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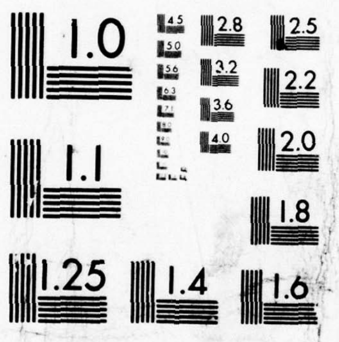
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

**HEAVY TRAFFIC RESULTS
FOR SINGLE SERVER QUEUES WITH
DEPENDENT (EARMA) SERVICE
AND INTERARRIVAL TIMES**

by

P. A. Jacobs

April 1979

NAVAL POSTGRADUATE SCHOOL
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20. ABSTRACT

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HEAVY TRAFFIC RESULTS FOR SINGLE SERVER QUEUES WITH DEPENDENT (EARMA) SERVICE AND INTERARRIVAL TIMES

by

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0. ABSTRACT

Models are given for sequences of correlated exponential interarrival and service times for a single server queue. These multivariate exponential models are formed as probabilistic linear combinations of sequences of independent exponential random variables and are easy to generate on a computer. Limiting results for customer waiting time under heavy traffic conditions are obtained for these queues. Heavy traffic results are useful for analyzing the effect of correlated interarrival and service times in queues on such quantities as queue length and customer waiting time. They can also be used to check simulation results.

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1. INTRODUCTION

Much of the work in queueing theory deals with models in which assumptions of independent service and interarrival times are made. One way to examine the effect of correlated interarrival and service times on measures of performance in queues is to consider models in which the interarrival and service times are correlated stationary sequences of random variables having given marginal distributions and compare results for these models with those for which the interarrival and service times are independent and identically distributed with the same marginal distributions.

In [3] Jacobs and Lewis introduced a scheme for generating sequences of dependent exponential random variables. The EARMA (1,1) (exponential mixed autoregressive moving average with both autoregression and moving average of order 1) model is defined as follows. Let $\{E_n\}$ be a sequence of independent exponential random variables with positive finite mean ν^{-1}. Let $\{J_n\}$ and $\{K_n\}$ be independent sequences of independent $\{0,1\}$-random variables with $P\{J_n=1\} = 1 - \beta$ and $P\{K_n=1\} = 1-\rho$ where $0 \leq \beta \leq 1$ and $0 \leq \rho < 1$ are given constants. We define

(1.1) $$X_n = \beta E_n + J_n A_{n-1}, \qquad n = 1,2,...$$

where

(1.2) $$A_n = \rho A_{n-1} + K_n E_n, \qquad n = 1,2,... .$$

We assume that A_0 is an independent exponential random variable with mean ν^{-1}. Under this assumption $\{X_n\}$ is a stationary sequence of dependent exponential random variables with mean ν^{-1}.

The dependence is not Markovian in general and as was shown in [3]

$$(1.3) \qquad \mathrm{Corr}(X_1, X_{1+n}) = \rho^{n-1}(1-\beta)[\beta(1-\rho) + (1-\beta)\rho] .$$

Note that if $\beta = 1$ or $\beta = 0$ and $\rho = 0$, then $\{X_n\}$ is a sequence of independent exponential random variables.

The EARMA(1,1) process is formed as a probabilistic linear combination of independent random variables. This technique can be extended to form sequences of multivariate exponential random variables. This idea was briefly explored in Lewis and Shedler [7]. These models have the advantage that the marginal distribution and correlation structure of the sequence are specified separately. In section 2 we will give several ways in which to generate sequences of dependent multivariate exponential random variables from probabilistic linear combinations of independent exponentials. These stationary multivariate EARMA sequences of dependent exponentials can be used as interarrival and service times in queues. Results for measures of performance for these queues can then be compared to queues with stationary independent sequences of interarrival and service times having the same marginal (exponential) distributions. Since the correlation structure in the multivariate EARMA sequences is specified by parameters independent of the marginal distribution, any difference between results for queues with these multivariate interarrival and service sequences and the corresponding queues with independent interarrival and service sequences having the same marginal distributions will be due to the correlation structure.

Suppose that the interarrival and service times of a single server queue form stationary multivariate EARMA sequences of dependent exponential random variables. Results of Loynes [9] give conditions for the existence of limiting distributions for the waiting time of a customer and the queue length. However, it seems to be difficult to obtain exact results even in the simplest queues with EARMA interarrival and service times (cf. Jacobs [4]).

As a result we turn our attention to the possibilities of obtaining limiting results of the heavy traffic type for queues with stationary multivariate EARMA sequences of dependent interarrival and service times. Results of this kind will indicate how dependence in queues introduced by the EARMA scheme affects quantities such as customer waiting time and queue length, at least under heavy traffic conditions. These limiting results can also be used to check simulations.

