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THE SOLUTION OF THE AXISYMMETRIC ELASTIC-PLASTIC TORSION OF A SHAFT USING VARIATIONAL INEQUALITIES

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ABSTRACT

The axisymmetric elastic-plastic torsion of a shaft subject to the von Mises yield criterion is considered. The problem is reformulated as a variational inequality and it is proved that the problem has a unique solution. Some properties of the solution are derived.

AMS (MOS) Subject Classifications: 35J20; 35J65; 35J70; 35R99; 49H05; 73E99.

Key Words: Torsion; elastic-plastic; axisymmetric; free boundary problem; variational inequalities; weighted Sobolev space; degenerate elliptic equation.

Work Unit Number 3 - Applications of Mathematics

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SIGNIFICANCE AND EXPLANATION

When an axially symmetric shaft is subjected to a small torque the shaft deforms elastically. As the torque increases, the maximum stress reaches the largest value permissible in elastic deformation and a plastic enclave forms and grows.

In this paper the problem of the elastic-plastic torsion of a shaft is reformulated as a variational inequality. This is mathematically equivalent to the principle of Haar and von Karman according to which the strain energy must be minimized subject to the constraint that the stress should not exceed its permissible limit.

The advantages of formulating the problem as a variational inequality are:

(i) The elastic and plastic regions are treated in a unified manner and there is no need to determine the boundary of the plastic region.

(ii) Mathematical questions, such as existence and uniqueness, are readily answered.

(iii) The variational inequality lends itself to numerical approximation.

We establish existence and uniqueness of the solution, and also obtain bounds for the size of the plastic region. Numerical results for a two-diameter shaft will be given in a later paper.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE SOLUTION OF THE AXISYMMETRIC ELASTIC-PLASTIC TORSION
OF A SHAFT USING VARIATIONAL INEQUALITIES

C. W. Cryer*

1. Classical formulation of the problem.

The problem to be considered is shown in Figure 1.1. Equal and opposite torques T are applied to the ends of a shaft of length L which is axially symmetric about the x_1-axis and has (variable) radius $R(x_1)$.

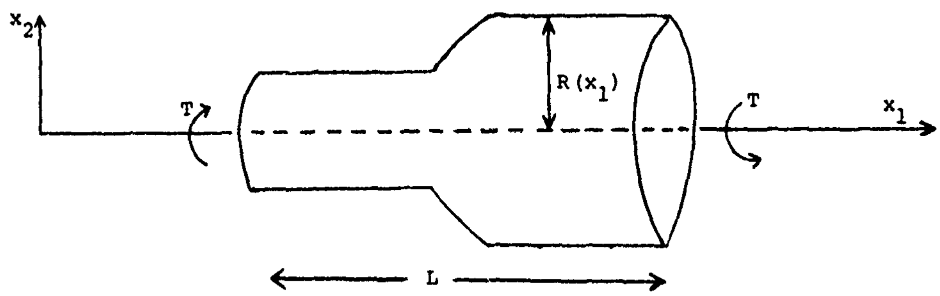

Figure 1.1: A circular shaft of varying diameter.

Because of axial symmetry it suffices to consider the problem in the two-dimensional domain

$$\Omega = \{x = (x_1, x_2) \; : 0 \le x_1 \le L; \quad 0 \le x_2 \le R(x_1)\} \quad , \tag{1.1}$$

corresponding to the cross-section of the shaft.

The boundary Γ of Ω consists of three parts: Γ_0, Γ_1, and $\Gamma_2 = \Gamma_{21} \cup \Gamma_{22}$ as shown in Figure 1.2. Γ_{21} and Γ_{22} are parallel to the x_2-axis. Γ_1 is the curve $x_2 = R(x_1)$, $0 \le x_1 \le L$. Γ_0 is a segment of the x_1-axis.

As regards the boundary Γ_1 it is assumed that:

(i) $R \in C^2(0,L)$, that is, R is twice continuously differentiable. This assumption allows us to prove that the solution is differentiable (see Theorem 5.9).

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(ii) $dR/dx_2 = d^2R/dx_2^2 = 0$ when $x_2 = 0$ and $x_2 = L$. This is true if the shaft has constant radius near its ends as often happens in practice. This assumption allows us to reflect Ω in Γ_{21} and Γ_{22} and obtain a smooth solution in the enlarged domain (see Lemma 5.1).

(iii) $dR/dx_2 \geq 0$, so that Γ_1 is of the form shown in Figure 1.2. This assumption allows us to conclude that $R(x_1) \geq R(0)$ for $x_1 \in [0,L]$. It also allows us to conclude that only one characteristic passes through each point on Γ_{21} (see Theorem 4.2).

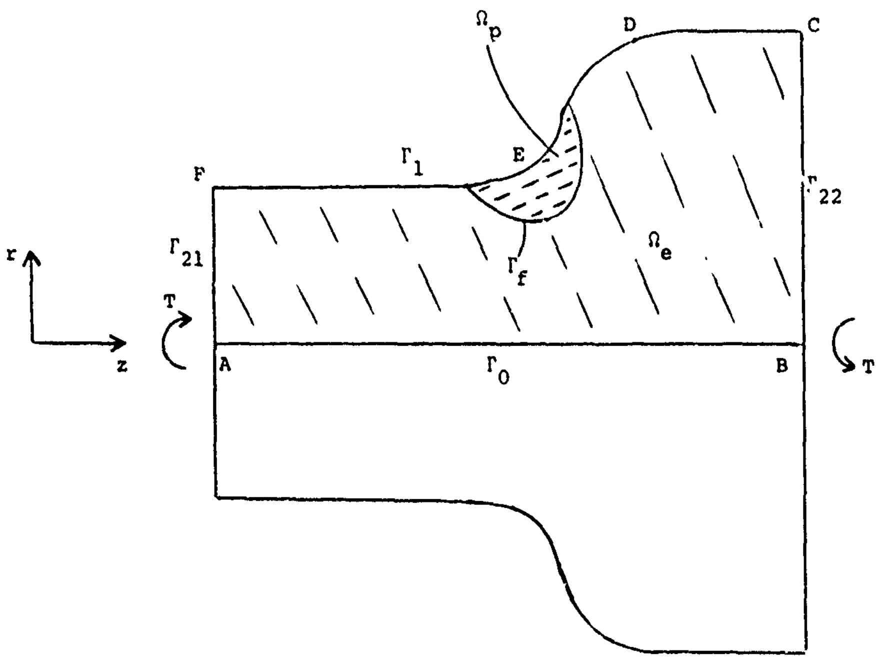

Figure 1.2: Cross-section of an axisymmetric shaft.

In analogy with the theory of torsion of prismatic bars due to Saint-Venant (Love [1944, p. 311]), it is assumed that the only non-zero stresses are shear stresses on the planes Ω. It can then be shown (Love [1944, p. 325], Eddy and Shaw [1949], Zienkiewicz and Cheung [1967]) that the problem reduces to finding a stress function u. The stress components $\tau_{r\theta} = \tau_{23}$ and $\tau_{z\theta} = \tau_{13}$ are given in terms of u by

$$\tau_{23} = - u_{,1}/(x_2)^2 \ , \qquad (1.2)$$
$$\tau_{13} = + u_{,2}/(x_2)^2 \ ,$$

where $u_{,i} = \partial u/\partial x_i$. The stress q is given by

$$q = [\tau_{13}^2 + \tau_{23}^2]^{1/2} = \frac{1}{(x_2)^2}[u_{,1}^2 + u_{,2}^2]^{1/2} = \frac{1}{(x_2)^2}|\text{grad } u| \quad . \tag{1.3}$$

When the torque T is small, the stresses are small and the response is elastic. As T increases a small plastic enclave forms. In general, Ω is divided into two subregions, the elastic region Ω_e and the plastic region Ω_p. The unknown free boundary between Ω_e and Ω_p is denoted by Γ_f (see Figure 1.2).

In Ω_e the material is elastic and u satisfies the differential equation

$$Au = -(u_{,i}/(x_2)^3)_{,i} \equiv -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\frac{1}{(x_2)^3}\frac{\partial u}{\partial x_i}\right), \quad \text{in } \Omega_e \quad . \tag{1.4}$$

The material is assumed to yield according to the criterion of von Mises; that is, the material yields when the stress q reaches the maximum permissible value k (a given constant). Thus,

$$|\text{grad } u| \le kx_2^2, \quad \text{in } \Omega_e \quad , \tag{1.5}$$

$$|\text{grad } u| = kx_2^2, \quad \text{in } \Omega_p \quad . \tag{1.6}$$

The boundary conditions for u on Γ are (see Figure 1.2):

$$u = 0, \quad \text{on } \Gamma_0 \quad , \tag{1.7}$$

$$u = T/2\pi \quad \text{on } \Gamma_1 \quad , \tag{1.8}$$

$$\frac{\partial u}{\partial n} = u_n = u_{,1} = 0, \quad \text{on } \Gamma_2 \quad . \tag{1.9}$$

Condition (1.7) arises from the axial symmetry of the problem. Condition (1.8) expresses the fact that the total torque is T and that there is no traction on the outer surface Γ_1. Condition (1.9) expresses the assumption that at the ends of the shaft the stresses correspond to a pure torque so that $\tau_{23} = u_{,1} = 0$.

The formulation of the problem is completed by the requirement that u and its first derivatives be continuous across Γ_f. The problem defined by (1.4) to (1.9) will be called the <u>Classical Problem</u>.

The remainder of the paper is organized as follows. In the remainder of this section we make some brief remarks about related work in the literature, and indicate the reasons for choosing the method of analysis used in this paper. In section 2 we introduce certain weighted Sobolev spaces; in section 3 we analyse the one-dimensional problem; in section 4 the classical problem is reformulated as a variational inequality; and in sections 5 and 6 the existence of a solution and various properties thereof are proved.

Numerical results will appear in a subsequent paper (Cryer [1979a]).

In recent years the elastic-plastic torsion of cylindrical bars has been intensively studied: see Ting [1973]; Lanchon [1974]; for other references see Cryer [1977, section I.5.3.1]. If the cross-section of the bar is denoted by $\hat{\Omega}$, then it is required to find a stress function ϕ such that

$$\hat{A}\phi = -\phi_{,11} - \phi_{,22} + 2\hat{\theta} = 0, \quad \text{in } \hat{\Omega}_e ,$$

$$|\mathrm{grad}\,\phi| = \hat{k}, \quad \text{in } \hat{\Omega}_p , \tag{1.10}$$

$$\phi = 0, \quad \text{on } \partial\hat{\Omega} .$$

Here, the constant $\hat{k}$ denotes the maximum stress, and the constant $\hat{\theta} > 0$ denotes the angle of twist per unit length of the bar, while $\hat{\Omega}_p$ and $\hat{\Omega}_e$ denote the plastic and elastic regions, respectively.

There are close similarities between the problem considered in this paper and the problem (1.10), but there are also two important differences:

(i) The differential operator A of (1.4) can be written in several forms.

$$-Au = \mathrm{div}(x_2^{-3}\,\mathrm{grad}\,u) , \tag{1.11}$$

$$-x_2^3 Au = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{3}{x_2}\frac{\partial u}{\partial x_2} , \tag{1.12}$$

$$-x_2^4 Au = x_2\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right] + \left[0\,\frac{\partial u}{\partial x_1} - 3\,\frac{\partial u}{\partial x_2}\right] , \tag{1.13}$$

but one cannot avoid the singularity at $x_2 = 0$.

(ii) The boundary conditions (1.7) through (1.9) are a combination of Dirichlet and Neumann conditions while the boundary conditions for (1.10) are Dirichlet.

