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A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW T--ETC(U)

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A PROOF OF THE CONVEXITY OF THE FREE
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RECTANGULAR DAM USING THE MAXIMUM
PRINCIPLE

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A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH
A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE

C. W. Cryer *

Technical Summary Report #1953
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ABSTRACT

We consider the steady two-dimensional flow under gravity of water from one reservoir to another reservoir through a porous rectangular isotropic homogeneous dam with impervious bottom. Using the maximum principle we give a proof of the convexity of the free boundary.

AMS (MOS) Subject Classifications: 35J65, 76S05

Key Words: Porous flow, Rectangular dam, Maximum principle,
Free boundary, Convexity

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SIGNIFICANCE AND EXPLANATION

We consider the steady two-dimensional flow under gravity of water from one reservoir (on the left) to a lower reservoir (on the right) through a porous rectangular isotropic homogeneous dam with impervious bottom. Because of gravity the water does not flow through the entire dam and the dam is dry near its upper right corner. The interface separating the dry and wet regions of the dam is a free boundary. Recently, Friedman and Jenkins have proved that the free boundary is convex. We give a different proof which uses only the maximum principle and its generalizations.

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A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH
A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE

C. W. Cryer*

1. The Dam Problem

We consider the following Dam Problem. Water flows steadily under gravity through a porous rectangular isotropic homogeneous dam ABCF with impervious bottom BC of length L from a reservoir of height H to a reservoir of height h (see Figure 1.1). The water-air interface AE is a free boundary which we denote by Γ .

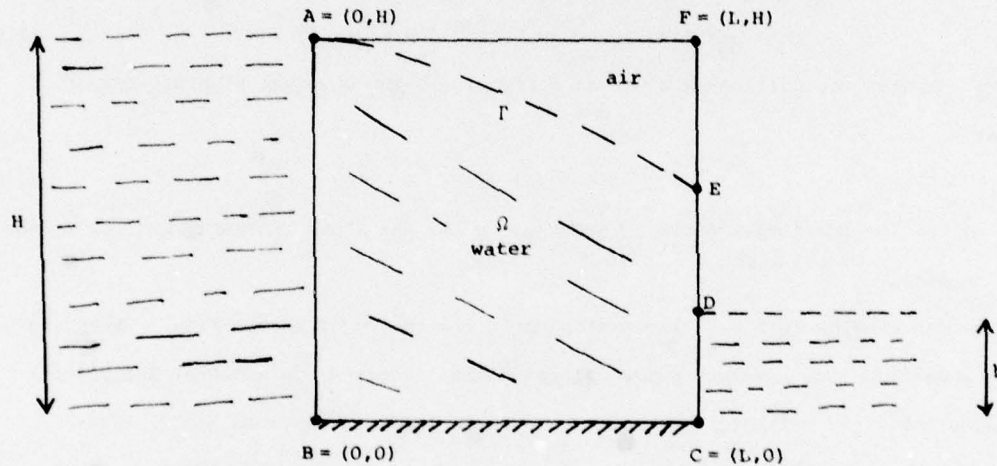


Figure 1.1: Flow through a rectangular dam.

The mathematical problem is as follows (Bear [1972], Baiocchi [1972], Baiocchi and Capelo [1978]):

Find functions $\varphi(x)$ (the height of the free boundary) and $u(x,y)$ (the hydraulic head) such that (from the equation of continuity and Darcy's law):

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$$\operatorname{div}(\operatorname{grad} u) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = u_{xx} + u_{yy} = \nabla^2 u = 0, \quad \text{in } \Omega, \quad (1.1)$$

together with the boundary conditions,

$$u = H, \quad \text{on } AB, \quad (\text{interface with water at rest}), \quad (1.2)$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } BC, \quad (\text{impervious boundary}), \quad (1.3)$$

$$u = h, \quad \text{on } CD, \quad (\text{interface with water at rest}), \quad (1.4)$$

$$u = y, \quad \text{on } DE, \quad (\text{interface with air}), \quad (1.5)$$

$$u = y, \quad \text{on } EA, \quad (\text{interface with air}), \quad (1.6)$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } EA, \quad (\text{streamline}). \quad (1.7)$$