In this paper we will show that Kingman's [5] result for the equilibrium waiting time W in heavy traffic holds for queues having multivariate EARMA interarrival and service times; that is, we will show under conditions detailed in Section 3, that if the traffic intensity $\lambda\mu^{-1}$ of the queue is close to 1, then the distribution of $(1 - \lambda\mu^{-1})W$ is approximately exponential. The mean of this exponential distribution will in general be different from the case of independent interarrival and service times. Positive cross-correlation between the interarrival and service times will tend to decrease the mean; while positive correlation within the interarrival and service time sequences will tend to increase it.

In Section 3 we will state the heavy traffic result more precisely and present several examples of its application. Its formulation for multivariate EARMA queues is very dependent on the fact that the marginal distribution and correlation structure are specified independently of one another in EARMA formulations. In Section 4 we will give the proof of the heavy traffic result.

2. QUEUES WITH EARMA INTERARRIVAL AND SERVICE TIMES

We will consider a single server queue at which customers arrive and are served in the order of their arrival. The 0th customer arrives at $t = 0$ and finds the server free. The arrival time, service time, and waiting time (excluding service) of the nth customer are denoted respectively by $T_n = \sum_{k=1}^{n} X_k$, S_n, and W_n. In this section we will present some examples of the use of the EARMA scheme to model dependent interarrival and service times in queues.

(2.1) EXAMPLE. Let $\{X_n\}$ be an EARMA(1,1) process with positive finite mean λ^{-1} and parameters $0 \le \beta_1 \le 1$ and $0 \le \rho_1 < 1$; let $\{S_n\}$ be an EARMA(1,1) process independent of $\{X_n\}$ with positive finite mean μ^{-1} and parameters $0 \le \beta_2 \le 1$ and $0 \le \rho_2 < 1$. Note that if $\beta_1 = 1$, $\beta_2 = 1$, or $\beta_1 = 1$, $\beta_2 = 0$, $\rho_2 = 0$, or $\beta_1 = 0$, $\rho_1 = 0$, $\beta_2 = 1$, or $\beta_1 = 0$, $\rho_1 = 0$ $\beta_2 = 0$, $\rho_2 = 0$, then the queue reduces to an M/M/1 queue. Other special cases include queues with Poisson arrivals and EARMA(1,1) service times independent of the arrival process, and queues with EARMA(1,1) interarrival times and independent exponential service times.

The sequence $\{(X_n, S_n)\}$ is stationary and $\{X_n\}$ and $\{S_n\}$ are independent. However the X_n's (respectively S_n's) are positively correlated with

(2.2) $\quad \mathrm{Cov}(X_n, X_{n+k}) = \lambda^{-2}\rho_1^{k-1}(1-\beta_1)[\beta_1(1-\rho_1) + (1-\beta_1)\rho_1]$

(respectively

(2.3) $\quad \mathrm{Cov}(S_n, S_{n+k}) = \mu^{-2}\rho_2^{k-2}(1-\beta_2)[\beta_2(1-\rho_2) + (1-\beta_2)\rho_2])$

from (1.2).

Let $\underline{\underline{F}}_m^1$ (respectively $\underline{\underline{F}}_m^2$) be the σ-algebra generated by $X_1,\ldots,X_m$ (respectively $S_1,\ldots,S_m$) and $\underline{\underline{G}}_{m+k}^1$ (respectively $\underline{\underline{G}}_{m+k}^2$) be the σ-algebra generated by $X_{m+k}, X_{m+k+1},\ldots$ (respectively $S_{m+k}, S_{m+k+1}, \ldots$). Let $L^2(\underline{\underline{F}}_m^i)$ (respectively $L^2(\underline{\underline{G}}_{m+k}^i)$) denote the collection of real-valued functions having finite second moment that are measurable with respect to $\underline{\underline{F}}_m^i$ (respectively $\underline{\underline{G}}_{m+k}^i$), $i = 1,2$. If $f_i \in L^2(\underline{\underline{F}}_m^i)$ and $g_i \in L^2(\underline{\underline{G}}_{m+k}^i)$, $i = 1,2$, then there is a k_0 independent of f_i, g_i, $i = 1, 2$, such that for $k \geq k_0$

$$
\begin{aligned}
(2.4) \quad & |E[(f_1f_2)(g_1g_2)] - E[f_1f_2]\,E[g_1g_2]| \\
&= |E[f_1g_1]\,E[f_2g_2] - E[f_1]\,E[f_2]\,E[g_1]\,E[g_2]| \\
&= |E[f_1g_1]\,\{E[f_2g_2] - E[f_2]\,E[g_2]\} \\
&\quad + E[f_2]\,E[g_2]\,\{E[f_1g_1] - E[f_1]\,E[g_1]\}|
\end{aligned}
$$