The singularity of the operator A is the most significant difference between the present problem and problem (1.10). There is an extensive literature on degenerate elliptic equations (Visik [1964], Oleinik and Radkevic [1973], Fichera [1956, 1960], Kohn and Nirenberg [1967, 1967a], Baouendi and Goulaouic [1972]).

Unfortunately, much of the literature is not applicable to the problem in hand. One reason for this is the following. The equation (1.13) is degenerate on Γ_0. However, the inner product of the coefficients of the first order derivatives with the outward normal on Γ_0 is

$$(0) \times (0) + (-3) \times (-1) = 3 \quad ,$$

which is positive so that boundary conditions must be imposed on Γ_0 (Fichera [1960], Oleinik and Radkevic [1973], Friedman and Pinsky [1973]). On the other hand, for the equation

$$x_2^{-2}\left[\frac{\partial}{\partial x_1}\left(x_2^3 \frac{\partial u}{\partial x_1} + \frac{\partial}{\partial x_2} \quad x_2^3 \frac{\partial u}{\partial x_2}\right)\right] = x_2\left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right] + \left[0 \frac{\partial u}{\partial x_1} + 3 \frac{\partial u}{\partial x_2}\right] = 0 \quad ,$$

the inner-product of the first order coefficients with the outward normal on Γ_0 is equal to -3 so that no boundary conditions can be imposed on Γ_0. This means that papers on degenerate elliptic equations in which only bounds on the absolute values of the coefficients of the equation are imposed (Murthy and Stampacchia [1968], Trudinger [1973]), cannot be of use in the present case.

However, the operator A gives rise to generalized axially symmetric potentials which have been extensively studied (see Weinstein [1953], Huber [1954, 1955], Quinn and Weinacht [1976], Quinn [1978], and the references below). Various methods have been used to study boundary value problems for generalized axially symmetric potentials:

(a) Maximum Principle. Jamet [1967, 1968], Parter [1965, 1965a], Lo [1973, 1976].

(b) Perturbation of Ω. The problem is considered in

$$\Omega_\epsilon = \{(x_1,x_2) \in \Omega : x_2 \geq \epsilon\} ,$$

and then the limit is taken as $\epsilon \to 0$. (Schechter [1960], Greenspan and Warten [1962]).

(c) Weighted Sobolev spaces. The problem is reformulated as a minimization problem in the space of functions u such that

$$\int_\Omega \frac{1}{x_2^3} (u_{,1}^2 + u_{,2}^2) dx_1 dx_2 < \infty .$$

(Leventhal [1973, 1975], Jakovlev [1966], Necas [1967, chapter 6]).

For the present problem the natural setting is a weighted Sobolev space, but we also use the maximum principle and perturbation of Ω.

Noncoercive variational inequalities have been considered by Lions and Stampacchia [1965], Lewy and Stampacchia [1971], and Deuel and Hess [1974], but none of these results are applicable to the problem considered here.

Acknowledgement

The author is grateful to Seymour Parter for drawing his attention to the work of Leventhal.

2. Some weighted Sobolev spaces.

Because of the term $1/(x_2)^3$ in the operator A defined by (1.4), it is necessary to introduce Sobolev spaces with a weight function

$$\rho(x) = \rho(x_1,x_2) = (x_2)^{-3} \quad . \tag{2.1}$$

There is an extensive literature on weighted Sobolev spaces (Necas [1967], Kudrjacev [1974], Kadlec [1966], Kadlec and Kufner [1966, 1967], Kufner [1965, 1965a, 1969], Jakovlev [1966]). The present problem presents several aspects which, taken together, are not covered in the literature:

(i) The weight function ρ involves the distance to the plane $x_2 = 0$.

(ii) $\rho = x_2^{-3}$ whereas most references consider the case $\rho = x_2^{\alpha}$, $\alpha \geq 0$.

(iii) The boundary conditions on $\partial\Omega$ are of the third kind (Dirichlet and Neumann).

The results of this section hold whenever Ω is of type $N^{(0),1}$, that is, Ω is a bounded domain whose boundary is Lipschitz continuous (Necas [1967, p. 55]). This condition is satisfied as long as Γ_1 consists of a finite number of Lipschitz continuous curves, without cusps, and is certainly satisfied when Ω is as in Figure 1.2.

$L^2(\Omega)$ and $W^{m,p}(\Omega)$ denote the usual Lebesgue spaces and Sobolev spaces defined over Ω.

We denote by $L = L^2_{\rho}(\Omega)$ the real linear space of real measurable functions v defined on Ω with finite norm

$$\|v;L\| = \|v;L^2_{\rho}(\Omega)\| = [\int_{\Omega} \rho(x)v^2\,dx]^{1/2} = [\int_{\Omega} x_2^{-3}v^2\,dx]^{1/2} \quad . \tag{2.2}$$

Thus, $v \in L$ iff $\rho^{1/2}v \in L^2(\Omega)$. We assert that L is complete. To see this let $\{v_n\}$ be a Cauchy sequence in L. Then $\{\rho^{1/2}v_n\}$ is a Cauchy sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, $\rho^{1/2}v_n \to u$ in $L^2(\Omega)$ for some $u \in L^2(\Omega)$. Thus $\rho^{1/2}v_n \to \rho^{1/2}v$ in $L^2(\Omega)$ where $v = u\rho^{-1/2} \in L$. That is, $v_n \to v$ in L, so that L is indeed complete.

We denote by $W = W^{1,2}_{\rho}(\Omega)$ the space of functions $v \in L$ with generalized derivatives $v_{,i} = D_i v$, $1 \leq i \leq 2$, which also belong to L. As norm we take

$$\|v;W\| = \|v;W^{1,2}_{\rho}(\Omega)\| = [\|v;L\|^2 + \sum_{i=1}^{2} \|D_i v;L\|^2]^{1/2} \quad . \tag{2.3}$$

We assert that W is a Banach space. To see this, let $\{v_n\}$ be a Cauchy sequence in W. Then, by the arguments of the previous paragraph, $\rho^{1/2}v_n \to \rho^{1/2}w \in L^2(\Omega)$ for some $w \in L$, while $\rho^{1/2}D_i v_n \to \rho^{1/2}w_i \in L^2(\Omega)$ for some $w_i \in L$, $1 \le i \le 2$. We must show that $w_i = D_i w$. To do so, choose a test function $\varphi \in \mathcal{D}(\Omega)$, the set of infinitely differentiable functions with compact support in Ω. By definition,

$$\int_\Omega (D_i v_n)\varphi\, dx = -\int_\Omega v_n (D_i \varphi)\, dx \quad .$$

Now φ has compact support on Ω. Thus $\varphi = 0$ outside some compact subset Ω_c of Ω. On Ω_c we have that, for some ϵ, $x_2 \ge \epsilon > 0$. Since $\rho^{1/2}v_n \to \rho^{1/2}w$ in $L^2(\Omega)$, we conclude that $v_n|\Omega_c \to w|\Omega_c$ in $L^2(\Omega_c)$. Similarly, $D_i v_n|\Omega_c \to w_i|\Omega_c$ in $L^2(\Omega_c)$. Thus,

$$\int_\Omega w(D_i\varphi)dx = \int_{\Omega_c} w(D_i\varphi)dx = \lim_{n\to\infty}\int_{\Omega_c} v_n(D_i\varphi)dx =$$

$$= -\lim_{n\to\infty}\int_{\Omega_c}(D_i v_n)\varphi\, dx = -\int_{\Omega_c} w_i\,\varphi\, dx =$$

$$= -\int_\Omega w_i\,\varphi\, dx \quad ,$$

and we conclude that indeed $w_i = D_i w$.

The preceding arguments used only the fact that ρ is continuous and positive in Ω. The arguments which follow use the fact that $\rho = x_2^{-3}$.

We denote by $V = V_\rho^{1,2}(\Omega)$ the set of real measurable functions v defined on Ω such that $x_2^{-1}v \in L$ and v has weak derivatives $D_i v \in L$. As norm, we take

$$\|v;V\| = \|v;V_\rho^{1,2}(\Omega)\| = [\|x_2^{-1}v;L\|^2 + \sum_{i=1}^{2}\|D_i v;L\|^2]^{1/2} \quad . \tag{2.4}$$

Using the arguments previously applied to W it follows that V is a Banach space.

If $v \in V$ then $v \in W$ and

$$\|v;W\| \le [\max_\Omega (1 + x_2)] \; \|v;V\| \quad , \tag{2.5}$$

so that V can be imbedded in W.

For small positive h let S_h be the strip

$$S_h = \{x \in \Omega : 0 < x_2 < h\} \ , \tag{2.6}$$

and let

$$\Omega_h = \{x \in \Omega : x_2 \geq h\} = \Omega / S_h \ . \tag{2.7}$$

Let $C_0^\infty(\Gamma_i) = C_0^\infty(R^2|\Omega;\Gamma_i)$ denote the set of restrictions to Ω of functions which are infinitely differentiable in R^2 and which vanish in some neighborhood of Γ_i. In particular, if $\varphi \in C_0^\infty(\Gamma_0)$ then φ vanishes in some S_h. We denote by ${}^0W = {}^0W_\rho^{1,2}(\Omega)$ the completion in W of $C_0^\infty(\Gamma_0)$, and set

$$\|w : {}^0W\| = [\sum_{i=1}^{2} \|D_i v;L\|^2]^{1/2} \ . \tag{2.8}$$

Theorems 2.2 and 2.3 below are based on results due to Kadlec and Kufner [1966].

We use the following inequality due to Hardy (Hardy, Littlewood, and Polya [1934, p.245]).

<u>Lemma 2.1.</u> (Hardy)

If $p > 1$, $\alpha < p - 1$, and $g(t)$ is a measurable function on $(0,\infty)$ such that

$$\int_0^\infty |g(t)|^p t^\alpha dt < \infty \ ,$$

then

$$\int_0^\infty \left[\int_0^t |g(s)|ds\right]^p t^{\alpha-p} dt \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty |g(t)|^p t^\alpha dt \ .$$

□

<u>Theorem 2.2.</u>

$$V = W \ .$$

The norms

$$\|w;V\|^2 = \int_\Omega \rho(x_2^{-2}w^2 + |\mathrm{grad}\ w|^2)dx \ ,$$

$$\|w;W\|^2 = \int_\Omega \rho(w^2 + |\mathrm{grad}\ w|^2)dx \ ,$$

and

$$\|w;W^0\|^2 = \int_\Omega \rho|\text{grad } w|^2\,dx \quad ,$$

are equivalent on W, and satisfy

$$\frac{4}{5}\,\|w;V\|^2 \le \|w;{}^0W\|^2 \le \|w;W\|^2 \le \max_\Omega\,(1 + x_2)^2\|w;V\|^2 \quad , \tag{2.9}$$

If $w \in W$ then $w(x_1,x_2) \to 0$ as $x_2 \to 0$ for almost all x_1. Indeed,

$$|w(x_1,x_2)| \le \frac{1}{\sqrt{5}}\left[\int_0^{x_2} \frac{1}{x_2^3}\,|D_2w(x_1,s)|^2ds\right]^{1/2} x_2^{5/2} \quad , \text{ a.e.} \tag{2.10}$$

Also,

$$\int_\Omega \frac{1}{x_2^5}\,w^2dx \le \frac{1}{4}\,\|w;{}^0W\|^2 \quad . \tag{2.11}$$

Proof: Let $w \in W$. Then w belongs to the Sobolev space $H^1(\Omega)$ and so $w(x_1,\cdot)$ is absolutely continuous as a function of x_2 for almost all x_1 (Morrey [1966, p. 66]). Thus,

$$w(x_1,t) - w(x_1,s) = \int_s^t D_2w(x_1,u)\,du \quad . \tag{*}$$

Furthermore, since $\|w;W\| < \infty$ it follows from Fubini's theorem that

$$\int_0^{R(x_1)} \frac{1}{x_2^3}\,|D_2w(x_1,u)|^2du < \infty \quad , \tag{**}$$

for almost all x_1. Thus, using Hölder's inequality,

$$|w(x_1,t) - w(x_1,s)| \le \int_s^t |D_2w(x_1,u)|.|u|^{-3/2}.|u|^{3/2}du \quad ,$$

$$\le \left[\int_s^t \frac{1}{u^3}\,|D_2w(x_1,u)|^2du\right]^{1/2}\left[\int_s^t |u|^3du\right]^{1/2} \quad ,$$

$$\le \frac{1}{\sqrt{5}}\left[\int_0^{x_2} \frac{1}{u^3}\,|D_2w(x_1,u)|^2du\right]^{1/2}|t-s|^{5/2} \quad , \tag{***}$$

from which we conclude that $w(x_1,0) = \lim_{s\to 0} w(x_1,s)$ exists for almost all x_1. However, from Fubini's theorem,

$$\int_0^{R(x_2)} \frac{1}{u^3} |w(x_1,u)|^2 du < \infty \quad , \quad \text{a.e.} \; ,$$

so that $w(x_1,0) = 0$ a.e. Indeed, we have (2.10).

