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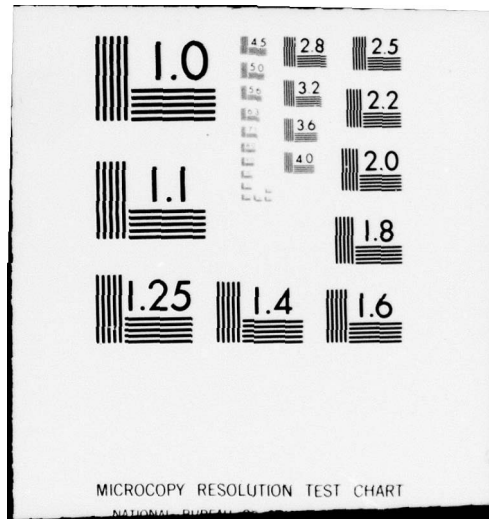
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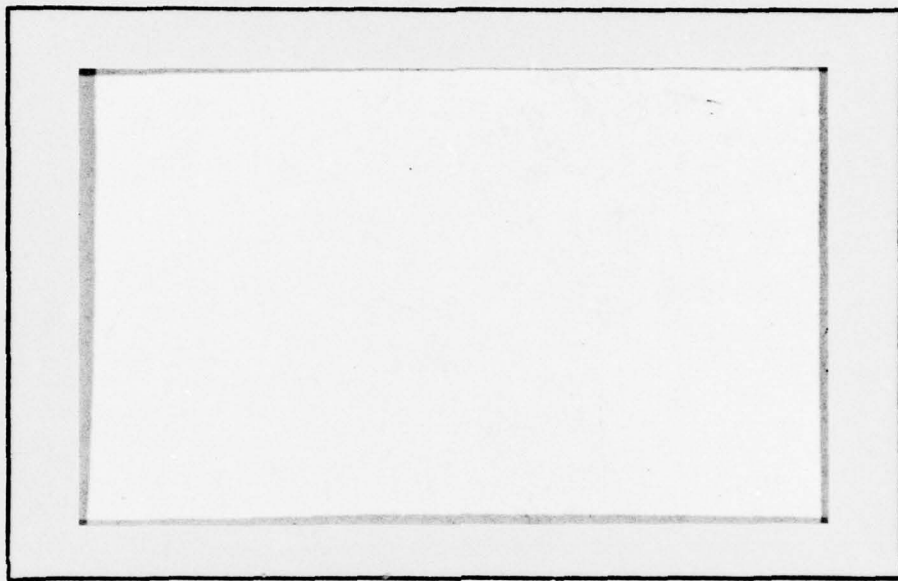
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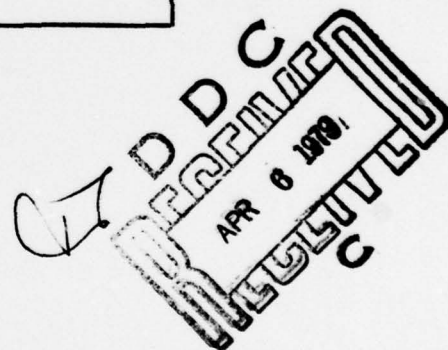
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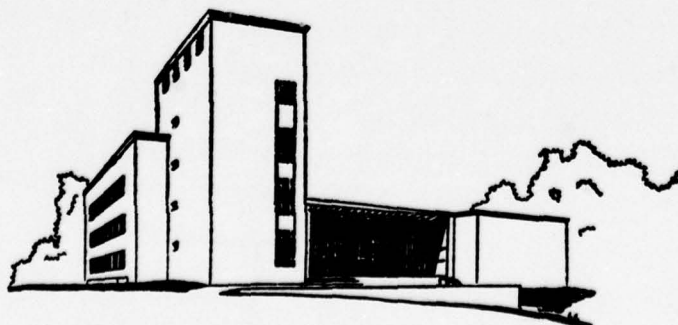
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6 INTEGER AND FRACTIONAL MATCHINGS.

by

10 Egon/Balas

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# Abstract

We examine the connections between maximum cardinality edge matchings in a graph and optimal solutions to the associated linear program, which we call maximum  $f$ -matchings (fractional matchings). We say that a maximum matching  $M$  separates an odd cycle with vertex set  $S$ , if  $M$  has no edge with exactly one end in  $S$ . An odd cycle is separable if it is separated by at least one maximum matching. We show that (1) a graph  $G$  has a maximum  $f$ -matching that is integer, if and only if it has no separable odd cycles; (2) the minimum number  $q$  of vertex-disjoint odd cycles for which a maximum  $f$ -matching has fractional components, equals the maximum number  $s$  of vertex-disjoint odd cycles, separated by a maximum matching; (3) the difference between the cardinality of a maximum  $f$ -matching and that of a maximum matching in  $G$  is one half times  $s$ ; (4) any maximum  $f$ -matching with fractional components for a minimum number  $s$  of vertex-disjoint odd cycles defines a maximum matching obtainable from it in  $s$  steps; and (5) if a maximum  $f$ -matching has fractional components for a set  $Q$  of odd cycles that is not minimum, there exists another maximum  $f$ -matching with fractional components for a minimum-cardinality set  $S$  of odd cycles, such that  $S \subset Q$ ,  $|Q \setminus S|$  is even, and the cycles in  $Q \setminus S$  are pairwise connected by alternating paths.