Here, Ω is the (unknown) domain,

$$\Omega = \{(x, y) : 0 < y < \varphi(x), 0 < x < L\}, \quad (1.8)$$

and $\frac{\partial}{\partial n}$ denotes the unit outward normal derivative. The physical significance of u is that

$$u = y + p/\rho g, \quad (1.9)$$

where g is the acceleration due to gravity, p is the fluid pressure, and ρ is the fluid density.

The dam problem is a well-known problem in the theory of porous flow. Cryer [1976, p. 54] summarizes the numerous numerical solutions. There are also three analytical solutions involving elliptic integrals: the first is due to Davison [1932, 1936, 1936a]; the second is due to Hamel [1934] and is described by Muskat [1937, p. 303] and Bear [1972, p. 398]; the third is due to Polubarinova-Kochina [1962, p. 284] (concerning misprints see Cryer [1976, p. 54]). Although these analytical solutions are rather complicated, they can, and have, been evaluated numerically.

Despite the fact that the analytical solution of the dam problem is known, the problem is still frequently considered in the literature because it serves as a useful model problem for porous flow free boundary problems. In a pioneering paper, Baiocchi [1972] reformulated the problem as a variational inequality for the function

$$w(x,y) = \int_y^{\varphi(x)} [u(x,t) - t] dt, \quad (1.10)$$

and derived many properties of w . Further properties of w and $u = y - w_y$ have been obtained by Caffarelli [to appear], Friedman [1976], Jensen [1977], and Friedman and Jensen [1977]. Aitchison [1972] gives an expansion for the solution near the separation point E , and Aitchison [1977] gives numerical solutions.

Recently, Friedman and Jensen [1977] have proved that Γ is convex under the following assumptions (which had previously been proved in the above-mentioned work of Friedman, Jensen, and Caffarelli):

$$u \in C(\bar{\Omega}) \cap C^2(\Omega), \quad (1.11)$$

$$\varphi \in C[0,L] \cap C^2(0,L). \quad (1.12)$$

$$\varphi \text{ is strictly monotone decreasing for } 0 < x < L, \quad (1.13)$$

$$u_y \text{ is bounded on } \Omega. \quad (1.14)$$

Making the same assumptions we give a different proof of convexity which uses only the maximum principle and its generalizations. Some preliminary results of general applicability are given in Section 2, and are then applied to the case in hand in Section 3.

The basic ideas in Section 3 have been known to us for almost twenty years and were originally motivated by proofs of convexity for fluid dynamics free boundary problems (Birkhoff and Zarantonello [1957, p. 84], Gilbarg [1960, p. 373]). Since the approach is effective in many fluid mechanics free boundary problems we hope that the same will be true for porous flow free boundary problems.

The present paper illustrates the power of the maximum principle as a tool for analyzing free boundary problems. It is appropriate to mention that the maximum principle was probably first applied to the dam problem by Davison [1936] and Shaw and Southwell [1941]. Some additional references to the application of the maximum principle to free boundary problems are given by Cryer [1977, Section II.13].

2. Preliminary Results on the Maximum Principle and Differential Geometry

We will require several versions of the maximum principle for a real function $u \in C(\bar{\Omega})$ which is harmonic in a bounded domain Ω with boundary $\partial\Omega$ in the complex $z = x + iy$ - plane. We assume that u is continuous in $\bar{\Omega}$ except perhaps at a finite number of points $z_1, z_2, \dots, z_m \in \partial\Omega$.