Applying Lemma 2.1 with $g = D_2 w$, $\alpha = -3$, and $p = 2$, we see that, for almost all x_1,

$$\int_0^{R(x_1)} \frac{1}{x_2^5} |w(x_1,x_2)|^2 dx_2 = \int_0^{R(x_1)} \frac{1}{x_2^5} \left| \int_0^{x_2} D_2 w(x_1,s) ds \right|^2 dx_2 \; ,$$

$$\leq \int_0^{R(x_1)} \frac{1}{x_2^5} \left[\int_0^{x_2} |D_2 w(x_1,s)| ds \right]^2 dx_2 \; ,$$

$$\leq \frac{1}{4} \int_0^{R(x_1)} \frac{1}{x_2^3} |D_2 w(x_1,x_2)|^2 dx_2 \; .$$

Integrating with respect to x_1 we obtain (2.11). The remainder of the Theorem now follows immediately.

□

Remark 2.1.

If $v = 0$ on $\partial\Omega$ then inequality (2.11) is related to the Poincaré inequality. For general mixed boundary conditions, one obtains an inequality such as (2.11) only when Ω satisfies certain restrictions (Stampacchia [1969, p. 145]).

□

Theorem 2.3. Given $v \in V$ and $\epsilon > 0$ there exists $\psi \in C_0^\infty(\Gamma_0)$ such that

$$\|v - \psi ; W\| \leq \epsilon \; .$$

If, in addition, $v - \gamma \in C_0^\infty(\Gamma_1)$ for some constant γ then ψ can be chosen so that $\psi - \gamma \in C_0^\infty(\Gamma_1)$.

Consequently, $V = W^0 = W$.

Proof:

Choose $f(t) \in C^\infty(R^1)$ such that $0 \le f(t) \le 1$, $f = 0$ for $t \le 1$, and $f = 1$ for $t \ge 2$. Such an f can be constructed using mollifiers. Let $c = \max |f'|$.

Let $F_h(x_1,x_2) = f(x_2/h)$, so that $0 \le F_h \le 1$,

$$F_h(x_1,x_2) = \begin{cases} 0, & \text{if } x_2 \le h \;, \\ 1, & \text{if } x_2 \ge 2h \;. \end{cases}$$

Then $F_h \in C_0^\infty(\Gamma_0)$ and

$$|D_i F_h| \le ch^{-1} \;.$$

For any $v \in V$ let $v_h = F_h v$. Then: $v_h \in V$; $v_h(x) = 0$ for $x \in S_h$; $v(x) = v_h(x)$ for $x \in \Omega_{2h}$; $|v(x) - v_h(x)| \le |v(x)|$. Also,

$$|D_i(v-v_h)|^2 \le [|(1 - F_h)D_i v| + |v\, D_i F_h|]^2 \;,$$

$$\le 2[|D_i v|^2 + v^2|D_i F_h|^2] \;,$$

so that

$$|D_i(v-v_h)|^2 = |D_i v|^2 \;, \text{ in } S_h \;,$$

$$|D_i(v-v_h)|^2 \le 2[|D_i v|^2 + v^2c^2/h^2] \;,$$

$$\le 2[|D_i v|^2 + 4v^2c^2/x_2^2], \text{ in } S_{2h}/S_h \;.$$

Thus, remembering that $v - v_h = 0$ in Ω_{2h},

$$\|(v-v_h);W\|^2 \le \int_{S_{2h}} \frac{1}{x_2^3} |v|^2 dx +$$

$$+ \sum_{i=1}^{2} 2\int_{S_{2h}} \left[\frac{1}{x_2^3} |D_i v|^2 + 4c^2 \frac{1}{x_2^5} |v|^2\right] dx \;.$$

Since $v \in V$ each integral is convergent. Since the measure of $S_{2h} \to 0$ as $h \to 0$, we conclude that

$$\|v-v_h;W\| \to 0, \text{ as } h \to 0 \;.$$

Choose $\epsilon > 0$, and then pick h so that

$$\|v-v_h;W\| \le \epsilon/2 \quad . \tag{*}$$

Since $v_h = 0$ in S_h we see that $v_h|\Omega_{h/2} \in W^{1,2}(\Omega_{h/2})$. But $\Omega_{h/2}$ satisfies the segment property (Adams [1975, p. 54]) and so there exists $w_h \in C_0^\infty(R^2)$ such that $w_h|\Omega_{h/2}$ is arbitrarily close to $v_h|\Omega_{h/2}$ in the $W^{1,2}(\Omega_{h/2})$ norm. Furthermore, remembering that $v_h = 0$ in S_h, examination of the proof of Theorem 3.18 of Adams shows that w_h may be chosen to be zero in a neighborhood U of the boundary component

$$\Gamma_{h/2} = \{x \in \Omega : x_2 = h/2\}$$

of $\Omega_{h/2}$, that is, $\psi = w_h|\Omega \in C_0^\infty(\Gamma_0)$. Since the norms $\|\cdot;W^{1,2}(\Omega_{h/2})\|$ and $\|\cdot;W_\rho^{1,2}(\Omega_{h/2})\|$ are equivalent on $\Omega_{h/2}$, we can choose w_h so that

$$\|\psi-v_h;W\| = \|(w_h-v_h)|\Omega_{h/2};W_\rho^{1,2}(\Omega_{h/2})\| \le \epsilon/2 \quad . \tag{**}$$

Combining (*) and (**)

$$\|\psi - v;W\| \le \epsilon \quad .$$

Next, let $v-\gamma \in C_0^\infty(\Gamma_1)$. Then $v_h = F_h v$ satisfies $v_h-\gamma \in C_0^\infty(\Gamma_1)$ and from the construction of w_h (Adams [1975, p. 55]) we can clearly choose w_h so that $\psi-\gamma \in C_0^\infty(\Gamma_1)$.

Since $\psi \in C_0^\infty(\Gamma_0)$, $\psi \in {}^0W$, and we conclude that ${}^0W$ is dense in V. Using Theorem 2.2 we have $V \subset {}^0W \subset W = V$.

$\square$

3. The one-dimensional problem.

It is instructive to consider the one-dimensional problem which arises when the shaft has constant diameter. In this case u depends only upon x_2. It is convenient to set $x_2 = r$. We normalize u and r so that the shaft has radius 1, and $u = 1$ on the outer surface of the shaft. To be consistent we should set $\Omega = (0,L) \times (0,1)$ but we set $\Omega = (0,1)$ since no confusion can arise. We look for a solution for which $\Omega_e = (0,\tau)$ and $\Omega_p = (\tau,1)$ for some constant τ.

Conditions (1.4) through (1.9) become:

$$Au = -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) = 0, \qquad 0 < r < \tau \ , \tag{3.1}$$

$$\left|\frac{\partial u}{\partial r}\right| = kr^2, \qquad \tau < r < 1 \ , \tag{3.2}$$

$$\left|\frac{\partial u}{\partial r}\right| \le kr^2, \qquad 0 < r < 1 \ , \tag{3.3}$$

$$u = 0, \qquad r = 0 \ , \tag{3.4}$$

$$u = 1, \qquad r = 1 \ . \tag{3.5}$$

Integrating (3.1) we see that

$$\frac{\partial u}{\partial r} = 4ar^3, \qquad 0 < r < \tau \ , \tag{3.6}$$

for some constant a. Integrating again we obtain

$$u = ar^4 + b, \qquad 0 < r < \tau \ ,$$

for some constant b. It follows from (3.4) that $b = 0$ so that

$$u = ar^4, \qquad 0 < r < \tau \ . \tag{3.7}$$

From (3.2),

$$\frac{\partial u}{\partial r} = \pm\, kr^2, \qquad \tau < r < 1 \ .$$

Since u is required to be continuously differentiable at $r = \tau$, the constants a and $\pm k$ must have the same sign, so that $\partial u/\partial r$ has the same sign throughout $(0,1)$. From (3.4) and (3.5) we see that $\partial u/\partial r$ must be positive.

Thus (3.2) becomes

$$\frac{\partial u}{\partial r} = kr^2, \qquad \tau < r < 1 \quad . \tag{3.8}$$

Integrating and using (3.5) we obtain

$$u = kr^3/3 + (1 - k/3), \qquad \tau < r < 1 \quad . \tag{3.9}$$

The expressions (3.7) and (3.9) involve two unknown constants τ and a. We determine these by requiring that u and u_r be continuous at $r = \tau$. From (3.6) and (3.8) we have

$$\frac{\partial u}{\partial r}(\tau - 0) = 4a\tau^3 = \frac{\partial u}{\partial r}(\tau + 0) = k\tau^2 \quad ,$$

so that

$$a = k/4\tau \quad . \tag{3.10}$$

From (3.7) and (3.9) we have

$$u(\tau - 0) = a\tau^4 = u(\tau + 0) = k\tau^3/3 + (1 - k/3) \quad .$$

Substituting from (3.10) and re-arranging, we obtain

$$\tau^3 = 12(k/3 - 1)/k \quad . \tag{3.11}$$

The solution τ of (3.11) depends upon the value of k. There are three possibilities:

1. $\underline{k < 3}$. Then $\tau < 0$. Physically this means that the torque T is too great and no solution exists.

2. $\underline{k > 4}$. Then

$$\tau = [4 - 12/k]^{1/3} > 1 \quad .$$

Physically this means that there is no plastic region, and the analysis must be modified. Setting $a = 1$ in (3.7), we obtain a solution $u = r^4$ of the elastic problem which satisfies the constraint (3.3) namely $\left|\frac{\partial u}{\partial r}\right| \leq kr^2$.

3. $\underline{3 < k < 4}$. Then

$$\tau = [4 - 12/k]^{1/3} \in (0,1) \tag{3.12}$$

and there is both an elastic region $\Omega_e = (0,\tau)$ as well as a plastic region $(\tau,1)$. From (3.7), (3.9), and (3.10),

$$u = \begin{cases} kr^4/4\tau, & \text{in } \Omega_e , \\ kr^3/3 + (1 - k/3), & \text{in } \Omega_p . \end{cases} \tag{3.13}$$

We now show that u, as given by (3.13), satisfies two alternative formulations of the problem.

