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RANDOM BIPARTITE GRAPHS: CONNECTEDNESS, ISOLATED NODES, DIAMETE--ETC(U)

APR 79 V KLEE, D LARMAN

N00014-67-A-0103-0003

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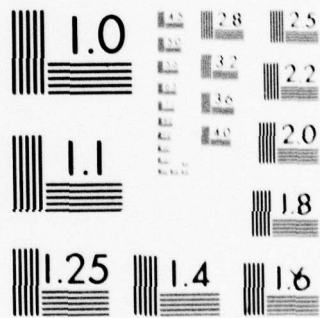
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RANDOM BIPARTITE GRAPHS: CONNECTEDNESS, ISOLATED NODES, DIAMETERS

BY

Victor Klee and David Larman

Technical Report No. 66

April 1979

Contract N00014-67-A-0103-0003

Project Number NR044 353

Department of Mathematics  
University of Washington  
Seattle, Washington 98195

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security Classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)  University of Washington	2a. REPORT SECURITY CLASSIFICATION <b>Unclassified</b>
	2b. GROUP

3. REPORT TITLE  
 ⑥ Random Bipartite Graphs: Connectedness, Isolated Nodes, Diameters.

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)  
 ⑨ Technical rept.

5. AUTHOR(S) (First name, middle initial, last name)  
 ⑩ Victor/Klee and David/Larman      ⑫ 25p.      ⑭ TR-66

6. REPORT DATE ⑪ Apr 1979	7a. TOTAL NO. OF PAGES 22	7b. NO. OF REFS 3
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8a. CONTRACT OR GRANT NO. ⑮ N00014-67-A-0103-0003	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 66
b. PROJECT NO. NR044 353	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
c.	
d.	

10. DISTRIBUTION STATEMENT Releasable without limitations on dissemination	<b>DISTRIBUTION STATEMENT A</b> Approved for public release; Distribution Unlimited
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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
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13. ABSTRACT

In addition to being of interest for its own sake, the study of random graphs provides the combinatorial foundation for investigations of the average-case behavior of various graph-theoretic algorithms. The present paper deals with the family  $B(m,n,E)$  of all labeled bipartite graphs that have  $m$  nodes in the first part and  $n$  nodes in the second part, with exactly  $E$  edges. The main result is that if the positive integers  $m(1), m(2), \dots, E(1), E(2), \dots$  are such that  $m(n) \leq n$ ,  $E(n) \leq m(n)n$ , and  $\liminf_{n \rightarrow \infty} E(n)/(n \log n) > 1$ , then the probability that a random member of  $B(m(n), n, E(n))$  is connected converges to 1 as  $n \rightarrow \infty$ . Results on isolated nodes and on diameters are also obtained.

370 275

LP



UNCLASSIFIED

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
bipartite connected diameter graph isolated node random graph sparse graph						

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DDC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification _____	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Avail and/or special
<b>A</b>	

DD FORM 1473 (BACK)  
1 NOV 65

S/N 0101-407-6001

Unclassified  
Security Classification

A-314-9

## RANDOM BIPARTITE GRAPHS: CONNECTEDNESS, ISOLATED NODES, DIAMETERS

VICTOR KLEE and DAVID LARMAN

Abstract Let  $\mathcal{B}(m,n,E)$  denote the family of all labeled bipartite graphs that have  $m$  nodes in the first part and  $n$  nodes in the second, with exactly  $E$  edges. If the positive integers  $m(1), m(2), \dots$  and  $E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)n$  for all  $n$ , and  $\liminf_{n \rightarrow \infty} E(n)/(n \log n) > 1$ , then the probability that a random member of  $\mathcal{B}(m(n), n, E(n))$  is connected converges to 1 as  $n \rightarrow \infty$ . Results on isolated nodes and on diameters are also obtained.

*as n approaches infinity*

Introduction

For  $1 \leq m \leq n < \infty$ , let  $\mathcal{B}(m,n)$  denote the family of all graphs with node-set  $\{1, \dots, m+n\}$ , each edge being of the form  $\{i, j\}$  for some  $i \in M = \{1, \dots, m\}$  and  $j \in N = \{m+1, \dots, m+n\}$ . In other words,  $\mathcal{B}(m,n)$  is the family of all labeled bipartite graphs that have  $m$  nodes in the small part and  $n$  nodes in the large part. For  $0 \leq E \leq mn$ , let  $\mathcal{B}(m,n,E)$  denote the family of all members of  $\mathcal{B}(m,n)$  that have exactly  $E$  edges. Note that  $|\mathcal{B}(m,n)| = 2^{mn}$  and  $|\mathcal{B}(m,n,E)| = \binom{mn}{E}$ .

All members of  $\mathcal{B}(m,n)$  are given the same weight, so the probability that a random member of  $\mathcal{B}(m,n,E)$  has property  $P$  is merely  $|\mathcal{B}_P(m,n,E)| / \binom{mn}{E}$ , where  $\mathcal{B}_P(m,n,E)$  is the set of all members of  $\mathcal{B}(m,n,E)$  that have  $P$ . In studying random bipartite graphs, it seems appropriate to focus on  $\mathcal{B}(m,n,E)$  rather than  $\mathcal{B}(m,n)$ , because graphs occurring in practical problems are apt to be sparse. For the many bipartite graphs that arise naturally in problems from operations research or computer science, a specific division of the nodes into two parts is usually imposed by the problem itself. Thus it is appropriate to focus on  $\mathcal{B}(m,n,E)$  rather than

the set of all bipartite members of  $\underline{G}(n,E)$ , where this denotes the family of all graphs with node-set  $\{1, \dots, n\}$  and exactly  $E$  edges.

Our main result deals with connectedness. It is a bipartite relative of the theorem of Erdős and Renyi [1] asserting that if  $\lambda$  is constant and

$E_\lambda(n) = \lfloor \frac{n}{2} \log n + \lambda n \rfloor$  for all  $n$ , then the probability that a random member of  $\underline{G}(n, E_\lambda(n))$  is connected converges to  $\exp(-e^{-2\lambda})$  as  $n \rightarrow \infty$ .

