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A DETERMINATION OF THE FIRST TWO MOMENTS OF THE ALE OUTPUT DETE--ETC(U)
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Technical Report 330

A DETERMINATION OF THE FIRST TWO MOMENTS OF THE ALE OUTPUT DETECTOR

M Shensa
(Hydrotronics, Incorporated)

30 November 1978

Final Report: July - September 1978

Prepared for
Naval Electronic Systems Command

NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

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AN ACTIVITY OF THE NAVAL MATERIAL COMMAND

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ADMINISTRATIVE INFORMATION

This report was prepared by Hydrotronics, Incorporated, under contract number N00123-76-C-1937. Work for this report was performed from July through September 1978. J. R. Zeidler, Head, Analysis Branch of the Fleet Engineering Department acted as Contract Monitor. The work presented in this report was part of NOSC Project Number 62711N-F11101-XF-111-011-00.

Released by
R. H. HEARN, Head
Electronics Division

Under authority of
D. A. KUNZ, Head
Fleet Engineering Department

ACKNOWLEDGMENT

The author wishes to thank J. R. Zeidler and J. T. Rickard for their many helpful comments.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NOSC TR-330	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A DETERMINATION OF THE FIRST TWO MOMENTS OF THE ALE OUTPUT DETECTOR		5. TYPE OF REPORT & PERIOD COVERED Final Report. July - Sept[redacted]978
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) M. Shensa		8. CONTRACT OR GRANT NUMBER(s) N00123-76-C-1937
9. PERFORMING ORGANIZATION NAME AND ADDRESS Hydrotronics, Inc. San Diego, California 92106		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62711N F11101 XF-111-011-00
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Ocean Systems Center San Diego, California 92152	39 p.	12. REPORT DATE 30 Nov[redacted]978
		13. NUMBER OF PAGES 37
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Naval Electronic Systems Command Washington, D.C. 20360		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Adaptive line enhancer
Discrete Fourier transform

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report deals with a computation of the first two moments of the output distribution of the adaptive line enhancer (ALE) detector. Its primary motivation is to provide a means of determining the detection characteristics of the integrated spectral output. Previous results have relied on assumptions regarding the output statistics themselves; i.e., that they be chi-square distributions. In this work, only two assumptions are made: that the input to the detector be a sinusoid in white Gaussian noise and that the discrete Fourier transform (DFT) length and ALE filter length be considerably smaller than tau, the adaptive time constant of the ALE filter.

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SUMMARY

OBJECTIVE

Determination of the first two moments of the ALE output detector statistics for sinusoids in white noise.

RESULTS

Moments were determined for the output, short-term integrated output, and long-term integrated output power spectra.

RECOMMENDATIONS

These results may be implemented on a computer to calculate ROC curves for the ALE output detector and thus aid in the determination of parameters such as μ, filter length, and FFT resolution.

CONTENTS

INTRODUCTION

This paper consists, basically, of a computation of the first two moments of the output distribution of the adaptive line enhancer (ALE) detector (figures 1 and 2). Its primary motivation is to provide a means of determining the detection characteristics of the integrated spectral output.

Previous results [1] have relied on assumptions regarding the output statistics themselves; i.e., that they be chi-square distributions. Although such an assumption greatly simplifies the analysis, it introduces an artificial relationship between the first two moments, which prevents a rigorous application even in the limiting Gaussian case typical of post-detection integration. In the present work we shall make only two assumptions; that the input to the detector be a sinusoid in white Gaussian noise, and that the discrete Fourier transform (DFT) length and ALE filter length be considered smaller than τ, the adaptive time constant of the ALE filter.

We then compute the moments of the ALE detector for three cases: (a) a single power spectrum; (b) a short term average of the output (integration time $< \tau$); and (c) long-term average (integration time $> \tau$). The distinction between the two types of averaging is related to the behavior of the converged ALE filter which is nearly constant over periods of time less than τ [2].

The above results are independent of any assumptions on the output statistics. If a model for the distribution is known, for example empirically, the receiver operating characteristic (ROC) of the detector may be computed. In particular, for second order distributions, the ROC curves are determined by the first two moments as follows. Let ρ_0 and ρ_1 be the normalized probability densities (mean zero and variance 1) corresponding to the output under the hypotheses signal, absent and signal present, respectively. Also, let μ_0, μ_1, σ_0^2 and σ_1^2 be their means and variances. Then the probabilities of detection and false alarm are given by

$$P_F = \int_{\frac{T-\mu_0}{\sigma_0}}^{\infty} \rho_0$$

$$P_D = \int_{\frac{T-\mu_1}{\sigma_1}}^{\infty} \rho_1 \quad . \qquad (1)$$

For example, in the case of a Gaussian distribution (which occurs for sufficently large integration times) for a given false alarm rate, the threshold, α, may be found in standard tables. Then

$$\frac{T - \mu_0}{\sigma_0} = \alpha \ , \tag{2}$$

which implies

$$\frac{T - \mu_1}{\sigma_1} = \frac{\alpha \sigma_0 + \mu_0 - \mu_1}{\sigma_1} \tag{3}$$

and P_D is also determined from tables.

MOMENTS OF INTEGRATED OUTPUT

Let y be the output of the detector pictured in figure 1 given by

$$y = |\mathfrak{F}(w * x)|^2 \tag{4}$$

where w and x are the weight vector and input vector of the ALE, * denotes convolution, and $\mathfrak{F}$ indicates a K point discrete Fourier transform (DFT) at frequency ω. For a sinusoid input,

$$x_\ell(k) = A \cos \omega (k - \ell) + n(k) \ . \tag{5}$$

The converged weight vector may be written [1,2,3]

$$w_\ell(k) = B \cos (\omega\ell + \phi) + n'_\ell(k) \ , \tag{6}$$

where n(k) is a Gaussian white noise process and $n'(k)$ is an independent stationary Gaussian vector process with a correlation time of the order τ. The individual components n'_ℓ are independent identically distributed random variables.

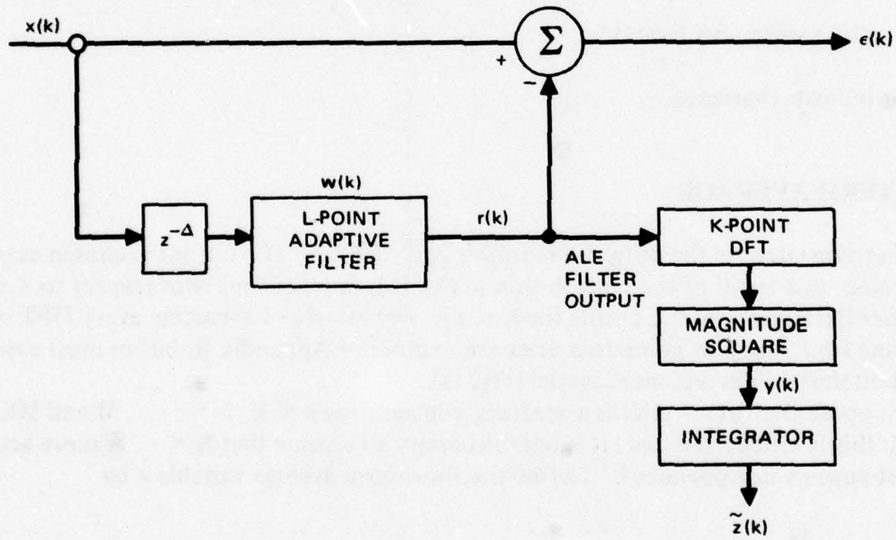

Figure 1. Block diagram of the ALE output detector.

The first moment of y was computed in [1]. The computation of the second moment is straightforward, although extremely tedious, and has been relegated to the Appendix. We therefore address ourselves here to a discussion of the effects of averaging. To do this, it is convenient to separate the effects of x and w through the use of conditional expectations.

Let $E(\cdot|w)$ denote conditional expectation, given w. Define

$$v = E(y|w)$$

$$\mu_y = E(y) \ . \tag{7}$$

Then v is a random variable dependent only on w and

$$E(v) = E(y) = \mu_y \ . \tag{8}$$

It follows that $E(y - v) = 0$, and

$$E\left(v(y-v)\right) = E\left[E\left(v(y-v)|w\right)\right] = E\left[vE(y-v|w)\right] = 0 \ .$$

Thus we may write

$$y = (y - v) + v$$

and

$$\text{var}(y) = \text{var}(y - v) + \text{var}(v) \ , \tag{9}$$

where var indicates variance.