Direct computation shows that

$$Au = \begin{cases} 0 , & \text{in } \Omega_e , \\ k/r^2 > 0, & \text{in } \Omega_p . \end{cases} \tag{3.14}$$

Let ψ be such that

$$\text{grad } \psi = \frac{\partial \psi}{\partial r} = kr^2, \qquad 0 < r < 1 , \tag{3.15}$$
$$\psi(1) = 1 ,$$

so that

$$\psi = kr^3/3 + (1 - k/3), \qquad 0 < r < 1 . \tag{3.16}$$

ψ is called the <u>obstacle</u>.

Noting from (3.11) that $k/3 - 1 = k\tau^3/12$, direct computation shows that

$$u-\psi = \begin{cases} k(r-\tau)^2(3r^2+2r\tau+\tau^2)/12\tau > 0, & \text{in } \Omega_e = (0,\tau) , \\ 0 , & \text{in } \Omega_p = (\tau,1) . \end{cases} \tag{3.17}$$

Combining (3.14) and (3.17) it follows that u satisfies the one-dimensional <u>Complementary Problem</u>:

$$\begin{aligned} Au \geq 0, &\quad \text{in } \Omega , \\ u - \psi \geq 0, &\quad \text{in } \Omega , \\ (Au)(u - \psi) = 0, &\quad \text{in } \Omega . \end{aligned} \tag{3.18}$$

Now, with the notation of section 2 let

$$V = V_\rho^{1,2}(\Omega) = {}^0W_\rho^{1,2}(\Omega) . \tag{3.19}$$

Set

$$K = \{v \in V : v(1) = 1 ; v(r) \geq \psi(r) \text{ for } r \in (0,1)\} . \tag{3.20}$$

$V \subset W^{1,2}(0,1)$ and so if $v \in V$ then v is equivalent to an absolutely continuous function. Thus, statements such as $v(1) = 1$ in the definition of K can be interpreted in the classical sense. Furthermore, since

$$\int_0^1 \frac{1}{r^3} v^3 dr < \infty \quad ,$$

we see that the condition

$$v(0) = 0 \tag{3.21}$$

is satisfied by all $v \in V$.

Let a be the bilinear function on $V \times V$,

$$a(u,v) = \int_0^1 \frac{1}{r^3} u_r(r) v_r(r) dr \quad . \tag{3.22}$$

Then, for any $v \in K$, and remembering that $u = ar^4$ in $(0,\iota)$,

$$a(u,v-u) = \int_0^1 \frac{1}{r^3} u_r(v_r - u_r) dr \quad ,$$

$$= \int_0^{\iota} \frac{1}{r^3} u_r(v_r - u_r) dr + \int_{\iota}^1 \frac{1}{r^3} u_r(v_r - u_r) dr \quad .$$

Integrating by parts,

$$a(u,v-u) = \left[(v-u)\frac{1}{r^3} u_r\right]_0^{\iota} + \int_0^{\iota} (v-u) Au \, dr +$$

$$+ \left[(v-u)\frac{1}{r^3} u_r\right]_{\iota}^1 + \int_{\iota}^1 (v-u) Au \, dr \quad .$$

Since $v(0) = u(0) = 0$, $v(1) = u(1) = 1$, $Au = 0$ in $\Omega_0 = (0,\iota)$, $u = \psi$ in $\Omega_p = (\iota,1)$, and u_r is continuous at $r = \iota$, we obtain

$$a(u,v-u) = \int_{\iota}^1 (v-\psi) Au \, dr \quad .$$

But, $v \in K$ so that $v \geq \psi$, and, by (3.14), $Au \geq 0$ in $\Omega_p = (\iota,1)$, so that u is a solution of the one-dimensional <u>Variational Inequality</u>: Find $u \in K$ such that

$$a(u,v-u) \geq 0, \quad \text{for all } v \in K. \tag{3.23}$$

4. The Two-dimensional Variational Inequality.

In the previous section it was shown by direct computation that the solution u of the classical one-dimensional elastic-plastic problem satisfies the one-dimensional complementarity problem (3.18) and the one-dimensional variational inequality (3.23). This suggests that we consider the corresponding two-dimensional problems.

The two-dimensional complementarity problem is very useful conceptually, and also very helpful when one considers numerical approximations. However, this problem gives rise to technical difficulties since it is necessary to carefully define the meaning of statements such as $(Au)(u - \psi) \geq 0$. This can be done, but we will not do so here.

In contrast, the two-dimensional variational inequality is relatively easy to apply since we can use the following fundamental result of Stampacchia [1964]:

Theorem 4.1: Let V be a real Hilbert space. Let a be a real bilinear operator on $V \times V$ such that a is coercive and continuous; that is, there are real strictly positive constants α_1 and α_2, such that

$$a(v,v) \geq \alpha_1 \|v\|^2, \quad \text{for} \quad v \in V \quad ,$$

$$|a(v,w)| \leq \alpha_2 \|v\| \, \|w\|, \quad \text{for} \quad v,w \in V \quad .$$

Let f be a real continuous linear functional on V. Let K be a closed convex non-empty subset of V. Then the variational inequality: Find $u \in K$ such that

$$a(u,v - u) \geq (f,v - u), \quad \text{for all} \quad v \in K \quad , \tag{4.1}$$

has a unique solution.

□

General references on variational inequalities include: Duvaut and Lions [1972], Glowinski, Lions, and Tremolieres [1976], Baiocchi [1978], Glowinski [1978], Kinderlehrer [1978], and Cryer [1977 section II.11, 1979].

In order to apply Theorem 4.1 to the problem in hand we must define V, a, K and f. In doing so, we have been guided by the work of Eddy and Shaw [1949], Brezis and Sibony [1971], and Leventhal [1973, 1975].

The space V is taken to be the space

$$V = W_\rho^{1,2}(\Omega) = {}^0W_\rho^{1,2}(\Omega) = {}^0W = W \tag{4.2}$$

defined and discussed in section 2. It was shown in Theorem 2.2 that there are several equivalent norms on V. Here we use the norm

$$\|v\| = \|v; {}^0W\| = [\int_\Omega \rho|\text{grad } v|^2 dx]^{1/2} \quad . \tag{4.3}$$

The bilinear operator a is defined on $V \times V$ by

$$a(v,w) = \int_\Omega \rho[v_{,1} w_{,1} + v_{,2} w_{,2}]dx \quad ,$$

$$= \int_\Omega \rho \text{ grad } v \text{ . grad } w \, dx \quad . \tag{4.4}$$

Since

$$|a(v,v)| = \|v\|^2 \quad ,$$

a is coercive, and since

$$|a(v,w)| \leq \|v\| \cdot \|w\| \quad ,$$

a is continuous.

The obstacle ψ is the solution of the initial value problem for a first order partial differential equation:

$$|\text{grad } \psi|^2 = k^2 x_2^4, \quad \text{in } \Omega \quad ,$$
$$\psi = T/2\pi, \quad \text{on } \Gamma_1; \quad \psi \leq T/2\pi \quad \text{in } \Omega \quad , \tag{4.5}$$

where the restraint $\psi \leq T/2\pi$ resolves the ambiguity in the sign of grad ψ.

The set K is defined by:

$$K = \{v \in V : v = T/2\pi \text{ on } \Gamma_1 \text{ (in the sense of } H^1(\Omega)) \quad ,$$
$$v \geq \psi \text{ a.e. (almost everywhere) in } \Omega\} \quad . \tag{4.6}$$

Here, the statement '$v = T/2\pi$ on Γ_1 in the sense of $H^1(\Omega)$' means that there exists a sequence of smooth functions $\{\varphi_k\}$ such that: (i) $\varphi_k \in V$; (ii) $\varphi_k = T/2\pi$ in a neighborhood of Γ_1; and (iii)

$$\|v_k - v\| \to 0 \quad \text{as} \quad k \to \infty \ .$$

K is closed and convex.

The boundary conditions (1.7) and (1.9) are incorporated into the definition of K: every $v \in V$ satisfies $v = 0$ on Γ_0 in a weak sense; and the condition $\partial u/\partial n = 0$ on Γ_2 is a 'natural' boundary condition in a variational formulation of the problem.

Finally, the functional f is zero in the present problem.

We claim that the Variational Inequality corresponding to the Classical Problem (1.4) - (1.9) is: Find $u \in K$ such that

$$a(u, v - u) \geq 0, \quad \text{for all} \quad v \in K \ , \tag{4.7}$$

where a and K are as defined in (4.4) and (4.6).

Before proceding further we need some information about the function ψ.

Theorem 4.2.

For $x \in \Omega$,

$$\psi(x_1, x_2) \leq g(x_2) = [kx_2^3/3 + T/2\pi - kR(0)^3/3] \ . \tag{4.8}$$

For $x \in \Gamma_0$,

$$\psi(x_1, 0) = \beta = g(0) \ . \tag{4.9}$$

Proof: ψ is defined by (4.5). On Γ_0,

$$|\text{grad}\ \psi| = kx_2^2 = 0 \ ,$$

so that $\psi = \beta$ on Γ_0 for some constant β.

To determine β we note that ψ satisfies the first order equation

$$F(x_1, x_2, \psi, p, q) \equiv p^2 + q^2 - k^2 x_2^4 = 0 \ , \tag{4.10}$$

where $p = \psi_{,1}$ and $q = \psi_{,2}$. The corresponding characteristic system of differential equations along a trajectory parameterised by s is (Courant and Hilbert [1962, p. 78]),

$$\frac{dx_1}{ds} = F_p = 2p \quad , \tag{4.11}$$

$$\frac{dx_2}{ds} = F_q = 2q \quad , \tag{4.12}$$

$$\frac{d\psi}{ds} = pF_p + qF_q = 2(p^2 + q^2) = 2k^2x_2^4 \quad , \tag{4.13}$$

$$\frac{dp}{ds} = -(pF_\psi + F_{x_1}) = 0 \quad , \tag{4.14}$$

$$\frac{dq}{ds} = -(qF_\psi + F_{x_2}) = 4k^2x_2^3 \quad . \tag{4.15}$$

We integrate this system starting at the point $(0,R(0))$ where

$$x_1(0) = 0,\ x_2(0) = R(0),\ p(0) = 0,\ q(0) = kx_2^2,\ \psi(0) = T/2\pi \quad . \tag{4.16}$$

From (4.14) we see that $p(s) \equiv 0$. It then follows from (4.11) that $x_1(s) \equiv 0$, and from (4.10) that $q = +kx_2^2$. We are thus integrating along Γ_{21} and we obtain the same value for ψ as for the corresponding one-dimensional problem.

By the appropriate modification of (3.16), we obtain

$$\psi(0,x_2) = g(x_2) \quad , \tag{4.17}$$

where

$$g(x_2) \doteq [kx_2^3/3 + T/2\pi - kR(0)^3/3] \quad . \tag{4.18}$$

In particular,

$$\beta = \psi(0,0) = T/2\pi - kR(0)^3/3 \quad . \tag{4.19}$$

It should be pointed out that there is a hidden complication in the above argument, because if we follow the same approach starting from the point $(L, R(L))$ we apparently obtain

$$\psi(L,0) = T/2\pi - kR(L)^3/3 \neq \beta \quad .$$

The explanation for this apparent paradox is that two or more characteristics may intersect. A more detailed study of ψ (Cryer [1979a]) shows that when x_2 is small, two characteristics pass through points $(L,x_2) \in \Gamma_{22}$. This does not happen on Γ_{21} because, as is readily seen from (4.11) and (4.14), if, as in Figure 1.2, $dR/dx_2 \geq 0$, the characteristics always have $dx_1/ds \geq 0$ and only the characteristic starting at $(0,R(0))$ passes through the point

$(0,x_2) \in \Gamma_{21}$.