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## INTEGER AND FRACTIONAL

### EDGE-MATCHINGS

by

Egon Balas

#### 1. Introduction

Let  $G = (V, E)$  be an undirected graph with  $|V| = m$  and  $|E| = n$ , let  $A$  be the vertex-edge incidence matrix of  $G$ , and  $e_p$  the  $p$ -vector of 1's. Consider the problem of finding a maximum-cardinality edge-matching in  $G$

$$EM(G) \max\{e_n x \mid Ax \leq e_m, x \in \{0,1\}^n\}$$

and the associated linear program

$$LEM(G) \max\{e_n x \mid Ax \leq e_m, x \geq 0\}.$$

We will call  $EM(G)$  the matching problem, and  $LEM(G)$  the fractional matching, or shortly the f-matching problem. A feasible solution to  $EM(G)$  will be called a matching, a feasible solution to  $LEM(G)$ , an f-matching. Clearly, every matching is an f-matching.

We wish to investigate the relationship between maximum matchings and maximum f-matchings in a graph  $G$ , i.e., between optimal solutions to  $EM(G)$  and  $LEM(G)$ . For a problem  $P$ , we denote by  $v(P)$  the value of (an optimal solution to)  $P$ . For vertex sets  $S, T \subseteq V$ , we denote by  $(S, T)$  the set of edges of  $G$  with one end in  $S$  and the other in  $T$ , and by  $\langle S \rangle$  the subgraph of  $G$  induced by  $S$ . Also,  $\bar{S} = V \setminus S$ .

An alternating path relative to a matching  $M$  is a path whose edges alternate between  $M$  and  $\bar{M}$ . An  $M$ -augmenting path is an alternating path whose end vertices are distinct and are not incident with  $M$ . A matching  $M$  in  $G$  is of maximum cardinality if and only if  $G$  contains no  $M$ -augmenting path (Berge [2]).

If  $G$  has no odd cycles (i.e., is bipartite),  $A$  is totally unimodular and every basic solution to  $LEM(G)$  is integer, i.e.,

$$(1) \quad v(LEM(G)) = v(EM(G)).$$

If  $G$  is not bipartite,  $LEM(G)$  has fractional basic solutions, and the convex hull of feasible integer solutions, i.e., the matching polytope, is defined (together with the constraints of  $LEM(G)$ ) by the inequalities

$$(2) \quad \sum_{e \in (S,S)} x_e \leq \frac{1}{2}(|S| - 1), \quad S \subseteq V, \quad |S| \geq 3 \text{ and odd}$$

(Edmonds [3]). Any basic feasible solution to  $LEM(G)$  has components equal to 0, 1 or  $\frac{1}{2}$  (Balinski [1]). Furthermore, if  $x_e = \frac{1}{2}$  then  $e$  belongs to some odd cycle  $C$ , and  $x_f = \frac{1}{2}, \forall f \in C$ .

Let  $M$  be a maximum matching in  $G$ . If  $G$  is bipartite,  $M$  is a maximum  $f$ -matching; but the converse is false, as illustrated by the graph in figure 1, where the maximum matching shown in heavy lines is also a maximum  $f$ -matching, in spite of the presence of a 5-cycle.

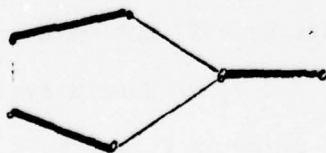


Fig. 1

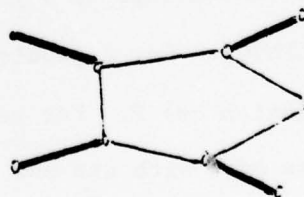


Fig. 2

It would seem that whether a maximum matching  $M$  is also a maximum  $f$ -matching, depends on whether  $M$  covers all vertices of every odd cycle; but this is not so. The matching in the graph of figure 2 leaves a vertex of an odd cycle exposed, yet it is a maximum  $f$ -matching. On the other hand, the

maximum matching  $M$  in the graph of figure 3a covers all vertices of the only odd cycle of the graph, yet it is not a maximum  $f$ -matching, as shown by the  $f$ -matching  $M'$  of figure 3b:  $|M| = 3$ , while the value of  $M'$  is  $3\frac{1}{2}$ .

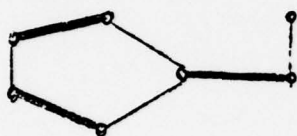


Fig. 3a

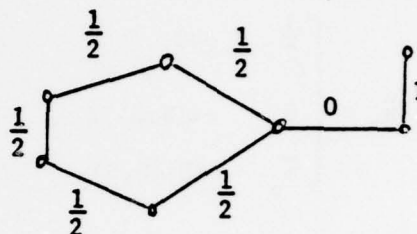


Fig. 3b

## 2. Separable Odd Cycles

The key concept for understanding the connection between maximum matchings and maximum  $f$ -matchings is the following.

Let  $M$  be a maximum matching and  $C$  an odd cycle in  $G$ , and let  $S$  be the vertex set of  $C$ . We will say that  $M$  separates  $C$  if

$$(3) \quad M \cap (S, \bar{S}) = \emptyset.$$

An odd cycle which is separated by at least one maximum matching in  $G$  will be called separable. If  $C$  is a separable odd cycle in  $G$ , with vertex set  $S$ , then  $G$  has a maximum matching of the form  $M = M_1 \cup M_2$ , where  $M_1$  and  $M_2$  are maximum matchings in  $\langle S \rangle$  and in  $\langle \bar{S} \rangle$  respectively. If  $C$  is a nonseparable odd cycle with vertex set  $S$ , then every maximum matching contains at least  $k + 1$  edges incident with  $S$ .