SHORT-TERM AVERAGE

Let y(m) denote the output due to the $2m^{th}$ DFT.† The output is chosen every other DFT because, as a result of the convolution in (4), it is independent with respect to x only every other DFT (every K + L points for $K < L$). The effects of averaging every DFT or of overlapping DFT's may be computed after the manner of Appendix B, but in most cases of practical interest will be inconsequential (cf(21)).

Suppose that $w(k + mK)$ is essentially constant for $k \leqslant K$, $m = 1, \ldots, M$ and $MK < \tau$. Although this is usually the case, it is only necessary to assume that $K < \tau$. A more accurate treatment appears in Appendix B. Define the short-term average variable z by

$$z = \frac{1}{M} \sum_{m=1}^{M} y(m)$$

$$= \frac{1}{M} \sum_{m=1}^{M} \left(y(m) - v\right) + v \ , \tag{10}$$

---

† actually $m \cdot \max(2K, K + L)/K$

which follows since v depends only on w and hence is independent of m. Let us determine the variance of the first term in (10):

$$E\left(\frac{1}{M}\Sigma\ (y(m)-v)^2\right) = \frac{1}{M^2}\sum_{m,r} E\,(y(m)-v)(y(r)-v)$$

$$= \frac{1}{M^2}\, E\left[\sum_{m,r} E\left((y(m)-v)(y(r)-v)|w\right)\right]$$

$$= \frac{1}{M^2}\, E\left[\sum_{m,r} \delta_{m,r}\, E\left((y(m)-v)^2|w\right)\right]$$

$$= \frac{1}{M}\, E(y-v)^2\ . \qquad (11)$$

Since the first term of (10) has zero mean, and since $E(v(y(m)-v)) = 0$, equations (10) and (11) yield

$$\mathrm{var}\,(z) = \frac{1}{M}\,\mathrm{var}\,(y-v) + \mathrm{var}\,(v)\ . \qquad (12)$$

From (9) and (12)

$$\mathrm{var}\,(z) = \mathrm{var}\,(y) + \left(\frac{1}{M} - 1\right)\,\mathrm{var}\,(y-v)\ . \qquad (13)$$

VARIANCE OF y–v

We now express the second term in (13) in a form suitable for computation. Since $E(y-v) = 0$,

$$\mathrm{var}\,(y-v) = E(y-v)^2$$

$$= E\left[E\left((y-v)^2\ |w\right)\right]$$

$$= E\,[\mathrm{var}_x(r)] \qquad (14)$$

where r is equal to y, considered as a random variable in x with w as a parameter. Note that the "mean" of r is $E(y|w) = v$, and both its "mean" and "variance" are functions of the random variable w. The subscript x is included to avoid confusion with expectation taken with respect to the entire sample space.

From (4), we see that for fixed w, $\mathfrak{F}(w * x)$ is complex Gaussian, and hence r is chi-square (possibly noncentral) of two degrees of freedom. Let the mean-squared and one-half the variance of the complex Gaussian process $\mathfrak{F}(w * x)$ be given by

$$\eta_r^2 = |E_x \mathfrak{F}(w*x)|^2$$

$$\lambda_r^2 = \frac{1}{2}\left(E_x|\mathfrak{F}(w*x)|^2 - \eta_r^2\right) . \qquad (15)$$

Then the mean and second moment of r are [3] (the noncentrality parameter is η_r^2/λ_r^2)

$$\text{mean}_x\, r = v = 2\,\lambda_r^2 + \eta_r^2$$

$$\text{var}_x\, r = 4\,\lambda_r^4 + 4\,\lambda_r^2\,\eta_r^2 . \qquad (16)$$

It follows from (16)

$$\text{var}_x\, r = v^2 - \eta_r^4 . \qquad (17)$$

Equations (14) and (17) yield

$$E\,[\text{var}\,(y - v)] = E(v^2) - E(\eta_r^4) . \qquad (18)$$

Also, from (9)

$$E[\text{var}\,(y - v)] = E(y^2) - E(v^2) . \qquad (19)$$

Adding these two expressions, we have

$$E[\text{var}\,(y - v)] = \frac{1}{2}\,E(y^2) - \frac{1}{2}\,E(\eta_r^4) . \qquad (20)$$

In this manner, we have removed the variable v from our calculations. The substitution of (20) in (13) yields

$$\text{var}(z) = E(y^2)\left(\frac{1}{2} + \frac{1}{2M}\right) - \mu_y^2 + E(\eta_r^4)\left(\frac{1}{2} - \frac{1}{2M}\right) . \qquad (21)$$

To evaluate this expression we need to know the first two moments of y ($E(y^2)$ and μ_y^2) and $E(\eta_r^4)$. The first moment is available in [1],† the second is calculated in Appendix A, and we proceed to determine $E(\eta_r^4)$.

† In the notation of [1], $E(y) = \sigma_u^2 + \sigma_v^2 + \bar{u}^2 + \bar{v}^2$.

CALCULATION OF $E(\eta_r^4)$

Let L be the length of the ALE. In order to simplify calculations, it is assumed that the analysis frequency ω is bin-centered, both with respect to the DFT and ALE; i.e., ωK = multiple of 2π and ωL = multiple of 2π. This assumption only affects terms of lower order in K and L (cf Appendix A), and hence will not change the final result.

For w fixed the mean of $\mathfrak{F}(w*x)$ is given by

$$E_x\left(\mathfrak{F}(w*x)\right) = E_x \sum_{\ell=1}^{L} \sum_{n=1}^{K} w_\ell x_\ell(k_0+n)e^{-i\omega n}$$

$$= \sum_{\ell=1}^{L} \sum_{n=1}^{K} w_\ell A \cos\left(\omega(n-\ell)+\phi\right) e^{-i\omega n}$$

$$= \sum_{\ell=1}^{L} w_\ell e^{-i\omega\ell} \sum_{n=1}^{K} A \cos\left(\omega(n-\ell)+\phi\right) e^{-i\,\omega(n-\ell)}$$

$$= \sum_{\ell=1}^{L} w_\ell e^{-i\omega\ell} \sum_{n'=1-\ell}^{K-\ell} A \cos(\omega n'+\phi)\, e^{-i\omega n'} .$$

Since K is a multiple of the period of $\cos(\omega n'+\phi)$, the second sum is independent of ℓ, and

$$E_x(\mathfrak{F}(w*x)) = \mathfrak{F}_L(w)\,\mathfrak{F}_K(A\cos(\omega n+\phi)) \quad . \tag{22}$$

Thus, from definitions (15) and (22)

$$\eta_r^2 = \frac{A^2K^2}{4}\left|\mathfrak{F}_L(w)\right|^2 \quad . \tag{23}$$

We now note that $|\mathfrak{F}_L(w)|^2$ considered as a function of w is a chi-square random variable with two degrees of freedom. Let

$$\gamma^2 = E(n_\ell'^2) \quad . \tag{24}$$

Then expressions (15) and (16) apply with

$$\lambda_w^2 = \frac{L\gamma^2}{2}$$

and

$$\eta_w^2 = \frac{L^2 B^2}{4} \quad .$$

Thus

$$E|\mathfrak{F}_L(w)|^4 = \text{second moment}$$

$$= 8\frac{L^2\gamma^4}{4} + 8\frac{L^3\gamma^2 B^2}{8} + \frac{L^4 B^4}{16} \quad . \tag{25}$$

Combining this with (23) we have

$$E(\eta_r^4) = \frac{A^4 K^4}{16}\left(2L^2\gamma^2 + L^3 B^2\gamma^2 + \frac{L^4 B^4}{16}\right) \quad . \tag{26}$$

LONG-TERM AVERAGE

Consider the average output over a time longer than τ

$$\tilde{z} = \frac{1}{NM}\sum_{\ell=1}^{NM} y_\ell \quad . \tag{27}$$

This may be rewritten, for $MK < \tau$,

$$\tilde{z} = \frac{1}{N}\sum_{s=1}^{N}\left(\frac{1}{M}\sum_{\ell=1}^{M} y(sM+\ell)\right)$$

$$= \frac{1}{N}\sum_{s=1}^{N} z(s) \quad . \tag{28}$$

The random variables y are a function of x and w. Since the same x(k) do not appear in $y(k_1)$ and $y(k_2)$ for $k_1 \neq k_2$, and since the time dependence of w(k) is exponential with a correlation time τ, z(s) has a time dependence of 2τ (w is squared in equation (4))

$$E\left(z(s_1)\,z(s_2)\right) \sim E(z^2)\,e^{-2|s_1-s_2|/\tau} \quad . \tag{29}$$

Consequently, if $N = T/\tau >> 1$, where T is the processing time,

$$\mathrm{var}(\tilde{z}) \sim \frac{1}{N}\,\mathrm{var}(z) \quad . \tag{30}$$

Roughly speaking, the long-term average may be considered an average of the short-term average, with independent samples at intervals of τ. A more exact treatment is found in Appendix B.