The Hopf Principle (Protter and Weinberger [1967, p. 65], Gilbarg and Trudinger [1977, p. 33]): Let $u(z) \leq M$ in Ω , and $u(z_0) = M$ at a point $z_0 \in \partial\Omega$, the boundary of Ω . Let (i) u be continuous at z_0 and (ii) there exist a ball $B \subset \Omega$ with $z_0 \in \partial B$.

Then the outer normal derivative of u at z_0 , if it exists, satisfies

$$\frac{\partial u}{\partial n}(z_0) > 0. \quad \square$$

The Maximum Principle states that if $u \in C(\bar{\Omega})$ and $u \leq M$ on $\partial\Omega$ then $u \leq M$ in Ω . In the present context, this principle is not adequate because we sometimes consider harmonic functions which are possibly not continuous at the boundary points $z_1 = A$, $z_2 = D$, and $z_3 = E$ (Figure 1.1). We will say that u satisfies the Generalized Maximum Principle if

$$u(z) \leq M, \quad z \in \partial\Omega; \quad z \neq z_j, \quad 1 \leq j \leq m, \quad (2.1)$$

implies

$$u(z) \leq M \quad \text{in } \Omega.$$

Remark 2.1

If $v(\zeta)$ is harmonic in the unit circle D and v is continuous on D except at $\zeta = 1$, it is not necessarily true that v satisfies the Generalized Maximum Principle as is shown by the following examples.

$$v(\zeta) = v(re^{i\theta}) = v(r, \theta) = \text{Real} \left(\frac{1 + \zeta}{1 - \zeta} \right) = \frac{(1 - r^2)}{1 + r^2 - 2r \cos \theta},$$

is a function which is defined on the unit circle D and $v = 0$ on ∂D except at the point $\zeta = +1$, but $v(r, 0) \rightarrow +\infty$ as $r \rightarrow 1$.

$$v(\zeta) = v(r, \theta) = \operatorname{Imag} \left(\frac{1 + \zeta}{1 - \zeta} \right)^2 = \frac{2r(1 - r^2) \sin \theta}{[1 + r^2 - 2r \cos \theta]^2}.$$

satisfies $v(r, \theta) \rightarrow 0$ as $r \rightarrow 1$ for all θ . \square

In order for the Generalized Maximum Principle to apply, u must satisfy some boundedness condition. In particular, it is known (Goluzin [1969, p. 267]) that if u is bounded above in Ω then u satisfies the Generalized Maximum Principle. The following lemma gives another condition.

Lemma 2.1

Let v be the harmonic conjugate of u in Ω . (v is well-defined and single-valued since, by assumption, Ω is a domain and hence simply connected.) Let $\partial\Omega$ be a rectifiable closed Jordan curve. If $|v| \leq M_V$ in Ω for some constant M_V then u satisfies the Generalized Maximum Principle.

Proof: Let $z = \omega(\zeta)$ denote the conformal mapping of the unit disk $D = \{\zeta : |\zeta| < 1\}$ in the ζ -plane onto Ω . The functions $u(z)$ and $v(z)$ are harmonic conjugates in Ω , and so the functions $u(\omega(\zeta))$ and $v(\omega(\zeta))$ are harmonic conjugates in D .

We recall (Goluzin [1969, p. 385]) that the Hardy-Lebesgue space h_2 consists of the class of functions ϕ defined on D , which are harmonic on D , and which satisfy

$$\int_0^{2\pi} |\phi(r, \theta)|^2 d\theta < m$$

for some constant m and all $r \in (0, 1)$.

Since v is bounded, $v(\omega(\zeta)) \in h_2$; hence, by the Riesz theorem, $u(\omega(\zeta)) \in h_2$ (Goluzin [1969, p. 392]).