Since

$$|\psi_{,2}| \le |\text{grad}\ \psi| = kx_2^2 \quad ,$$

we see that

$$|\psi(x_1,x_2) - \psi(x_1,0)| = \int_0^{x_2} |\psi_{,2}| dx_2 \le kx_2^3/3 \quad .$$

Thus,

$$\psi(x_1,x_2) \le \beta + kx_2^3/3 = g(x_2) \quad .$$

□

Remark 4.1.

At first sight it may seem surprising that ψ is constant along Γ_0 on which no conditions were imposed. This can be understood more clearly after considering the detailed calculation of ψ as done by Cryer [1979a].

Alternatively, since $|\text{grad}\ \psi| \le kx_2^2$, we know that

$$\|\psi;{}^0W\|^2 = \int_\Omega \rho|\text{grad}\ \psi|^2 dx < \infty \quad .$$

It is known (Kadlec and Kufner [1966, p. 469], Leventhal [1973, Lemma 6.2]) that this implies that ψ is constant on Γ_0.

Theorem 4.3.

Let

$$k_0 = (3T/2\pi) R(0)^{-3} \quad . \tag{4.20}$$

If $k < k_0$ then K is empty and the variational inequality (4.7) has no solution.

If $k \ge k_0$ then (4.7) has a unique solution u.

Proof: If $k < k_0$ then, from (4.8) and (4.9), $\psi = \beta > 0$ on Γ_0. Thus, if $v \in K$,

$$\|v;W\|^2 \ge \int_\Omega \rho v^2 dx \ge \int_\Omega \rho\psi^2 dx = +\infty \quad .$$

Hence, K is empty, and (4.7) has no solution.

If $k \geq k_0$ then $\beta \leq 0$, and $v = \max(0,\psi) \in K$. Since K is not empty, it follows from Theorem 4.1 that the variational inequality (4.7) has a unique solution.

□

Remark 4.2.

In section 3 for the case $T = 2\pi$ and $R(x_1) \equiv 1$, we saw that there were three possibilities: $k < 3$ (no solution); $3 \leq k \leq 4$ (an elastic-plastic solution); $k > 4$ (an elastic solution). In Theorem 4.3 we only distinguish between two possibilities: $k < k_0$ (no solution); $k \geq k_0$ (either an elastic-plastic solution or an elastic solution).

Further properties of the solution of (4.7) are discussed in the next two sections; these properties are such that they justify our claim that the variational inequality (4.7) is an appropriate extension of the Classical Problem.

5. Regularity of the solution u of the variational inequality.

We assume henceforth that

$$k > k_0 = \left(\frac{3T}{2\pi}\right) R(0)^{-3} \quad , \tag{5.1}$$

and set

$$h_0 = [R(0)^3 - 3T/2\pi k]^{1/3} > 0 \quad . \tag{5.2}$$

We prove that u, the solution of the variational inequality (4.7), is regular by first proving that u is regular in the strip S_{h_0} near Γ_0, and then proving that u is regular in Ω_{h_0}, where S_{h_0} and Ω_{h_0} are as in (2.6) and (2.7).

We recall certain properties of the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$ which are proved, for example, by Gilbarg and Trudinger [1977, Chapter 7 and p. 168].

If $v, w \in H^1(\Omega)$ then $\max(v,w) \in H^1(\Omega)$ where $\max(v,w)$ is defined by

$$\max(v,w)(x) = \max(v(x),w(x)) \quad . \tag{5.3}$$

If $v \in H^1(\Omega)$ then, by definition,

$$\sup_{\Omega} v = \inf\{s \in R^1 : v(x) \leq s \text{ a.e. in } \Omega\} \quad , \tag{5.4}$$

$$\sup_{\partial\Omega} v = \inf\{\ell \in R^1 : v(x) - \ell \leq 0 \text{ on } \partial\Omega\} \quad . \tag{5.5}$$

where

$$u(x) - \ell \leq 0 \text{ on } \partial\Omega \text{ iff } \max\{u - \ell, 0\} \in H_0^1(\Omega) \quad . \tag{5.6}$$

As a preliminary step in the analysis we show that it is possible to enlarge the domain Ω by reflection in the vertical sides so as to avoid the difficulties associated with Γ_2. This is a well-known trick for handling Neumann boundary conditions (see, for example, Baiocchi, Comincioli, Magenes and Pozzi [1973], p. 25 footnote 33]). The arguments are elementary and rather tedious but we are not aware of any detailed treatment in the literature.

Let $\Omega^0 = \Omega$. Let Ω^1 be the reflection of Ω^0 in Γ_{21} and set $\tilde{\Omega} = \Omega^0 \cup \Gamma_{21} \cup \Omega^1$ with boundary $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_{21} \cup \tilde{\Gamma}_{22}$ (see Figure 5.1). Let $\tilde{u}$ be defined on $\tilde{\Omega}$ by reflection:

$$\tilde{u}(x) = \begin{cases} u(+x_1, x_2), & x \in \Omega^0 , \\ u(-x_1, x_2), & x \in \Omega^1 . \end{cases} \tag{5.7}$$

The spaces $\tilde{V}$, $\tilde{W}$, ${}^0\tilde{W}$, and the convex set $\tilde{K}$ are defined for $\tilde{\Omega}$ in the same way that they were previously defined for Ω in section 2 and (4.6). That is, $\tilde{V} = V_\rho^{1,2}(\tilde{\Omega})$, $\tilde{W} = W_\rho^{1,2}(\tilde{\Omega})$, ${}^0\tilde{W} = {}^0W_\rho^{1,2}(\tilde{\Omega})$, while $\tilde{K}$ is the subset of $\tilde{V}$ consisting of those functions which are greater or equal to $\tilde{\psi}$ (the reflection of ψ) in $\tilde{\Omega}$ and are equal to $T/2\pi$ on $\tilde{\Gamma}_1$.

Lemma 5.1.

$\tilde{u} \in \tilde{K}$ and $\tilde{u}$ is the unique solution of the variational inequality: Find $\tilde{u} \in \tilde{K}$ such that

$$\tilde{a}(\tilde{u}, \tilde{v}-\tilde{u}) \geq 0, \quad \text{for all} \quad v \in \tilde{K} , \tag{5.8}$$

where

$$\tilde{a}(\tilde{v},\tilde{w}) = \int_{\tilde{\Omega}} \rho \text{ grad } \tilde{v} \cdot \text{grad } \tilde{w} \, dx .$$

Furthermore,

$$\tilde{u}|_\Omega = u .$$

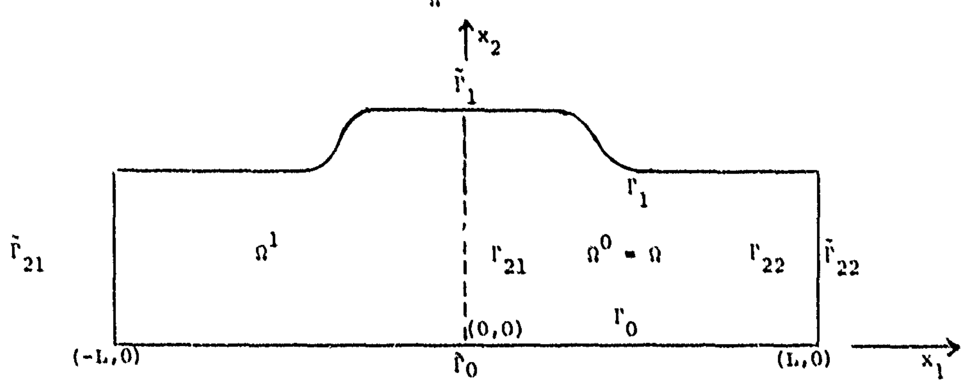

Figure 5.1: The enlarged domain $\tilde{\Omega} = \Omega^0 \cup \Gamma_{21} \cup \Omega^1$.

Proof: We first show that $\tilde{u}$ has weak derivatives in $L^2(\tilde{\Omega})$:

$$\tilde{u}_{,1}(x_1,x_2) = \begin{cases} + u_{,1}(+x_1,x_2), & \text{in } \Omega^0 , \\ - u_{,1}(-x_1,x_2), & \text{in } \Omega^1 , \end{cases} \tag{*}$$

$$\tilde{u}_{,2}(x_1,x_2) = +\, u_{,2}(|x_1|,x_2), \quad \text{in } \tilde{\Omega} \quad . \qquad (**)$$

The values of $\tilde{u}_{,i}$ need not be defined on Γ_{21} since it is a set of measure zero.

We introduce the strips parallel to Γ_{21}:

$$\tilde{T}_d = \{x \in \tilde{\Omega} ; |x_1| \leq 2d\} \quad ,$$

$$T_d = \tilde{T}_d \cap \Omega \quad .$$

For any $d \in (0,L/2)$ let g_d be a cut-off function with the following properties:

(a) $g_d \in C_0^\infty(R^n)$,

(b) $|g_{d,1}| \leq 2d^{-1}$, $g_{d,2} = 0$,

(c) g_d is symmetric about $x_1 = 0$,

(d) $g_d = 1$ if $|x_1| \leq d$ and $g_d = 0$ if $x \notin \tilde{T}_d$.

For any $\varphi \in C_0^\infty(\tilde{\Omega})$,

$$\varphi = \varphi_d^1 + \varphi_d^0 + \varphi_d$$

where

$$\varphi_d = g_d\varphi, \; \varphi_d^0 \in C_0^\infty(\Omega^0), \text{ and } \varphi_d^1 \in C_0^\infty(\Omega^1) \quad .$$

For $i = 1,2$, and $\tilde{u}_{,i}$ defined by (*) and (**)

$$\int_{\tilde{\Omega}} \tilde{u}_{,i} \, \varphi_d \, dx = \int_{\tilde{T}_d} \tilde{u}_{,i} \, \varphi_d \, dx \to 0 \quad \text{as} \quad d \to 0 \quad ,$$

because φ_d is bounded, $\tilde{u}_{,i} \in L^2(\tilde{\Omega})$, and the measure of $\tilde{T}_d$ goes to zero as $d \to 0$. For the same reason,

$$\int_{\tilde{\Omega}} \tilde{u} \, \varphi_{d,2} \, dx = \int_{\tilde{\Omega}} \tilde{u} \, g_d \, \varphi_{,2} \, dx \to 0 \quad \text{as} \quad d \to 0 \quad .$$

Finally,

$$\int_{\tilde{\Omega}} \tilde{u} \, \varphi_{d,1} \, dx = \int_{\tilde{\Omega}} \tilde{u} \, g_d \, \varphi_{,1} \, dx + \int_{\tilde{\Omega}} \tilde{u} \, g_{d,1} \, \varphi \, dx \quad ,$$

$$= I_d^{(1)} + I_d^{(2)}, \quad \text{say.}$$

As before, $I_d^{(1)} \to 0$ as $d \to 0$. Using the symmetry of g_d and $\tilde{u}$,

$$I_d^{(2)} = \int_{T_d} u(x)\ g_{d,1}(x)\ [\varphi(x_1,x_2) - \varphi(-x_1,x_2)]dx \to 0 \quad \text{as} \quad d \to 0$$

since $|g_{d,1}| \leq 2d^{-1}$, and $|\varphi(x_1,x_2) - \varphi(-x_1,x_2)| \leq 2d \sup_{\Omega} |\varphi,_1(x)|$, on T_d.