SUMMARY OF RESULTS

Let us write the weight update algorithm for the ALE in the following form

$$w_\ell(k+1) = w_\ell(k) + 2\mu\, \epsilon(k)\, x_\ell(k) \quad , \tag{31}$$

where $x_\ell(k)$ is defined in (5), and $\epsilon(k)$ is the error at time k as pictured in figure 2. Define the following quantities

$$\nu^2 = \text{input noise power} = E(n_\ell^2)$$

$$\text{SNR} = \text{total signal power}/\nu^2$$

$$a^* = \frac{L}{2} \frac{\text{SNR}}{1 + \frac{L}{2}\text{SNR}}$$

$$\gamma^2 = E(n'^2)$$

$$\sim \mu \nu^2 \left(1 + (1 - a^*)\, \text{SNR}\right) \quad , \tag{32}$$

where n and n′ are the noise processes of equations (5) and (6), and SNR is the input signal to noise ratio.

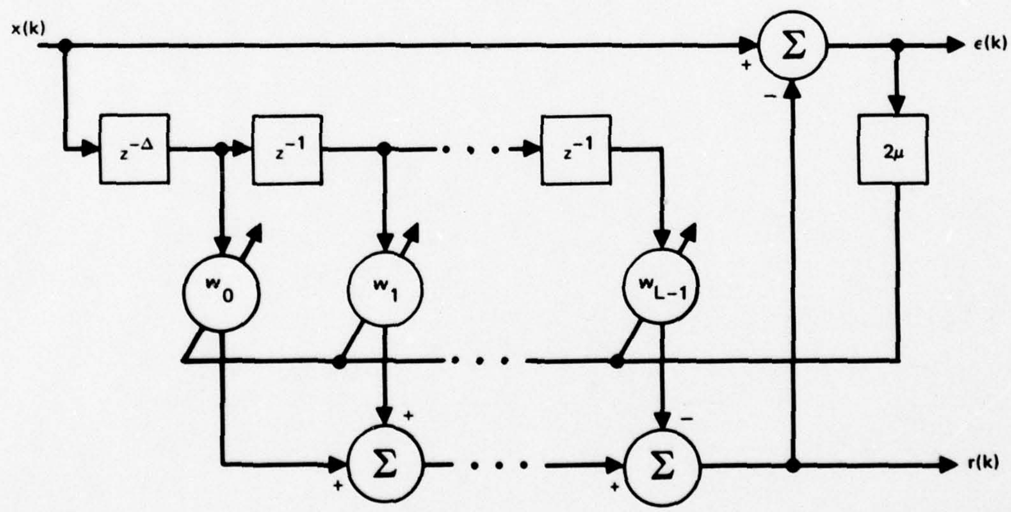

Figure 2. Block diagram of the Adaptive Line Enhancer (ALE).

It then follows [2,3] that w_ℓ has the form (6) and

$$A^2 = 2\nu^2 \, SNR$$

$$E(n_\ell'^2) = \gamma^2$$

$$B^2 = \frac{4}{L^2}(a^*)^2 \qquad (33)$$

$$\tau = \frac{1}{2\mu \nu^2 \left(1 + \frac{L}{2} SNR\right)}$$

where

$$L = \text{ALE filter length} .$$

Let K be the DFT length in the ALE output detector pictured in figure 2 and assume that max $(2K, K + L) << \tau$. Then a single output of the detector, y, with no averaging has mean [1]

$$\mu_y = E(y)$$

$$= \gamma^2 \nu^2 LK + \frac{\gamma^2 A^2 LK^2}{4} + \frac{A^2 B^2 K^2 L^2}{16} + B^2 \nu^2 D$$

where

$$D = \begin{cases} \frac{K^2 L}{4} - \frac{K^3}{12} & K \leq L \\ \frac{L^2 K}{4} - \frac{L^3}{12} & K \geq L \end{cases} . \qquad (34)$$

Its variance, as computed in Appendix A, is

$$\text{var}(y) = E(y^2) - \mu_y^2 \qquad (35)$$

where

$$E(y^2) = \gamma^4 A^2 \nu^2 D_1 + \gamma^4 \nu^4 D_2 + B^4 A^2 \nu^2 D_3$$
$$+ B^4 \nu^4 D_4 + B^2 \gamma^2 A^2 \nu^2 D_5 + B^2 \gamma^2 \nu^4 D_6$$
$$+ \gamma^4 A^4 D_7 + B^2 \gamma^2 A^4 D_8 + B^4 A^4 D_9 , \qquad (36)$$

and

$$D_1 = \begin{cases} K^2[2KL^2 - L^3/3] , & K \geq L \\ K^2[KL^2 + K^2L - K^3/3] , & K \leq L \end{cases} \tag{37}$$

$$D_2 = \begin{cases} 4K^2L^2 - \frac{4}{3}KL^3 + \frac{L^4}{3} , & K \geq L \\ 2K^2L^2 + \frac{4}{3}K^3L - \frac{K^4}{3} , & K \leq L \end{cases} \tag{38}$$

$$D_3 = \begin{cases} \frac{K^2L^2}{16}[KL^2 - L^3/3] , & K \geq L \\ \frac{K^2L^2}{16}[K^2L - K^3/3] , & K \leq L \end{cases} \tag{39}$$

$$D_4 = \begin{cases} \frac{1}{8}[KL^2 - L^3/3]^2 , & K \geq L \\ \frac{1}{8}[K^2L - K^3/3]^2 , & K \leq L \end{cases} \tag{40}$$

$$D_5 = \begin{cases} \frac{LK^2}{4}[4KL^2 - L^3] & K \geq L \\ \frac{LK^2}{4}[3K^2L + KL^2 - K^3] & K \leq L \end{cases} \tag{41}$$

$$D_6 = \begin{cases} 2K^2L^3 - KL^4 + \frac{7}{60}L^5 , & K \geq L \\ -\frac{L^5}{60} - \frac{KL^4}{6} + \frac{3}{2}K^2L^3 - \frac{13}{6}K^3L^2 + \frac{8}{3}LK^4 - \frac{7}{10}K^5 , & K \leq L \leq 2K \\ L^2K^3 + \frac{2}{3}K^4L - \frac{17}{30}K^5 , & 2K \leq L \end{cases} \tag{42}$$

$$D_7 = K^4L^2/8 \tag{43}$$

$$D_8 = K^4L^3/16 \tag{44}$$

$$D_9 = K^4L^4/256 \tag{45}$$

Now, suppose the output is averaged over a processing time T

$$\tilde{z} = \frac{1}{T} \int_0^T y \, dt \sim \frac{1}{N} \sum_{i=1}^{N} y(i) \tag{46}$$

where

$$N = T/\max(2K, K + L) \quad . \tag{47}$$

Then

$$\mathrm{var}(\tilde{z}) = \frac{1}{2N} \left(E(y^2) - E(\eta_r^4) \right)$$
$$+ q \left(\frac{1}{2} E(y^2) - \mu_y{}^2 + \frac{1}{2} E(\eta_r^4) \right) \tag{48}$$

where

$E(\eta_r^4)$ is given by (26)

$$E(\eta_r^4) = \gamma^4 A^4 D_7 + B^2 \gamma^2 A^4 D_8 + B^4 A^4 D_9 \quad , \tag{49}$$

and q is, from (B-8),

$$q = \frac{\tau}{T^2} \left| T + \frac{\tau}{2} (e^{-2T/\tau} - 1) \right| \quad . \tag{50}$$

Note that for large T

$$q \sim \frac{\tau}{T} \qquad T >> \tau \quad . \tag{51}$$

If we write

$$N_0 = \frac{1}{q}$$

$$M = N/N_0 \tag{52}$$

then

$$\mathrm{var}(\tilde{z}) \sim \frac{1}{N_0}\left[E(y^2)\left(\frac{1}{2}+\frac{1}{2M}\right) - \mu_y^2 + E(\eta_r^4)\left(\frac{1}{2}-\frac{1}{2M}\right)\right] \tag{53}$$

which is (30).