Our methods are in part inspired by theirs.

**THEOREM 1** If the positive integers  $m(1), m(2), \dots$  and  $E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)n$  for all  $n$ , and  $\liminf_{n \rightarrow \infty} E(n)/(n \log n) > 1$ , then the probability that a random member of  $\underline{B}(m(n), n, E(n))$  is connected converges to 1 as  $n \rightarrow \infty$ .

For any finite family  $\underline{G}$  of graphs, let  $K(\underline{G})$  denote the probability that a random member of  $\underline{G}$  is connected. For each positive integer  $r$ , let  $C_r(\underline{G})$  denote the probability, for a random member  $G$  of  $\underline{G}$ , that each component of  $G$  has at least  $r$  nodes, and let  $D_r(\underline{G})$  denote the probability that  $G$  is of diameter  $\leq r$ . If all members of  $\underline{G}$  have precisely  $s$  nodes, and if we follow the usual convention that disconnected graphs are of infinite diameter, then

$$C_2 \geq C_3 \geq \dots \geq C_{\lfloor (s+1)/2 \rfloor} = \dots = C_s = K = D_{s-1} \geq \dots \geq D_3 \geq D_2.$$

Our second result concerns  $C_2(\underline{B}(n, n, E))$ . It implies that Theorem 1's conclusion fails if  $E$  increases much less rapidly than is required in the hypothesis of Theorem 1.

**THEOREM 2** If the positive integers  $E(1), E(2), \dots$  are such that  $E(n) \leq n^2$  for all  $n$ , and  $\lim_{n \rightarrow \infty} (E(n)/n) - \log n = \lambda < \infty$ , then the probability that a random member of  $\underline{B}(n, n, E(n))$  has no isolated node converges to  $\exp(-2e^{-\lambda})$  as  $n \rightarrow \infty$ .



Our third result deals with diameters.

**THEOREM 3** If the positive integers  $m(1), m(2), \dots$  are such that  $m(n) \leq n$  for all  $n$ , and  $\lim_{n \rightarrow \infty} (\log n)/m(n) = 0$ , then the probability that a random member of  $B(m(n), n)$  is of diameter 3 converges to 1 as  $n \rightarrow \infty$ .

There is a large gap between this result and Theorem 1, which concerns the probability that the diameter is finite. Most of the gap is filled by the following conjecture, which we have not proved.

**CONJECTURE** If the positive integers  $r, m(1), m(2), \dots, E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)n$  for all  $n$ , and

$$\lim_{n \rightarrow \infty} E(n)^{2r-2}/m(n)^{r-1}n^r = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (E(n)^{2r}/m(n)^r n^{r+1}) - \log n = \infty,$$

then the probability that a random member of  $B(m(n), n, E(n))$  is of diameter  $2r$  or  $2r+1$  converges to 1 as  $n \rightarrow \infty$ .

Elementary Estimates

This section collects some elementary estimates that are used throughout the paper and are henceforth referred to by number. We use the combinatorial inequalities

$$\binom{n}{k} \leq \frac{n^k}{k!} \quad \text{for } 1 \leq k \leq n \quad (1)$$

$$\binom{m-w}{N} / \binom{m}{N} \leq \left(1 - \frac{w}{m}\right)^N \quad \text{for } 1 \leq N \leq m-w \text{ and } 0 \leq w < m, \quad (2)$$

the analytic inequality

$$1 + x \leq e^x \quad \text{for all } x, \quad (3)$$

and the fact that for  $0 \leq |x| < y$ ,

$$\log\left(1 - \frac{x}{y}\right)^{y-x} = (y-x) \left(-\sum_{k=1}^{\infty} \frac{x^k}{ky^k}\right) = -x + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \frac{x^k}{y^{k-1}}. \quad (4)$$

When the functions  $\alpha$  and  $\beta$  are defined for positive integers,

$$\alpha \sim \beta \text{ means } \lim_{n \rightarrow \infty} \alpha(n)/\beta(n) = 1$$

and  $\alpha \prec \beta$  means  $\alpha(n) < \beta(n)$  for all sufficiently large  $n$ .

We use the Stirling-de Moivre estimate,

$$n! \sim \sqrt{2\pi n} n^{n+1/2} / e^n \quad (5)$$

and the following consequences of (4) and (5):

if  $0 \leq s \leq o(N)$  then

$$\frac{N!}{(N-s)!} \sim N^s \exp\left(-\sum_{k=2}^{\infty} \frac{1}{k(k-1)} \frac{s^k}{N^{k-1}}\right) \text{ as } N \rightarrow \infty; \quad (6)$$

if  $0 \leq w \leq o(m)$  then

$$\frac{(m-w)!}{m!} \sim m^{-w} \exp\left(\sum_{k=2}^{\infty} \frac{1}{k(k-1)} \frac{w^k}{m^{k-1}}\right) \text{ as } m \rightarrow \infty. \quad (7)$$



Since

$$\binom{m-w}{N-s} / \binom{m}{N} = \frac{(m-w)!}{m!} \frac{N!}{(N-s)!} \frac{(m-N)!}{(m-N-w+s)!},$$

it follows from (6) and (7) that if  $0 \leq w \leq o(m)$ ,  $0 \leq s \leq o(N)$  and  $0 \leq w-s \leq o(m-N)$ , then

$$\binom{m-w}{N-s} / \binom{m}{N} \sim \left(\frac{N}{m}\right)^s \left(1 - \frac{N}{m}\right)^{w-s} \exp\left(\sum_{k=2}^{\infty} \frac{1}{k(k-1)} \left(\frac{w^k}{m^{k-1}} - \frac{s^k}{N^{k-1}} - \frac{(w-s)^k}{(m-N)^{k-1}}\right)\right) \quad (8)$$

as  $m \rightarrow \infty$ ,  $N \rightarrow \infty$ ,  $m - N \rightarrow \infty$ .

