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A COMPARISON OF SAMPLING PLANS FOR BAYESIAN ESTIMATION OF A POI--ETC(U)
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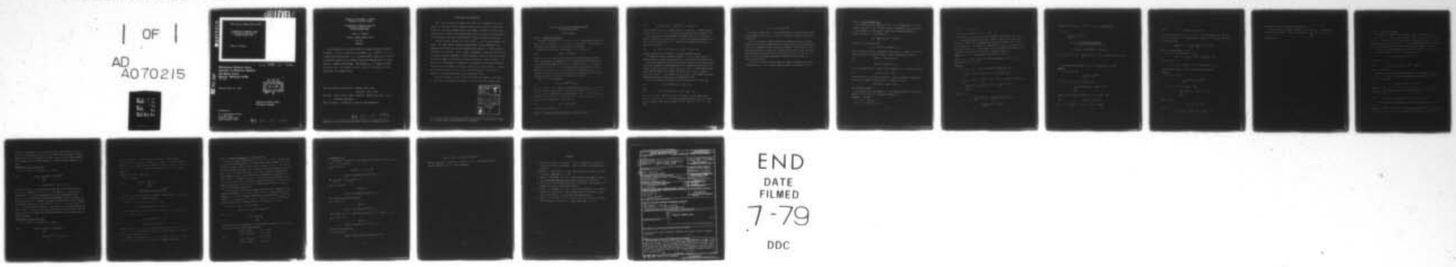

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A COMPARISON OF SAMPLING PLANS FOR BAYESIAN ESTIMATION OF A POISSON PROCESS RATE

Robert L. Wardrop

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ABSTRACT

Bayes estimation of the arrival rate of a Poisson process is studied in this paper. For any loss function in the family $L_p = (\theta-\hat{\theta})^2\theta^{-p}$, $-\infty < p < \infty$, a simple sequential procedure τ_p is introduced which, based on the criterion of minimizing expected cost (estimation error plus sampling cost), is either optimal or asymptotically optimal. The procedure τ_p is compared to Type I and II censoring - the comparison should be useful to experimenters choosing between the three sampling plans.

AMS (MOS) Subject Classifications - Primary: 62L12, 62F15.

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Key Words - Poisson process, Bayes estimation, stopping time, Types I and II censoring, martingale.

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SIGNIFICANCE AND EXPLANATION

When trying to explain or analyze events that occur randomly in time or in space, one tends first to test whether the events are governed by a Poisson distribution. The classic example concerns the number of soldiers killed per year from the kick of a mule in the Prussian Army in the early 1800's, and applications have continued to this day in many different contexts, military and otherwise.

Usually, the mean arrival or occurrence rate, θ, of a Poisson process is unknown. This paper derives optimal and approximately optimal procedures for sampling from a Poisson process and estimating the value of θ. Three classes of procedures are considered. First is "Type I censoring" in which the length of time the Poisson process is observed is fixed in advance. Second is "Type II censoring" in which the number of arrivals or occurrences observed is fixed in advance. The final class of procedures is "sequential" in the sense that neither the length of observation nor the number of occurrences observed are fixed in advance. Instead, the outcome of the process prior to any point in time may be used to decide whether to continue observing the process beyond that time.

The best procedures in each of the three classes are compared. The results are useful to the experimenter who wants to choose an efficient sampling plan.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A COMPARISON OF SAMPLING PLANS FOR BAYESIAN ESTIMATION OF A POISSON PROCESS RATE

Robert L. Wardrop

Section 1: Introduction and Notations

Conditional on the value of $\theta > 0$, let $X(t)$, $t \geq 0$, be a Poisson process with arrival rate θ. Set $t_0 = 0$, and for $i = 1,2,\ldots$, let $t_i = \inf\{t : X(t) = i\}$ be the time of the ith arrival. Bayes estimation of θ will be studied in this paper.