Let $\phi(r, \theta) = \phi(\zeta) = u(\omega(\zeta))$. Since $\phi \in h_2$ the boundary values $\phi(1, \theta)$ are defined almost everywhere and Poisson's formula holds (Goluzin [1969, p. 391, Theorem 3], Rudin [1966, p. 232]); that is,

$$\phi(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \phi(1, \theta) \frac{1 - r^2}{1 - 2r \cos \theta + r^2} d\theta.$$

Let u satisfy (2.1). Since $\partial\Omega$ is a rectifiable Jordan curve, the mapping ω associates sets of measure zero on $\partial\Omega$ with sets of measure zero on ∂D (Goluzin [1969, p. 420, Theorem 2]). Thus $|\phi(1, \theta)| \leq M$ a.e. on ∂D . Remembering that the weight

function in Poisson's formula is non-negative, it follows from Poisson's formula above that $|\phi| \leq M$ on D . \square

Remark 2.2

If u and v are harmonic conjugates and v is bounded then u need not be bounded as shown by the example

$$u + iv = i \ln(x + iy) . \quad \square$$

Further comments on the Generalized Maximum Principle will be found in the appendix.

In the dam problem we are given the values of u and u_n on Γ . The following lemma summarizes the relationships between the derivatives of a function on a curve. The lemma differs from the usual Frenet formulas (Eisenhart [1940, p. 25]) in that the curvature κ may be either positive or negative.

Lemma 2.2

Let the boundary $\partial\Omega$ of Ω be defined parametrically by $x = \xi(s)$, $y = \eta(s)$, where s denotes arc length along $\partial\Omega$ in the positive direction so that Ω is to the left of $\partial\Omega$ (Figure 2.1).

Let $\kappa = d\theta/ds = \dot{\theta}$ denote the signed curvature, so that if κ is positive then Ω is convex. Let \underline{t} be the unit tangent, and \underline{n} the unit outward normal. Then

$$\begin{aligned} \underline{t} &= (t_1, t_2) = (\dot{\xi}, \dot{\eta}) , \\ \underline{n} &= (n_1, n_2) = (\dot{\eta}, -\dot{\xi}) , \\ \dot{\xi}^2 + \dot{\eta}^2 &= 1, \quad \ddot{\xi}\dot{\xi} + \ddot{\eta}\dot{\eta} = 0 , \\ \kappa &= \ddot{\xi}\dot{\eta} - \dot{\xi}\ddot{\eta} , \\ \ddot{\xi} &= -\kappa\dot{\eta}, \quad \ddot{\eta} = +\kappa\dot{\xi} , \\ \frac{d\underline{t}}{ds} &= -\kappa\underline{n}, \quad \frac{d\underline{n}}{ds} = +\kappa\underline{t} . \end{aligned}$$

Let $\psi = \psi(x, y)$ be twice continuously differentiable. We denote by ψ_s the derivative of ψ along $\partial\Omega$; that is,

$$\psi_s = \frac{d\psi(\xi(s), \eta(s))}{ds} .$$

We denote by ψ_n and ψ_t the directional derivatives of ψ along \underline{n} and \underline{t} ; that is,

$$\varphi_n = n_1 \frac{\partial \varphi}{\partial x} + n_2 \frac{\partial \varphi}{\partial y}, \quad \varphi_t = t_1 \frac{\partial \varphi}{\partial x} + t_2 \frac{\partial \varphi}{\partial y}.$$

Similarly,

$$\varphi_{nt} = n_1 t_1 \frac{\partial^2 \varphi}{\partial x^2} + (n_1 t_2 + n_2 t_1) \frac{\partial^2 \varphi}{\partial x \partial y} + n_2 t_2 \frac{\partial^2 \varphi}{\partial y^2},$$