Proof of Theorem 1

LEMMA 1 Suppose the positive integers  $m(1), m(2), \dots$  and  $E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)$  for all  $n$ , and  $\lim_{n \rightarrow \infty} E(n)/(n \log n) > 1$ . Then for all sufficiently large  $n$  the quantity

$$\alpha(s, t) = \binom{m}{s} \binom{n}{t} \binom{mn - (n-t)s - (m-s)t}{E} / \binom{mn}{E}$$

is less than the reciprocal of

- (a)  $\lfloor s/2 \rfloor! \lfloor t/2 \rfloor!$  when  $1 \leq s \leq m/2$  and  $1 \leq t \leq n/2$   
 (b)  $(m+n) \lfloor (m-s)/4 \rfloor! \lfloor t/4 \rfloor!$  when  $m/2 < s \leq m$  and  $1 \leq t \leq n/2$   
 (c)  $(m+n) \lfloor s/4 \rfloor! \lfloor (n-t)/4 \rfloor!$  when  $1 \leq s \leq m/2$  and  $n/2 < t \leq n$   
 (d)  $\lfloor (m-s)/2 \rfloor! \lfloor (n-t)/2 \rfloor!$  when  $m/2 < s \leq m$  and  $n/2 < t \leq n$ .

Proof. Since the reasoning for (c) and (d) is essentially the same as that for (b) and (a) respectively, only (a) and (b) are discussed. Let the constants  $\xi$  and  $\zeta$  be such that

$$1 < \xi < \zeta < \infty \text{ and } \xi n \log n < E(n) < \zeta n \log n \quad (9)$$

and let 
$$\rho = (\xi - 1)/2\zeta. \quad (10)$$

Let 
$$b = (n-t)s + (m-s)t \quad (11)$$

and 
$$\gamma = \binom{mn-b}{E} / \binom{mn}{E} \leq (2,3) \exp(-Eb/mn), \quad (12)$$

so that 
$$\alpha(s, t) = \binom{m}{s} \binom{n}{t} \gamma. \quad (13)$$

To establish (a), note that by (1) and (11)-(13),

$$\alpha(s, t) \leq \frac{1}{s!t!} e^{\beta(s, t)} \quad (14)$$

with 
$$\beta(s, t) = s \log m + t \log n - Eb/mn. \quad (15)$$

Since  $\beta(s,t) = s(\log m - E/m) + t(\log n - E/n + 2Es/mn)$ ,

it follows from (9) that for all sufficiently large  $n$ ,

$$\begin{aligned} \beta(s,t) &< s(\log m - \frac{\xi n \log n}{m}) + t(\log n - \xi \log n + 2\zeta \frac{s}{m} \log n) \\ &\leq s(1 - \xi) \log n + t(1 - \xi + 2\zeta \frac{s}{m}) \log n \end{aligned}$$

for all  $s$  and  $t$  and hence  $\beta(s,t) < 0$  if  $s < \rho m$ . Similarly,

$$\beta(s,t) = s(\log m - \frac{E}{m} + \frac{2Et}{mn}) + t(\log n - \frac{E}{n}),$$

so (9) implies that for all sufficiently large  $n$ ,

$$\beta(s,t) < s(\log m - \frac{\log n}{m}(\xi n - 2\zeta t)) + t(1 - \xi) \log n$$

for all  $s$  and  $t$  and hence  $\beta(s,t) < 0$  if  $t < \rho n$ . That settles (a) if  $s < \rho m$  or  $t < \rho n$ .

To complete the proof of (a) there remains the case in which

$$\rho m \leq s \leq m/2 \quad \text{and} \quad \rho n \leq t \leq n/2. \quad (16)$$

Recalling that the left side of (5) always exceeds the right, we see that

$$\frac{1}{s!t!} < \exp(s + t - (s+\frac{1}{2})\log m - (t+\frac{1}{2})\log n + A_1) \quad (17)$$

where  $A_1 = -\log(2\pi) + (s+\frac{1}{2})\log(m/s) + (t+\frac{1}{2})\log(n/t)$

and hence  $A_1 \leq B_1 = -\log(2\pi) - (s+t+1)\log \rho$ . (18)

By (14), (15) and (17),

$$\alpha(s,t) \leq \exp(s + t - \frac{1}{2}(\log m + \log n) + B_1 - Eb/mn), \quad (19)$$

where by (9), (11) and (16) it is true with  $1 < \xi' < \xi$  that



$$\begin{aligned}
 Eb/mn &> \frac{\xi'}{m}(\log n)((n-t)s + (m-s)t) \\
 &\geq \frac{s\xi'}{2m}\log n + \frac{t\xi'}{2}\log n \geq \frac{\xi'}{2}(s+t)\log n.
 \end{aligned} \tag{20}$$

Hence by (18)-(20) and the fact that  $\xi' > \frac{1}{2}(\xi'+1)$ ,

$$a(s,t) \ll \exp(-\frac{1}{2}(\xi'+1)(s+t)\log n).$$

Since  $\frac{1}{2}(\xi'+1) > \frac{1}{2}$ , it then follows with the aid of (5) that

$$a(s,t) \ll \frac{1}{[s/2]!} \frac{1}{[t/2]!}.$$

That completes the discussion of (a), and we turn to (b).