Assume that the loss incurred from estimating θ by $\hat{\theta}$ is given by

$$L_p(\theta,\hat{\theta}) = (\theta-\hat{\theta})^2\theta^{-p} \quad , \tag{1.1}$$

for some p, $-\infty < p < \infty$. Loss functions of the form (1.1) have been proposed by numerous authors, including Hodges and Lehmann (1953) ($p = 1$), Dvoretzky, Kiefer and Wolfowitz (1951) ($p = 1$), El-Sayyad and Freeman (1973) ($p = 0,1$, or 2), Novic (1977) ($0 \leq p \leq 3$), and Shapiro and Wardrop (1977, 1978) ($0 \leq p \leq 3$). These papers present some justification of various choices of p, especially $p = 0,1$, or 2. Results will be obtained for all real p because this does not increase the difficulty of the proofs and, more importantly, because it provides additional insight into the behavior of the sampling rules considered.

Assume θ has prior distribution

$$\lambda_p(\theta) = \Gamma(\alpha_0)^{-1}\beta_0^{\alpha_0}\theta^{\alpha_0-1}\exp(-\beta_0\theta) \quad ,$$

with $\beta_0 > 0$ and $\alpha_0 \geq 0 \vee p$. (Many of the results obtained are true with $\alpha_0 < p$, this will be discussed again in Sections 3 and 4.) Denote this distribution by $\Gamma(\alpha_0,\beta_0)$. For $t \geq 0$, let $\mathcal{F}(t)$ denote the sigma-algebra of events generated by $\{X(s),\ 0 \leq s \leq t\}$. The posterior distribution of θ given $\mathcal{F}(t)$ is $\Gamma(\alpha(t),\beta(t))$, $\alpha(t) = \alpha_0 + X(t)$ and $\beta(t) = \beta_0 + t$. For loss L_p, the Bayes estimator of θ given $\mathcal{F}(t)$ is

$$\hat{\theta}_p(t) = (\alpha(t)-p)(\beta(t))^{-1} \quad , \tag{1.2}$$

and the posterior expected loss is

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(1.3) $$E(L_p(\theta,\hat{\theta}_p(t))|F(t)) = \beta(t)^{p-2}\Gamma(\alpha(t) + 1 - p)\Gamma(\alpha(t))^{-1} .$$

It can be shown that (1.2) and (1.3) remain true with t replaced by σ, a stopping time with respect to $F(t)$, $t \geq 0$, if $F(\sigma)$ is given its usual definition (Shapiro and Wardrop (1977)).

For a precise method to discourage sampling indefinitely, let $c_A \geq 0$ be the cost of observing one arrival of the process, let $c_T \geq 0$ be the cost of observing the process for one unit of time, and assume that $c_A \vee c_T > 0$. The total cost of observing the process for t units of time is defined as

(1.4) $$C(t) = C_p(t) = \beta(t)^{p-2}\Gamma(\alpha(t) + 1 - p)\Gamma(\alpha(t))^{-1} + c_A X(t) + c_T t .$$

Different sampling plans will be compared on the basis of expected total cost.

To motivate later results, note that the values of c_A and c_T represent the cost of sampling measured in the units of the loss function L_p (this is clear from the definition of C). Intuitively, asymptotic results should be obtained by letting the cost of sampling relative to the cost of estimation error decrease,because this will encourage longer observation of the process. One way to achieve this is to let c_A and c_T tend to 0. Another approach (following El-Sayyad and Freeman (1973)) would be to define total cost as $C^*(t) = DE(L_p|F(t)) + c_A X(t) + c_T t$ and let $D \to \infty$ while holding c_A and c_T fixed. The two methods are obviously mathematically equivalent; in this paper the first method will be used.

The total cost function may be written in two other ways which will be useful later:

(1.5) $$C(t) = E(\theta^{1-p}|F(t))\beta(t)^{-1} + c_A X(t) + c_T t ,$$

and

(1.6) $$C(t) = E(\theta^{2-p}|F(t))(\alpha(t) + 1 - p)^{-1} + c_A X(t) + c_T t .$$

