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EQUIVALENT CIRCUIT ANALYSIS
OF A WT-2 PIEZOELECTRIC TRANSDUCER

by
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UNCLASSIFIED

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INTRODUCTION

An equivalent circuit is invaluable in the design of a sound source and in the prediction of its behavior when subject to an acoustic load.

A piezoelectric transducer may actually be represented by means of two different equivalent circuits each of which will have the same impedance characteristics as the source.

The attached computations show in detail how to compute the parameters of the two equivalent circuits for the WT-2 (1200) piezoelectric transducer designed by P. Weber. The impedance of one of the two equivalent circuits is evaluated at various frequencies for the unloaded condition and the impedance loop thus obtained is compared with the source in-air measurements. Adjustments are made in the circuit to make its impedance match the actual source characteristics as closely as possible.

The circuit is then modified by the addition of the theoretical acoustic water load and the impedances of this circuit are compared with measurements on the source in water.

Good agreement between theory and measurements was obtained in the analysis of the WT-2 (1200) and WT-2 (400) transducers.
ANALYSIS OF WT-2 (1200)

MASS = 12.8" = 5.8 kg

3 - PZT-4 CRYSTALS IN PARALLEL
1/8" WALK 1/2" OD, 21/4" LONG

K = 1300  k_31 = .33

Y_{R} = 15 \times 10^{6} \text{ psi} = 10.3 \times 10^{10} \text{ N/m}^{2}

Y_{SC} = 8.2 \times 10^{6} \text{ N/m}^{2}

PARAMETERS FOR EQUIVALENT CIRCUITS:

\[ C_{E} = n K_{fr} \epsilon \frac{A}{t} = 3 \cdot 1300 \cdot 8.85 \times 10^{-12} \frac{\pi (1.25) 2.25}{\frac{1}{8} \cdot 39.4} = 0.0745 \mu F \]

\[ C'_{M} = \frac{k}{A V_{SC}} = \frac{6.75 \times 39.4 \cdot 4}{\pi (1.50^2 - 1.25^2) 8.2 \times 10^{-10}} = 0.0060 \cdot 10^{-6} \text{ N/m} \]

\[ C_{R} = \frac{k}{A V_{R}} = \frac{8.75 \times 39.4 \cdot 4}{\pi (3.8^2) 10.3 \cdot 10^{-10}} = 0.0304 \cdot 10^{-6} \text{ N/m} \]

\[ C_{S} = \frac{C_{E} C_{M}'}{C_{E} + C_{M}'} = \frac{0.0304 (0.0060) \cdot 10^{-6}}{0.0364} = 0.0050 \cdot 10^{-6} \text{ N/m} \]

\[ a = \frac{C_{S}}{C_{M}'} = \frac{0.0304}{0.0060} = 5.06 \]

\[ \frac{k^2}{1 - k^2} = \frac{a}{a + 1} \left( \frac{k^2}{1 - k^2} \right) \quad k' = 0.303 \]

This value of k is theoretical. Experience has shown that actual value is lower. One must know k to solve for C_E', C_M', N and N.
If the air impedance loop of the source is available one may solve for the actual $k$ directly from the indicated antiresonance (max impedance point if damping is small). This will be done below.

The two equivalent circuits of the transducer in air are given below:

\[
C_e = k_{105} + \frac{A}{e} (\mu_0)
\]

\[
C_e' = \frac{k_e e A (1-k^2)}{(\mu_0)}
\]

Eliminating the transformers and combining $C_m$ and $C_r$ into $C_s$ we obtain:
APPLICABLE RELATIONS FOR ABOVE CIRCUITS:

\[
\begin{align*}
N^2 &= \frac{C_e - C_e'}{C_{m}} \\
N^2 &= \frac{C_m - C_m'}{C_e} \\
N^1 N^2 &= k^4
\end{align*}
\]

Both circuits are identical when viewed from the terminals and have the same resonance and antiresonance:

Resonance (if \( R_L \) is small) is the minimum impedance point. At that frequency \( C_S N^2 \) resonates with \( \frac{M}{N^2} \):

Thus:

\[
\frac{1}{\omega_C C_S N^2} = \omega_r M N^2
\]

\[
\omega_r = \frac{1}{\sqrt{C_S M}} = \sqrt{\frac{1}{C_m} + \frac{1}{C_r}}
\]

Note: \( \omega_r \) does not depend on \( k \)

Antiresonance (if \( R_L \) is small) is the maximum impedance point. At that frequency \( C_S N^2 \) and \( \frac{M}{N^2} \) are in parallel resonance with \( C_e' \):

Thus:

\[
\frac{1}{\omega_C C_e'} = \frac{w_r M}{N^2} - \frac{1}{\omega_C C_S N^2}
\]

And:

\[
\omega_r = \sqrt{\frac{1}{N^2 \left[ \frac{1}{C_i} + \frac{1}{C_S N^2} \right]}} = \sqrt{\frac{1}{C_m (1-k^2)} + \frac{1}{C_r}}
\]