Since, by (1),

$$\binom{m}{s} \binom{n}{t} \leq \frac{m^{m-s}}{(m-s)!} \frac{n^t}{t!},$$

we conclude from (12)-(13) that

$$a(s,t) \leq \frac{1}{(m-s)!t!} e^{\lambda(s,t)} \tag{21}$$

where

$$\lambda(s,t) = (m-s)\log m + t \log n - \frac{E}{mn}((n-t)s + (m-s)t).$$

Note that for each fixed  $s$ ,  $\lambda(s,t)$  is an increasing function of  $t$ , and for each fixed  $t \leq n/2$ ,  $\lambda(s,t)$  is a decreasing function of  $s$ . It follows that

$$\begin{aligned}
 \lambda(s,t) &\leq \lambda(s, n/2) \leq (m-s)\log m - \frac{n}{2} \left( \frac{E}{m} - \log n \right) \\
 &\ll (m-s)\log m - \frac{n}{2}(\xi-1)\log n
 \end{aligned} \tag{22}$$

and

$$\begin{aligned}
 \lambda(s,t) &\leq \lambda(m/2, t) \leq \frac{m}{2} \log m + t \log n - \frac{1}{2}E \\
 &\ll \frac{m}{2} \log m + t \log n - \frac{1}{2}\xi n \log n.
 \end{aligned} \tag{23}$$

Now suppose that  $s$  and  $t$  are in the ranges associated with (b). If, moreover,

$$m-s \leq (\xi-1)m/4 \quad \text{or} \quad t \leq (\xi-1)n/4 \quad (24)$$

then it follows with the aid of (22) and (23) that

$$\lambda(s,t) \leq \frac{1-\xi}{4} n \log n,$$

whence it is true for all sufficiently large  $n$  that

$$e^{\lambda(s,t)} < \frac{1}{m+n}$$

and the desired conclusion follows from (21). If, on the other hand, (24) fails, then it follows from (5) that

$$\frac{1}{(m-s)!t!} < \exp((m-s) + t - (m-s+\frac{1}{2})\log m - (t+\frac{1}{2})\log n + B_2)$$

where

$$B_2 = -\log(2\pi) - (m-s+t+1)\log(\xi-1).$$

Using this in (15), and noting that  $Eb/mn \geq E/2$ , we have

$$\begin{aligned} \alpha(s,t) &< \exp((m-s) + t - \frac{1}{2}\log m - \frac{1}{2}\log n - B_2 - E/2) \\ &< (5) \frac{1}{m+n} \frac{1}{[(m-s)/4]! [t/4]!}. \end{aligned}$$

That completes the discussion of (b) and of Lemma 1.

**LEMMA 2** Let  $B'(m,n,E)$  denote the set of all members of  $B(m,n,E)$  that have a component with more than

$$K(m,n,E) = m + n - 2(2E/(m+n))^{\frac{1}{2}}$$

nodes. If the positive integers  $m(1), m(2), \dots$  and  $E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)n$  for all  $n$ , and  $\lim_{n \rightarrow \infty} E(n)/(n \log n) > 1$ ,



then the probability that a random member of  $B(m(n), n, E(n))$  belongs to  $B'(m(n), n, E(n))$  converges to 1 as  $n \rightarrow \infty$ .

Proof. We show first that

each member  $G$  of  $B(m, n, E)$  has a component with at least  $K(m, n, E)$  nodes, (25)

Indeed, if  $r$  is the number of components of  $G$  that have more than one node, and the  $i^{\text{th}}$  of these components has  $k_i$  nodes in the set  $M$  and  $\ell_i$  nodes in the set  $N$ , then  $k_i \geq 1 \leq \ell_i$ ,

$$\sum_{i=1}^r k_i \leq m, \quad \sum_{i=1}^r \ell_i \leq n, \quad \text{and} \quad \sum_{i=1}^r k_i \ell_i \geq E.$$

Since  $r \leq m$ ,

$$\max_{1 \leq i \leq r} k_i \ell_i \geq \frac{E}{m} \geq \frac{2E}{m+n} = \left(\frac{K}{2}\right)^2$$

and hence

$$\max_{1 \leq i \leq r} (k_i + \ell_i) \geq K.$$

It follows from (25) that

$$|B(m, n, E) - B'(m, n, E)| \leq \sum_{K \leq s+t \leq m+n-K} \binom{m}{s} \binom{n}{t} \binom{mn - (n-t)s - (m-s)t}{E}, \quad (26)$$

where it is understood that  $s, t \geq 1$ . To establish (26), consider a graph  $G \in B(m, n, E) - B'(m, n, E)$  that has a largest component intersecting  $M$  and  $N$  in sets  $S$  and  $T$  respectively, with  $|S| = s$  and  $|T| = t$ . Then  $K \leq s+t \leq m+n-K$  and the pair  $(S, T)$  can be chosen in  $\binom{m}{s} \binom{n}{t}$  ways. For each such choice there are at most

$$\binom{mn - (n-t)s - (m-s)t}{E}$$

ways of choosing the edges of  $G$ , because no edge can join  $S$  to  $N-T$  or  $T$  to  $M-S$ .

From (26) and Lemma 1 it follows that

$$\begin{aligned}
 & \frac{|B(m,n,E) - B'(m,n,E)|}{|B(m,n,E)|} \leq \sum_{K \leq s+t \leq m+n-K} \alpha(s,t) \\
 & < 2 \sum_{s \geq 1 \leq t, K \leq s+t} \frac{1}{[s/2]! [t/2]!} + \frac{1}{n} \sum_{m/2 < s \leq m, 1 \leq t} \frac{1}{[(m-s)/4]! [t/4]!} \\
 & \quad + \frac{1}{n} \sum_{n/2 < t \leq n, 1 \leq s} \frac{1}{[s/4]! [(n-t)/4]!} \\
 & < 2 \left( \sum_{s=1}^{\infty} \frac{1}{[s/2]!} \right) \left( \sum_{t=\lfloor K/2 \rfloor}^{\infty} \frac{1}{[t/2]!} \right) \\
 & \quad + \frac{2}{n} \left( \sum_{s=1}^{\infty} \frac{1}{[s/4]!} \right) \left( \sum_{t=1}^{\infty} \frac{1}{[t/4]!} \right) \\
 & < 2(2e) \left( 2 \sum_{r=\lfloor K/4 \rfloor}^{\infty} \frac{1}{r!} \right) + \frac{2}{n} (4e)^2 \rightarrow 0
 \end{aligned}$$

because  $K > (E/n)^{1/2} \rightarrow \infty$  as  $n \rightarrow \infty$ . That settles Lemma 2.