$$\varphi_{ns} = \frac{d}{ds} (\varphi_n(\xi(s), \eta(s))).$$

Then, on $\partial\Omega$,

$$\varphi_t = \varphi_s, \quad \varphi_{tt} = \varphi_{ss} + \kappa \varphi_n,$$

$$\varphi_{nt} = \varphi_{ns} - \kappa \varphi_s. \quad \square$$

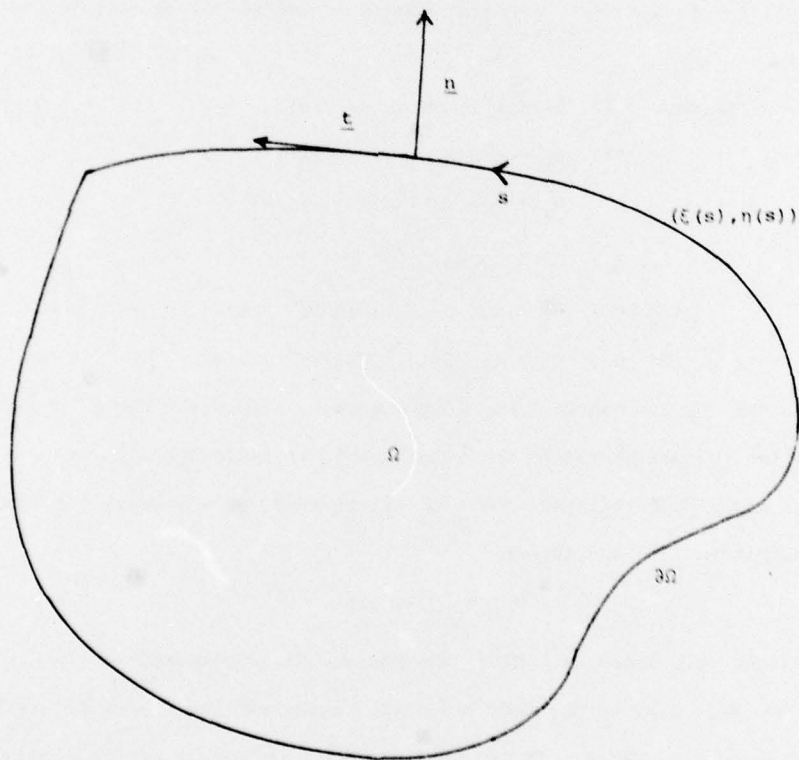


Figure 2.1: The curve $\partial\Omega$.

3. Convexity of the Free Boundary

The basic idea is to consider the function

$$U = u - y/2, \quad (3.1)$$

where u is the solution of the Dam Problem. We begin by obtaining estimates for U , u , and their derivatives, all of which are harmonic functions.

The solution u can be reflected by symmetry across BC (Courant and Hilbert [1962, p. 272]) so that u is infinitely differentiable near BC . It thus follows from the basic regularity theory for elliptic equations (Gilbarg and Trudinger [1977, Section 6.4]) that u is twice continuously differentiable on $\bar{\Omega}$ except possibly at the points $z_1 = A$, $z_2 = D$, and $z_3 = E$. In this section, statements about the values of the derivatives of u and U on $\partial\Omega$ should be understood to exclude the points A , D , and E .

Using (1.2) through (1.7) direct computation yields:

$$\begin{aligned} \text{On EA: } u_x &= +\varphi'(x)/[1 + (\varphi'(x))^2] < 0, \\ u_y &= +(\varphi'(x))^2/[1 + (\varphi'(x))^2] > 0, \\ Q^2 &\equiv |\text{grad } U|^2 = 1/4. \end{aligned} \quad (3.2)$$

$$\text{On ABCD: } u_y = 0, \quad Q^2 = |\text{grad } U|^2 \geq 1/4, \quad (3.3)$$

$$\text{On DE: } u_y = 1, \quad Q^2 = |\text{grad } U|^2 \geq 1/4. \quad (3.4)$$

The fact that Q is constant on Γ means that in porous flow Q plays a role which is similar to that played by the velocity q in fluid dynamics.

