\alpha T} - \frac{2\left(1-e^{-2i\alpha T}\right)}{(2i\alpha T)^2} \right) \right)^2} << 1 \,. \tag{D-27}$$

The random process parameters of interest are P and αT. Note that increasing P decreases the left-hand side of Equation (D-27) while increasing αT increases the left-hand side of Equation (D-27). The left-hand side of Equation (D-27) is plotted in Figure D-1 for representative values of P and αT. Only values of P and αT for which Equation (D-27) is satisfied are considered in Chapter II par. 2.4.3.

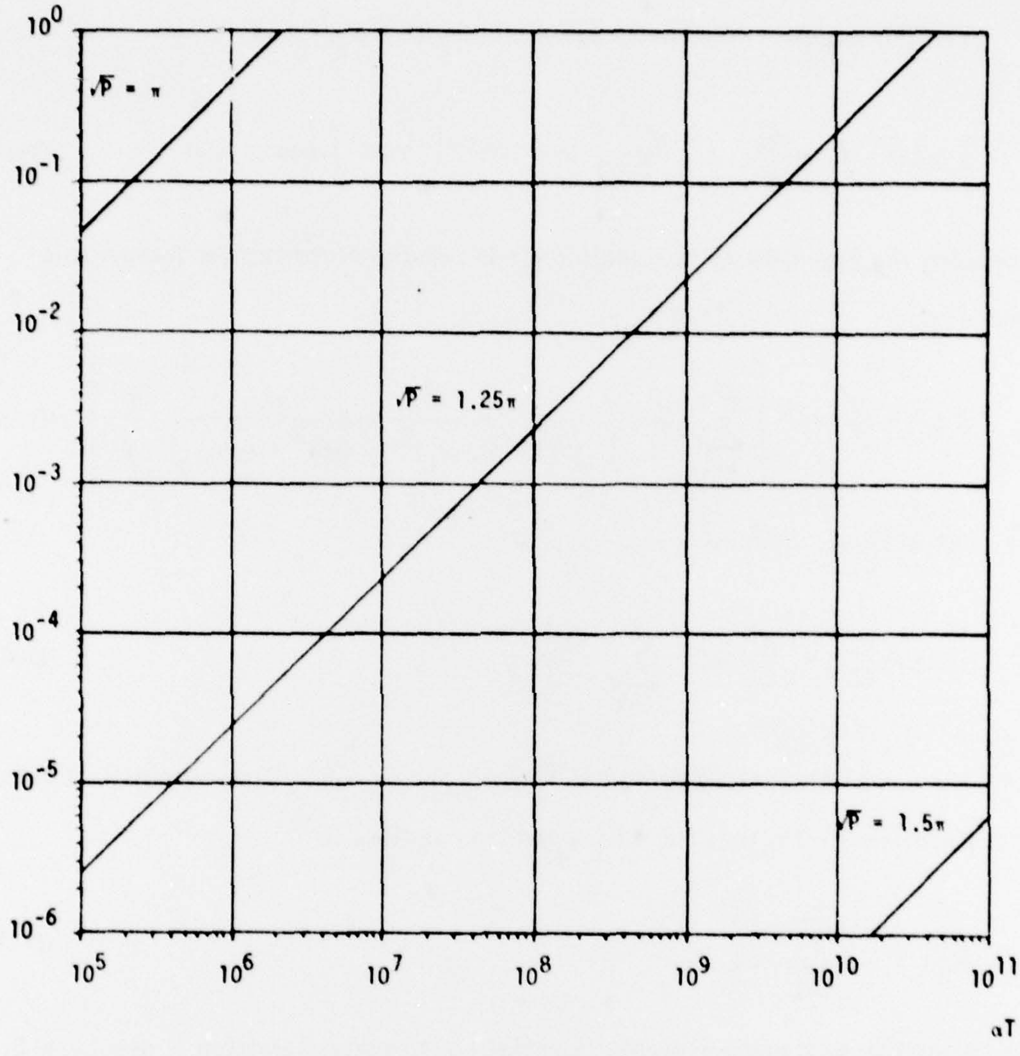

Figure D-1. Plot of

$$\frac{\displaystyle\sum_{i=1}^{\infty} \frac{(-P)^i(2^i-2)}{i!}\left[\frac{1}{i\alpha T} - \frac{(1-e^{-i\alpha T})}{(i\alpha T)^2}\right]}{\left(\displaystyle\sum_{i=1}^{\infty} \frac{P^{2i}}{2i!}\left[\frac{1}{i\alpha T} - \frac{2\,(1-e^{-2i\alpha T})}{(2i\alpha T)^2}\right]\right)^2}$$

versus αT parametric in $\sqrt{P}$

The power spectral density associated with $\overline{K}(t_1, t_2)$ is

$$\overline{S}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} A^2 \frac{e^{-P}}{2} \left[\exp\left(Pe^{-\alpha|\tau|}\right) - 1 \right] \cos \omega_o \tau \, d\tau . \qquad \text{(D-28)}$$

Expanding the exponential in Equation (D-28) and performing the integration results in

$$\overline{S}(\omega) = \frac{A^2}{2} e^{-P} \left[\sum_{i=1}^{\infty} \frac{P^i}{i!} \left(\frac{i\alpha}{(i\alpha)^2 + (\omega - \omega_o)^2} + \frac{i\alpha}{(i\alpha)^2 + (\omega + \omega_o)^2} \right) \right] . \qquad \text{(D-29)}$$

The peak of $\overline{S}(\omega)$, assuming $\alpha << \omega_o$, is

$$|\overline{S}(\omega)|_{MAX} = \frac{A^2}{2\alpha} e^{-P} \sum_{i=1}^{\infty} \frac{P^i}{i!i} . \qquad \text{(D-30)}$$

The LEC condition applies provided $|\overline{S}(\omega)|_{MAX} << \frac{N_o}{2}$. Consequently, it follows from Equation (D-30) that the LEC condition applies if

$$\frac{A^2}{N_o \alpha} e^{-P} \sum_{i=1}^{\infty} \frac{P^i}{i!i} << 1. \qquad \text{(D-31)}$$

The inequality in Equation (D-31) is satisfied throughout Section II par. 2.4.3.

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