**THEOREM 1** If the positive integers  $m(1), m(2), \dots$  and  $E(1), E(2), \dots$  are such that  $m(n) \leq n$  and  $E(n) \leq m(n)n$  for all  $n$ , and

$$\liminf_{n \rightarrow \infty} E(n)/(n \log n) > 1,$$

then the probability  $P_1(m(n), n, E(n))$  that a random member of  $B(m(n), n, E(n))$  is connected converges to 1 as  $n \rightarrow \infty$ .

**Proof.** Consideration of the natural subgraph correspondence shows that if  $F \leq E$  then  $P_1(m, n, F) \leq P_1(m, n, E)$ . Hence we may assume without loss of generality that the sequence  $(E(n)/(n \log n))_{n=1,2,\dots}$  converges to a real number  $\lambda > 1$ . In view of Lemma 2, it suffices to show that  $Q(m, n, E) \rightarrow 0$ , where  $Q(m, n, E)$  is the probability that a random member of  $B(m, n, E)$  is disconnected and belongs to  $B'(M, N, E)$ .

Note first that

$$Q(m,n,E) \leq \sum_{1 \leq s+t \leq K} \binom{m}{s} \binom{n}{t} \left[ \sum_{r=0}^{st} \binom{st}{r} \left( \frac{m-s}{E-r} \right)^{m-s} \left( \frac{n-t}{E-r} \right)^{n-t} \right] \left( \frac{E}{E-r} \right)^r. \quad (27)$$

For consider a disconnected member  $G$  of  $B'(m,n,E)$  that has a largest component intersecting  $M$  and  $N$  in sets  $U$  and  $V$  respectively, where

$|U| = m-s$ ,  $|V| = n-t$ , and  $1 \leq s+t \leq K$ . The pair  $(U,V)$  can be chosen in

$\binom{m}{s} \binom{n}{t}$  ways. For each such choice, and for each possible number  $r$  of edges  $G$

connecting  $M-U$  to  $N-V$ , there are  $\binom{st}{r}$  ways of choosing those edges and

$\binom{m-s}{E-r} \binom{n-t}{E-r}$  ways of choosing the remaining edges.

Observe next that

$$\begin{aligned} \binom{m-s}{E-r} \binom{n-t}{E-r} &\leq (8) \left( \frac{E}{mn} \right)^r \left( 1 - \frac{E}{mn} \right)^{sn+tm-st-r} \\ &\leq (3) \left( \frac{E}{mn} \right)^r \exp\left(-\frac{E}{mn}(sn+tm-st-r)\right) \\ &\leq (a) \left( \frac{E}{mn} \right)^r \exp(-\mu(s+t)\log n) \end{aligned} \quad (28)$$

with

$$\mu = \frac{1}{2}(1+\lambda).$$

Here the first step requires not only (8) but also the sort of argument used in proving Lemma 7 of [2] to show that the asymptotic convergence depends only on  $n$ .

Using (28) in (27) yields

$$\begin{aligned} Q(m,n,E) &\leq \sum_{1 \leq s+t \leq K} \binom{m}{s} \binom{n}{t} (\exp(-\mu(s+t)\log n) \left( \sum_{r=0}^{st} \binom{st}{r} \left( \frac{E}{mn} \right)^r \right)) \\ &\leq (1) \sum_{1 \leq s+t \leq K} \frac{1}{s!} \frac{1}{t!} \exp(-(\mu-1)(s+t)\log n) \left( 1 + \frac{E}{mn} \right)^{st} \\ &\leq (3) \sum_{1 \leq s+t \leq K} \frac{1}{s!} \frac{1}{t!} \exp(-(\mu-1)(s+t)\log n + \frac{Est}{mn}). \end{aligned} \quad (29)$$



Now

$$\frac{\text{Est}}{mn} \prec \frac{2\lambda sK \log n}{m} \leq \frac{4\lambda}{m} \left(\frac{2E}{n}\right)^{\frac{1}{2}} (s+t) \log n$$

where

$$\frac{4\lambda}{m} \left(\frac{2E}{n}\right)^{\frac{1}{2}} \rightarrow 0 \dots$$

Thus it follows from (23) that

$$\begin{aligned} Q(m, n, E) &\prec \sum_{1 \leq s+t \leq K} \frac{1}{s!} \frac{1}{t!} \exp(-\frac{1}{2}(\mu-1)(s+t) \log n) \\ &\prec n^{-(\frac{1}{2})(\mu-1)} e^2 \rightarrow 0. \end{aligned}$$

The proof of Theorem 1 is complete.

Proof of Theorem 2

LEMMA 3 The number of members of  $B(m,n,E)$  that have no isolated node is

$$\sum_{k=0}^m \sum_{\ell=0}^n (-1)^{k+\ell} \binom{m}{k} \binom{n}{\ell} \binom{m-k}{E} \binom{n-\ell}{E} \quad (30)$$

Proof. Consider an arbitrary member  $G$  of  $B(m,n,E)$  that has  $p$  isolated nodes in  $\{1, \dots, m\}$  and  $q$  isolated nodes in  $\{m+1, \dots, m+n\}$ . For each choice of  $k$  nodes in  $\{1, \dots, m\}$  and  $\ell$  nodes in  $\{m+1, \dots, m+n\}$ , the number of members of  $B(m,n,E)$  that have these  $k+\ell$  points among their isolated nodes is  $\binom{m-k}{E} \binom{n-\ell}{E}$ .