For $t \geq 0$, let $Y(t) = E(\theta^{1-p}|F_t)$ and $Z(t) = E(\theta^{2-p}|F(t))$. By a well known theorem $\{Y(t), t \geq 0\}$ and $\{Z(t), t \geq 0\}$ are uniformly integrable martingales. Thus $Y(\infty) = \lim_{t\to\infty} Y(t)$ and $Z(\infty) = \lim_{t\to\infty} Z(t)$ exist almost surely and $Y(\infty) = \theta^{1-p}$ and $Z(\infty) = \theta^{2-p}$. For $b > -\alpha_0$, define

$$\nu_b = E(\theta^b) = \Gamma(\alpha_0+b)\Gamma(\alpha_0)^{-1}\beta_0^{-b} \quad . \tag{1.7}$$

In Section 2 a stopping time τ will be defined and shown to be either optimal or asymptotically optimal for all p, c_A, c_T, α_0 and β_0. In addition, the limiting form of $E(C(\tau))$ will be given.

In Section 3, nonsequential sampling plans will be considered, namely Type I censoring (observing the process for a predetermined length of time) and Type II censoring (observing the process until a predetermined number of arrivals are observed). Using the criterion of minimizing expected total cost, the Bayes Type I censoring (B1) and Type II censoring (B2) procedures are obtained explicitly along with their respective expected costs (V_1 and V_2). The values V_1 and V_2 are compared asymptotically to determine the cases in which B1 is superior (inferior) to B2.

In Section 4, the results of Sections 2 and 3 are combined to determine how much better τ performs than B1 or B2. An explicit asymptotic measure of the improvement is given.

Section 2: Sequential Sampling Plans

All stopping times are with respect to $\{F(t),\ t \geq 0\}$. The stopping time σ is called optimal if, and only if $E(C(\sigma)) = \inf E(C(\rho))$ with the infimum taken over all stopping times ρ. In such problems, it is often useful to compute the infinitesimal generator of the stochastic process $C(t)$. It is defined to be

$$AC(t) = \lim_{h \downarrow 0} h^{-1}[E(C(t+h) - C(t)|F(t))] \quad .$$

Using (1.5), it is easy to show that

$$AC(t) = -\beta(t)^{-2}Y(t) + c_A\alpha(t)\beta(t)^{-1} + c_T \quad ,$$

for $Y(t)$ defined in Section 1. Intuitively, as long as $AC(t) < 0$, sampling should continue since the total cost is "expected" to decrease. A natural stopping time to consider is

(2.1) $$\tau = \text{least } t \geq 0 \text{ such that } AC(t) \geq 0 \quad ,$$

or

$$Y(t)\beta(t)^{-2} \leq c_A\alpha(t)\beta(t)^{-1} + c_T \quad .$$

For p an integer, the rule τ is easy to use; for example if $p = 0$, τ stops the first time

(2.2) $$\alpha(t)\beta(t)^{-2} \leq c_A\alpha(t) + c_T\beta(t) \quad .$$

The left side of (2.2) is the posterior variance of θ while the right side is approximately the total cost of sampling. In fact, for any p, τ stops the first time

$$E(L_p(\theta,\hat{\theta})|F(t)) \leq c_A\alpha(t) + c_T\beta(t) \quad ,$$

which generalizes the above.

The following result on the optimality of τ was obtained independently, using different methods of proof, by Novic (1977) and Shapiro and Wardrop (1977).

Theorem 2.1. In the cases

(i) $0 \leq p < 1$ and $c_T = 0$,

(ii) $1 \leq p \leq 2$ and all c_A, c_T, or

(iii) $2 < p \le 3$ and $c_A = 0$,

τ given by (2.1) is optimal for all $\beta_0 > 0$ and $\alpha_0 \ge p$.

For situations not covered by Theorem 2.1, including the interesting case $p = 0$ and $c_T > 0$, the optimal stopping time is not known. Moreover, there are not any general existing results on the limiting form of the expected total cost of the optimal rule. In this section it will be shown that τ is asymptotically optimal for all p, c_A, c_T, β_0 and $\alpha_0(\ge p)$, and the limiting form of $E(\mathcal{C}(\tau))$ will be obtained. First a lower bound for the asymptotic expected total cost of any sequence of stopping times is obtained in Theorem 2.2.

Lemma 2.1. For $U, V > 0$ random variables, $\min_V E(UV^{-1} + V) = 2E(U^{1/2})$.