Since it was assumed ((1.14)) that u_y is bounded, we may apply the Generalized Maximum Principle to u_y and obtain

$$0 \leq u_y \leq 1, \quad \text{in } \Omega. \quad (3.5)$$

Now consider u . Since $u \in C(\bar{\Omega})$ the Maximum Principle implies that u attains its extrema on $\partial\Omega$, but, by the Hopf Principle, these extrema cannot be attained on BC because $\partial u/\partial n = 0$ there. It has been assumed that Γ is monotonically decreasing, so that $h \leq u \leq H$ on Γ . It follows that $h \leq u \leq H$ on $CDEAB$ and so

$$h \leq u \leq H \quad \text{in } \Omega. \quad (3.6)$$

Next, we consider u_x . By the Hopf Principle applied to u on AB and CD we see that

$$u_x < 0 \text{ on } AB \cup CD, \quad (3.7)$$

where we have used the fact that u can be reflected across BC . The boundary conditions for u together with (3.6) imply that the harmonic function $u - y$ is non-negative on $\bar{\Omega}$. But $u - y = 0$ on DE and hence, by the Hopf Principle,

$$u_x < 0, \text{ on } DE. \quad (3.8)$$

On BC we have $u_y = 0$; noting (3.5), the Hopf principle implies that $(u_y)_y \geq 0$. That is,

$$(u_x)_x = -(u_y)_y \leq 0, \text{ on } BC.$$

But u is smooth on BC , $u_x(B) < 0$, and $u_x(C) < 0$. Thus,

$$u_x < 0, \text{ on } BC. \quad (3.9)$$

The functions u_x and $-u_y$ are harmonic conjugates in Ω and u_y is bounded. Applying Lemma 2.1 and noting (3.2) as well as (3.7) through (3.10), we conclude that

$$U_x = u_x < 0 \text{ in } \Omega. \quad (3.10)$$

Thus, $u_x < 0$ in Ω since otherwise the strong maximum principle (Courant and Hilbert [1962, p. 326]) implies that $u_x \equiv 0$ in Ω , which is not possible. Hence

$$|\text{grad } U|^2 \geq u_x^2 > 0 \text{ in } \Omega. \quad (3.11)$$

Remark 3.1

U decreases monotonically on ABC and increases monotonically on CDE . The fact that $|\text{grad } U|^2 > 0$ in Ω is also a consequence of a result of Walsh [1950, p. 318, last paragraph] on the critical points of harmonic functions. \square

Since $U_x < 0$ in Ω the function

$$\ln(-U_x + iU_y) = \frac{1}{2} \ln[U_x^2 + U_y^2] + i \arctan(-U_y/U_x), \quad (3.12)$$

is regular in Ω and has bounded imaginary part in Ω . From Lemma 2.1 and (3.2) through (3.4) we conclude that

$$Q^2 = U_x^2 + U_y^2 \geq 1/4, \text{ in } \Omega. \quad (3.13)$$

Finally, since $\ln Q$ is harmonic in Ω and Q attains its minimum on Γ , the Hopf Principle implies that

$$\frac{\partial}{\partial n} \ln Q = \frac{1}{Q} \frac{\partial Q}{\partial n} < 0, \text{ on } \Gamma. \quad (3.14)$$

We now use Lemma 2.2 to express $\partial Q / \partial n$ in terms of the boundary data for u on Γ and the curvature κ of Γ . Direct computation shows that, on Γ ,

$$\begin{aligned} U &= \eta/2; \quad U_t = U_s = \dot{\eta}/2; \\ U_n &= u_n - \frac{1}{2} n_1 \frac{\partial y}{\partial x} - \frac{1}{2} n_2 \frac{\partial y}{\partial y} = + \dot{\xi}/2, \\ U_{tt} &= U_{ss} + \kappa U_n = \ddot{\eta}/2 + \kappa \dot{\xi}/2 = \kappa \dot{\xi}, \\ U_{nt} &= U_{ns} - \kappa U_s = \ddot{\xi}/2 - \kappa \dot{\eta}/2 = -\kappa \dot{\eta}, \\ U_{nn} &= -U_{tt} = -\kappa \dot{\xi}. \end{aligned}$$

Thus, since $Q^2 = U_n^2 + U_t^2$,

$$\begin{aligned} Q Q_n &= U_n U_{nn} + U_t U_{nt} \\ &= -\kappa \dot{\xi}^2/2 - \kappa \dot{\eta}^2/2 \\ &= -\kappa/2. \end{aligned} \quad (3.15)$$

Together, (3.14) and (3.15) show that $\kappa > 0$ so that Γ is convex.