SIMULATION

M. Dentino [5] provides several Monte Carlo simulations of the ALE for a sinusoid in white Gaussian noise and plots their ROC curves. Figure 3 contains the results of one of these simulations, for which the conditions $\max(2K, K+L) \ll \tau$ and $\tau \ll T$ are satisfied. The corresponding ROC curve, computed using the formulae of this report under the assumption of a Gaussian output, appears in the same figure. The results are in excellent agreement. Note that in this case $K = L$ and $\tau/K \sim 72$ and $T/\tau \sim 8.5$.

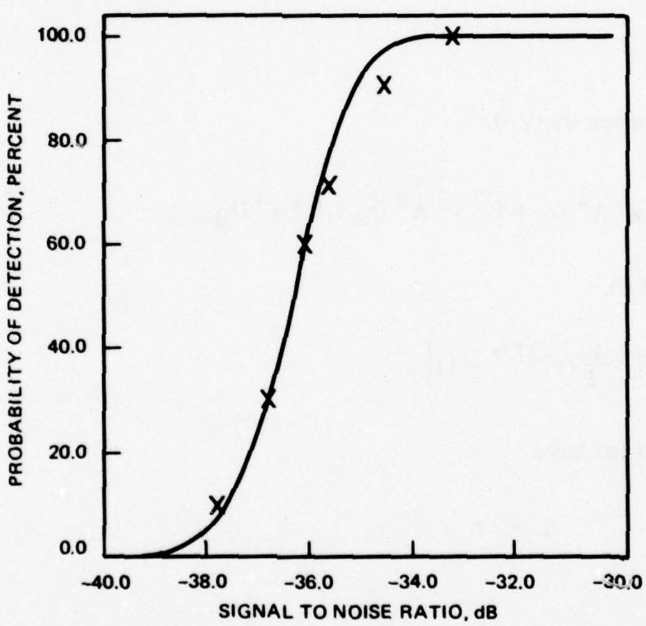

Figure 3. Comparison of computed ROC curve (solid line) to Monte Carlo simulation (X) for $P_F = 10^{-4}$, $T = 629{,}760$, $\mu\nu^2 = 7.6 - 10^{-6}$, and $K = L = 1024$.

MOMENTS RATIOS – NOISE ONLY

For noise only, we have

$$\mu_y = \gamma^2 \nu^2 LK$$

$$E(y^2) = \gamma^4 \nu^4 \begin{cases} 4K^2L^2 - \frac{4}{3}KL^3 + L^4/3 , & K \geqslant L \\ 2K^2L^2 + \frac{4}{3}K^3L - K^4/3 , & K \leqslant L . \end{cases}$$

Thus

$$\frac{\mu_y{}^2}{\mathrm{var}(y)} = \begin{cases} \frac{1}{3} & K >> L \\ \frac{1}{2} & K = L \\ 1 & K << L . \end{cases}$$

Since this ratio is 1 for a chi-square distribution of two degrees of freedom, the output distribution is in general not chi-square as hypothesized in [1]. Nor is it K_0 (ratio 1/3) for K = L as maintained in [6].

APPENDIX A: $E(y^2)$

Our starting point is equations (5) and (6)

$$\widetilde{x}_{k-\ell} = x_\ell(k) = A \cos \omega(k-\ell) + n(k-\ell)$$

$$E\Big(n(\ell)\, n(m)\Big) = \nu^2\, \delta_{\ell m}$$

$$\widetilde{w}_\ell = w_\ell(k) = B \cos(\omega\ell + \phi) + n'_\ell \qquad \text{(A-1)}$$

$$E(n'_\ell\, n'_m) = \gamma^2\, \delta_{\ell m}$$

$$E(n\, n') = 0$$

where $w_\ell(k)$ is assumed independent of k for max (2K, K + L) << τ (see (B-2)).

Note that

$$\mathfrak{F}(w * x) = \sum_{k=1}^{K} e^{-i\omega k} \sum_{\ell=1}^{L} \widetilde{w}_\ell\, \widetilde{x}_{k-\ell}$$

$$= \sum_{k} \sum_{\ell} e^{-i\omega\ell}\, \widetilde{w}_\ell\, e^{-i\omega(k-\ell)}\, \widetilde{x}_{k-\ell}$$

$$= \sum_{\ell=1}^{L} e^{-i\omega\ell}\, \widetilde{w}_\ell \sum_{s=1-\ell}^{K-\ell} e^{-i\omega s}\, \widetilde{x}_s\ .$$

Thus

$$E(y^2) = E|\mathfrak{F}(w * x)|^2$$

$$= E\Bigg[\sum_{\ell_1,\ell_2,\ell_3,\ell_4=1}^{L} w_{\ell_1} w_{\ell_2} w_{\ell_3} w_{\ell_4}\, e^{i(\ell_1-\ell_2+\ell_3-\ell_4)\omega} \cdot \sum_{s_1=1-\ell_1}^{K-\ell_1} \sum_{s_2=1-\ell_2}^{K-\ell_2} \sum_{s_3=1-\ell_3}^{K-\ell_3} \sum_{s_4=1-\ell_4}^{K-\ell_4} x_{s_1} x_{s_2} x_{s_3} x_{s_4} \cdot e^{i(s_1-s_2+s_3-s_4)\omega}\Bigg]\ . \qquad \text{(A-2)}$$

The products of w and x each have three non-zero types of terms.

$$
\begin{aligned}
&\text{type 1:} \quad \cos \quad \cos \quad \cos \quad \cos \\
&\text{type 2:} \quad \cos \quad \cos \quad n \quad n \qquad \text{(6 terms)} \\
&\text{type 3:} \quad n \quad n \quad n \quad n \ .
\end{aligned} \tag{A-3}
$$

All terms containing an odd number of noise terms are zero since $E(n) = E(n') = 0$. An even product of n's will produce a delta function. The second type will contain $6 = \binom{4}{2}$ terms. We shall denote the product of a "w term" of type i with an "x term" of type j by (i; j).

TERMS OF TYPE (w;2)

An x term of type two contains the sum

$$\left[\sum_{s_k=1-\ell_k}^{K-\ell_k} \sum_{s_j=1-\ell_j}^{K-\ell_j} \delta(s_k - s_j)\, e^{\pm i(s_k \pm s_j)}\right]$$

$$\cdot\, \nu^2 A^2 \sum_{s_n} \sum_{s_m} (\cos s_n \omega)(\cos s_m \omega)\, e^{\pm i(s_n \pm s_m)\,\omega}$$

$$= \nu^2 A^2 \frac{K^2}{4} \sum_{K-\max(\ell_j,\ell_k)}^{K-\min(\ell_j,\ell_k)} e^{\pm i(s_k \pm s_k)} \ .$$

The exponential is zero only for (j,k) = (1,2), (2,3), (3,4) or (1, 4). For a non-zero exponential the sum is of the order 1, otherwise it is $\max(K - |\ell_j - \ell_k|, 0)$. Thus

$$(w;2) = \frac{\nu^2 A^2 K^2}{4} \begin{cases} O(1), & (j,k) = (1,3) \text{ or } (2,4) \\ \max(K - |\ell_j - \ell_k|, 0)\ , & \text{other} \end{cases} \ . \tag{A-4}$$

TERMS OF TYPE (w;3)

The fourth moment of the Gaussian process is $E(n^4) = 3\,\nu^4$. Let β be the number of terms for which $s_1 = s_2 = s_3 = s_4$ (can be shown to be $K - \max_i \ell_i + \min_i \ell_i$). Then

$$(w,3) = \sum_{s_1 \neq s_3} \delta(s_1 - s_2)\, \delta(s_3 - s_4)\, \nu^4$$

$$+ \sum_{s_1 \neq s_2} \delta(s_1 - s_4)\, \delta(s_2 - s_3)\, \nu^4$$

$$+ \sum_{s_1 \neq s_2} \delta(s_1 - s_3)\, \delta(s_2 - s_4)\, \nu^4\, e^{2i\omega(s_3 - s_4)}$$

$$+ 3\, \beta \nu^4 \quad . \tag{A-5}$$

As will be seen in the final results, individual terms give contributions proportional to powers of K and L. In any sum such as (A-5), terms of lower order may be neglected. For example in (A-4), after summation over w (i.e., ℓ_j and ℓ_k), the upper term is of order K^2 whereas the lower is of order K^2L or K^3. In particular, oscillatory terms such as the third term of (A-5) will be of lower order. (The reader should see [1] for more information.) In what follows, lower order terms will be pointed out and discarded without further comment.