Proof: For $x, y > 0$, $g(x) = yx^{-1} + x$ achieves a unique minimum of $2y^{1/2}$ at $x = y^{1/2}$. Therefore $EE(UV^{-1} + V|U) \ge 2E(U^{1/2})$.

Theorem 2.2.

(i) If $c_A = 0$, let $\sigma = \sigma(c_T)$ be any family of stopping times, then

$$\liminf_{c_T \to 0} c_T^{-1/2} E(\mathcal{C}(\sigma)) \ge 2EY(\infty)^{1/2} = 2\Gamma(\alpha_0 + (1-p)/2)\beta_0^{(p-1)/2}\Gamma(\alpha_0)^{-1} .$$

(ii) If $c_A > 0$, write $c_T = ac_A (a \ge 0)$. Let $\sigma = \sigma(c_A)$ be any family of stopping times, then

$$\liminf_{\substack{c_A \to 0 \\ c_T = ac_A}} c_A^{-1/2} E(\mathcal{C}(\sigma)) \ge 2E(aY(\infty) + Z(\infty))^{1/2} = 2E(a\theta^{1-p} + \theta^{2-p})^{1/2} .$$

Proof: For (i),

$$\liminf_{c_T \to 0} c_T^{-1/2} E(\mathcal{C}(\sigma)) = \liminf_{c_T \to 0} E(Y(\sigma)c_T^{-1/2}\beta_\sigma^{-1} + c_T^{1/2}\beta_\sigma)$$

$$\ge \liminf_{c_T \to 0} 2EY(\sigma)^{1/2} \ge 2EY(\infty)^{1/2} ,$$

by Lemma 2.1 and the fact that $\{Y(t)^{1/2},\ 0 \le t \le \infty\}$ is a supermartingale.

For (ii)

$$\liminf_{c_A \to 0,\, c_T = ac_A} c_A^{-1/2} E(\ (\sigma)) =$$

$$\liminf E\left\{\frac{(a\beta(\sigma)+\alpha(\sigma)+1-p)\Gamma(\alpha(\sigma)+1-p)}{c_A^{1/2}(a\beta(\sigma)+\alpha(\sigma)+1-p)\beta(\sigma)^{2-p}\Gamma(\alpha(\sigma))}\right.$$

$$\left. + c_A^{1/2}(\alpha(\sigma) + 1-p + a\beta(\sigma))\right\} = \liminf E(UV^{-1} + V)$$

with $U = aY(\sigma) + Z(\sigma)$ and $V = c_A^{1/2}(a\beta(\sigma) + \alpha(\sigma) + 1 - p)$. The result now follows as in (i).

To obtain the limiting value of $E(C(\tau))$, the following result is needed.

<u>Lemma 2.2</u>.

(i) If $c_A = 0$, then

$$\lim_{c_T \to 0} c_T^{1/2} E\tau = E\theta^{(1-p)/2} .$$

(ii) If $c_A > 0$, and $c_T = ac_A (a \ge 0)$, then

$$\lim_{\substack{c_A \to 0 \\ c_T = ac_A}} c_A^{1/2} E(a\beta(\tau) + \alpha(\tau)) = E(a\theta^{1-p} + \theta^{2-p})^{1/2} .$$

<u>Proof</u>: In case (i), from the definition of τ, $Y(\tau) \le c_T\beta(\tau)^2$. Thus,

(2.3) $$c_T^{1/2} E(\beta(\tau)) \ge E(Y(\tau)^{1/2}) .$$

For $\epsilon > 0$, on the set $\tau > \epsilon$, $Y(\tau-\epsilon) > c_T\beta(\tau-\epsilon)^2$, yielding

(2.4) $$c_T^{1/2} E(\beta(\tau-\epsilon)) < E(Y(\tau-\epsilon)^{1/2}) \quad .$$

It is easy to see that as $c_T \to 0$, $\tau \to \infty$ and $Y(\tau) \to Y(\infty)$ with probability one. Moreover, both $Y(\tau-\epsilon)^{1/2}$ and $Y(\tau)^{1/2}$ are bounded above by $(\sup_{t\geq 0} Y(t))^{1/2}$. This latter random variable is integrable because

$$P((\sup_{t\geq 0} Y(t))^{1/2} > a) = \lim_{T\to\infty} P(\sup_{0\leq t\leq T} Y(t) > a^2) \leq a^{-2} E(Y_0) \quad .$$

Part (i) now follows from (2.3) and (2.4).