Thus, for $i = 1,2$, and $\tilde{u},_i$ defined by (*) and (**),

$$I_i = \int_{\tilde{\Omega}} \tilde{u},_i\ \varphi\, dx = \int_{\Omega^0} \tilde{u},_i\ \varphi_d^0\, dx + \int_{\Omega^1} \tilde{u},_i\ \varphi_d^1\, dx + o(1) \quad ,$$

$$= \int_{\Omega^0} u,_i\ \varphi_d^0\, dx + (-1)^i \int_{\Omega^1} u,_i(-x_1,x_2)\, \varphi_d^1(x) dx + o(1) \quad ,$$

$$= \int_{\Omega} u,_i(x)\, [\varphi_d^0(+x_1,x_2) + (-1)^i\ \varphi_d^1(-x_1,x_2)]dx + o(1) \quad .$$

But $u,_i$ is the weak derivative of u on Ω, so that

$$I_i = - \int_{\Omega} u(x)\, [\varphi_{d,i}^0(+x_1,x_2) + \varphi_{d,i}^1(-x_1,x_2)]dx + o(1) \quad ,$$

$$= - \int_{\Omega^0} \tilde{u}\ \varphi_{d,i}^0\, dx - \int_{\Omega^1} \tilde{u}\ \varphi_{d,i}^1\, dx + o(1) \quad ,$$

$$= - \int_{\tilde{\Omega}} \tilde{u}\ \varphi,_i\, dx + o(1) \quad .$$

We conclude that the functions $u,_i$ as defined by (*) and (**) are indeed the weak derivatives of $\tilde{u}$.

Clearly, $\tilde{u} \in \tilde{V}$ and $\|\tilde{u};\tilde{V}\| = 2^{1/2}\|u;V\|$.

Finally, we note that if $\tilde{v} \in \tilde{K}$ then $v^0, v^1 \in K$ where

$$v^0 = \tilde{v}|_{\Omega} \quad ,$$

$$v^1(x_1,x_2) = \tilde{v}(-x_1,x_2) \quad , \quad x \in \Omega \quad .$$

For any $\tilde{v} \in \tilde{K}$,

$$\tilde{a}(\tilde{u},\tilde{v}-\tilde{u}) = \int_{\Omega^0} \rho \text{ grad } \tilde{u} \cdot \text{grad } (\tilde{v}-\tilde{u})dx + \int_{\Omega^1} \rho \text{ grad } \tilde{u} \quad \text{grad } (\tilde{v}-\tilde{u})dx \quad .$$

Making the substitution $x_1 = -x_1$ in the second integral, we obtain,

$$\tilde{a}(\tilde{u},\tilde{v}-\tilde{u}) = \int_{\Omega^0} \rho \text{ grad } u \cdot \text{grad}(v^0-u)dx + \int_{\Omega^0} \rho \text{ grad } u \cdot \text{grad } (v^1-u)dx \quad ,$$

$$= a(u,v^0-u) + a(u,v^1-u) \quad ,$$

$$\geq 0 \quad ;$$

since u solves the variational inequality (4.7). That is, $\tilde{u}$ solves the variational inequality (5.8). From Theorem 4.1, we see that the solution of the variational inequality (5.8) is unique, and the lemma follows.

Remark 5.1.

We can also reflect Ω in Γ_{22} and obtain results analogous to those of Lemma 5.1.

Theorem 5.2.

u is non-negative a.e. in Ω. That is,

$$u = \max(u,0) \quad .$$

Proof: The proof is a modification of the proof of the weak maximum principle in Gilbarg and Trudinger [1977, p. 168].

Let

$$\Omega_+ = \{x \in \Omega : u(x) \geq 0\} \quad ,$$

$$\Omega_- = \{x \in \Omega : u(x) < 0\} \quad ,$$

$$v = \max(u,0) \quad .$$

Then $v \in H^1(\Omega)$. Furthermore, $v = 0$ and hence $|\text{grad } v| = 0$ on Ω_-. Thus,

$$\int_\Omega \rho[x_2^{-2}v^2 + |\text{grad } v|^2]dx = \int_{\Omega_+} \rho[x_2^{-2}u^2 + |\text{grad } u|^2]dx \quad ,$$

and we conclude that $v \in V$.

Similarly,

$$a(v,v) = \int_\Omega \rho|\text{grad } v|^2dx \leq \int_{\Omega_+} \rho|\text{grad } u|^2dx = a(u,u) \quad . \qquad (*)$$

Obviously, $v(x) \geq u(x) \geq \psi(x)$ a.e.

We now show that $v = T/2\pi$ on Γ_1 in the sense of $H^1(\Omega)$. Since $u = T/2\pi$ on Γ_1 in the sense of $H^1(\Omega)$, there is a sequence $\{\varphi_k\}$ with $\varphi_k \to u$ in V, and $\varphi_k - T/2\pi \in C_0^\infty(\Gamma_1)$.

Let

$$v_k = \max(\varphi_k, 0) \quad .$$

Then v_k belongs to V and $v_k = T/2\pi$ in some neighborhood of Γ_1 (but $v_k - T/2\pi \notin C_0^\infty(\Gamma_1)$). Since $\varphi_k \to u$ in V we know that $\varphi_k(x) \to u(x)$ a.e. and that the norms $\|\varphi_k\|$ are bounded. Consequently, $v_k(x) \to v(x)$ a.e. and the norms $\|v_k\|$ are bounded. V is a Hilbert space. In a Hilbert space bounded sets are weakly sequentially compact, so there exists a subsequence $\{v_k'\}$, which converges weakly to some $v' \in V$. Weak convergence in V implies weak convergence in $L^2(\Omega)$ which in turn implies pointwise convergence a.e. Thus, $v_k'(x) \to v'(x)$ a.e., from which it follows that $v(x) = v'(x)$ a.e. and hence that $v = v'$. Taking finite convex linear combinations of the v_k' we obtain a sequence $\{\hat{v}_k\}$ which converges in norm to $v' = v$. Each $\hat{v}_k$ is a finite linear combination of the v_k, so $\hat{v}_k = T/2\pi$ in some neighborhood of Γ_1. Finally, applying Theorem 2.3, we approximate $\hat{v}_k$ by ψ_k where $\psi_k \in V$ and $\psi_k - T/2\pi \in C_0^\infty(\Gamma_1)$. Since $\psi_k \to v$, we conclude that $v = T/2\pi$ in the sense of $H^1(\Omega)$.

In summary, $v \in K$.

Now, u solves the variational inequality (4.7), and so

$$a(u,v-u) = -\int_{\Omega_-} \rho|\text{grad } u|^2 dx \leq 0 \quad ,$$

which implies that $a(u,v-u) = 0$. But then, by (*),

$$\begin{aligned} a(v-u,\ v-u) &= a(v,v) - a(u,u) - 2a(u,\ v-u) \\ &= a(v,v) - a(u,u) \quad , \\ &\leq 0 \quad , \end{aligned}$$

so that $v = u$.

□

Remark 5.2.

Parter [1965, p.281] gives an example involving generalized axially symmetric potentials where the maximum principle does not apply. In Parter's example, however, the region Ω is symmetric about Γ_0 and so the line of degeneracy is contained in Ω. In the present paper the line of degeneracy is on the boundary of Ω.

Theorem 5.3.

In the strip S_{h_0}, u satisfies the differential equation $Au = 0$ in the weak sense; that is,

$$\int_\Omega \rho \text{ grad } u \cdot \text{grad } \varphi \, dx = 0 \quad , \tag{5.9}$$

for any $\varphi \in C_0^\infty(S_{h_0})$.

Proof: If $k > k_0$ then, from Theorem 4.2,

$$\psi(x_1,x_2) \le g(x_2) < 0$$

in the strip S_{h_0}.

By Theorem 5.2 we know that $u \ge 0$ a.e. in Ω. Thus, $u \ge 0 > \psi$ a.e. in S_{h_0}. More specifically, given a compact subset G of S_{h_0} there exists $\epsilon > 0$ such that $u \ge \psi + \epsilon$ a.e. in G. For any $\varphi \in C_0^\infty(G)$ choose $\delta > 0$ so that $|\delta\varphi| < \epsilon$. Then, $v_+ = u + \delta\varphi \in K$ and $v_- = u - \delta\varphi \in K$. Hence,

$$a(u, v_+-u) = a(u, v_--u) = \int_\Omega \rho \text{ grad } u \cdot \text{grad}(\delta\varphi)dx = 0 \quad .$$

□

Remark 5.3.

Theorems 5.2 and 5.3 depend on Theorem 4.2 which assumes the specific geometry of Figure 1.2 to evaluate ψ. If Γ_1 is not as shown in Figure 1.2, let

$$\bar{R} = R(\bar{x}_1) = \min\{R(x_1) : 0 \le x_1 \le L\} \quad .$$

We believe that Theorem 5.3 remains true if in the definition of h_0, $R(0)$ is replaced by $\bar{R}$. The proof would require a detailed study of the function ψ in the case of a general domain, along the lines of the study by Ting [1966] for the case of the torsion of a prismatic bar.

Remark 5.4.

Theorem 5.3 provides a bound for the size of the plastic region. This is particularly interesting because in the numerical computations of Eddy and Shaw [1949] the plastic region dips down near the corner E on Γ_1 (see Figure 1.2), and it is far from clear that the plastic region will not grow very rapidly as the torque increases. For the second problem considered by Eddy and Shaw [1949], $k = 49$, $T = 6349 \times 2\pi$, $R(0) = 8$. In their numerical calculations

$$\bar{x}_2 = \min\{x_2 : (x_1,x_2) \in \Omega_p\} \doteq 6\cdot 95 \quad .$$

From (5.1), (5.2) and Theorem 5.3, $k_0 \doteq 37\cdot 20$, and

$$\bar{x}_2 \geq h_0 \doteq 5\cdot 13 \quad .$$

Remark 5.5.

The fact that there is an elastic strip near Γ_0 as long as $k > k_0$ is analogous to the situation for the elastic-plastic torsion of prismatic bars where an elastic core also remains until the entire bar becomes plastic (Lanchon [1974]).

Remark 5.6.

Our analysis is not adequate to handle the limiting case $k = k_0$. We conjecture that if $k = k_0$ then $u = \psi$ for $0 \leq x_1 \leq \bar{x}_1$, where

$$\bar{x}_1 = \max\{x_1 : R(x) = R(0) \quad \text{for} \quad 0 < x < x_1\} \quad .$$

Theorem 5.3 asserts that $Au = 0$ in the weak sense in S_{h_0}. We may thus expect that u is regular in S_{h_0}. This does not appear to follow from known results about elliptic equations, and we therefore prove this by modifying the corresponding proof for uniformly elliptic equations. We use Gilbarg and Trudinger [1977] as a basic reference, since this is a comprehensive and readily accessible test.

The basic idea is to obtain bounds for the differences of the solution u and then proceed to the limit.

The difference quotient in the x_1 direction is defined by:

$$\Delta_1^h v(x) = \Delta^h v(x) = \Delta v(x) = \frac{v(x+he_1) - v(x)}{h}, \ h \neq 0 \quad , \tag{5.10}$$

where e_1 is the unit vector in the x_1 direction. If $v \in V$, then the difference quotient $\Delta^h v$ is defined on Ω^h,

$$\Omega_1^h = \Omega^h = \{x \in \Omega : x + te_1 \in \Omega \quad \text{for} \quad t \in (0,h]\} \quad . \tag{5.11}$$

As is customary, the weak derivatives of u are denoted by $D_i u$, $D_{ij} u$, etc.

Lemma 5.4.