The third and fourth terms of (A-5) are of lower order. Also, the effects of setting $s_1 \neq s_3$ and $s_1 \neq s_2$ in the first two may be neglected. Hence,

$$(w;3) = \nu^4 \Big[\max\Big(0, (K - |\ell_1 - \ell_2|)(K - |\ell_3 - \ell_4|)\Big) + \max\Big(0, (K - |\ell_1 - \ell_4|)(K - |\ell_2 - \ell_3|)\Big)\Big] \quad . \tag{A-6}$$

TERM (3, 2)

There are three significant non-zero terms of type 3 in w. Consider first $\ell_1 = \ell_2$; $\ell_3 = \ell_4$, $\ell_1 \neq \ell_3$. This gives rise to

$$\gamma^4 \Big(\sum_{\ell_1 \neq \ell_3} (2\, K + 2 \max(0, K - |\ell_1 - \ell_3|) \Big) \frac{A^2 K^2}{4} \nu^2 \quad ,$$

which contains (see (A-4)) four contributions from x-terms plus two contributions from x which are of lower order and have been ignored. The term $\ell_1 = \ell_4$, $\ell_2 = \ell_3$, $\ell_1 \neq \ell_2$ is the same as above, whereas the term $\ell_1 = \ell_3$, $\ell_2 = \ell_4$, $\ell_1 \neq \ell_2$ is

$$\gamma^4 \left[\sum_{\ell_1 \neq \ell_2} e^{i(2\ell_1 - 2\ell_2)\omega}\, 4 \max(0, K - |\ell_1 - \ell_2|) \right] \frac{A^2 K^2}{4} \nu^2 \quad .$$

This is of lower order and may be neglected. The two contributing terms are identical, hence,

$$(3,2) = \frac{\gamma^4 A^2 K^2 \nu^2}{4} \left[2\Big(\sum_{\ell_1,\ell_3} \big(2K + 2 \max (0, K - |\ell_1 - \ell_3|)\big)\Big)\right] . \tag{A-7}$$

We note that

$$\sum_{\ell_1,\ell_3} \max (0, K - |\ell_1 - \ell_3|) = \begin{cases} \displaystyle\sum_{\ell_1,\ell_3} (K - |\ell_1 - \ell_3|) & K \geq L \\ \displaystyle\sum_{|\ell_1 - \ell_3| \leq K} (K - |\ell_1 - \ell_3|) & K \leq L \end{cases} .$$

Also,

$$\sum_{\ell_1,\ell_2} |\ell_1 - \ell_2| = \sum_{\ell_2} \left(\sum_{\ell_1=\ell_2}^{L} (\ell_1 - \ell_2) + \sum_{\ell_1=1}^{\ell_2} (\ell_2 - \ell_1) \right)$$

$$= \sum_{\ell_2} \left(\frac{(L - \ell_2)(L - \ell_2 + 1)}{2} + \frac{(\ell_2 - 1)\,\ell_2}{2} \right)$$

$$= \sum_{\ell_2} \frac{1}{2} \left(L^2 - 2 L \ell_2 + 2 \ell_2^2 + L - 2 \ell_2 \right)$$

$$= \frac{L^3}{2} - \frac{L^3}{2} + \frac{L^3}{3} + \text{lower order terms} \sim \frac{L^3}{3} . \tag{A-8}$$

Thus,

$$\sum_{\ell_1,\ell_3} (K - |\ell_1 - \ell_3|) = K L^2 - L^3/3 .$$

The situation is somewhat more complicated for $K \leq L$. We first calculate some identities. Ignoring lower order terms, we have for $L \geq K$

$$\sum_{|\ell_j-\ell_k|<K} 1 = \sum_{\ell_j,\ell_k=1}^{K} 1 + \sum_{\ell_j=K+1}^{L} \sum_{\ell_k=\ell_j-K}^{\ell_j} 1 + \sum_{\ell_k=K+1}^{K} \sum_{\ell_j=\ell_k-K}^{\ell_k} 1$$

$$= K^2 + \sum_{\ell_j=K+1}^{L} K + \sum_{\ell_k=K+1}^{L} K$$

$$= K^2 + K(L-K) + K(L-K) + \text{lower order}$$

$$= 2K\,L - K^2 \qquad \text{(A-9)}$$

$$\sum_{|\ell_j-\ell_k|<K} |\ell_j-\ell_k| = \sum_{\ell_j\ell_k}^{K} |\ell_j-\ell_k| + 2 \sum_{\ell_j=K+1}^{L} \sum_{\ell_j-\ell_k=0}^{K} (\ell_j-\ell_k) + \text{lower order}$$

$$= \frac{K^3}{3} + 2 \sum_{\ell_j=K+1}^{L} \frac{K^2}{2} + \text{lower order}$$

$$= \frac{K^3}{3} + K^2L - K^3 = K^2L - \frac{2}{3}K^3 \; . \qquad \text{(A-10)}$$

Substitution of these expressions in (A-7) yields

$$(3,2) = \gamma^4 A^2 \nu^2 K^2 \begin{cases} 2\,K\,L^2 - L^3/3 & K \geqslant L \\ K\,L^2 + K^2L - K^3/3 & K \leqslant L \end{cases} \; . \qquad \text{(A-11)}$$

TERM (3,3)

As in the previous case we consider the separate W terms. For $\ell_1 = \ell_2$, $\ell_3 = \ell_4$, $\ell_1 \neq \ell_3$ we have

$$\gamma^4 \nu^4 \sum_{\ell_1 \neq \ell_3} \left[K^2 + \left(\max\left(0, K - |\ell_1 - \ell_3|\right)\right)^2\right] \; .$$

For $\ell_1 = \ell_4$; $\ell_2 = \ell_3$; $\ell_1 \neq \ell_2$, the result is the same. For $\ell_1 = \ell_3$; $\ell_2 = \ell_4$; $\ell_1 \neq \ell_2$, a factor of $e^{2i\omega(\ell_1-\ell_2)}$ enters, and hence it is of lower order.

Let us evaluate $\Sigma |\ell_1 - \ell_3|^2$. For $K \geqslant L$, we have

$$\sum_{\ell_1,\ell_3} |\ell_1 - \ell_3|^2 = \sum_{\ell_1,\ell_3} (\ell_1 - \ell_3)^2$$

$$= \sum_{\ell_1\ell_3} \ell_1^2 - 2\,\ell_1\,\ell_3 + \ell_3^2$$

$$= 2\,\frac{L^4}{3} - 2\left(\frac{L^2}{2}\right)^2 + \text{lower order}$$

$$= L^4\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{L^4}{6} \qquad \text{(A-12)}$$

and for $K \leqslant L$

$$\sum_{|\ell_j-\ell_k|<K} |\ell_j - \ell_k|^2 = \sum_{\ell_j\ell_k}^{K} |\ell_j - \ell_k|^2 + 2 \sum_{\ell_j=K+1}^{L} \sum_{\ell_j-\ell_k=0}^{K} (\ell_j-\ell_k)^2 + \text{lower order}$$

$$= \frac{K^4}{6} + 2(L - K)(K^3/3) + \text{lower order}$$

$$= \frac{K^4}{6} + \frac{2}{3}K^3L - \frac{K^4 2}{3} = \frac{2}{3}K^3L - \frac{K^4}{2} \quad . \qquad \text{(A-13)}$$