For case (ii), the definition of τ gives

$$aY(\tau) + Z(\tau) \leq c_A(a\beta(\tau) + \alpha(\tau))^2 + c_A(1-p)(a\beta(\tau) + \alpha(t))$$

and on the set $\tau > \epsilon > 0$,

$$aY(\tau-\epsilon) + Z(\tau-\epsilon) < c_A(a\beta(\tau-\epsilon) + \alpha(\tau-\epsilon))^2 + c_A(1-p)(a\beta(\tau-\epsilon) + \alpha(\tau-\epsilon)) \quad .$$

The remainder of the proof is similar to case (i) and will not be given.

<u>Theorem 2.3</u>.

(i) If $c_A = 0$, then

$$\lim_{c_T\to 0} c_T^{-1/2} E(C(\tau)) = 2E(\theta^{(1-p)/2}) \quad .$$

(ii) If $c_A > 0$, and $c_T = ac_A (a\geq 0)$, then

$$\lim_{\substack{c_A\to 0\\ c_T = ac_A}} c_A^{-1/2} E(C(\tau)) = 2E(a\theta^{1-p} + \theta^{2-p})^{1/2} \quad .$$

<u>Proof</u>: For (i),

$$C(\tau) = Y_\tau \beta_\tau^{-1} + c_T \tau \leq 2c_T \beta_\tau \quad ,$$

and for (ii),

$$C(\tau) = Y(\tau)\beta(\tau)^{-1} + c_A(a\tau + X(\tau)) \leq 2c_A(a\beta(\tau) + \alpha(\tau)) \quad .$$

The desired results follow from Theorem 2.2 and Lemma 2.2.

In view of the results presented in Theorem 2.2 and 2.3, say that τ is asymptotically optimal. Note that if both c_A and c_T are positive then $\lim\limits_{\substack{c_A \to 0 \\ c_T = ac_A}} c_A^{-1/2} E(\mathcal{C}(\tau))$ must be computed numerically.

Section 3. Types I and II Censoring

By Theorem 2.1, the optimal sequential procedure is type I censoring if either $c_T = p = 0$ or $c_A = 0$ and $p = 1$, and it is type II censoring if either $c_T = 0$ and $p = 2$ or $c_A = 0$ and $p = 3$ (the cases with $p = 1,2$ were obtained by El-Sayyad and Freeman (1973)). In some applications it may not be feasible to use τ, and the experimenter must choose either Type I or II censoring. The results of this section should be helpful in making that choice.

The Bayes type I censoring procedure (B1) is that t^* which minimizes $E(\mathcal{C}(t))$ over all $t \geq 0$. Using representation (1.5) for $\mathcal{C}(t)$, it is easy to verify

$$t^* = \{[\nu_{1-p}(c_A\nu_1 + c_T)^{-1}]^{1/2} - \beta_0\}^+ ,$$

and for $t^* > 0$,

$$V_1 = E(\mathcal{C}(t^*)) = 2[\nu_{1-p}(c_A\nu_1 + c_T)]^{1/2} - \beta_0(c_A\nu_1 + c_T) , \tag{3.1}$$

with ν_b given by (1.7).