$$(2.5)\qquad \le E[|f_1 g_1|]\ 5\rho_2^{k/2}\ E[f_2^2]^{1/2}\ E[g_2^2]^{1/2}$$

$$+ E[|f_2|]\ E[|g_2|]\ 5\rho_1^{k/2}\ E[f_1^2]^{1/2}\ E[g_1^2]^{1/2}$$

$$\le 5[\rho_1^{k/2} + \rho_2^{k/2}]\ E[(f_1 f_2)^2]^{1/2}\ E[(g_1 g_2)^2]^{1/2}$$

where (2.4) follows from the independent of $\{X_n\}$ and $\{S_n\}$ and (2.5) follows from (6.5) of Jacobs and Lewis [3]. Hence, if f (respectively g) has finite second moment and is measurable with respect to the product σ-algebra $\underline{\underline{F}}_m^1 \times \underline{\underline{F}}_m^2$ (respectively $\underline{\underline{G}}_{m+k}^1 \times \underline{\underline{G}}_{m+k}^2$), then for k sufficiently large

$$(2.6)\qquad |E[fg] - E[f]\,E[g]| \le 5E[f^2]^{1/2}\ E[g^2]^{1/2}\ [\rho_1^{k/2} + \rho_2^{k/2}].$$

Hence, the sequence $\{(X_n, S_n)\}$ is asymptotically uncorrelated in the sense of Rosenblatt [10] and is ϕ-mixing in the sense of Billingsley [1].

(2.7) EXAMPLE. In Lewis and Shedler [7] one of the schemes that is proposed for cross-correlating the arrival process and the service times is the following. Assume $\{X_n\}$ and $\{E_n\}$ are independent sequences of independent exponential random variables with positive finite means λ^{-1} and μ^{-1} respectively. The nth interarrival time is assumed to be X_n and hence the arrival process is Poisson with rate λ. Let the nth service time

$$(2.8)\qquad S_n = \beta E_n + J_n(\lambda\mu^{-1} B_n)$$

where

$$B_n = \rho B_{n-1} + K_n X_n \tag{2.9}$$

and β, ρ, $\{J_n\}$ and $\{K_n\}$ are as in (1.1) and (1.2) so that $\{S_n\}$ and $\{X_n\}$ are now cross-correlated sequences. We will assume that B_0 has an exponential distribution with mean λ^{-1}.

Since E_n and B_n are independent, $\{S_n\}$ is a stationary sequence of exponential random variables with mean μ^{-1}. If $\beta = 1$, then the queue reduces to an M/M/1 queue. The S_n's can be shown to be positively correlated with

$$\mathrm{Cov}(S_n, S_{n+k}) = \mu^{-2}(1-\beta)^2 \rho^k, \quad k = 1,2,\ldots . \tag{2.10}$$

Further, the nth service time S_n is positively correlated with the interarrival times $X_n, X_{n-1}, \ldots, X_1$ with

$$\mathrm{Cov}(S_n, X_{n-k}) = (\lambda\mu)^{-1}(1-\beta)(1-\rho)\rho^k \tag{2.11}$$

for $k = 0,\ldots,n-1$.

Note that $(X_n, S_n, \ldots, X_{n+k}, S_{n+k})$ is a function of $(B_{n-1}, X_n, E_n, J_n, K_n, X_{n+1}, E_{n+1}, J_{n+1}, K_{n+1}, \ldots, X_{n+k}, E_{n+k}, J_{n+k}, K_{n+k})$. Hence, since the sequence $\{B_n\}$ is stationary so is the sequence $\{(X_n, S_n)\}$. Further, by arguments similar to those used in obtaining (6.5) of Jacobs and Lewis [3], if f and g are as in (2.6), then for k sufficiently large

$$|E[fg] - E[f]\,E[g]| \le 5\rho^{k/2}\, E[f^2]^{1/2}\, E[g^2]^{1/2} \tag{2.12}$$

so that the sequence $\{(X_n, S_n)\}$ is ϕ-mixing.

There are many ways to cross-correlate the interarrival and service times using the EARMA scheme beyond those given above and in Lewis and Shedler [7]. We will present one more, somewhat more general than that of Example (2.7) in which the service times were coupled back over the previous interarrival times.

(2.13) EXAMPLE. Let $\{E_n\}$, $\{\mathcal{E}_n\}$, and $\{C_n\}$ be independent sequences of independent exponential random variables having respective positive finite means λ^{-1}, μ^{-1}, and 1. Let

$$X_n = \beta E_n + J_n(\lambda^{-1}A_n) \tag{2.14}$$

$$S_n = \gamma \mathcal{E}_n + I_n(\mu^{-1}A_n) \tag{2.15}$$

with

$$A_n = \rho A_{n-1} + K_n C_n \tag{2.16}$$

where β, ρ, $\{J_n\}$ and $\{K_n\}$ are as in (1.1) and (1.2) and $\{I_n\}$ is a sequence of independent binary random variables independent of everything with $P\{I_n=1\} = 1 - P\{I_n=0\} = (1-\gamma)$ for some fixed constant $0 \le \gamma \le 1$. Again we assume that A_0 is an independent random variable having an exponential distribution with mean 1. Then $\{(E_n, \mathcal{E}_n, A_n, I_n, J_n)\}$ is a stationary sequence of random variables and, hence, the same is true of $\{(X_n, S_n)\}$.