Theorem 1.  $G$  has a maximum  $f$ -matching that is integer if and only if  $G$  has no separable odd cycle.



Proof. Necessity. Let  $M$  be a maximum matching in  $G$  which separates an odd cycle  $C$  with vertex set  $S$ , and  $|S| = 2k + 1$ . Since  $M$  is maximal and  $M \cap (S, \bar{S}) = \emptyset$ ,  $M$  contains  $k$  edges of  $(S, S)$ . Let  $\bar{x}$  be the incidence vector of  $M$ , and define  $\hat{x}$  by

$$\hat{x}_e = \begin{cases} \frac{1}{2} & e \in C \\ 0 & e \in (S, S) \setminus C \\ \bar{x}_e & e \in E \setminus (S, S) \end{cases}$$

Clearly,  $\hat{x}$  is a feasible solution to  $LEM(G)$ , and

$$\begin{aligned} v(LEM(G)) &\geq e_n \hat{x} \\ &= |M| - k + \frac{1}{2}(k + 1) > v(EM(G)), \end{aligned}$$

hence  $G$  has no maximum  $f$ -matching that is integer.

Sufficiency. Let  $v(LEM(G)) > v(EM(G))$ , and let  $\hat{x}$  be a maximum  $f$ -matching in  $G$  having fractional components for a minimum number of odd cycles, say  $C_1, \dots, C_q$ . For  $i = 1, \dots, q$ , let  $S_i$  be the vertex set of  $C_i$ , with  $|S_i| = 2k_i + 1$ . Clearly,  $S_i \cap S_j = \emptyset$ ,  $\forall i, j \in \{1, \dots, q\}$ . Further, let  $S = \bigcup_{i=1}^q S_i$ . The components  $\hat{x}_e$  of  $\hat{x}$  such that  $\hat{x}_e = \frac{1}{2}$  define a matching  $M'$  in  $\langle \bar{S} \rangle$  which is clearly maximum, or else  $\hat{x}$  itself could not be a maximum  $f$ -matching.

For  $i = 1, \dots, q$ , let  $M_i$  be a set of  $k_i$  mutually nonadjacent edges of  $C_i$ , i.e., a maximum matching in  $\langle S_i \rangle$ . Then the edge set

$$(4) \quad M = M' \cup M_1 \cup \dots \cup M_q$$

is a matching in  $G$  such that  $M \cap (S_i, \bar{S}_i) = \emptyset$ ,  $i = 1, \dots, q$ .

Next we prove by contradiction that  $M$  is of maximum cardinality, which implies that the odd cycles  $C_1, \dots, C_q$  are all separable and thus completes the proof of the theorem.

Suppose  $M$  is not maximum. Then there exists an  $M$ -augmenting path  $P$  in  $G$ . Moreover, we claim that the maximum matchings  $M_i$  in  $\langle S_i \rangle$ ,  $i = 1, \dots, q$ , which are not unique, can always be chosen such that  $P$  has no edge in common with any of the cycles  $C_i$ ,  $i = 1, \dots, q$ . For suppose  $P$  contains an edge  $(u, w) \in C_i$  for some  $i \in \{1, \dots, q\}$ . Since  $C_i$  has only one vertex exposed with respect to  $M$ , say  $t$ , at least one of the end vertices of the path  $P$ , say  $v$ , is not contained in  $C_i$ . Of the vertices  $u$  and  $w$ , let  $w$  be the one lying between  $u$  and  $v$  in  $P$ . Then the maximum matching  $M_i$  in  $\langle S_i \rangle$  can be replaced by  $\tilde{M}_i$ , containing those  $k_i$  mutually nonadjacent edges of  $C_i$  that leave exposed  $w$  rather than  $t$ . This replaces the matching  $M$  in  $G$  by

$$\tilde{M} = (M \setminus M_i) \cup \tilde{M}_i,$$

such that  $|\tilde{M}| = |M|$ . The portion of  $P$  between  $t$  and  $w$  can then be removed, and the remaining portion, between  $w$  and  $v$ , is an  $\tilde{M}$ -augmenting path  $\tilde{P}$  not containing  $(u, w)$ . If  $\tilde{P}$  still has some edge in common with  $C_i$ , the above procedure can be repeated. After a finite number of applications of this procedure, we obtain a matching  $\hat{M}$  in  $G$  such that  $|\hat{M}| = |M|$ ,

$$\hat{M} = M' \cup \tilde{M}_1 \cup \dots \cup \tilde{M}_q$$

(where each  $\tilde{M}_i$  is a maximum matching in  $\langle S_i \rangle$ ), and an  $\hat{M}$ -augmenting path  $\hat{P}$ , not containing any edge of any  $C_i$ ,  $i = 1, \dots, q$ . This proves that w.l.o.g., the path  $P$  can be assumed to have no edge in common with any  $C_i$ ,  $i = 1, \dots, q$ .

If we now reverse the assignment of edges in  $P$  with respect to  $M$ , we obtain a matching  $M^*$  in  $G$ , such that  $|M^*| = |M| + 1$ . But from  $M^*$  one can construct an  $f$ -matching  $\bar{x}$  in  $G$  that has fractional components for fewer odd cycles than  $\hat{x}$ , contrary to our assumptions on  $\hat{x}$ .