Thus

$$(3,3) = 2\,\gamma^4\nu^4 \left[K^2 + \sum_{|\ell_1-\ell_3|\leqslant K} \left(K^2 - 2K|\ell_1-\ell_3| + |\ell_1 - \ell_3|^2\right)\right]$$

$$= \gamma^4\nu^4 \begin{cases} 4\,K^2L^2 - \frac{4}{3}K\,L^3 + \frac{L^4}{3}\;, & K \geqslant L \\ 2\,K^2L^2 + \frac{4}{3}K^3L - \frac{K^4}{3}\;, & K \leqslant L \end{cases} \qquad . \qquad \text{(A-14)}$$

TERM (1,2)

The type 1 terms contain four cosines. The summation over two of them gives

$$B^2\ \Sigma(\cos \ell_m\omega)(\cos \ell_n\omega)\cos(\ell_m \pm \ell_n)\omega = B^2 L^2/4.$$

From (A-4), we see that only four of the type two terms are significant. *Each* of these gives a contribution of (from (A-4))

$$\frac{B^2 L^2}{4} \sum_{\ell_j,\ell_k} (\cos \omega\, \ell_j)(\cos \omega\, \ell_k)(\cos \omega(\ell_j - \ell_k))$$

$$\cdot \max(0, K - |\ell_j - \ell_k|)\frac{A^2 K^2 \nu^2}{4} \quad .$$

We now use the identity

$$(\cos a)(\cos(a \pm b))(\cos b) = \frac{1}{4} + \frac{\cos 2(a \pm b)}{4} + \frac{\cos 2a}{4} + \frac{\cos 2b}{4} \quad . \qquad \text{(A-15)}$$

All terms but the 1/4 will be of lower order. Thus the above becomes

$$\frac{B^2 A^2 \nu^2 K^2 L^2}{16}\ \frac{1}{4} \sum_{\ell_j,\ell_k} \max(0, K - |\ell_j - \ell_k|)$$

which with (A-9) and (A-10) yields

$$(1,2) = \frac{B^4 A^2 \nu^2 K^2 L^2}{16} \begin{cases} KL^2 - L^3/3\ , & K \geq L \\ K^2 L - K^3/3\ , & K \leq L \end{cases} \quad . \qquad \text{(A-16)}$$

TERM (1,3)

Proceeding as for term (1,2) and using (A-6), we get two terms of the form

$$B^4 \nu^4 \left\{ \left[\sum_{\ell_1,\ell_2} (\cos \ell_1\omega)(\cos \ell_2\omega)(\cos(\ell_1 - \ell_2)\omega \max(0, K - |\ell_1 - \ell_2|) \right]^2 \right.$$

$$\left. - \left[\sum_{\ell_1,\ell_2} (\cos \ell_1\omega)(\cos \ell_2\omega) \sin(\ell_1 - \ell_2)\omega \max(0, K - |\ell_1 - \ell_2|) \right]^2 \right\}$$

since

$$\cos\omega(\ell_1 - \ell_2 + \ell_3 - \ell_4) = \cos(\ell_1 - \ell_2)\omega \cos(\ell_3 - \ell_4)\omega$$

$$- \sin(\ell_1 - \ell_2)\omega \sin(\ell_3 - \ell_4)\omega \quad .$$

Because of the sine, the second term is of lower order. Also, according to (15), the first term equals 1/4 plus terms of lower order. Thus

$$(1,3) = 2\,B^4\nu^4 \Big(\frac{1}{4} \sum_{\ell_1,\ell_2} \max(0, K - |\ell_1 - \ell_2|)\Big)^2$$

$$= \frac{B^4\nu^4}{8} \begin{cases} (KL^2 - L^3/3)^2 & K \geq L \\ (K^2L - K^3/3)^2 & K \leq L \end{cases} \quad . \tag{A-17}$$

TERM (2,2)

As a consequence of (A-4), we are only concerned with the sets of indices j,k = 1,2 or 1,4 or 2,3 or 3,4. There will be four terms in which the indices of the two cosines in w coincide with j and k, four terms in which they are disjoint, and sixteen terms in which one index coincides and the other does not. We note that

$$\sum_{\ell_m,\ell_n} \delta(\ell_m - \ell_n)\, e^{i(\ell_m - \ell_n)}\, \gamma^2 = L\gamma^2 \ .$$

Thus (2,2) will contain

$$4 \sum_{j,k} (\cos\omega\ell_j)(\cos\ell_k\omega) \cos\omega(\ell_j - \ell_k) \max(0, K - |\ell_j - \ell_k|)\, \frac{A^2K^2\nu^2}{4}\, L\gamma^2 B^2$$

$$+ 4 \sum_{\ell_j,\ell_k,\ell_m,\ell_n} (\cos\ell_m\omega)(\cos\ell_n\omega) \cos(\ell_m - \ell_n)\omega\, \delta(\ell_j - \ell_k) \max(0, K - |\ell_j - \ell_k|)\, \frac{A^2K^2\nu^2}{4}\, \gamma^2 B^2$$

$$+ 16 \sum_{\ell_j,\ell_k,\ell_m,\ell_n} (\cos\ell_j\omega)(\cos\ell_m\omega) \cos(\ell_j \pm \ell_m)\omega \max(0, K - |\ell_j - \ell_k|)\, \delta(\ell_n - \ell_k) \cos(\ell_n \pm \ell_k)\omega$$

$$\cdot \frac{A^2K^2\nu^2}{4}\, \gamma^2 B^2 \ .$$

From (A-8) and (A-15), the first term is

$$B^2 \gamma^2 A^2 \nu^2 \frac{K^2 L}{4} \begin{cases} KL^2 - L^3/3 , & L \leq K \\ K^2 L - K^3/3 , & K \leq L \end{cases} . \qquad \text{(A-18)}$$

The second term is

$$\frac{B^2 \gamma^2 A^2 \nu^2 K^2}{4} K L^3 . \qquad \text{(A-19)}$$

The third term yields

$$16 \frac{L}{2} \sum_{\ell_j, \ell_k, \ell_n} (\cos^2 \ell_j \omega) \max(0, K - |\ell_j - \ell_k|) \, \delta(\ell_n - \ell_k) \cos(\ell_n \pm \ell_k)$$

$$\cdot \frac{A^2 K^2}{4} \nu^2 \gamma^2 B^2$$

$$= 8L \sum_{\ell_j, \ell_k} (\cos^2 \ell_j \omega) \max(0, K - |\ell_j - \ell_k|) \cos(\ell_k \pm \ell_k) \frac{A^2 K^2}{4} \nu^2 \gamma^2 B^2 .$$

Since those terms containing a plus sign are of lower order, four terms remain:

$$4L \sum_{\ell_j, \ell_k} (\cos^2 \ell_j \omega) \max(0, K - |\ell_j - \ell_k|) \frac{A^2 K^2}{4} \nu^2 \gamma^2 B^2 .$$

Also $\cos^2(\ell_j \omega) = 1/2 + (\cos 2\ell_j \omega)/2$; so that neglecting lower order terms, we have

$$B^2 \gamma^2 A^2 \nu^2 \frac{K^2 L}{2} \begin{cases} KL^2 - L^3/3 , & K \geq L \\ K^2 L - K^3/3 , & K \leq L \end{cases} . \qquad \text{(A-20)}$$

Equations (A-18), (A-19) and (A-20) yield

$$(2,2) = B^2 \gamma^2 A^2 \nu^2 \frac{K^2 L}{4} \begin{cases} 4\, KL^2 - L^3 , & K \geq L \\ 3\, K^2 L + KL^2 - K^3 , & K \leq L \end{cases} . \qquad \text{(A-21)}$$

TERM (2,3)

There will be four terms in which the cosine terms of w match at least one pair of indices in (A-6) and two terms in which they do not. The first four give

$$4B^2\gamma^2\nu^2\left[\sum_{\ell_1,\ell_2,\ell_3}(\cos\ell_1\omega)(\cos\ell_2\omega)\cos(\ell_1-\ell_2)\omega\right.$$

$$\left.\cdot\left\{K\max(0,K-|\ell_1-\ell_2|)+\max(0,K-|\ell_1-\ell_3|)\max(0,K-|\ell_2-\ell_3|)\right\}\right].$$

The usual identity (A-15) yields

$$B^2\gamma^2\nu^4\left[\sum_{\ell_1,\ell_2}LK\max(0,K-|\ell_1-\ell_2|)\right.$$

$$\left.+\sum_{\ell_1,\ell_2,\ell_3}\max(0,K-|\ell_1-\ell_3|)\max(0,K-|\ell_2-\ell_3|)\right]\quad. \tag{A-22}$$

The other terms are

$$2B^2\gamma^2\nu^2\left[\sum_{\ell_1,\ell_2,\ell_3}(\cos\ell_1\omega)(\cos\ell_3\omega)\cos(\ell_1+\ell_3)\omega\cos(2\ell_2\omega)\right.$$

$$\cdot\left\{\max(0,K-|\ell_1-\ell_2|)\max(0,K-|\ell_3-\ell_2|)+\max(0,K-|\ell_1-\ell_2|)\right.$$

$$\left.\left.\cdot\max(0,K-|\ell_2-\ell_3|)\right\}\right]$$

which is of lower order because of the $\cos(2\ell_2\omega)$ factor.