The Bayes type II censoring procedure is that integer n^* which minimizes $E(\mathcal{C}(t_n))$ over $n = 0,1,2,\ldots$. Treating n as a continuous variable and using representation (1.6) for $\mathcal{C}(t)$, it is easy to show

$$n^* = \{[\nu_{2-p}(c_A + c_T\nu_{-1})^{-1}]^{1/2} - (\alpha_0 + 1-p)\}^+ , \tag{3.2}$$

provided that either $\alpha_0 > 1$ or $c_T = 0$. If $\alpha_0 \leq 1$ and $c_T > 0$, then $n^* = 0$ since $c_T E(t_1) = \infty$. For the remainder of this section assume that either $\alpha_0 > 1$ or $c_T = 0$. If $n^* > 0$, then set

$$V_2 = E(\mathcal{C}(t_{n^*})) = 2[\nu_{2-p}(c_A + c_T\nu_{-1})]^{1/2} - (\alpha_0 + 1-p)(c_A + c_T\nu_{-1}) . \tag{3.3}$$

For the remainder of the section assume c_A and c_T are small enough to insure n^*, $t^* > 0$.

Remark 3.1. The value n^* given by (3.2) is not necessarily an integer, so, strictly speaking, B2 is either $[n^*]$ or $[n^* + 1]$, $[\cdot]$ the greatest integer function. Also, the expected total cost of B2 is not exactly V_2 if n^* is not an integer. For given values of α_0, β_0, p, c_A and c_T, one may compare V_1 and V_2 (or V_2's 'exact' version) as an aid in choosing between B1 and B2. In the remainder of the section V_1 and V_2 will be compared asymptotically as sampling costs go to zero (the effect of n^* not being an integer disappears in the limit).

Comparison of V_1 and V_2 when $c_A = 0$.

If $p = 2$, then $V_1 = V_2$ for all $c_T > 0$. Define

$$(3.4) \qquad R_A(p,\alpha_0) = \lim_{\substack{c_T \to 0 \\ c_A = 0}} V_1 V_2^{-1} = [\nu_{1-p}\nu_{-1}^{-1}\nu_{2-p}^{-1}]^{1/2}$$

$$= [(\alpha_0-1)(\alpha_0+1-p)]^{-1}, \quad \text{for } \alpha_0 > p \vee 1 ,$$

by (3.1), (3.3) and (1.7). It is not difficult to show that result (3.4) remains true for $\alpha_0 > (p-1) \vee 1$. Moreover, if $0 \vee (p-1) < \alpha < 1$, then $R_A(p,\alpha_0) = 0$. Clearly $R_A(p,\alpha_0) < 1$ if and only if $p < 2$. In words, asymptotically B1 gives a lower expected cost than B2 iff $p < 2$. As shown above, the asymptotic expected savings in using B1 instead of B2 can be 100% (when $R_A = 0$). Conversely, the asymptotic expected savings in using B2 instead of B1 can approach 100% when $R_A \to \infty$ (e.g. take $p = 3$, $\alpha_0 = 2 + \epsilon$, $\epsilon > 0$; then $R_A = [(1+\epsilon)\epsilon^{-1}]^{1/2} \to \infty$ as $\epsilon \to 0$). Thus, depending on the values of p and α_0 there can be a tremendous difference between V_1 and V_2. If $R_A(p,\alpha_0)$ is near unity then the difference (based on expected total cost) between B1 and B2 is not dramatic and the experimenter may choose to use the procedure which is easier to implement.

Comparison of V_1 and V_2 when $c_T = 0$.

If $p = 1$, then $V_1 = V_2$ for all $c_A > 0$. Define

$$(3.5) \qquad R_T(p,\alpha_0) = \lim_{\substack{c_A \to 0 \\ c_T = 0}} V_1 V_2^{-1} = (\nu_{1-p}\nu_1\nu_{2-p}^{-1})^{1/2}$$

$$= [\alpha_0(\alpha_0+1-p)^{-1}]^{1/2} , \quad \text{for } \alpha_0 > p \vee 0 ,$$

by (3.1, (3.3) and (1.7). It is not difficult to show that (3.5) remains true for $\alpha_0 > (p-1) \vee 0$ (for B2 one need not require $\alpha_0 > 1$ since $c_T = 0$). Clearly $R_T(p,\alpha_0) < 1$ if and only if $p < 1$ and R_T takes on all values in $(0,\infty)$ (e.g. $R_T \to \infty$ as $\alpha_0 \downarrow (p-1) > 0$; for $p = 0$, $R_T \to 0$ as $\alpha_0 \to 0$). The discussion given above for $c_A = 0$ is also relevent in this case.

