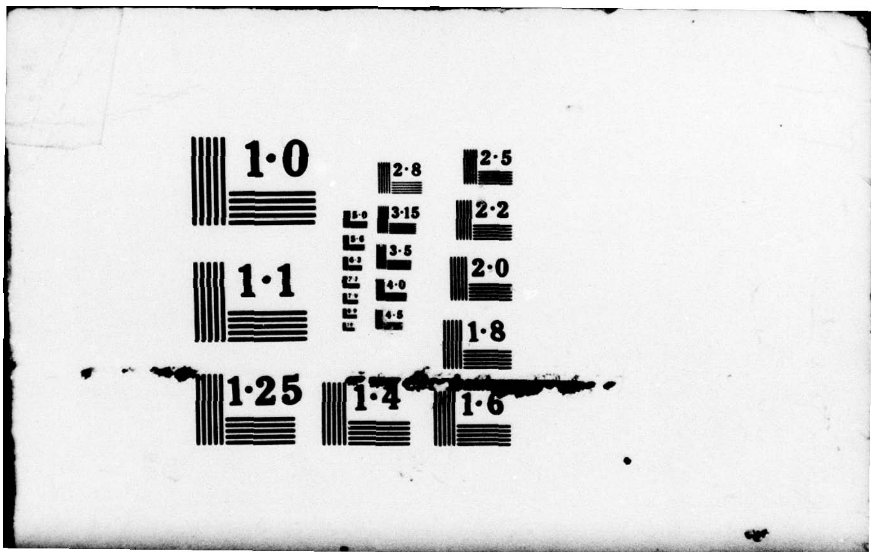
AD-A067 506 BROWN UNIV PROVIDENCE R I DIV OF APPLIED MATHEMATICS F/G 20/4
MOTION OF A VORTEX BLOTCH,(U)
NOV 78 C H SU N00014-76-C-0924
UNCLASSIFIED NL

| OF |
ADA
067506

END
DATE
FILMED
6-79
DDC


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·2
2·5
2·8
3·15
3·5
4·0
4·5

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LEVEL II (12)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Motion of a Vortex Blotch		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) C. H. Su		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-76-C-0924
9. PERFORMING ORGANIZATION NAME AND ADDRESS Div. of Applied Mathematics Brown University Providence, R.I. 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR062-555
11. CONTROLLING OFFICE NAME AND ADDRESS Fluid Dynamics Branch Office of Naval Research Arlington, VA. 22217		12. REPORT DATE (11) Nov[redacted] 1978
		13. NUMBER OF PAGES 7
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (12) 8p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) Preprint to the Physics of Fluids		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Vortex Invariant motion		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We derive an equation for invariant curves in the motion of a vortex blotch. Kirchhoff's elliptical vortex is shown to satisfy the equation exactly. We also find that perturbations to a circular vortex are invariant configurations to the (third) order of approximation calculated here.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Motion of a Vortex Blotch

C. H. Su
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

Abstract

We derive an equation for invariant curves in the motion of a vortex blotch. Kirchhoff's elliptical vortex is shown to satisfy the equation exactly. We also find that perturbations to a circular vortex are invariant configurations to the (third) order of approximation calculated here.

Motion of a Vortex Blotch

by

C. H. Su
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

We consider the motion of vortex blotches, a two dimensional incompressible flow with zero vorticity except inside certain closed curves. The streamfunction of such a flow is[1]

$$\psi(\underset{\sim}{x},t) = +\frac{1}{2\pi}\iint d^2x_1\ \omega(\underset{\sim}{x};t)\ln|\underset{\sim}{x}-\underset{\sim}{x}_1| \quad (1)$$

where t for time and $\underline{x}$ denotes a two-dimensional position vector, and the integral is over areas where the vorticity $\omega(\underset{\sim}{x},t)$ is different from zero. We consider, for simplicity, such areas as a single closed curve C of constant vorticity within. The velocity field is then given by

$$\underset{\sim}{u}(x,t) = -\frac{\omega}{2\pi}\oint_C \hat{t}(\underset{\sim}{x}_1)\ \ln|\underset{\sim}{x}-\underset{\sim}{x}_1|d\ell_1 \quad (2)$$

where $\hat{t}$ is a unit tangent counter-clockwise along the curve. Since fluid particles on the curve C remain there all the time, the motion of a vortex blotch is completely described by the motion of these particles. Denoting the position vector of points on curve C by $\underset{\sim}{x} = [x(t,s),y(t,s)]$, where s is a parametric coordinate on C, we have the equation of motion as

$$\frac{\partial \underset{\sim}{x}}{\partial t} = -\frac{\omega}{2\pi}\oint_C \hat{t}(\underset{\sim}{x}_1)\ln|\underset{\sim}{x}-\underset{\sim}{x}_1|ds_1 \quad (3)$$

where $\underset{\sim}{x}_1 = [x(t,s_1),y(t,s_1)]$.

It can be shown, for any C that the following integrals along C are constant in time :

$$\text{Area within} = \frac{1}{2}\oint (x\,dy - y\,dx),$$

$$\text{Centroid} = \left(\frac{1}{2}\oint x^2 dy,\ -\frac{1}{2}\oint y^2\,dx\right),$$

$$\text{Moment of inertia} = \frac{1}{3}\oint (x^3 dy - y^3 dx).$$

In this note we concern ourselves with the question of invariant curves, i.e. those curves which do not change their shape in time. The motion of aninvariant curve must be a combination of solid translation, solid rotation, and sliding along the curve. The translational motion is prohibited by the invariant of centroid as listed herein. The condition for invariant curves can be expressed as in a certain rotating frame of reference, the normal velocity of every point on the curve should vanish. Mathematically this becomes, after using (3) in a rotating frame of reference about the centroid of C with an angular speed Ω,

$$\Omega \hat{t}(\underset{\sim}{x}) \cdot \underset{\sim}{x} = \frac{\omega}{2\pi} \oint \ln |\underset{\sim}{x} - \underset{\sim}{x}_1| \, \hat{n}(\underset{\sim}{x}) \cdot \hat{t}(\underset{\sim}{x}_1) ds_1 \qquad (4)$$

when $\hat{n}$ is an outward unit normal.