To construct  $\bar{x}$ , first note that since  $P$  contains no edge of any  $C_i$  and is an alternating path,  $P$  contains no edge of  $\langle S \rangle$ , and the only vertices of  $P$  contained in  $S$  are one, or possibly both, end vertices of  $P$ . The fact that  $P$  has at least one vertex in  $S$  follows from the maximality of the matching  $M' = M \cap (\bar{S}, \bar{S})$  in  $\langle \bar{S} \rangle$ .

Let  $v_i$  and  $v_j$  be the two end vertices of  $P$ . We consider two cases.

Case 1.  $v_i \in S_i$  for some  $i \in \{1, \dots, q\}$ ,  $v_j \notin S$ . Let  $e_1, \dots, e_{2k+1}$  be the edges of  $C_i$ , and let  $e_1$  and  $e_{2k+1}$  be incident with  $v_i$ . Define  $\bar{x}$  by

$$\bar{x}_e = \begin{cases} 1 - \hat{x}_e & e \in P \\ 1 & e \in \{e_2, e_4, \dots, e_{2k}\} \\ 0 & e \in \{e_1, e_3, \dots, e_{2k+1}\} \\ \hat{x}_e & \text{otherwise} \end{cases}$$

Case 2.  $v_i \in S_i$  and  $v_j \in S_j$  for some  $i, j \in \{1, \dots, q\}$ . Let  $e_r$ ,  $r = 1, \dots, 2k_i + 1$ , and  $d_s$ ,  $s = 1, \dots, 2k_j + 1$ , be the edges of  $C_i$  and  $C_j$  respectively, where  $e_1$  and  $e_{2k_i+1}$  are incident with  $v_i$ , and  $d_1$  and  $d_{2k_j+1}$  are incident with  $v_j$ . Then define  $\bar{x}$  by

$$\bar{x}_e = \begin{cases} 1 - \hat{x}_e & e \in P \\ 1 & e \in \{e_2, e_4, \dots, e_{2k_i}\} \cup \{d_2, d_4, \dots, d_{2k_j}\} \\ 0 & e \in \{e_1, e_3, \dots, e_{2k_i+1}\} \cup \{d_1, d_3, \dots, d_{2k_j+1}\} \\ \hat{x}_e & \text{otherwise.} \end{cases}$$

In Case 1,  $\bar{x}$  has fractional components for  $q - 1$  odd cycles, and  $e_n \bar{x} = e_n \hat{x} + \frac{1}{2}$ , since the reversal of edge assignments in  $P$  adds 1 to, while the change of values for  $e \in C_i$  subtracts  $\frac{1}{2}$  from, the value of  $e_n \hat{x}$ . In case 2,



$\bar{x}$  has fractional components for  $q - 2$  odd cycles, and  $e_n \bar{x} = e_n \hat{x}$ . In both cases the existence of  $\bar{x}$  contradicts our assumption that  $\hat{x}$  is a maximum  $f$ -matching which has fractional components for a minimum number ( $q$ ) of odd cycles.

Thus  $M$  is a maximum cardinality matching in  $G$ , which separates the odd cycles  $C_1, \dots, C_q$ .

The graphs in figures 1 and 2 both have maximum  $f$ -matchings that are integer, since in each case the unique maximum matching shown in heavy lines does not separate the unique odd cycle of the graph, hence the latter has no separable odd cycle. The graph of figure 3, on the other hand, has a maximum matching (other than the one shown in figure 3a) which separates the odd cycle: it contains two edges of the cycle, and a third edge not incident with any vertex of the cycle. Therefore there exists no maximum  $f$ -matching that is integer.

It is interesting to note that if  $LEM(G)$  has an integer optimum, it does not follow that the linear program dual to  $LEM(G)$ , namely the fractional vertex covering problem  $LVC(G)$ , also has an integer optimum. In other words, the absence of a separable odd cycle in  $G$  does not guarantee that the cardinality of a maximum edge matching is equal to that of a minimum vertex cover. For example the graph of figure 4 has no separable odd cycle and therefore it has a maximum  $f$ -matching of value 3 that is integer. On the other hand, the cardinality of a minimum vertex cover is 4, and the fractional vertex cover that assigns value  $\frac{1}{2}$  to every vertex  $v$  is the only one with value 3.

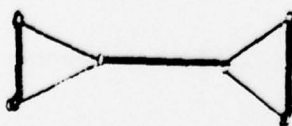


Fig. 4a

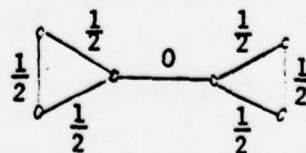


Fig. 4b

The proof of Theorem 1 is constructive in the sense that it gives a procedure for obtaining a maximum matching in an arbitrary graph  $G$  from a maximum  $f$ -matching in  $G$  that satisfies an additional requirement (having fractional components for a minimum number of odd cycles). It turns out that this procedure can be reversed, to obtain a maximum  $f$ -matching from a maximum matching that, again, satisfies an additional requirement: to separate a maximum number of disjoint odd cycles. (Two odd cycles are disjoint if they have no common vertex.) The next Corollary and its proof state this relationship more precisely.

For a maximum matching  $M$ , let  $\sigma(G, M)$  be the maximum number of disjoint odd cycles separated by  $M$ . For a maximum  $f$ -matching  $x$ , let  $\gamma(G, x)$  be the number of odd cycles for which  $x$  has fractional components. Finally, let  $\mathcal{M}$  be the set of maximum matchings, and  $\mathcal{X}$  the set of maximum  $f$ -matchings in  $G$ .