Comparison of V_1 and V_2 when $c_A, c_T > 0$.

For $a > 0$ define

$$R(p,\alpha_0,a\beta_0) = \lim_{\substack{c_A \to 0 \\ c_T = ac_A}} V_1 V_2^{-1} = \tag{3.6}$$

$$\left[(\alpha_0 + a\beta_0)\left\{(\alpha_0+1-p)(1+a\beta_0(\alpha_0-1)^{-1}\right\}^{-1}\right]^{1/2} ,$$

for $\alpha_0 > p \vee 1$, by (3.1), (3.3) and (1.7). Result (3.6) remains true for $\alpha_0 > (p-1) \vee 1$, and if $(0 \vee (p-1)) < \alpha_0 < 1$, then $R(p,\alpha_0,a\beta_0) = 0$.

Clearly, $R > 1$ if $p \geq 2$ and $R < 1$ if $p \leq 1$. Set

$$p^* = (\alpha_0 - 1 + 2a\beta_0)(\alpha_0 - 1 + a\beta_0)^{-1} \quad (1 < p^* < 2) \ .$$

Simple algebra gives $R = 1$ if $p = p^*$, and $R > 1$ $(R < 1)$ if $p > p^*$ $(p < p^*)$. If $1 < p < 2$, and $\alpha_0 > 1$, set

$$\alpha_0^* = 1 + a\beta_0(2-p)(p-1)^{-1} \ , \text{ and}$$

$$(a\beta_0)^* = (\alpha_0 - 1)(p-1)(2-p)^{-1} \ .$$

Simple algebra gives $R = 1$ if $\alpha_0 = \alpha_0^*$, equivalently $a\beta_0 = (a\beta_0)^*$, and $R > 1$ $(R < 1)$ if $\alpha_0 > \alpha_0^*$ $(\alpha_0 < \alpha_0^*)$ or, equivalently, $a\beta_0 < (a\beta_0)^*$ $(a\beta_0 > (a\beta_0)^*)$. Finally, it is easy to see that R takes on all values in $[0,\infty)$.

In summary, for given values of p, c_A, c_T, α_0 and β_0, B1 and B2 can be compared readily, either using exact values of V_1 and V_2 or an asymptotic approximation.

Section 4. An Asymptotic Comparison of τ with B1 and B2

Analogous to the earlier definitions of V_1 and V_2, let $V_0 = E(\mathcal{C}(\tau))$. The exact value of V_0 must be computed numerically, but its limiting value is given in Theorem 2.3. The class of sequential procedures includes B1 and B2; hence, $\lim V_0V_i^{-1} \leq 1$, for $i = 1,2$, with the limit taken as sampling costs tend to zero. Thus, based on the criterion of expected total cost, neither B1 nor B2 are ever better than τ in the limit. However, in real applications, other considerations (such as case of implementation) are important and an experimenter may prefer B1(B2) if $V_0V_1^{-1}(V_0V_2^{-1})$ is near unity. In this section the limiting value of $V_0V_i^{-1}$, $i = 1,2$, will be given in the case of exactly one sampling cost positive. In the case of both sampling costs positive the limit of V_0 must be computed numerically; hence it is not a convenient case to determine general patterns.

It is not difficult to see that the conclusions of Theorem 2.2 and 2.3 remain true if the hypotheses are weakened to $\alpha_0 > 0 \vee (p-1)$. In fact, if $c_T = 0$, then the hypotheses can be weakened to $\alpha_0 > 0 \vee (p-2)$. Similarly, results given on the limiting form of V_2 can be extended to $\alpha_0 > j \vee (p-2)$, where $j = 1$ if $c_T > 0$ and $j = 0$ if $c_T = 0$. As mentioned in Section 3, results given on the limiting form of V_1 require only $\alpha_0 > 0 \vee (p-1)$.