If $\gamma = 1$, $\beta = 1$ or $\gamma = 1$, $\beta = 0$, $\rho = 0$ or $\gamma = 0$, $\beta = 1$, $\rho = 0$, then the queue reduces to an M/M/1 queue. If $\beta = 1$, $\rho = 0$ then the queue reduces to a special case of Example (2.7). Straightforward manipulation shows that in general the interarrival times X_n are positively correlated with

(2.17) $$\mathrm{Cov}(X_n, X_{n+k}) = \lambda^{-2}(1-\beta)^2\rho^k, \qquad k = 1,2,\ldots.$$

The service times S_n are also positively correlated with

(2.18) $$\mathrm{Cov}(S_n, S_{n+k}) = \mu^{-2}(1-\gamma)^2\rho^k, \qquad k = 1,2,\ldots .$$

The interarrival times and service times are positively cross-correlated with

(2.19) $$\mathrm{Cov}(X_n, S_{n+k}) = (1-\beta)(1-\gamma)(\lambda\mu)^{-1}\rho^k, \; k = 0,1,\ldots ;$$

and

(2.20) $$\mathrm{Cov}(X_{n+k}, S_n) = (1-\beta)(1-\gamma)(\lambda\mu)^{-1}\rho^k, \; k = 0,1,\ldots .$$

Finally, by the arguments used in obtaining (6.5) in Jacobs and Lewis [3], if f and g are as in (2.6), then for k sufficiently large

(2.21) $$|E[fg] - E[f]\,E[g]| \leq 5\rho^{k/2}\, E[f^2]^{1/2}\, E[g^2]^{1/2} .$$

so that the sequence $\{(X_n, S_n)\}$ is ϕ-mixing.

3. A HEAVY TRAFFIC RESULT FOR EARMA QUEUES

Let $\{X_n\}$ (respectively $\{S_n\}$) be the sequence of interarrival times (respectively service times) for a single server queue with first-in-first-out service discipline.

Put $U_n = S_n - X_n$ and assume that $\{U_n\}$ is a stationary sequence with $E[U_n] < 0$. Let W_n be the waiting time of the nth customer and $W = \sup_{n \geq 0} \sum_{k=1}^{n} U_k$ (with the convention that $\sum_{n=1}^{0} = 0$). Loynes [9] has shown that if the sequence $\{U_n\}$ is metrically transitive, then the distribution of W_n tends in the limit to the distribution function of W. Since it is usually difficult to obtain the distribution of W analytically, particularly when the service and interarrival times are dependent, we will obtain a heavy traffic limiting result for it. In this section we will give the statement of the result as well as some examples of its use leaving the proof to the next section.

For each $\alpha > 0$ let Q_α be a single server queue with FIFO queueing discipline having stationary sequences of interarrival times $\{X_n(\alpha)\}$ with positive finite mean $\lambda(\alpha)^{-1}$ and service times $\{S_n(\alpha)\}$ with positive finite mean $\mu(\alpha)^{-1}$ such that, if $U_n(\alpha) = S_n(\alpha) - X_n(\alpha)$, then $\{U_n(\alpha)\}$ is a stationary sequence of random variables. Let

(3.1) $\quad E[U_n(\alpha)] = \nu(\alpha) < 0; \; \mathrm{Var}[U_n(\alpha)] = s^2(\alpha)$;

$$(3.2)\qquad \sigma^2(\alpha) = \operatorname{Var} U_1(\alpha) + 2\sum_{n=1}^{\infty} \operatorname{Cov}(U_1(\alpha), U_{1+n}(\alpha));$$

$$(3.3)\qquad d(\alpha) = |\nu(\alpha)|/\sigma(\alpha) \quad\text{and}\quad n(\alpha) = d(\alpha)^{-2}.$$

We will assume the following:

(3.4) ASSUMPTION. There exists a distribution function G so that $\lim_{\alpha\to\infty} P\{[U_n(\alpha) - \nu(\alpha)]^2 \le x\} = G(x)$ for all continuity points x of G; $\lim_{\alpha\to\infty} \nu(\alpha) = 0$; and $0 < \lim_{\alpha\to\infty} s^2(\alpha) = s^2 < \infty$.

(3.5) ASSUMPTION. $0 < \lim_{\alpha\to\infty} \sigma^2(\alpha) = \sigma^2 < \infty$.