If this count is repeated over all possible choices of  $k$  nodes in  $\{1, \dots, m\}$  and  $\ell$  nodes in  $\{m+1, \dots, m+n\}$ , the number  $\sum_{k=0}^m \sum_{\ell=0}^n \binom{m-k}{E} \binom{n-\ell}{E}$  is obtained and  $G$  is counted  $\binom{p}{k} \binom{q}{\ell}$  times. Thus  $G$  is counted

$$\tau(G) = \sum_{k=0}^p \sum_{\ell=0}^q (-1)^{k+\ell} \binom{p}{k} \binom{q}{\ell}$$

times in (30). Since  $\sum_{k=0}^p (-1)^k \binom{p}{k} = 0$  if  $p > 0$  and  $\sum_{\ell=0}^q (-1)^\ell \binom{q}{\ell} = 0$  if  $q > 0$ ,

$\tau(G)$  is 0 or 1 according as  $p+q > 0$  or  $p = q = 0$ .

LEMMA 4 Let  $\theta_n(k, \ell) = \binom{n}{k} \binom{n}{\ell} \binom{(n-k)(n-\ell)}{E(n)} / \binom{n^2}{E(n)}$ . If the positive integers  $E(1), E(2),$

$\dots$  are such that  $E(n) \leq n^2$  for all  $n$  and  $\lim_{n \rightarrow \infty} (E(n)/n) - \log n = \lambda < \infty$ , then

for all sufficiently large  $n$  it is true that  $\theta_n(k, \ell) \leq 1/[k/4]![\ell/4]!$  whenever

$k, \ell$  and  $n$  are nonnegative integers with  $k \leq n$ ,  $\ell \leq n$ ,  $k+\ell \geq 2n^{1/3}$ , and

$E(n) \leq (n-k)(n-\ell)$ .



Proof. Note first that

$$\begin{aligned} \Theta_n(k, \ell) &\leq (1,2) \frac{n^k n^\ell}{k! \ell!} \left(1 - \frac{n(k+\ell) - k\ell}{n^2}\right)^E \\ &\leq (3) \frac{1}{k! \ell!} \exp\left((k+\ell) \log n - (k+\ell) \frac{E}{n} + \frac{k\ell}{n} \frac{E}{n}\right) \\ &< \frac{1}{k! \ell!} \exp\left(2(1-\lambda)(k+\ell) + \frac{k\ell}{n} \log n\right), \end{aligned}$$

where the third inequality is a consequence of the facts that  $k\ell/n < k+\ell$  and  $\lambda-1 < (E/n) - \log n < \lambda+1$ .

By symmetry we may assume  $k \leq \ell$ , whence  $\ell \geq n^{1/3}$ . If  $k \leq n^{1/6}$  then  $(k\ell \log n)/n \leq n^{1/6} \log n$ , and with the aid of Stirling's formula (5) it follows that for all sufficiently large  $n$ ,

$$\begin{aligned} \Theta_n(k, \ell) &< \frac{1}{k!} \exp(-\log \sqrt{2\pi} - (\ell + \frac{1}{2}) \log \ell + \ell + 2(1-\lambda)(k+\ell) + n^{1/6} \log n) \\ &\leq \frac{1}{k!} \frac{1}{\lfloor \ell/2 \rfloor!}. \end{aligned}$$

If  $k > n^{1/6}$  we may apply Stirling's formula to both  $k!$  and  $\ell!$  to obtain, for suitable constants  $A, B$  and  $C$  and for all sufficiently large  $n$ ,

$$\begin{aligned} \Theta_n(k, \ell) &\leq \exp\left(\left(\frac{k\ell}{n} - k - \ell\right) \log n + Ak + B\ell + C\right) \\ &\leq \exp(-\ell \log n + Ak + B\ell + C) \\ &\leq \frac{1}{\lfloor \ell/2 \rfloor!} \leq \frac{1}{\lfloor k/4 \rfloor!} \frac{1}{\lfloor \ell/4 \rfloor!}. \end{aligned}$$

**THEOREM 2** If the positive integers  $E(1), E(2), \dots$  are such that  $E(n) \leq n^2$  for all  $n$  and  $\lim_{n \rightarrow \infty} (E(n)/n) - \log n = \lambda < \infty$ , then the probability  $P_2(n)$  that a random member of  $B(n, n, E(n))$  has no isolated node converges to  $\exp(-2e^{-\lambda})$  as  $n \rightarrow \infty$ .

Proof. Note that

$$\exp(-2e^{-\lambda}) = \sum_{s=0}^{\infty} (-1)^s 2^s e^{-\lambda s} / s!$$

while it follows from Lemma 3 that

$$P_2(n) = \sum_{s=0}^{2n} (-1)^s f_n(s) \quad \text{with} \quad f_n(s) = \sum_{k+\ell=s} o_n(k, \ell). \quad (31)$$

Hence for each pair of positive integers  $n$  and  $s'$ ,

$$\begin{aligned} & |P_2(n) - \exp(-2e^{-\lambda})| \\ & \leq \sum_{s=0}^{s'} |f_n(s) - 2^s e^{-\lambda s} / s!| + \left| \sum_{s=s'+1}^{2n} (-1)^s f_n(s) \right| \\ & \quad + \left| \sum_{s=\lceil 2n^{1/3} \rceil}^{2n} (-1)^s f_n(s) \right| + \left| \sum_{s=s'+1}^{\infty} (-1)^s 2^s e^{-\lambda s} / s! \right|. \end{aligned} \quad (32)$$

To prove Theorem 2 we show that for each  $\epsilon > 0$  there exists  $s'$  and  $n'$  such that each of the summands on the right of (32) is less than  $\epsilon$  for all  $n \geq n'$ .