Let $v \in V$ and $h > 0$. Let $\Omega' \subset \Omega_1^h \cap \Omega_1^{-h}$. Then $\Delta v = \Delta_1^h v \in L_\rho^2(\Omega')$ and

$$\|\Delta_1^h v \; ; \; L_\rho^2(\Omega')\| \le \|D_1 v \; ; \; L_\rho^2(\Omega)\| \quad .$$

Proof: The proof is a modification of the proof of Lemma 7.23 of Gilbarg and Trudinger [1977].

We begin by assuming that $v \in C_0^\infty(\Gamma_0)$. If $x \in \Omega_1^h$, then

$$\rho^{1/2}(x)\Delta v(x) = \rho^{1/2}(x)\frac{[v(x+he_1) - v(x)]}{h} \quad ,$$

$$= \frac{1}{h}\int_0^h \rho^{1/2}(x) D_1 v(x+te_1)\,dt \quad ,$$

so that, using the Cauchy-Schwarz inequality,

$$\rho(x)[\Delta v(x)]^2 \le \frac{1}{h}\int_0^h \rho(x)(D_1 v(x+te_1))^2 dt \quad .$$

Since $\rho(x+te_1) = \rho(x) = x_2^{-3}$,

$$\|\Delta v \; ; \; L_\rho^2(\Omega')\|^2$$

$$= \int_{\Omega'} \rho(x)[\Delta v(x)]^2 dx \quad ,$$

$$\le \frac{1}{h}\int_{\Omega'} dx [\int_0^h \rho(x+te_1)(D_1 v(x+te_1))^2 dt] \quad ,$$

$$= \frac{1}{h}\int_0^h dt \int_{\Omega'} \rho(x+te_1)\,(D_1 v(x+te_1))^2 dx \quad ,$$

$$\le \frac{1}{h}\int_0^h \|D_1 v \; ; \; L_\rho^2(\Omega)\|^2 dt \quad ,$$

and the lemma follows for $v \in C_0^\infty(\Gamma_0)$.

But, by Theorem 2.3 there exists $\psi \in C_0^\infty(\Gamma_0)$ such that $\|v-\psi \; ; \; W\| \le \epsilon$ for any $\epsilon > 0$.

Thus,

$$\|\Delta v\ ;\ L^2_\rho(\Omega')\|$$

$$\le \|\Delta\psi\ ;\ L^2_\rho(\Omega')\| + 2\epsilon/h\ ,$$

$$\le \|D_1\psi\ ;\ L^2_\rho(\Omega)\| + 2\epsilon/h\ ,$$

$$\le \|D_1 v\ ;\ L^2_\rho(\Omega)\| + 2\epsilon/h + \epsilon\ .$$

Letting $\epsilon \to 0$ the lemma follows.

□

Using the arguments used to prove Lemma 7.24 of Gilbarg and Trudinger [1977] we obtain

Lemma 5.5.

Let $v \in L^2_\rho(\Omega)$ and let $\Omega' \subset\subset \Omega$. If for $h <$ distance $(\Omega', \partial\Omega)$ we have

$$\|\Delta^h_1 v\ ;\ L^2_\rho(\Omega')\| \le c\ ,$$

then v has a weak derivative $D_1 v$ which satisfies

$$\|D_1 v\ ;\ L^2_\rho(\Omega)\| \le c\ .$$

□

Theorem 5.6.

Let $u \in V$ satisfy $Au = 0$ in S_{h_0} in the weak sense. Let

$$S = \{\underline{x} = (x_1, x_2)\ :\ L/4 < x_1 < 3L/4 \text{ and } 0 < x_2 < h_0/2\}\ .$$

Then:

(i) $u|S \in C^\infty(S)$.

(ii) $u,_{11}|S$ and $u,_{12}|S$ belong to $L^2_\rho(S)$.

(iii) $\int_S \frac{1}{x_2} u,_{22}^2\, dx < \infty$.

(iv) $u \in H^2(S)$. u can be extended as a continuous function to $\bar{S}$, and $u = 0$ on $\partial S \cap \Gamma_0$.

(v) $u = x_2^4 v$, where v is analytic in $\bar{S}$.

__Proof:__ The proof is a modification of the proofs of Theorem 8.8 and 8.9 of Gilbarg and Trudinger [1977].

__Proof of (i):__ Statement (i) follows from Corollary 8.11 of Gilbarg and Trudinger [1977].

__Proof of (ii):__ We denote by $C^1(\partial S_{h_0}/\Gamma_0)$ the set of functions which are continuously differentiable in Ω, vanish outside S_{h_0}, and vanish in some neighborhood of $\partial S_{h_0}/\Gamma_0$.

Let $\eta \in C^1(\partial S_{h_0}/\Gamma_0)$ be such that

(a) $\eta = 1$ for $x \in S$,

(b) $|D_1\eta|, |D_2\eta| \leq c_1$ for some constant c_1. η is readily constructed as the product of two one-dimensional cut-off functions.

For small positive h, set

$$v = \eta^2 \Delta_1^h u \equiv \eta^2 \Delta u \quad ,$$

$$N^2 = \sum_{i=1}^{2} \|\eta D_i \Delta u \; ; \; L_\rho^2(\Omega)\|^2 \quad .$$

Then, for $h < \text{dist}(\text{supp}\,\eta, \partial S_{h_0}/\Gamma_0)$,

$$N^2 = \sum_{i=1}^{2} \int_\Omega \rho (D_i \Delta u)(\eta^2 D_i \Delta u)\, dx \quad ,$$

$$= \sum_{i=1}^{2} \int_\Omega \Delta(\rho D_i u)(D_i v - 2\, D_i \eta \Delta u)\, dx \quad ,$$

$$= - \sum_{i=1}^{2} \int_\Omega [(\rho D_i u) \Delta_1^{-h} D_i v + 2\rho\eta D_i \eta (\Delta D_i u) \Delta u]\, dx \quad , \qquad (*)$$

where we have used the identity, valid for any $f \in C^1(\partial S_{h_0}/\Gamma_0)$, $g \in V$, and sufficiently small h,

$$\int_\Omega (\Delta_1^h f) g \, dx = - \int_\Omega f \Delta_1^{-h} g \, dx \quad .$$

Since $Au = 0$ weakly in S_{h_0}, we know that for any $v \in C_0^\infty(S_{h_0})$,

$$\sum_{i=1}^{2} \int_{S_{h_0}} \rho (D_i u) D_i v \, dx = 0 \quad .$$

Now $u \in V$ and thus, by Theorem 2.3, u is the limit in W of functions $\varphi_j \in C_0^\infty(\Gamma_0)$. Consequently, $\Delta_1^{-h} v = \Delta_1^{-h}(\eta^2 \Delta_1^h u)$ is the limit in W of the functions $\Delta_1^{-h}(\eta^2 \Delta_1^h \varphi_j) \in C_0^\infty(S_{h_0})$. By proceeding to the limit we find that

$$\sum_{i=1}^{2} \int_{S_{h_0}} \rho(D_i u)(D_i \Delta_1^{-h} v)dx = 0 \quad ,$$

so that the first term on the right of (*) is zero.

Thus, using the Cauchy-Schwarz inequality and Lemma 5.4,

$$N^2 \le 2c_1 \sum_{i=1}^{2} \int_{S_{h_0}} \rho|\eta \Delta D_i u| \; |\Delta u| dx \quad ,$$

$$\le 2c_1 (2^{1/2} N) \| D_1 u \; ; \; L_\rho^2(\Omega) \| \quad ,$$

and hence

$$N \le 4c_1 \|u\| \quad .$$

This bound holds for all sufficiently small h. In consequence, appealing to Lemma 5.5 and (i) above, we see that $u_{,11}|S$ and $u_{,21}|S = u_{,12}|S$ belong to $L_\rho^2(S)$.

<u>Proof of (iii)</u>: From Theorem 8.8 of Gilbarg and Trudinger we know that $u_{,22}$ exists in S and satisfies

$$u_{,22} = - u_{,11} + 3u_{,2}/x_2 \quad .$$

a.e. in S. If $\rho = x_2^{-1}$ then $u_{,22}$ and $u_{,2}/x_2$ belong to $L_\rho^2(S)$, and thus so does $u_{,22}$.

<u>Proof of (iv)</u>: It follows from (ii) and (iii) that u belongs to the Sobolev space $H^2(S)$. From the Sobolev embedding theorems $u \in C(\bar{S})$ (Adams [1975, Theorem 5.4]).

Furthermore, since $u \in V$, it follows from Theorem 2.2 that

$$u(x_1, x_2) \to 0 \quad \text{as} \quad x_2 \to 0$$

for almost all x_1. Since u is continuous on $\bar{S}$ we conclude that $u = 0$ on $\partial S \cap \Gamma_0$.

Proof of (v): This is an immediate consequence of Theorem 2 of Huber [1954].

□

Theorem 5.6 informs us that u is well-behaved near Γ_0. Away from Γ_0 the operator A is well-behaved. There are many results on the regularity of solutions of variational inequalities for coercive operators (Lewy and Stampacchia [1969], Frehse [1972], Gerhardt [1973], Brezis and Kinderlehrer [1974]). However, there is a difficulty to be overcome before these results can be applied: the function ψ is not smooth. This is because when we integrate along the characteristics of ψ as in Theorem 4.2 we find that certain points in Ω lie on two characteristics. This is best seen by considering Γ_{22}. It follows from an analysis of the characteristic equations (4.11) to (4.15) that if $dR/dx_1(x_1) > 0$ then the characteristic starting at $(x_1, R(x_1))$ intersects Γ_{22} (Cryer [1979a]). On the other hand, the characteristic starting at $(L, R(L))$ coincides with Γ_{22}. At points which lie on two characteristics, ψ must be taken to be the larger of the two values obtained by integrating along the characteristics.

The motivation for the following arguments is as follows. We cannot prove directly that $u \in H^{2,p}(\Omega)$ because $\psi \notin H^{2,p}(\Omega)$. However, the discontinuities of ψ occur in the upper right corner of Ω where in general the material is elastic and $u > \psi$. We therefore seek to replace ψ by a smooth function ψ_c which agrees with ψ when $u = \psi$.

Let u_e denote the solution of the elastic problem corresponding to the elastic-plastic problem. That is, $Au_e = 0$ in Ω and u_e satisfies the boundary conditions (1.7) to (1.9). u_e satisfies $u \in K_e$ and

$$a(u_e, v_e - u_e) = 0 \quad \text{for} \quad v_e \in K_e \quad , \tag{5.12}$$

where

$$K_e = \{v_e \in V : v_e = T/2\pi \quad \text{on} \quad \Gamma_1 \quad \text{in the sense of} \quad H^1(\Omega)\} \quad . \tag{5.13}$$

As in Theorem 5.6 we conclude that u_e is smooth in some S_h. From the standard theory of elliptic equations we conclude that u_e is at least twice continuously differentiable in Ω_h, so that $u_e \in C^2(\bar{\Omega})$.

Lemma 5.7.

$$u \geq u_e \quad .$$

Proof: See Stampacchia [1965] and Cryer and Dempster [1978]. Let $\zeta = \max(u, u_e)$. The theorem will be true if we can prove that $u = \zeta$.