Note: \( \omega_r \) depends on \( k \)

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Thus for this source

Resonance

\[ M = \frac{58}{2} \times 4 \]

\[ \omega_r = \frac{1}{\sqrt{2.9 \times 0.0050 \times 10^{-6}}} = 8300; f_r = 1320 \text{ cps} \]

Actual = 1318 cps

Anti-resonance

\[ \omega_a = \sqrt{\frac{2.9}{10^4 \times 0.0060 (1-k^2) + 0.0304}} = 8650; f_a = 1380 \text{ cps} \]

Actual = 1335 cps

Value of \( k \) must be lowered to obtain better agreement.

\[ \sqrt{\frac{10^4}{0.0060 (1-k^2)} + \frac{10^6}{0.0304}} = 8400 \]

\[ k = 0.152 \]

Circuit parameters:

\[ C_E = 0.0745 \mu F \]

\[ C'_E = C_E (1-k^2) = 0.0726 \mu F \]

\[ C_M = 0.0060 \times 10^{-6} \frac{M}{\text{newt}} \]

\[ C'_M = C_M (1-k^2) = 0.0585 \times 10^{-6} \frac{M}{\text{newt}} \]

\[ C_R = 0.0304 \times 10^{-6} \frac{M}{\text{newt}} \]

\[ C_S = 0.0050 \times 10^{-6} \frac{M}{\text{newt}} \]

\[ N'^2 = \frac{C_E - C'_E}{C'_M} = \frac{0.0745 - 0.0726}{0.0060} = 0.317 \frac{\text{newt}^2}{\text{volt}^2} \]

\[ N^2 = \frac{N'^2}{k^2} = 0.00168 \frac{\text{volt}^2}{\text{newt}^2} \]
At antiresonance the measured air impedance circle dia = 5000 $\Omega$. From this information one can solve for $R_L$. For the first circuit it can be shown that at antiresonance the resistive component is

$$\frac{Z_s^2 + (R_L N^2)^2}{R_L N^2} = \frac{Z_s^2}{R_L N^2} = 5000 \Omega$$

$$Z_s = \omega (0.00487 - \frac{10^6}{\omega \cdot 18.1}) \quad Z_{s,1335} = 34.44 \Omega \quad R_L N^2 = 237 \Omega$$

$$R_L = 141 \frac{k\Omega}{\sec}$$

![Circuit Diagram](image)

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<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$Z_s$</th>
<th>$Z_{CM}$</th>
<th>$\left(\frac{R}{R^2 + Z_s^2}\right)$</th>
<th>$\frac{Z_s}{R^2 + Z_s^2}$</th>
<th>$\left(\frac{1}{Z_{CM} - Z_s - j\omega C}\right)$</th>
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These points agree well with measurements.
WT-2 (1800) IN WATER

\[ \lambda @ 1200 = 4' \]

\[ k = \frac{\pi}{4} \frac{3}{12} = 0.393 \sim \] 

\[ Z_{ac} = \rho \sigma A \left( 0.075 + j \cdot 3.25 \right) \]

\[ Z_{ac} = 1500 \times 1000 \frac{\pi}{4} \frac{(3)^2}{(39.4)^2} \left( 0.075 + j \cdot 3.25 \right) \]

\[ = 2050 \ \frac{\text{kg}}{\text{sec}} + j \cdot 8890 \ \frac{\text{kg}}{\text{sec}} \]

\[ \lambda @ 1200 \psi = \frac{8890}{2 \pi (1200)} = 1.18 \]

EQUIVALENT CIRCUIT IN WATER

\[ M = 2.9 + \frac{1.18}{2} = 3.49 \ \text{kg} \]

\[ R_L + R_{ac} = 141 + \frac{2050}{2} = 1166 \ \frac{\text{kg}}{\text{sec}} \]

**Antiresonance:**

\[ \omega_{ar} \left( 0.00586 \right) = \frac{10^6}{\omega_{ar} 18.1} + \frac{10^6}{3.48} \]

\[ \omega_{ar} = 7650 \ \text{far} = 1220 \]

**Circle Dia:**

\[ Z_s = 0.00586 \left( 7650 \right) - \frac{10^6}{7650 \left( 18.1 \right)} = 37.7 \]

\[ \frac{Z_s^2}{R} = \frac{(37.7)^2}{1.96} = 725 \ \Omega \]

\[ \omega_{r} = \frac{1}{\sqrt{11.0 \left( 0.00586 \right) 10^6}} \]

\[ f_r = 1205 \ \text{cps} \]

**This agrees well with measurements**
<table>
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<tr>
<th>f</th>
<th>Zs</th>
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TYP WATER LOOP

TYP AIR LOOP

TYP MEASURED LOOPS

WT-2 (1200)
SL 10-12-64