For $-\infty < r < \infty$ and $b > 0 \vee (r-1)$, define

$$H(b,r) = \Gamma(b + (1-r)/2)\Gamma(b)^{-1/2}\Gamma(b+1-r)^{-1/2} .$$

For $i = 1,2$, set

$$Q_p(T,i) = \lim_{\substack{c_A \to 0 \\ c_T = 0}} V_0V_i^{-1}$$

and define $Q_p(A,i)$ in the analogous way. It follows easily from Theorem 2.2 and 2.3, formulas (1.7), (3.1) and (3.3) and the remarks above that

$$
\begin{aligned}
Q_p(A,1) &= H(\alpha_0,p) \quad , & \alpha_0 &> 0 \vee (p-1) \\
Q_p(A,2) &= H(\alpha_0-1,p-2) & \alpha_0 &> 1 \vee (p-1) \\
&= 0 & 1 \geq \alpha_0 &> 0 \vee (p-1) \qquad (4.1)\\
Q_p(T,1) &= H(\alpha_0+1,p+1) & \alpha_0 &> 0 \vee p-1 \\
Q_p(T,2) &= H(\alpha_0,p-1) & \alpha_0 &> 0 \vee p-2 \quad .
\end{aligned}
$$

Some Examples of (4.1)

For simplicity, only compare τ to the "better" of B1 and B2 (i.e. B1 if $p \leq 1$, B2 if $p \geq 2$, see Section 3).

$p = 0$: In this case

$$Q_0(T,1) = 1 \ , \ \alpha_0 > 0, \text{ and}$$

$$Q_0(A,1) = \Gamma(\alpha_0 + \tfrac{1}{2})\Gamma(\alpha_0)^{-1/2}\Gamma(\alpha_0+1)^{-1/2}, \ \alpha_0 > 0 \ .$$

Note $\lim_{\alpha_0 \downarrow 0} Q(A,1) = 0$ and $\lim_{\alpha_0 \to \infty} Q(A,1) = 1$.

$p = 1$: In this case,

$$Q_1(A,1) = 1 \ ,$$

$$Q_1(T,1) = \Gamma(\alpha_0 + \tfrac{1}{2})\Gamma(\alpha_0)^{-1/2}\Gamma(\alpha_0+1)^{-1/2}, \ \alpha_0 > 0 \ ,$$

and the remarks of the previous case apply.

$p = 2$: In this case,

$$Q_2(T,2) = 1 \ ,$$

$$Q_2(A,2) = \Gamma(\alpha_0 - \tfrac{1}{2})\Gamma(\alpha_0-1)^{-1/2}\Gamma(\alpha_0)^{-1/2}, \ \alpha_0 > 1 \ ,$$

and the remark of case $p = 0$ apply with the modification $\alpha_0 \downarrow 1$ instead of $\alpha_0 \downarrow 0$.

$p = 3$: In this case,

$$Q_3(A,2) = 1$$

$$Q_3(T,2) = \Gamma(\alpha_0 - \tfrac{1}{2})\Gamma(\alpha_0-1)^{-1/2}\Gamma(\alpha_0)^{-1/2}, \ \alpha_0 > 1 \ ,$$

and the previous remark applies.

$p < 0$: In this case,

$$Q_p(T,1) = \Gamma(\alpha_0+1-p/2)\Gamma(\alpha_0+1)^{-1/2}\Gamma(\alpha_0+1-p)^{-1/2}, \ \text{and}$$

$$Q_p(A,1) = \Gamma(\alpha_0 + (1-p)/2)\Gamma(\alpha_0)^{-1/2}\Gamma(\alpha_0+1-p)^{-1/2} \quad .$$

Note that $\lim_{p\to-\infty} Q_p(T,1) = 1$ for all $\alpha_0 > 0$. Thus if $c_T = 0$, τ is little better than Bl even for a vague prior, when p is large and negative.

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Bayes estimation of the arrival rate of a Poisson process is studied in this paper. For any loss function in the family L_p, ~~$= (\theta-\hat{\theta})^2\theta^{-p}, -\infty < p < \infty$~~, a simple sequential procedure τ_p is introduced which, based on the criterion of minimizing expected cost (estimation error plus sampling cost), is either optimal or asymptotically optimal. The procedure τ_p is compared to Type 1 and ~~II~~ censoring - the comparison should be useful to experimenters choosing between the three sampling plans.

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