Corollary 1.1.

$$\max_{M \in \mathcal{M}} \sigma(G, M) = \min_{x \in \mathcal{X}} \gamma(G, x).$$

Proof. In the sufficiency part of the proof of Theorem 1, a maximum  $f$ -matching  $\hat{x}$  such that

$$\gamma(G, \hat{x}) = q = \min_{x \in \mathcal{X}} \gamma(G, x)$$

was used to construct the matching defined by (4), which was shown to be maximum in  $G$ , and which separates  $q$  cycles of  $G$ . We claim that  $q = \max_{M \in \mathcal{M}} \sigma(G, M)$ .

For suppose not, then there exists a maximum matching  $M^*$  in  $G$  that separates  $p \geq q + 1$  odd cycles  $C_1, \dots, C_p$ . For  $i = 1, \dots, p$ , let  $S_i$  be the vertex set of

$C_i$ ,  $|S_i| = 2k_i + 1$ , let  $S = \bigcup_{i=1}^p S_i$ , and let  $\bar{x}$  be the incidence vector of  $M^*$ .

Then setting  $\tilde{x}_e = \frac{1}{2}$  for  $e \in C_i$ ,  $i = 1, \dots, p$ ,  $\tilde{x}_e = \bar{x}_e$  otherwise, defines an f-matching  $\tilde{x}$  such that  $e_n \tilde{x} = |M^*| + \frac{1}{2} p$ . On the other hand,  $e_n \tilde{x} = |M| + \frac{1}{2} q$ , where  $M$  is the maximum matching defined by (4). Since  $|M| = |M^*|$  and  $p \geq q + 1$ , this contradicts the assumption that  $\hat{x}$  is a maximum f-matching in  $G$ .||

Next we turn to the relationship between the value of a maximum f-matching and that of a maximum matching in an arbitrary graph  $G$ . Let  $\Delta(G)$  denote the difference of these two values, i.e.,

$$(5) \quad \Delta(G) = v(\text{LEM}(G)) - v(\text{EM}(G)).$$

It is not hard to derive bounds on  $\Delta(G)$ . If  $C$  is an odd cycle with vertex set  $S$ , clearly  $\Delta(< S >) = \frac{1}{2}$ . Since in the absence of odd cycles in  $G$ ,  $\Delta(G) = 0$ , if  $\gamma(G)$  is the number of odd cycles in  $G$ , it is easy to see that

$$(6) \quad \Delta(G) \leq \frac{1}{2} \gamma(G).$$

This also yields a bound on  $\Delta(G)$  independent of the number of odd cycles. Indeed, since a cycle has at least 3 edges,  $\gamma(G) \leq \frac{1}{3} |E|$ , i.e.,

$$(7) \quad \Delta(G) \leq \frac{1}{6} |E|,$$

a bound which is attained for any graph  $G$  which is the union of triangles.

Further, since  $\frac{1}{3} |E| \leq v(\text{EM}(G))$  obviously holds for any graph  $G$ , from

(7) one also has

$$(8) \quad \frac{v(\text{LEM}(G))}{v(\text{EM}(G))} \leq \frac{3}{2}.$$

The bounds (6), (7) and (8) are more or less trivial. The proof of Theorem 1 yields a considerably stronger bound, namely the actual value of  $\Delta(G)$ .

We define the separation number  $\sigma(G)$  of a graph  $G$  as the maximum number of disjoint odd cycles separated by any maximum matching in  $G$ , i.e.,

$$\sigma(G) = \max_{M \in \mathcal{M}} \sigma(G, M).$$

Corollary 1.2.  $\Delta(G) = \frac{1}{2} \sigma(G).$

Proof. Let  $\hat{x}$  be a maximum  $f$ -matching in  $G$  such that  $\gamma(G, \hat{x}) = \sigma(G)$ . Then the matching  $M$  defined by (4) with respect to  $\hat{x}$  is maximum, as shown in the proof of Theorem 1. Clearly,  $|M| = e_n \hat{x} - \frac{1}{2} \sigma(G).$

The difference between the value of a maximum  $f$ -matching and that of a maximum matching can be considerably smaller than  $\frac{1}{2} \gamma(G)$ , the bound given by (6), since the separation number  $\sigma(G)$  of a graph can be much smaller than the number  $\gamma(G)$  of its odd cycles. Figure 5 shows a graph  $G$  with  $\sigma(G) = 0$  and  $\gamma(G) = 2p$ , where  $p$  can be made arbitrarily large (in the figure,  $p = 5$ ). The unique maximum matching, which is also a maximum  $f$ -matching in  $G$ , is shown in heavy lines.

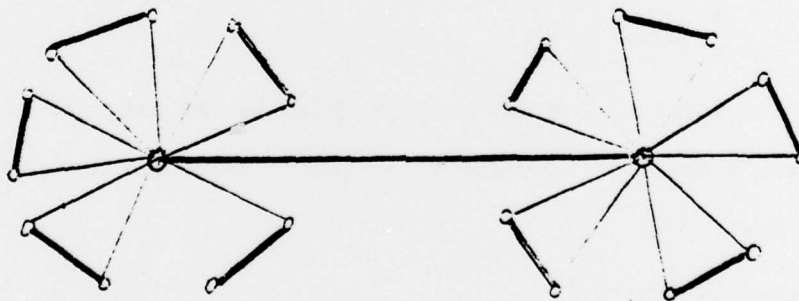


Fig. 5

For  $S \subseteq V$ , we denote by  $N(S)$  the set of vertices in  $\bar{S}$  adjacent to some vertex in  $S$ . For  $v \in V$ , we denote by  $I(v)$  the set of edges incident with  $v$  in  $G$ .