(3.6) ASSUMPTION. If f (respectively g) is a function with finite second moment measurable with respect to the σ-algebra generated by $U_1(\alpha),\ldots,U_m(\alpha)$ (respectively $U_{m+k}(\alpha), U_{m+k+1}(\alpha),\ldots$), then

$$(3.7)\qquad |E[fg] - E[f]\,E[g]| \le \phi(k)\,E[f^2]^{1/2}\,E[g^2]^{1/2}$$

for some decreasing function ϕ bounded above by 1 not depending on α satisfying

$$(3.8)\qquad \sum_{n=1}^{\infty} \phi(n) < \infty.$$

Assumption (3.4) concerns the convergence of the distribution of $U_n^2(\alpha) = [S_n(\alpha)-X_n(\alpha)]^2$ and its first two moments as $\alpha\to\infty$. Assumption (3.6) is roughly a condition that the sequences $\{U_n(\alpha),\ n=1,2,\ldots\}$

be ϕ-mixing uniformly over all α. This assumption guarantees that the series in (3.2) converges and

$$\lim_{n \to \infty} \frac{1}{n} E[(\sum_{k=1}^{n} (U_k(\alpha) - \nu(\alpha)))^2] = \sigma^2(\alpha).$$

Assumption (3.5) concerns the convergence of this normalized variance term as $\alpha \to \infty$.

Put

$$Z_\alpha(t) = d(\alpha)\, \sigma(\alpha)^{-1} \sum_{k=1}^{[n(\alpha)t]} U_k(\alpha), \qquad t \geq 0$$

where $[x]$ is the largest integer less than or equal to x. Let $\mathbb{D}([0,\infty))$ be the collection of real-valued, right continuous functions on $[0,\infty)$ which have left limits everywhere endowed with the metric given in Lindvall [8].

(3.9) THEOREM. If (3.4)-(3.6) are satisfied, then the process $Z_\alpha = \{Z_\alpha(t); t \geq 0\}$ converges weakly in $\mathbb{D}([0,\infty))$ to a Brownian motion with negative unit drift.

The proof of this result will be given in the next section.

Let

$$W(\alpha) = \sup_{n \geq 0} \sum_{k=1}^{n} U_k(\alpha) .$$

Then it follows from Theorem (3.9) that

$$\lim_{\alpha \to \infty} P\left\{\frac{d(\alpha)}{\sigma(\alpha)} W(\alpha) \le x\right\} = \lim_{\alpha \to \infty} P\{\sup_{t \ge 0} Z_\alpha(t) \le x\}$$

$$= 1 - \exp\{-2x\}$$

where the last equality follows from the continuous mapping theorem (cf. Billingsley [1]). Thus we have the following result.

(3.10) THEOREM. If (3.4)-(3.6) hold, then

$$\lim_{\alpha \to \infty} P\{|\nu(\alpha)|\, W(\alpha) \le x\} = 1 - \exp\{-2\sigma^{-2}x\} .$$

Hence, if the traffic intensity $\lambda\mu^{-1}$ of a queue is less than but close to 1 and $\{U_n\}$ is a stationary sequence satisfying (3.7) and (3.8), then the distribution of W is approximately exponential with mean $-\frac{1}{2}\sigma^2 E[U_1]^{-1}$ where

$$\sigma^2 = \text{Var}[U_1] + 2 \sum_{n=1}^{\infty} \text{Cov}(U_1, U_{1+n}) .$$

Thus positive cross correlation between the interarrival and service times will tend to decrease the average mean waiting time; while positive correlation within the interarrival time or service time sequence will tend to increase the mean waiting time.

(2.11) **EXAMPLE.** For each $\alpha > 0$, let Q_α be a queue as in Example (2.1) with $\{X_n(\alpha)\}$ (respectively $\{S_n(\alpha)\}$) forming an EARMA(1,1) process with parameters $\lambda(\alpha)$, β_1 and ρ_1 (respectively $\mu(\alpha)$, β_2 and ρ_2). Assume $\lambda(\alpha) < \mu(\alpha)$ for each α, but $\lim_{\alpha\to\infty} \lambda(\alpha) = \lim_{\alpha\to\infty} \mu(\alpha) = m$; then hypotheses (3.4)-(3.6) are satisfied and Theorem (3.10) holds with

$$\sigma^2 = 2m^{-2}\{1 + (1-\rho_1)^{-1}(1-\beta_1)[(1-\rho_1)\beta_1 + (1-\beta_1)\rho_1]$$

$$+ (1-\rho_2)^{-1}(1-\beta_2)[(1-\rho_2)\beta_2 + (1-\beta_2)\rho_2]\} .$$

Hence, if the traffic intensity $\lambda\mu^{-1}$ of a queue with independent EARMA(1,1) services and interarrival times is less than but close to 1, then the distribution of the equilibrium waiting time W is approximately exponential with mean $-\frac{1}{2}\sigma^2 E[U_1]$ where $E[U_1] = \mu^{-1} - \lambda^{-1}$ and σ^2 is given by

$$\sigma^2 = \lambda^{-2}\{1 + 2(1-\rho_1)^{-1}(1-\beta_1)[(1-\rho_1)\beta_1 + (1-\beta_1)\rho_2]\}$$

$$+ \mu^{-2}\{1 + 2(1-\rho_2)^{-1}(1-\beta_2)[(1-\rho_2)\beta_2 + (1-\beta_2)\rho_2]\} .$$

Thus the dependence in this queue tends to increase the mean customer waiting time at least in close to heavy traffic conditions.