In terms of the parametric representation of the curve

$$x = x(s)\ ,\quad y = y(s)\ \text{ and } x_1 = x(s_1)\ ,\ y_1 = y(s_1)$$

we have

$$\lambda \frac{d}{ds}(x^2+y^2) = \oint \ln| \underset{\sim}{x} - \underset{\sim}{x}_1| \, (\dot{y}\dot{x}_1 - \dot{x}\dot{y}_1) ds_1 \qquad (5)$$

where a dot denotes differentiation with its argument and $\lambda = \Omega\pi/\omega$.

Equation (5) takes the form of the eigenvalue problems; all the eigenfunctions represent the possible invariant curves, and the corresponding eigenvalue give the speed of the rotating frame wherein the curve remains fixed in time.

Equation (5) is invariant under a uniform stretching, and a solid rotation. Therefore the size and the orientation of the figure are immaterial as one would expect. It is, however, not a translational invariant. The origin ($x = 0$, $y = 0$) has been taken at the centroid of the figure. This introduces two integral constraints

$$\oint x^2 \, dy = 0 \quad \text{and} \quad \oint y^2 \, dx = 0 \tag{6}$$

to the solution of (5).

We are looking for periodic solutions of $x(s)$ and $y(s)$. It is natural to express these in Fourier series.

$$\begin{aligned} x(\theta) &= \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) , \\ y(\theta) &= \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta) , \end{aligned} \tag{7}$$

where we choose θ instead of s, ranges from 0 to 2π as the parameter. The logarithmic term under the integral sign of (5) make the integral quite difficult to evaluate. However, if one takes only one value of n in (7), one can solve (5) explicitly. Now for a single value of n, Eq. (7) represents an ellipse. Realizing that (5) is indifferent to the orientation we take (7) for a single value of n to be simply

$$x = a \cos \theta , \quad y = b \sin \theta .$$

This solves (5) provided $\lambda = \pi ab/(a+b)^2$. The speed of rotating frame of reference is then given by

$$\Omega = \frac{\omega ab}{(a+b)^2}, \tag{8}$$

which checks with Kirchhoff's solution of elliptical vortex[1].

In a polar coordinate, a curve is represented by $r = r(\theta)$, and Eq. (5) can be put into the

$$2\lambda r \frac{dr}{d\theta} = \frac{1}{2r}\frac{dr}{d\theta}\int_0^{2\pi} d\phi\, r^2(\phi)$$

$$- \frac{d}{d\theta}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\frac{1}{r^{n-1}}\int_0^{2\pi} d\phi\, r^{n+1}(\phi)\cos(n-1)(\theta-\phi), \tag{9}$$

where we have used a series expansion for the logarithmic term in (5) as follows:

$$\ln[r^2 + r_1^2 - 2rr_1\cos\mu]^{1/2} = \ln r - \sum_{n=1}^{\infty}\left(\frac{r_1}{r}\right)^n \frac{\cos n\mu}{n}, \tag{10}$$

with $r = r(\theta)$, $r_1 = r(\phi)$ and $\mu = \theta-\phi$.

We next consider a perturbation solution to (9) corresponding to some small distortion of a circular vortex. It is found that the solution can be represented as follows:

$$r(\theta) = a_o+\varepsilon a_1 \cos m\theta+\varepsilon^2 a_2 \cos 2m\theta+\varepsilon^3 a_3 \cos 3m\theta+0(\varepsilon^4) \qquad (11)$$

$$\lambda = \lambda_o+\varepsilon^2\lambda_2 \qquad (12)$$

where m is an integer taking to be greater than one in order for (11) to satisfy (6). We consider that $\varepsilon << 1$ so that (11) represents some small distortion of a circular vortex. Substituting (11) and (12) into (9), and equating the coefficient of each power of ε (up to ε^3 in the present case) to zero. We obtain

$$\lambda_o = \frac{\pi(m-1)}{2m} \qquad (13)$$

$$\lambda_2 = -\frac{\pi(m-1)}{4}\left(\frac{a_1}{a_o}\right)^2 \qquad (14)$$

$$\frac{a_2}{a_o} = \frac{2m-1}{4}\left(\frac{a_1}{a_o}\right)^2 \qquad (15)$$

$$\frac{a_3}{a_o} = \frac{(3m-1)(m-1)}{8}\left(\frac{a_1}{a_o}\right)^3 \qquad (16)$$

The first expression above checks with the calculation for the infinitesimal perturbation of a circular vortex as given in Lamb[1]. The approximate invariant as given by (11) and (12) can be improved by adding higher terms in ε. A systematic procedure for higher order invariant will be presented elsewhere.

References

1. H. Lamb, Hydrodynamics (Dover, New York, 1932), 6th Ed. pp. 219 and 232.

2. O. D. Kellogg, Foundations of Potential Theory. (Dover, New York, 1953), p. 145.

Acknowledgment

This research was sponsored by the Office of Naval Research, Fluid Dynamics Program and the National Science Foundation ENG 77-00982.