Corollary 1.3. Every graph  $G$  has a maximum  $f$ -matching  $\hat{x}$  such that

$$(9) \quad \hat{x}_e = \begin{cases} \frac{1}{2} & \forall e \in C_1, & i = 1, \dots, \sigma(G) \\ 0 & \forall e \in (S_i, V) \setminus C_1, & i = 1, \dots, \sigma(G) \\ 0 \text{ or } 1 & \forall e \in (\bar{S}, \bar{S}) \end{cases}$$

and

$$(10) \quad \sum_{e \in I(v)} \hat{x}_e = 1, \quad \forall v \in N(S),$$

where  $C_1, \dots, C_{\sigma(G)}$  is a maximum-cardinality set of disjoint separable odd cycles in  $G$ ,  $S_i$  is the vertex set of  $C_i$ ,  $i = 1, \dots, \sigma(G)$ , and  $S = S_1 \cup \dots \cup S_{\sigma(G)}$ .

Conversely, every feasible solution  $\hat{x}$  of  $LEM(G)$  that satisfies (9) is a maximum  $f$ -matching in  $G$ , and also satisfies (10).

Proof. From Corollary 1.1,  $G$  has a maximum  $f$ -matching  $\hat{x}$  such that  $\gamma(G, \hat{x}) = \sigma(G)$ . Such an  $\hat{x}$  clearly satisfies (9). Conversely, if  $\hat{x}$  satisfies (9), then the matching  $M$  defined by (4) relative to  $\hat{x}$  is maximum in  $G$ , hence  $e_n \hat{x} = |M| + \frac{1}{2} \sigma(G) = v(EM(G)) + \frac{1}{2} \sigma(G)$  and thus  $\hat{x}$  is maximal.

To show by contradiction that every maximum  $f$ -matching that satisfies (9) also satisfies (10), suppose (10) is violated for some  $v \in N(S)$ . Let  $u \in S_i$  be the vertex of  $S$  adjacent to  $v$ , and let  $e_1, \dots, e_{2k_i+1}$  be the edges of  $C_i$ , with  $e_1$  and  $e_{2k_i+1}$  incident with  $u$ .

Since (10) is violated for  $v$ , either  $\hat{x}_e = 0, \forall e \in I(v)$ , or  $\hat{x}_{e_*} = \frac{1}{2}$  for some  $e_* \in I(v)$  and  $\hat{x}_e = 0, \forall e \in I(v) \setminus \{e_*\}$ . In the first case, the vector  $\tilde{x}$  defined by

$$\tilde{x}_e = \begin{cases} 1 & e = (u, v) \\ 1 & e \in \{e_2, e_4, \dots, e_{2k_i}\} \\ 0 & e \in \{e_1, e_3, \dots, e_{2k_i+1}\} \\ \hat{x}_e & \text{otherwise} \end{cases}$$

is an  $f$ -matching with  $e_n \tilde{x} = e_n \hat{x} + \frac{1}{2}$ . In the second case,  $\bar{x}$  defined by  $\bar{x}_e = \tilde{x}_e, \forall e \in E \setminus \{e_*\}, \bar{x}_{e_*} = 0$ , is an  $f$ -matching with  $e_n \bar{x} = e_n \hat{x}$ , but with fractional components for  $\gamma(G, \hat{x}) - 1$  odd cycles. In both cases the outcome contradicts the assumption that  $\hat{x}$  is a maximum  $f$ -matching with fractional components for a minimum number of odd cycles. ||

While every graph has a maximum  $f$ -matching  $\hat{x}$  satisfying (9) (and hence (10)), it is not true that every (basic) maximum  $f$ -matching satisfies (9). In case of alternative optima there may be only one that satisfies the condition of Corollary 1.3. The graph  $G$  of figure 4, for instance, has the maximum  $f$ -matching shown in 4a, which satisfies (9), since  $G$  has no separable odd cycle; but it also has the one shown in 4b, which does not satisfy the condition of Corollary 1.3.

### 3. Generating Maximum Matchings from Maximum $f$ -Matchings

Let  $\hat{x}$  be a basic maximum  $f$ -matching, let

$$C(\hat{x}) = \{C_1, \dots, C_q\}$$

be the set of odd cycles  $C_i$  such that  $\hat{x}_e = \frac{1}{2}, \forall e \in C_i$ , and for  $i = 1, \dots, q$ , let  $S_i$  be the vertex set of  $C_i$ .

We will denote by  $\mathcal{F}(\hat{x})$  the family of matchings associated with  $\hat{x}$ , introduced in the proof of Theorem 1, i.e., the set of all matchings of the form

$$M = M' \cup M_1 \cup \dots \cup M_q,$$

where  $M' = \{e \in E | \hat{x}_e = 1\}$ , and for  $i = 1, \dots, q$ ,  $M_i$  is a maximum matching in  $\langle S_i \rangle$ .

We define two operations on the components of  $\hat{x}$  associated with certain edge sets of  $G$ .

By complementing on  $E' \subseteq E$ , we mean replacing  $\hat{x}_e$  by  $\hat{x}'_e = 1 - \hat{x}_e$ ,  $\forall e \in E'$ .

By alternate rounding on  $C_i \in \mathcal{C}(\hat{x})$ ,  $C_i = \{e_1, \dots, e_{2k_i+1}\}$ , we mean replacing  $\hat{x}_e$ ,  $e \in C_i$ , by  $\hat{x}'_e = 0$ ,  $e \in \{e_1, e_3, \dots, e_{2k_i+1}\}$ , and  $\hat{x}'_e = 1$ ,  $e \in \{e_2, \dots, e_{2k_i}\}$ .