The first step is to produce positive integers  $n_3$  and  $s_1$  such that

$$f_n(s) > f_n(s+1) \quad \text{whenever} \quad n \geq n_3 \quad \text{and} \quad s_1 \leq s \leq \lceil 2n^{1/3} \rceil. \quad (33)$$

To do this, first apply (8) to fixed  $k$  and  $\ell$  with  $k+\ell \leq \lceil 2n^{1/3} \rceil$  to obtain

$$\begin{aligned} & \left( \frac{(n-k)(n-\ell)}{E} \right) / \left( \frac{n^2}{E} \right) - \left( 1 - \frac{E}{n^2} \right) \frac{(k+\ell)(k-\ell)}{2n^4} \exp\left( \frac{(k+\ell)n - k\ell}{2n^4} + \dots \right) \\ & \quad - \exp(-E(k+\ell)/n). \end{aligned} \quad (34)$$

Then analyze the arguments leading to (8) to verify that the convergence in (34) is uniform over all  $(k, \ell)$  with  $k+\ell \leq \lceil 2n^{1/3} \rceil$ . (The details are similar to those in the proof of Lemma 7 in [2].) Hence there exists  $n_1$  such that

$$\frac{1}{2} e^{-E(k+\ell)/n} \leq \left( \frac{(n-k)(n-\ell)}{E} \right) / \left( \frac{n^2}{E} \right) \leq 2e^{-E(k+\ell)/n}$$

whenever  $n \geq n_1$  and  $k+\ell \leq \lceil 2n^{1/3} \rceil$ . Also, there exists  $n_2$  such that

$$\frac{1}{2t!} \leq \binom{n}{t}/n^t \leq \frac{2}{t!}$$

whenever  $n \geq n_2$  and  $t \leq 2n^{1/3}$ . Thus for  $n \geq \max\{n_1, n_2\}$  and  $k+l \leq \lfloor 2n^{1/3} \rfloor$ ,

$$\Theta_n(k, l) \in [\frac{1}{4}, 4] e^{-E(k+l)/n} n^{k+l}/k!l! \quad (35)$$

With 
$$\sum_{k+l=s} \frac{1}{k!l!} = \frac{2^s}{s!} \quad (36)$$

it follows from (31) and (35)-(36) that

$$f_n(s) \in [\frac{1}{4}, 4] 2^s e^{-Es/n} n^s/s! .$$

and hence

$$f_n(s+1) < 2^{s+1} \cdot 4e^{-E(s+1)/n} n^{s+1}/(s+1)! < 2^s e^{-Es/n} n^s/4s! < f_n(s)$$

whenever  $s \geq 32e^{2\lambda}$ . Thus (33) holds with

$$s_1 = 32e^\lambda \quad \text{and} \quad n_3 = \max\{n_1, n_2, s_1^3/8\}.$$

For fixed  $k$  and  $l$  it follows with the aid of (34) that as  $n \rightarrow \infty$ ,

$$k!l! \Theta_n(k, l) - n^{k+l} \exp(-E(k+l)/n) = \exp((k+l)(\log n - E/n)) - e^{-\lambda(k+l)}$$

and hence for each fixed  $s$  it follows with the aid of (36) that

$$\lim_{n \rightarrow \infty} f_n(s) = \sum_{k+l=s} e^{-\lambda(k+l)}/k!l! = 2^s e^{-\lambda s}/s! . \quad (37)$$

Now choose  $s' \geq s_1$  such that

$$|\sum_{s=s'+1}^{\infty} (-1)^s 2^s e^{-\lambda s}/s!| < \epsilon \quad (38)$$

and

$$2^{s'+1} e^{-\lambda(s'+1)}/(s'+1)! < \epsilon \quad (39)$$

By Lemma 4,

$$|\sum_{s=\lfloor 2n^{1/3} \rfloor}^{2n} (-1)^s f_n(s)| \leq \sum_{s=\lfloor 2n^{1/3} \rfloor}^{2n} \sum_{k+l=s} 1/[k/4]! [l/4]!$$

$$\leq (\sum_{s=\lfloor n^{1/3}/2 \rfloor}^{\infty} 1/s!)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Hence there exists  $n_4 \geq n_3$  such that



$$\left| \sum_{s=\lceil 2n^{1/3} \rceil}^{2n} (-1)^s f_n(s) \right| < \epsilon \text{ for all } n \geq n_4. \quad (40)$$

By (37) and (39) there exists  $n' \geq n_4$  such that

$$\sum_{s=0}^{s'} \left| f_n(s) - 2^s e^{-\lambda s} / s! \right| < \epsilon \text{ for all } n \geq n' \quad (41)$$

and

$$f_n(s'+1) < \epsilon \text{ for all } n \geq n'. \quad (42)$$

Since, by (33), it is true for each  $n \geq n'$  that  $f_n(s)$  decreases as  $s$  increases in the range from  $s'$  to  $\lceil 2n^{1/3} \rceil$ , it follows with the aid of (42)

that

$$\left| \sum_{s=s'+1}^{\lceil 2n^{1/3} \rceil} (-1)^s f_n(s) \right| \leq f_n(s'+1) < \epsilon. \quad (43)$$

The desired conclusion then follows from (38), (40), (41) and (43).

Proof of Theorem 3

LEMMA 5 If A is the adjacency matrix of a graph G in  $B(m,n)$ , and  $A^2 = (b_{ij})$ , then the probability that  $b_{ij} = 0$  is  $(\frac{3}{4})^n$  for  $1 \leq i < j \leq m$  and  $(\frac{3}{4})^m$  for  $m+1 \leq i < j \leq m+n$ .

Proof. There is an  $m \times n$  matrix  $Q = (q_{ij})$  of 0's and 1's such that

$$A = \begin{bmatrix} 0 & Q \\ Q^T & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} QQ^T & 0 \\ 0 & Q^TQ \end{bmatrix}.$$

Then 
$$b_{ij} = q_{i1}q_{j1} + \dots + q_{in}q_{jn},$$

and  $b_{ij} = s \geq 1$  if and only if there is a choice of  $s$  indices  $k_n$  such that

$$m < k_1 < k_2 < \dots < k_s \leq n, \quad q_{ik_1} = q_{jk_1} = \dots = q_{ik_s}q_{jk_s} = 1,$$

and at least one of  $q_{i\ell}$  and  $q_{j\ell}$  is 0 for each

$$\ell \in \{m+1, \dots, m+n\} - \{k_1, \dots, k_s\}.$$

Now suppose that  $1 \leq i < j \leq m$ , so that the indices  $k_1, \dots, k_s$  may be chosen

in  $\binom{n}{s}$  ways. For each such choice there are  $3^{n-s}$  choices for the values of

$q_{i\ell}$  and  $q_{j\ell}$ , and  $2^{(m-2)n}$  choices for the values of the  $q_{uv}$  with  $u \notin \{i, j\}$ .