Now (as in the proof of Theorem 5.2) $\zeta \in K$ and since u satisfies (4.7),

$$\begin{aligned} &a(u-\zeta,\ u-\zeta) \\ &\quad = a(u,\ u-\zeta) + a(\zeta,\ \zeta-u) \quad , \\ &\quad \leq + a(\zeta,\ \zeta-u) \quad , \\ &\quad = a(u_e,\ \zeta-u) + a(u_e-\zeta,\ u-\zeta) \quad . \end{aligned}$$

But,

$$a(u_e,\ \zeta-u) = a(u_e,\ v_e-u_e) = 0 \quad ,$$

where

$$v_e = u_e + \zeta - u \in K_e \quad .$$

Furthermore, either $\zeta = u$ or $\zeta = u_e$ and so

$$a(u_e-\zeta,\ u-\zeta) = 0 \quad .$$

Thus, $a(u-\zeta,\ u-\zeta) \leq 0$ and we conclude that $u = \zeta$.

□

Definition 5.1.

ψ satisfies Condition C if there exists $\psi_c \in H^{2,\infty}(\Omega)$ such that $\psi_c = \psi$ whenever $u_e \leq \psi$ and $\psi \leq \psi_c \leq u_e$ whenever $u_e > \psi$.

Remark 5.7.

Condition C can be checked knowing only ψ and u_e, both of which can be evaluated fairly easily and do not depend upon u. Condition C is satisfied in some practical cases (Cryer [1979a]).

If ψ satisfies Condition C we introduce the variational inequality with unique solution u_c: Find $u_c \in K_c$ such that

$$a(u_c,\ v_c-u_c) \geq 0, \quad \text{for } v_c \in K_c \quad , \tag{5.14}$$

where

$$K_c = \{v_c \in K_e : v_c \geq \psi_c\} \quad . \tag{5.15}$$

Lemma 5.8.

If ψ satisfies Condition C then $u = u_c$.

Proof: u and u_c are the unique solutions of the variational inequalities (4.7) and (5.14), respectively.

Let $v_c \in K_c$. Then $v_c \in K$ since $v_c \geq \psi_c \geq \psi$. Thus,

$$a(u, v_c - u) \geq 0 \quad \text{if} \quad v_c \in K_c \quad .$$

Furthermore, $u \geq \psi_c$ because either $u_e \leq \psi$ in which case $u \geq \psi = \psi_c$ or $u_e > \psi$ in which case $u \geq u_e \geq \psi_c$.

We conclude that u also solves the variational inequality (5.14), so that $u = u_c$.

□

Theorem 5.9.

If ψ satisfies Condition C then $u = u_c \in H^{2,p}(\Omega) \cap C^{\infty}(S_{h_0/2})$, for any $p \in (1,\infty)$.

Outline of Proof: It was shown in Theorem 5.6 that $u|S \in C^{\infty}(S)$ where

$$S = \{\underline{x} = (x_1, x_2) : L/4 < x_1 < 3L/4, \quad 0 < x_2 < h_0/2\} \quad .$$

The restrictions on the length of S can be easily removed by enlarging Ω to $\tilde{\Omega}$ by reflection as in Lemma 5.1. We conclude that $u \in C^{\infty}(S_{h_0/2})$. By Lemma 5.8, $u = u_c$.

Now let

$$K_1 = \{v_1 \in H^1(\Omega_{h_0/2}) : v_1 = u = u_c \text{ in the sense of } H^1(\Omega) \text{ on}$$

$$\partial S_{h_0/2} \cap \partial\Omega_{h_0/2};\ v_1 = T/2\pi \text{ on } \Gamma_1 \text{ in the sense of } H^1(\Omega); v_1 \geq \psi_c\} \quad ,$$

and let a_1 be defined on $H^1(\Omega_{h_0/2}) \times H^1(\Omega_{h_0/2})$ by

$$a_1(v_1, w_1) = \int_{\Omega_{h_0/2}} \rho\, D_i v_1 D_i w_1 dx \quad .$$

The variational inequality: Find u_1 K_1 satisfying

$$a_1(u_1, v_1-u_1) \geq 0, \quad \text{for all } v_1 \quad K_1 \quad , \qquad (*)$$

has a unique solution u_1.

Now let $u = u|\Omega_{h_0/2}$. Then, $u_1 \in K_1$. Furthermore, for $v \in K_1$ let $v \in K$ be obtained by setting $v(\underline{x}) = u(\underline{x})$ for $\underline{x} \in S_{h_0/2}$. Then

$$a_1(u, v-u) = a(u, v-u) \geq 0 \quad ,$$

so that u also solves the variational inequality (*). Thus $\tilde{u} = u|\Omega_{h_0/2} = u_1$.

Since $u|S_{h_0/2}$ is smooth, it only remains to show that the solution u_1 of (*) belongs to $H^{2,p}(\Omega)$. From Condition C we know that $\psi_c \in H^1(\Omega) \cap H^{2,p}(\Omega)$. Furthermore, a_1 is well-behaved on $\Omega_{h_0/2}$. The regularity results in the literature (Lewy and Stampacchia [1969, Theorem 3.1], Brezis and Stampacchia [1968, Corollary II.3], Stampacchia [1973, Theorem 6.4]) are not immediately applicable because they consider the case $K_1 = H_0^1(\Omega_{h_0/2})$. It is, however, clear that the arguments can be modified so as to show that $u_1 \in H^{2,p}(\Omega_{h_0/2})$.

□

6. Bounds for grad u.

We introduce the elliptic operator

$$Mv = \operatorname{div}(x_2 \operatorname{grad} v) \quad . \tag{6.1}$$

Lemma 6.1.

Let $v \in C^3(\Omega_h) \cap C^1(\bar{\Omega}_h)$ satisfy $Av = 0$ in Ω_h. Let $w = \frac{|\operatorname{grad} v|}{x_2^2}$. Then

$$\sup_{\bar{\Omega}_h} w = \sup_{\partial\Omega_h} w \quad .$$

Proof: Since $Av = 0$ in Ω_h we have, using summation notation,

$$\frac{1}{x_2^3} v_{,ii} = \frac{3}{x_2^4} v_{,2} \quad , \tag{6.2}$$

so that, by differentiation,

$$v_{,iij} = (\frac{3}{x_2} v_{,2})_{,j} \quad . \tag{6.3}$$

Let

$$w = \frac{|\operatorname{grad} v|^2}{x_2^4} = \frac{[v_{,j} v_{,j}]}{x_2^4} \quad . \tag{6.4}$$

Then $w \in C^2(\Omega_h) \cap C^0(\bar{\Omega}_h)$. Also,

$$\begin{aligned} Mw &= (x_2(v_{,j}v_{,j}/x_2^4)_{,i})_{,i} \quad , \\ &= (2v_{,j}v_{,ij}/x_2^3)_{,i} - 4(v_{,j}v_{,j}/x_2^4)_{,2} \quad , \\ &= 2(v_{,ij})^2/x_2^3 + 2v_{,j}v_{,iij}/x_2^3 - \\ &\quad - 6v_{,j}v_{,2j}/x_2^4 - 8v_{,j}v_{,2j}/x_2^4 + \\ &\quad + 16v_{,j}v_{,j}/x_2^5 \quad . \end{aligned}$$

Using (6.3) to replace $v_{,iij}$ and collecting terms,

$$Mw = 2(v_{,1j})^2/x_2^3 + 6v_{,j}v_{,2j}/x_2^4 -$$

$$- 6(v_{,2})^2/x_2^5 - 6v_{,j}v_{,2j}/x_2^4 - 8v_{,j}v_{,2j}/x_2^4 +$$

$$+ 16(v_{,j})^2/x_2^5 \ .$$

$$= 2(v_{,1j})^2/x_2^3 - 8v_{,j}v_{,2j}/x_2^4 -$$

$$- 6(v_{,2})^2/x_2^5 + 16(v_{,j})^2/x_2^5 \ .$$

Thus,

$$Mw = 2(v_{,1j})^2/x_2^3 + 2(v_{,2})^2/x_2^5 + 8(v_{,1})^2/x_2^5 +$$

$$+ 2x_2[v_{,2j}/x_2^2 - 2v_{,j}/x_2^3]^2 \ ,$$

$$\geq 0 \ . \tag{6.5}$$

Since M is an elliptic operator in Ω_h we conclude from the maximum principle (Gilbarg and Trudinger [1977, p. 31]) that w attains its maximum on the boundary of Ω_h.

□

Theorem 6.2.

If $k > k_0$, if ψ satisfies Condition C, and if u is the solution of the variational inequality (4.7) then

$$w = \frac{|\text{grad } u|}{x_2^2} \leq k, \quad \text{in } \Omega \ . \tag{6.6}$$

Proof: We know from Theorem 5.9 that $u \in H^{2,p}(\Omega)$. In particular $u \in C^1(\bar{\Omega})$. Let

$$\Omega_+ = \{x \in \Omega : u > \psi\} \ ,$$

$$\Omega_0 = \{x \in \Omega : u = \psi\} \ .$$

In Ω_0 we have that grad u = grad ψ so that, from the definition of ψ,

$$w = k, \quad \text{in } \Omega_0 \ . \tag{6.7}$$

From Theorem 5.6 we know that $u = x_2^4 v$ in S_{h_0}. Thus, for some $h_1 \geq h_0$ we have

$$w < k/2, \text{ in } S_{h_1} \quad . \tag{6.8}$$

Now consider the set $\hat{\Omega} = \Omega_+/S_{h_1}$. Applying Lemma 6.1 we conclude that

$$\max_{\hat{\Omega}} w = \max_{\partial\hat{\Omega}} w \quad .$$

Now, $\partial\hat{\Omega}$ consists of several components which we consider in turn.

(i) $\hat{\Gamma}_0 = \partial\hat{\Omega} \cap \partial S_{h_1}$. Using (6.8) we conclude that $w < k$ on $\hat{\Gamma}_0$.

(ii) $\hat{\Gamma}_2 = \partial\hat{\Omega} \cap \Gamma_2$. Since we can use Theorem 5.1 to enlarge Ω, we know that

$$\max_{\hat{\Gamma}_2} w \leq \max_{\hat{\Omega}} w \quad .$$

(iii) $\hat{\Gamma}_+ = \partial\hat{\Omega} \cap \partial\Omega_0$. Since w is continuous, it follows from (6.7) that

$$\max_{\hat{\Gamma}_+} w = k \quad .$$

(iv) $\hat{\Gamma}_1 = \partial\hat{\Omega} \cap \Gamma_1$. On Γ_1 we have $u = \psi = T/2\pi$. Since $Au = 0$ in $\hat{\Omega}$, and $u \leq T/2\pi$ on $\partial\hat{\Omega}$, it follows from the maximum principle applied to $Au = 0$ that $u \leq T/2\pi$ on $\hat{\Omega}$. Now consider a point on $\hat{\Gamma}_1$.

Let t and n denote the normal and tangential directions, so that

$$|\text{grad } u|^2 = u_n^2 + u_t^2 \quad .$$

Since $u = T/2\pi$ on Γ_0, we have $u_t = 0$. On the other hand, along the inward normal n we have

$$\psi \leq u \leq T/2\pi \quad .$$

Remembering that $\psi = T/2\pi$ on Γ_1, we conclude that

$$|\text{grad } u|^2 = |u_n|^2 \leq |\psi_n|^2 = kx_2^2 \quad ,$$

so that $w \leq k$ on $\hat{\Gamma}_1$.

□

Remark 6.1.

Theorem 6.1 is analogous to the result of Brezis and Sibony [1971] for the elastic plastic torsion of prismatic bars. They showed that the solution ϕ of the corresponding obstacle problem satisfies the condition $|\text{grad}\ \phi| \leq \tilde{k}$.

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The axisymmetric elastic-plastic torsion of a shaft subject to the von Mises yield criterion is considered. The problem is reformulated as a variational inequality and it is proved that the problem has a unique solution. Some properties of the solution are derived.

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