(3.12) EXAMPLE. For each $\alpha > 0$, let Q_α be a queue as in Example (2.7) with parameters $E[X_n(\alpha)] = \lambda(\alpha)^{-1}$, β, and ρ and $E[S_n(\alpha)] = \mu(\alpha)^{-1}$. Again we will assume $\lambda(\alpha) < \mu(\alpha)$ for each α but $\lim_{\alpha\to\infty} \lambda(\alpha) = \lambda = \lim_{\alpha\to\infty} \mu(\alpha) = \mu = m$. Hence (3.4)-(3.6) are satisfied and Theorem (3.10) holds with

$$\sigma^2 = \lambda^{-2} + \mu^{-2} - 2(1-\beta)(1-\rho)(\lambda\mu)^{-1}$$
$$+ 2\rho(1-\rho)^{-1} [(1-\beta)^2\mu^{-2} - (1-\beta)(1-\rho)(\lambda\mu)^{-1}]$$
$$= \lambda^{-2} + \mu^{-2}[1 + 2(1-\beta)^2 \rho(1-\rho)^{-1}] - 2(\lambda\mu)^{-1} (1-\beta)$$
$$= 2m^{-2}[1 + (1-\beta)^2 \rho(1-\rho)^{-1} - (1-\beta)] .$$

The quantity $\sigma^2 - \sigma^2_{ind}$ can be either positive or negative where $\sigma^2_{ind} = \lambda^{-2} + \mu^{-2} = 2m^{-2}$. It is negative if $\rho < (2-\beta)^{-1}$ and nonnegative otherwise. Hence, if ρ is sufficiently less than $(2-\beta)^{-1}$, then the average customer waiting time in the queue will tend to be less than the waiting time in the independent case at least in close to heavy traffic conditions. It will tend to be greater if ρ is sufficiently greater than $(2-\beta)^{-1}$.

(3.13) EXAMPLE. For each $\alpha > 0$, let Q_α be a queue as in Example (2.13) with parameters $\lambda(\alpha)$, $\mu(\alpha)$, ρ, β, and γ. Assume $\lim_{\alpha\to\infty} \lambda(\alpha) = \lambda = \lim_{\alpha\to\infty} \mu(\alpha) = \mu = m$ as before. Then (3.4)-(3.6) are satisfied and Theorem (3.10) holds with

$$\sigma^2 = \mu^{-2} + \lambda^{-2} - 2(\lambda\mu)^{-1}\,(1-\beta)(1-\gamma)$$
$$+ 2\rho(1-\rho)^{-1}\{\mu^{-1}(1-\gamma) - \lambda^{-1}(1-\beta)\}^2$$

$$= 2m^{-2}[1-(1-\beta)(1-\gamma) + \rho(1-\rho)^{-1}\{(\beta-\gamma)\}^2] .$$

The difference $\sigma^2 - \sigma^2_{ind}$ can be either positive or negative; ($\sigma^2_{ind} = \lambda^{-2} + \mu^{-2} = 2m^{-2}$). The difference will be negative if $\rho < (1-\beta)(1-\gamma)/[(1-\beta)(1-\gamma) + (\beta-\gamma)^2]$ and nonnegative otherwise. Hence under close to heavy traffic conditions, the average customer waiting time will tend to be less than the independent case if $(1-\beta)(1-\gamma)/[(1-\beta)(1-\gamma) + (\beta-\gamma)^2]$ is sufficiently greater than ρ. It will be greater than the independent case if the same quantity is sufficiently less than ρ.

4. PROOF OF THE MAIN RESULT

In this section we will give a proof of Theorem (3.9). We will assume that (3.4)-(3.6) are satisfied. Let

$$Y_\alpha(t) = \frac{d(\alpha)}{\sigma(\alpha)} \sum_{k=1}^{[n(\alpha)t]} [U_k(\alpha) - \nu(\alpha)] .$$

By Theorem 3' of Lindvall [8] and the continuous mapping theorem, Theorem (3.9) will be proved if for each $a > 0$ we can show the following result.

(4.1) THEOREM. The process $\{Y_\alpha(t);\ 0 \le t \le a\}$ converges weakly on $\mathbb{D}([0,a])$ to Brownian motion.

($\mathbb{D}([0,a])$ is the space of real-valued right continuous functions on $[0,a]$ with left-hand limits endowed with the Skorohod J_1 topology.)

We will prove Theorem (4.1) for $a = 1$ by a series of lemmas. The proof for arbitrary $a > 0$ is similar.

(4.2) LEMMA. The process $\{Y_\alpha(t);\ 0 \le t \le 1\}$ has asymptotically independent increments.