In order to complete the evaluation of (2,3) we need some additional summation identities. For $K \geq L$, with lower order terms omitted,

$$\sum_{\ell_1,\ell_2,\ell_3} |\ell_1-\ell_3|\,|\ell_2-\ell_3| = \sum_{\ell_1,\ell_3}\left[\sum_{\ell_2=1}^{\ell_3}(\ell_3-\ell_2)|\ell_1-\ell_3| + \sum_{\ell_2=\ell_3}^{L}(\ell_2-\ell_3)|\ell_1-\ell_3|\right]$$

$$= \sum_{\ell_1,\ell_3} |\ell_1-\ell_3|\left(\frac{1}{2}\right)\left(L^2-2L\ell_3+2\ell_3^2\right)$$

$$= \sum_{\ell_3} \frac{1}{4}\left(L^2-2L\ell_3+2\ell_3^2\right)^2$$

$$= \frac{1}{4}\sum_{\ell_3}\left(L^4+8L^2\ell_3^2+4\ell_3^4-4L^3\ell_3-8L\ell_3^3\right)$$

$$= \frac{1}{4}\left(L^5+\frac{8}{3}L^5+\frac{4}{5}L^5-2L^5-2L^5\right)$$

$$= \frac{7}{60}L^5 \ . \qquad \text{(A-23)}$$

(Note that these summations are easily performed using the formula

$$\sum_{\ell=a}^{b} \ell^n = \frac{1}{n+1}\left(b^{n+1}-a^{n+1}\right) + \text{lower order terms} \ ,$$

which may be proved by considering integral approximations.)

For $K \leqslant L$, let $|\ell_1-\ell_3| = s$ and $|\ell_2-\ell_3| = m$.

Then, since the factors $\max(0, |\ell_i-\ell_j|)$ restrict the summations to $|s| \leqslant K$, $|m| \leqslant K$, for $\ell_1 \geqslant \ell_3$ s ranges from 0 to $\min(K, L-\ell_3)$; and for $\ell_1 \leqslant \ell_3$ it ranges from 0 to $\min(K, \ell_3-1)$. Thus

$$\sum_{\substack{s \leq K \\ m \leq K}} = \sum_{\ell_3=1}^{L} \left(\sum_{s=0}^{\min(K,L-\ell_3)} + \sum_{s=0}^{\min(K,\ell_3-1)} \right) \left(\sum_{m=0}^{\min(K,L-\ell_3)} + \sum_{m=0}^{\min(K,\ell_3-1)} \right)$$

$$= \sum_{\ell_3=1}^{L} \left(\sum_{s=0}^{\min(K,L-\ell_3)} + \sum_{s=0}^{\min(K,\ell_3-1)} \right)^2 . \qquad \text{(A-24)}$$

We now must consider the two "sub-cases," $2K \leq L$ and $K \leq L \leq 2K$. For $2K \leq L$, we have from (A-24)

$$\sum_{\substack{s \leq K \\ m \leq K}} = \sum_{\ell_3=1}^{K} \left(\sum_{s=0}^{K} + \sum_{s=0}^{\ell_3-1} \right)^2 + \sum_{\ell_3=K+1}^{L-K+1} \left(\sum_{s=0}^{K} + \sum_{s=0}^{K} \right)^2$$

$$+ \sum_{\ell_3=L-K}^{L} \left(\sum_{s=0}^{L-\ell_3} + \sum_{s=0}^{K} \right)^2$$

$$= \sum_{\ell_3=1}^{K} (K+\ell_3)^2 + \sum_{\ell_3=K+1}^{L-K} (2K)^2 + \sum_{\ell_3=L-K}^{L} (L-\ell_3+K)^2$$

$$= \sum_{\ell_3=1}^{K} K^2 + 2K\ell_3 + \ell_3^2 + 4K^2(L-2K)$$

$$+ \sum_{\ell_3=L-K}^{L} (L-\ell_3)^2 + 2K(L-\ell_3) + K^2$$

$$= K^3 + K^3 + \frac{K^3}{3} + 4K^2L - 8K^3 + \frac{K^3}{3} + K^3 + K^3$$

$$= 4K^2L - \frac{10}{3}K^3 . \qquad \text{(A-25)}$$

Also,

$$\sum_{\substack{s \leq K \\ m \leq K}} s = \sum_{\ell_3=1}^{K} \left(\frac{K^2}{2} + \frac{\ell_3^2}{2}\right) (K + \ell_3) + \sum_{\ell_3=K+1}^{L-K+1} K^2\, 2K$$

$$+ \sum_{\ell_3=L-K}^{L} \left(\frac{(L-\ell_3)^2}{2} + \frac{K^2}{2}\right) (L - \ell_3 + K)$$

$$= \sum_{\ell_3=1}^{K} \left(\frac{K^3}{2} + \frac{K^2}{2}\ell_3 + \frac{K\ell_3^2}{2} + \frac{\ell_3^3}{2}\right)$$

$$+ 2K^3 (L - 2K) + \sum_{\ell_3=L-K}^{L} \left[\frac{(L-\ell_3)^3}{2} + \frac{(L-\ell_3)^2}{2} K + \frac{K^2}{2} (L-\ell_3) + \frac{K^3}{2}\right]$$

$$= \frac{K^4}{2} + \frac{K^4}{4} + \frac{K^4}{6} + \frac{K^4}{8} + 2K^3L - 4K^4 + \frac{K^4}{8} + \frac{K^4}{6} + \frac{K^4}{4} + \frac{K^4}{2}$$

$$= 2\, K^3L - \frac{23}{12} K^4 \qquad \text{(A-26)}$$

and

$$\sum_{\substack{s \leq K \\ m \leq K}} sm = \sum_{\ell_3=1}^{K} \left(\frac{K^2}{2} + \frac{\ell_3^2}{2}\right)^2 + \sum_{\ell_3=K+1}^{L-K} (K^2)^2 + \sum_{\ell_3=L-K}^{L} \left(\frac{(L-\ell_3)^2}{2} + \frac{K^2}{2}\right)^2$$

$$= \sum_{\ell_3=1}^{K} \left(\frac{K^4}{4} + \frac{K^2\ell_3^2}{2} + \frac{\ell_3^4}{4}\right) + K^4(L-2K)$$

$$+ \sum_{\ell_3=L-K}^{L} \left(\frac{(L-\ell_3)^4}{4} + \frac{K^2(L-\ell_3)^2}{2} + \frac{K^4}{4}\right)$$

$$= \frac{K^5}{4} + \frac{K^5}{6} + \frac{K^5}{20} + K^4L - 2K^5 + \frac{K^5}{20} + \frac{K^5}{6} + \frac{K^5}{4} = K^4L - \frac{16}{15} K^5 \ . \qquad \text{(A-27)}$$