From the results of the previous section it follows that one way of finding a maximum matching in a graph is to find a maximum  $f$ -matching  $\hat{x}$  with a minimum-cardinality  $\mathcal{C}(\hat{x})$  and then use alternative rounding on the odd cycles in  $\mathcal{C}(\hat{x})$  to generate a matching  $M \in \mathcal{M}(\hat{x})$ .

Remark 1. The matchings in  $\mathcal{M}(\hat{x})$  are of maximum cardinality if and only if  $\mathcal{C}(\hat{x})$  is of minimum cardinality.

Proof. From Corollaries 1.1 and 1.2. ||

Next we address the problem of finding a maximum  $f$ -matching  $\hat{x}$  with a minimum-cardinality  $\mathcal{C}(\hat{x})$ .

Lemma 1. If  $\hat{x}$  is a basic maximum  $f$ -matching and  $q = |\mathcal{C}(\hat{x})| > \sigma(G)$ , then

$$q \equiv \sigma(G) \pmod{2}.$$

Proof. Let  $M \in \mathcal{M}(\hat{x})$ . Then  $e_n \hat{x} = |M| + \frac{1}{2} q$ , and from Corollary 1.2,  $e_n \hat{x} = v(EM(G)) + \frac{1}{2} \sigma(G)$ . Since both  $|M|$  and  $v(EM(G))$  are integer, subtracting the second equation from the first one produces

$$\frac{1}{2}(q - \sigma(G)) \equiv 0 \pmod{1}$$

and multiplication by 2 yields the congruence in the Lemma. ||

We define an alternating path relative to an  $f$ -matching  $\hat{x}$  as a path whose edges  $e$  alternate between  $\hat{x}_e = 0$  and  $\hat{x}_e = 1$ . We say that a path  $P$  connects two odd cycles  $C_1, C_2$ , if  $P$  connects a vertex of  $C_1$  to a vertex of  $C_2$ . A vertex  $v$  of  $G$  is exposed relative to  $\hat{x}$  if  $\hat{x}_e = 0$  for all edges  $e$  incident with  $v$ .



**Theorem 2.** Let  $\hat{x}$  be a basic maximum  $f$ -matching such that  $q = |C(\hat{x})| > \sigma(G)$ . Then  $G$  contains  $\alpha = \frac{1}{2}(q - \sigma(G))$  vertex-disjoint alternating paths  $P_k$  relative to  $\hat{x}$ , each of which connects two odd cycles  $C_{i_k}, C_{j_k} \in C(\hat{x})$ , and the cycles  $C_{i_k}, C_{j_k}, k = 1, \dots, \alpha$ , are all distinct. Furthermore, alternate rounding on the odd cycles  $C_{i_k}, C_{j_k}$ , and complementing on the paths  $P_k, k = 1, \dots, \alpha$ , produces a maximum  $f$ -matching  $\bar{x}$  with  $|C(\bar{x})| = \sigma(G)$ .

**Proof.** Let  $M^*$  be a maximum matching in  $G$  and  $M \in \mathcal{F}(\hat{x})$ . Since  $|M| = e_n \hat{x} - \frac{1}{2} q$  and  $|M^*| = e_n \hat{x} - \frac{1}{2} \sigma(G)$  (Corollary 1.2),  $|M^*| - |M| = \frac{1}{2}(q - \sigma(G)) = \alpha$ . It is therefore known (see Theorem 1 of [4]) that  $G$  contains  $\alpha$  vertex-disjoint  $M$ -augmenting paths,  $\pi_1, \dots, \pi_\alpha$ .

We show by contradiction that each  $\pi_k$  connects two cycles,  $C_{i_k}, C_{j_k} \in C(\hat{x}), i_k \neq j_k$ . Suppose this is false; then at least one end-vertex of  $\pi_k$ , say  $u$ , is in  $\bar{S}$ , where  $S = \bigcup_{i=1}^q S_i$ . Since  $\langle \bar{S} \rangle$  contains no  $M'$ -augmenting path (or else  $\hat{x}$  would not be a maximum  $f$ -matching),  $\pi_k$  is incident with some vertex of  $S$ . Let  $v$  be the first vertex of  $S$  encountered when  $\pi_k$  is traversed starting from  $u$ , and let  $\pi'_k$  be the subpath of  $\pi_k$  connecting  $u$  to  $v$ . Clearly,  $\pi'_k$  is an alternating path relative to  $\hat{x}$ . Then using alternate rounding on  $C_{i_1}$ , the cycle whose vertex set  $S_{i_1}$  contains  $v$ , and complementing on  $\pi'_k$ , produces an  $f$ -matching  $\bar{x}$  such that  $e_n \bar{x} = e_n \hat{x} + \frac{1}{2}$ , contrary to the assumption that  $\hat{x}$  is maximum.

This proves that both end-vertices of  $\pi_k$  belong to  $S$ . Further, since each  $S_i$  contains only one vertex exposed relative to  $M$ , the sets  $S_{i_k}, S_{j_k}$  containing the two end-vertices of  $\pi_k$  are distinct - hence so are the two cycles  $C_{i_k}, C_{j_k}$ ; and for the same reason (that each  $S_i$  contains only one exposed vertex) the  $\alpha$  pairs of odd cycles  $C_{i_k}, C_{j_k}$ , connected by the  $\alpha$  paths  $\pi_k$  are all distinct.