<u>Proof</u>. Let $0 \le s_1 \le t_1 < s_2 \le t_2 < \cdots < s_r \le t_r \le 1$. Let C_i be the event $\{Y_\alpha(t_i) - Y_\alpha(s_i) \le x_i\}$ for real x_i. Then C_i is an event in the σ-algebra generated by

$U_{[n(\alpha)s_i]+1}, \ldots, U_{[n(\alpha)t_i]}$. If δ is the smallest difference $s_i - t_{i-1}$, then $[n(\alpha)s_i] + 1 - [n(\alpha)t_{i-1}] \geq [n(\alpha)\delta]$. By (3.7)

$$|P(\bigcap_{i=1}^{r} C_i) - \prod_{i=1}^{r} P(C_i)| \leq r\phi([n(\alpha)\delta]) .$$

Since $\delta > 0$, $r\phi([n(\alpha)\delta])$ tends to zero as $\alpha \to \infty$ and the result follows.

In the following results, let $D_i(\alpha) = U_i(\alpha) - \nu(\alpha)$ and $V_m(\alpha) = \sum_{i=1}^{m} D_i(\alpha)$. By (3.4)-(3.6) the following result holds.

(4.3) LEMMA.

a) $\frac{1}{k} E[V_k(\alpha)^2] \leq 2 \sum_{j=0}^{\infty} |E[D_1(\alpha)\ D_{1+j}(\alpha)]|$;

b) $\lim_{\alpha \to \infty} n(\alpha)^{-1} E[V_{n(\alpha)}(\alpha)^2] = \sigma^2$;

c) If D_1 is bounded by C, then $E[V_{n(\alpha)}(\alpha)^4] \leq K_\phi C^4\ n(\alpha)^2$ where K_ϕ depends on ϕ alone.

(4.4) LEMMA. The sequence of random variables $\{n(\alpha)^{-1}(V_{n(\alpha)}(\alpha))^2;\ \alpha \geq 0\}$ is uniformly integrable.

Proof. By (3.4), $\{D_1^2(\alpha);\ \alpha \geq 0\}$ is uniformly integrable. The remainder of the proof is very similar to that of (20.48) on page 176 of Billingsley [1] and will be omitted.

(4.5) LEMMA. For each positive ε there exists a γ, $\gamma > \sigma$ and an integer n_0 such that $n(\alpha) \geq n_0$ implies

$$P\{\max_{i \leq n(\alpha)} |V_i(\alpha)| \geq \gamma\sqrt{n(\alpha)}\} \leq \frac{\varepsilon}{\gamma^2} .$$

<u>Proof</u>. Put $C_i^*(\alpha) = \sum_{j=1}^{i} |D_j(\alpha)|$. By (3.4), there exists an increasing sequence of integers m_i such that $n(\alpha) \geq m_i$ implies $P\{|D_1(\alpha)| \geq \gamma \sqrt{n(\alpha)}/i^2\} \leq (\gamma\, n(\alpha) i^2)^{-1}$ for each positive γ. Let $p(\alpha) = i$ if $m_i \leq n(\alpha) < m_{i+1}$; then $p(\alpha)$ tends to infinity as $\alpha \to \infty$ but

$$(4.6) \quad \lim_{\alpha \to \infty} n(\alpha)\, P\{C^*_{p(\alpha)}(\alpha) \geq \gamma\sqrt{n(\alpha)}\}$$

$$\leq \lim_{\alpha \to \infty} n(\alpha)\, p(\alpha)\, P\{|D_1(\alpha)| \geq \gamma \sqrt{n(\alpha)}\, p(\alpha)^{-1}\}$$

$$\leq \lim_{\alpha \to \infty} n(\alpha)\, p(\alpha)\, P\{|D_1(\alpha)| \geq \gamma \sqrt{n(\alpha)}\, p(\alpha)^{-2}\}$$

$$\leq \lim_{\alpha \to \infty} n(\alpha)\, p(\alpha)\, [\gamma\, n(\alpha)\, p(\alpha)^2]^{-1} = 0 .$$

We can further choose $p(\alpha)$ so that $p(\alpha) \leq n(\alpha)$. Fix $\varepsilon > 0$. By the previous lemma we can choose γ so that $\gamma > \sigma$ and

$$P\{\max_{i \le n(\alpha)} |V_i(\alpha)| \ge 3\gamma \sqrt{n(\alpha)}\}$$

$$\le P\{|V_{n(\alpha)}(\alpha)| \ge \gamma \sqrt{n(\alpha)}\}$$

$$+ [n(\alpha) + p(\alpha)]\, P\{C^*_{p(\alpha)}(\alpha) \ge \gamma \sqrt{n(\alpha)}\}$$

$$+ \sum_{i=1}^{n(\alpha)-p(\alpha)-1} P(\Lambda_i(\alpha)[P\{|V_{n(\alpha)}(\alpha) - V_{i+p(\alpha)}(\alpha)| \ge \gamma \sqrt{n(\alpha)}\} + \phi(p(\alpha))]$$

$$\le 2\varepsilon\gamma^{-2} + \phi(p(\alpha)) + [n(\alpha) + p(\alpha)]\, P\{C^*_{p(\alpha)}(\alpha) \ge \gamma \sqrt{n(\alpha)}\}$$

by (4.7) and the fact that the $\Lambda_i(\alpha)$ are disjoint. Hence by (4.6) and (3.8) for α large

$$P\{\max_{i \le n(\alpha)} |V_i(\alpha)| \ge 3\gamma \sqrt{n(\alpha)}\} \le 3\varepsilon\gamma^{-2}.$$