Thus for $2K \leqslant L$, from (A-25), (A-26), and (A-27),

$$\sum_{\substack{s\leqslant K\\ m\leqslant K}} (K-s)(K-m) = 4\,K^4L - \frac{10}{3}K^5 - 4K^4L + \frac{23}{6}K^5 + K^4L - \frac{16}{15}K^5$$

$$= K^4L - \frac{17}{30}K^5 \quad . \tag{A-28}$$

Similar techniques are used for $K \leqslant L \leqslant 2K$:

$$\sum_{\substack{s\leqslant K\\ m\leqslant K}} = \sum_{\ell_3=1}^{L-K} \left(\sum_{s=0}^{K} + \sum_{s=0}^{\ell_3-1} \right)^2 + \sum_{\ell_3=L-K}^{K} \left(\sum_{s=0}^{L-\ell_3} + \sum_{s=0}^{\ell_3-1} \right)^2$$

$$+ \sum_{\ell_3=K}^{L} \left(\sum_{s=0}^{L-\ell_3} + \sum_{s=0}^{K} \right)^2$$

$$= \sum_{\ell_3=1}^{L-K} (K+\ell_3)^2 + \sum_{\ell_3=L-K}^{K} L^2 + \sum_{\ell_3=K}^{L} (L+K-\ell_3)^2$$

$$= \frac{1}{3}(L^3-K^3) + L^2(2K-L) + \frac{1}{3}(L^3-K^3)$$

$$= 2L^2K - L^3/3 - \frac{2}{3}K^3 \quad . \tag{A-29}$$

Also,

$$\sum_{\substack{s\leqslant K\\ m\leqslant K}} s = \sum_{\ell_3=1}^{L-K} \left(\frac{K^2}{2} + \frac{\ell_3^2}{2} \right)(K+\ell_3) + \sum_{\ell_3=L-K}^{K} \left(\frac{(L-\ell_3)^2}{2} + \frac{\ell_3^2}{2} \right) L$$

$$+ \sum_{\ell_3=K}^{L} \left(\frac{(L-\ell_3)^2}{2} + \frac{K^2}{2} \right)(L+K-\ell_3)$$

$$= \sum_{\ell_3=1}^{L-K} \left(\frac{K^3}{2} + \frac{K^2}{2}\ell_3 + \frac{K\ell_3^2}{2} + \frac{\ell_3^3}{2} \right) + \frac{L}{6}\left(K^3 - (L-K)^3 - (L-K)^3 + K^3 \right)$$

$$+ \sum_{\ell_3=K}^{L} \frac{(L-\ell_3)^3}{2} + K\frac{(L-\ell_3)^2}{2} + \frac{(L-\ell_3)K^2}{2} + \frac{K^3}{2} ,$$

which after more calculation,

$$= -LK^3 - \frac{K^4}{4} + \frac{11}{4}L^2K^2 - \frac{3}{2}KL^3 + L^4/3 \quad . \tag{A-30}$$

Finally,

$$\sum_{\substack{s \leq K \\ m \leq K}} sm = \sum_{\ell_3=1}^{L-K} \left(\frac{K^2}{2} + \frac{\ell_3^2}{2} \right)^2 + \sum_{\ell_3=L-K}^{K} \left(\frac{(L-\ell_3)^2}{2} + \frac{\ell_3^2}{2} \right)^2$$

$$+ \sum_{\ell_3=K}^{L} \left(\frac{(L-\ell_3)^2}{2} + \frac{K^2}{2} \right)^2 ,$$

which after some computation yields,

$$\sum_{\substack{s \leq K \\ m \leq K}} sm = LK^4 - \frac{8K^5}{15} - \frac{7}{6}K^2L^3 + \frac{1}{3}K^3L^2 + \frac{1}{2}L^4K - \frac{L^5}{60} \quad . \tag{A-31}$$

Combining (A-29), (A-30), and (A-31), we have for $K \leq L \leq 2K$,

$$\sum_{\substack{s \leq K \\ m \leq K}} (K-s)(K-m) = -\frac{L^5}{60} - \frac{KL^4}{6} + \frac{9}{6}K^2L^3 - \frac{19}{6}K^3L^2$$

$$+ 3LK^4 - \frac{7}{10}K^5 \quad . \tag{A-32}$$

We now substitute (A-9), (A-10), (A-23), (A-28), and (A-32) into (A-22) to get

$$(2,3) = B^2\gamma^2\nu^4 \begin{cases} 2K^2L^3 - KL^4 + \frac{7}{60}L^5 \;, & K \geqslant L \\ -\frac{L^5}{60} - \frac{KL^4}{6} + \frac{3}{2}K^2L^3 - \frac{13}{6}K^3L^2 + \frac{8}{3}LK^4 - \frac{7}{10}K^5 \;, & \\ & 2K \geqslant L \geqslant K \\ L^2K^3 + \frac{2}{3}K^4L - \frac{17}{30}K^5 \;, & L \geqslant 2K \;. \end{cases} \tag{A-33}$$

TERMS (1,1) + (2,1) + (3,1)

We note that since the cosine function is periodic, the summation in (A-2) for x of type 1 may be rewritten (see (22)) so that the product formula for the convolution holds. Thus,

$$(1,1) + (2,1) + (3,1) = E\,|\mathfrak{F}(w)|^4\;|\mathfrak{F}(A\cos\omega k)|^4 \;.$$

The second factor is $(AK/2)^4$. The first is given by (25). Consequently

$$(1,1) + (2,1) + (3,1) = \frac{A^4K^4}{16}\left(2\,L^2\,\gamma^4 + L^3\,B^2\,\gamma^2 + \frac{L^4B^4}{16}\right) \;. \tag{A-34}$$

RESULT

The final result is given by

$$E(y^2) = (1,1) + (2,1) + (3,1) + (1,2) + (2,2) + (3,2) + (1,3) + (2,3) + (3,3) \;,$$

all of which have been computed above.

APPENDIX B: TIME AVERAGING THE WEIGHT VECTOR

At low signal-to-noise ratios, the eigenvalues of the correlation matrix of the input to the ALE will be approximately equal, and the system of linear difference equations (31) will exhibit a single time constant τ given by (33) [1,7]. Then, with equation (6) as an initial condition

$$w_\ell(t) = n'_\ell(0)\, e^{-t/\tau} + c_t \tag{B-1}$$

where c_t is independent of $n'_\ell(0)$ since it depends only on inputs after $t = 0$. Since $n'_\ell(t)$ differs from $w_\ell(t)$ only by a non-random quantity (6), and since $E\, n'_\ell(t) = 0$,

$$E(n'_\ell(0)\, n'_\ell(t)) = E\left(n^2_\ell(0)\right) e^{-t/\tau} = \gamma^2 e^{-t/\tau} \qquad t > 0 \quad . \tag{B-2}$$

This dependence is also mentioned in [6].

Let us now consider the time average of w,

$$q' = \frac{1}{T} \int_0^T w_\ell(t)\, dt \quad .$$

The variance of q is given by

$$\mathrm{var}(q') = \frac{1}{T^2} E\left[\int_0^T \int_0^T n'_\ell(t)\, n'_\ell(s)\, ds dt\right]$$

$$= \frac{\gamma^2}{T^2} \int_0^T \int_0^T e^{-|t-s|/\tau}\, ds\, dt$$

$$= \frac{2\tau}{T^2} \gamma^2 \left[T + \tau(e^{-T/\tau} - 1)\right] \quad . \tag{B-3}$$

For large T

$$\lim_{T\to\infty} \mathrm{var}(q') = \gamma^2 \frac{2\tau}{T} \quad . \tag{B-4}$$

Note that the variable v appearing in equation (7) is homogeneous in w_j of order 2 (i.e., depends on $w_j w_k$), and thus by factoring out $(e^{-t/\tau})^2$ we have

$$E(v(t)\, v(0)) = E(v^2(0))\, e^{-2t/\tau} \quad . \tag{B-5}$$

Also, relationship (11) is valid for all M. Hence, if

$$\tilde{z} = \frac{1}{N} \sum_{n=1}^{N} y(n) \quad , \tag{B-6}$$

$$\text{var}(\tilde{z}) = \frac{1}{N} \text{var}(y - v) + q \, \text{var}(v) \tag{B-7}$$

where

$$q = \frac{\tau}{T^2} \left[T + \frac{\tau}{2} \left(e^{-2T/\tau} - 1 \right) \right] \tag{B-8}$$

and

$$T = N \cdot \max(2K, K + L) \tag{B-9}$$

since y(n) is output every max (K, L + K)/K DFT's.

Combining equation (B-9) with the results of this report's second section, we have a more accurate version of equation (30):

$$\text{var}(\tilde{z}) = \frac{1}{2N} \left(E(y^2) - E(\eta_r^4) \right) + q \left(\frac{1}{2} E(y^2) - \mu_y^2 + \frac{1}{2} E(\eta_r^4) \right) . \tag{B-10}$$

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