Appendix: Remarks on the Generalized Maximum Principle

It is natural to try to weaken the assumptions of Lemma 2.1 because this would simplify applications to free boundary problems. For example, in this paper the assumption that U_y is bounded (see (1.14)) was only needed so that Lemma 2.1 could be applied. The proof that U_y is bounded (Friedman [1976]) is not straightforward, and it would be nice if it could be avoided.

The requirement in Lemma 2.1 that v be bounded can be replaced by the weaker condition that $v(\omega(\zeta)) \in h_2$, but this is not a very convenient hypothesis to check. Alternatively, one could require that $u + iv \in E_1(\Omega)$ (Goluzin [1969, p. 438], Priwalow [1956, p. 188]).

In applications we will usually know that

$$\iint_{\Omega} (u^2 + v^2) dx dy < \infty, \quad (\text{A.1})$$

so that u and v both belong to the space $L_2H(\Omega)$ which consists (Hille [1962, p. 325]) of functions ϕ which are harmonic in Ω and satisfy

$$\iint_{\Omega} \phi^2 dx dy < \infty. \quad (\text{A.2})$$

It is tempting to conjecture that if $u, v \in L_2H(\Omega)$ then $u, v \in h_2$, but this is not true as the following example shows. Let $\Omega = D$, so that $\zeta = z$. Set

$$u(z) + iv(z) = \sum_{n=1}^{\infty} z^n / \sqrt{n}. \quad (\text{A.3})$$

Expanding u and v in Fourier series and using the orthogonality relations of $\sin n\theta$ and $\cos n\theta$ we see that

$$\int_0^{2\pi} u^2(r, \theta) d\theta = \int_0^{2\pi} v^2(r, \theta) d\theta = \sum_{n=1}^{\infty} r^{2n}/n. \quad (\text{A.4})$$

Thus $u, v \in L_2H(D)$ but $u, v \notin h_2$. Furthermore, using a result given by Titchmarsh [1939, p. 163, Problem 15], we see that

$$f(z) = f(re^{i\theta}) = u + iv = \int_0^{\infty} \frac{t^{-1/2}}{e^t - z} dt, \quad (\text{A.5})$$

so that $f(z)$ is continuous on \bar{D} except at $z = 1$.

Gehring [1957] (see also Tsuji [1959, p. 186]) uses the maximal theorem of Hardy and Littlewood to prove that if $u \in L_2H(D)$ then, for almost all θ ,
 $u(z) = o(|1 - |z||^{-1/2})$ as $z \rightarrow e^{i\theta}$ in any fixed Stolz domain with vertex $e^{i\theta}$.
Unfortunately, this result is not quite strong enough to prove that u is bounded, even if we assume that u is bounded on $\partial\Omega$ except at $z = 1$. Otherwise, by mapping D onto the right half plane we could conclude that a harmonic function on the right half plane which has bounded limits on the imaginary axis, and *bounded growth near the real axis*, must be bounded. This would constitute a substantial strengthening of the Phragmen-Lindelöf Principle (Protter and Weinberger [1967, p. 94]).

In conclusion, we observe that the behaviour of solutions of elliptic equations near corners has been considered by Oddson [1968], Kondrat'ev [1967], Miller [1967, 1971] and Grisvard [1969]. These results are not immediately applicable to the present problem because they require knowledge of the behaviour of the free boundary Γ near its endpoints. Of course, once it is known that Γ is differentiable at its endpoints then this work will yield useful information about the behaviour of the solution can be obtained.

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