PROOF OF THEOREM (4.1). For $a = 1$: By Lemma (4.5) the sequence $\{Y_\alpha\}$ is tight on $\mathbb{D}[0,1]$. Further $\lim_{\alpha\to\infty} E[Y_\alpha(t)] = 0$ and

$$\lim_{\alpha\to\infty} E[Y_\alpha(t)^2] = \lim_{\alpha\to\infty} (\sqrt{n(\alpha)}\,\sigma(\alpha))^{-2}\, E\left[\left(\sum_{k=1}^{[n(\alpha)t]} D_k(\alpha)\right)^2\right] = t$$

by (3.5). Since $Y_\alpha(t)^2 = [\sigma^2(\alpha)\, n(\alpha)]^{-1}\, V_{[n(\alpha)t]}(\alpha)^2$, $\{Y^2_\alpha(t)\}$ is uniformly integrable for each t by Lemma (4.4). By Lemma (4.2) $\{Y_\alpha(t);\ 0 \le t \le 1\}$ has asymptotically independent increments. The result now follows from Theorem (19.2) of Billingsley [1].

$$(4.7) \qquad P\{|V_{k(\alpha)}(\alpha)| > \gamma\sqrt{k(\alpha)}\} < \varepsilon\gamma^{-2}$$

for all α. Let

$$\Lambda_i(\alpha) = \{\max_{j<i} |V_j(\alpha)| < 3\gamma\sqrt{n(\alpha)} \le |V_i(\alpha)|\} \; ;$$

then

$$P\{\max_{i \le n(\alpha)} |V_i(\alpha)| \ge 3\gamma\sqrt{n(\alpha)}\}$$

$$\le P\{|V_{n(\alpha)}(\alpha)| \ge \gamma\sqrt{n(\alpha)}\}$$

$$+ \sum_{i=1}^{n(\alpha)-1} P(\Lambda_i(\alpha) \cap \{|V_{n(\alpha)} - V_i(\alpha)| \ge 2\gamma\sqrt{n(\alpha)}\}) \; .$$

The summation is bounded above by

$$\sum_{i=1}^{n(\alpha)-p(\alpha)-1} P\{|V_i(\alpha) - V_{i+p(\alpha)}(\alpha)| \ge \gamma\sqrt{n(\alpha)}\}$$

$$+ \sum_{i=1}^{n(\alpha)-p(\alpha)-1} P(\Lambda_i(\alpha) \cap \{|V_{n(\alpha)}(\alpha) - V_{i+p(\alpha)}(\alpha)| > \gamma\sqrt{n(\alpha)}\})$$

$$+ \sum_{i=n(\alpha)-p(\alpha)}^{n(\alpha)-1} P\{|V_{n(\alpha)}(\alpha) - V_i(\alpha)| \ge \gamma\sqrt{n(\alpha)}\} \; .$$

Each term in the first and third of these sums is bounded above by $P\{C^*_{p(\alpha)}(\alpha) \ge \gamma\sqrt{n(\alpha)}\}$. Since $\Lambda_i(\alpha)$ is measurable with respect to the σ-algebra generated by $U_1(\alpha),\ldots, U_i(\alpha)$ by (3.7)

5. EXTENSIONS

This paper presented a heavy traffic result for the limiting distribution of the customer waiting time in single server queues with dependent interarrival and service times. There are many different ways in which to model dependent interarrival and service times in queues using the EARMA scheme. Some other ways are given in Lewis and Shedler [7]. Properties of these multivariate exponential models will be given elsewhere.

Work is currently going on in the simulation of these queues. Some simulation results have been reported for the queue of Example (2.7) with $\beta = 0.5$, $\mu = 4$ and various values of ρ and λ in Boonsong [2]. They seem to indicate that the heavy traffic result approximates the average customer waiting time fairly well for a traffic intensity of .9 and quite well for a traffic intensity of .95. However, the distribution of W in a simulation with $\rho = 0.25$ and traffic intensity .995 shows a large amount of underdispersion relative to an exponential distribution. Thus, it seems that the convergence of the waiting time distribution is very slow.

It is expected that further simulations will give more indication of the validity of using the heavy traffic result to approximate the average customer waiting time, or quantiles of the waiting time distribution under less than heavy traffic conditions. These results will be reported on elsewhere.

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