Thus the total number of adjacency matrices  $A$  of graphs  $G \in B(m,n)$  such that

the  $(i,j)$  entry of  $A^2$  is  $\geq 1$  is

$$3^n 2^{(m-2)n} \sum_{s=1}^n \binom{n}{s} 3^{-s} = 3^n 2^{(m-2)n} \left( \left( \frac{4}{3} \right)^n - 1 \right) = 2^{mn} \left( 1 - \left( \frac{3}{4} \right)^n \right).$$



Since  $|B(m,n)| = 2^{mn}$ , this is the desired value. The same argument applies when  $m+1 \leq i < j \leq m+n$ .

**THEOREM 3** If the positive integers  $m(1), m(2), \dots$  are such that  $m(n) \leq n$  for all  $n$ , and  $\lim_{n \rightarrow \infty} (\log n)/m(n) = 0$ , then the probability  $P(m(n), n)$  that a random member of  $B(m(n), n)$  is of diameter 3 converges to 1 as  $n \rightarrow \infty$ .

**Proof.** For any two nodes  $x$  and  $y$  of a graph  $G$ , let  $\delta_G(x,y)$  denote the number of edges in the shortest path joining  $x$  to  $y$ , with  $\delta_G(x,y) = \infty$  if there is no such path. Then  $\delta(G)$ , the maximum of  $\delta_G(x,y)$  as  $(x,y)$  ranges over all pairs of nodes of  $G$ , is the diameter of  $G$ .

Let  $B'(m,n)$  denote the set of all graphs  $G \in B(m,n)$  such that for each pair of nodes in the set  $M = \{1, \dots, m\}$  (resp.  $N = \{m+1, \dots, m+n\}$ ) there is at least one common  $G$ -neighbor in the set  $N$  (resp.  $M$ ). Then  $\delta(G) = 3$  whenever  $G \in B'(m,n)$  and  $G$  is not a complete bipartite graph. Plainly  $\delta_G(x,y) = 2$  when  $x$  and  $y$  are both in  $M$  or both in  $N$ . Now consider an arbitrary pair of nodes  $x \in M$  and  $y \in N$  that are not joined by an edge of  $G$ . Then of course  $\delta_G(x,y) \geq 3$ . Let  $x' \in M - \{x\}$ , let  $y' \in N$  be a common neighbor of  $x$  and  $x'$ , and let  $x'' \in M$  be a common neighbor of  $y$  and  $y'$ . Then  $(x, y', x'', y)$  describes a path of length 3 from  $x$  to  $y$ . Hence  $\delta(G) = 3$ .

Plainly

$$P(m,n) \geq 1 - Q(m,n) - R(m,n) - 2^{-mn},$$

where  $Q(m,n)$  (resp.  $R(m,n)$ ) is the probability, for a random  $G \in B(m,n)$ , that some two nodes in  $M$  (resp.  $N$ ) fail to have a common neighbor in  $N$  (resp.  $M$ ).

By the preceding lemma,

$$Q(m,n) \leq \binom{m}{2} \left(\frac{3}{4}\right)^n < n^2 \left(\frac{3}{4}\right)^n \rightarrow 0$$

and

$$R(m,n) \leq \binom{n}{2} \left(\frac{3}{4}\right)^m < n^2 \left(\frac{3}{4}\right)^m.$$

Now pick  $\epsilon > 0$  such that  $e^{2\epsilon} < 4/3$ . By hypothesis,  $\log n < \epsilon m$ , whence

$$n^2 \left(\frac{3}{4}\right)^m = (e^{\log n})^2 \left(\frac{3}{4}\right)^m < \left(\frac{3}{4} e^{2\epsilon}\right)^m \rightarrow 0$$

and  $P(m,n) \rightarrow 1$ . That completes the proof.

Theorem 3 is a bipartite analogue of Moon's and Moser's observation [3] that if  $\underline{G}(n)$  is the family of all graphs with node-set  $\{1, \dots, n\}$ , then the probability that a random member of  $\underline{G}(n)$  is of diameter 2 converges to 1 as  $n \rightarrow \infty$ . The conjecture stated in the introduction is a bipartite analogue of the theorem, established in [2], that if  $d \geq 2$  and the positive integers  $E(1), E(2), \dots$  are such that  $E(n)^{d-1}/n^d \rightarrow 0$  and  $E(n)^d/n^{d+1} - \log n \rightarrow \infty$  as  $n \rightarrow \infty$ , then the probability that a random member of  $\underline{G}(n, E(n))$  is of diameter  $d$  converges to 1 as  $n \rightarrow \infty$ . Probably the conjecture can be proved by adapting the computations in [2], but that would be a task of considerable technical difficulty.

For each  $G \in \underline{B}(m, n)$ , let  $G'$  (resp.  $G''$ ) denote the graph whose node-set is  $M$  (resp.  $N$ ) and whose edges are those pairs  $\{x, y\}$  of nodes for which  $\delta_G(x, y) = 2$ .

If  $1 \leq m \leq n$  and  $\lim_{n \rightarrow \infty} (\log n)/m = 0$  then the expected number of edges of

$G'$  (resp.  $G''$ ), for a random member  $G$  of  $\underline{B}(m, n, E)$ , is of the order of

$E^2/n$  resp.  $\langle E^2/m \rangle$  as  $n \rightarrow \infty$ . If the appropriate independence (or asymptotic independence) results could be established, then  $G'$  and  $G''$  could be treated as random members of  $\underline{G}(m)$  and  $\underline{G}(n)$  respectively and the conjecture would follow from the result of [2]. Even lacking this independence, the methods of [2] might be applicable.

## References

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