Now using alternate rounding on the  $2\alpha$  odd cycles connected by the paths  $\pi_k$  reduces the value of the f-matching for every such cycle by  $\frac{1}{2}$ , i.e., for every pair of cycles  $C_{i_k}, C_{j_k}$  connected by a path  $\pi_k$ , by 1. On the other hand, using complementing on the paths  $\pi_k$  increases the value of the f-matching by 1 for every path  $\pi_k$ . Thus the alternate rounding and complementing produces an f-matching  $\bar{x}$  of the same value as  $\hat{x}$ , hence maximum; and it reduces the number of odd cycles for which the f-matching has fractional components, by

$$2\alpha = q - \frac{1}{2} \sigma(G).$$

Two cycles  $C_i, C_j$  with vertex sets  $S_i, S_j$  will be called adjacent, if there exists  $u \in S_i$  and  $v \in S_j$  such that  $(u, v) \in E$ . Since the edge  $(u, v)$  is an alternating path relative to  $\hat{x}$ , we have

Corollary 2.1. If two odd cycles  $C_i, C_j \in C(\hat{x})$  are adjacent, with  $u \in C_i, v \in C_j$  and  $(u, v) \in E$ , then alternate rounding on  $C_i$  and  $C_j$  and complementing on  $(u, v)$ , produces a maximum f-matching  $\bar{x}$  such that  $|C(\bar{x})| = |C(\hat{x})| - 2$ .

Also, from Theorem 2 we have

Corollary 2.2. If  $|C(\hat{x})| > \sigma(G)$ , there exists a maximum f-matching  $\bar{x}$  such that  $C(\bar{x}) \subset C(\hat{x})$  and  $|C(\bar{x})| = \sigma(G)$ .

Theorem 2 can be used to obtain from an arbitrary basic maximum f-matching, one with fractional components for a minimum number of odd cycles. The alternating paths relative to  $\hat{x}$  can be found by a scanning and labeling procedure of the type used for finding augmenting paths relative to a matching. The complexity of such a scanning and labeling routine is  $O(m^2)$  for each path found (where  $m = |V|$ ), and since there are  $\alpha = \frac{1}{2}(q - \sigma(G))$  paths to be found, obtaining from  $\hat{x}$  a maximum f-matching  $\bar{x}$  such that  $|C(\bar{x})| = \sigma(G)$  requires at most  $O(m^2 \cdot \alpha)$  steps.

The problem of finding a maximum  $f$ -matching in an arbitrary graph  $G$  is equivalent to the problem of finding a maximum matching in a bipartite graph  $\hat{G} = (V_1 \cup V_2, E_1 \cup E_2)$ , defined as follows. For every vertex  $v_i \in V$ ,  $\hat{G}$  has a pair of vertices  $v_{i1} \in V_1$  and  $v_{i2} \in V_2$ . For every edge  $(v_i, v_j) \in E$ ,  $G$  has a pair of edges  $(v_{i1}, v_{j2}) \in E_1$  and  $(v_{i2}, v_{j1}) \in E_2$ . We say that the vertices  $v_{i1}$  and  $v_{i2}$ , as well as the edges  $(v_{i1}, v_{j2})$  and  $(v_{i2}, v_{j1})$ , are copies of each other. If we associate the 0-1 vectors  $y^1$  and  $y^2$  with the edge sets  $E_1$  and  $E_2$  respectively, then a matching  $(\hat{y}^1, \hat{y}^2)$  in  $\hat{G}$  defines an  $f$ -matching  $\hat{x}$  in  $G$  via the rule

$$\hat{x}_i = \begin{cases} 1 & \text{if } \hat{y}_i^1 = \hat{y}_i^2 = 1 \\ 0 & \text{if } \hat{y}_i^1 = \hat{y}_i^2 = 0 \\ \frac{1}{2} & \text{if } \hat{y}_i^1 = 0, \hat{y}_i^2 = 1, \text{ or } \hat{y}_i^1 = 1, \hat{y}_i^2 = 0. \end{cases}$$

Obviously, a maximum matching  $M$  in  $\hat{G}$  then defines a maximum  $f$ -matching of cardinality  $\frac{1}{2}|M|$  in  $G$ . Figure 6 shows a graph  $G$  with a maximum  $f$ -matching, and the associated graph  $\hat{G}$  with the corresponding maximum matching.

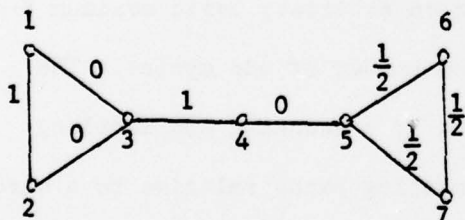


Fig. 6a

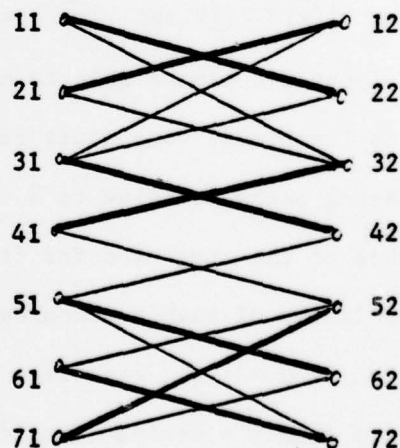


Fig. 6b

A bipartite graph of the above type was introduced by Edmonds and Pulleyblank (see Nemhauser and Trotter [6]) as an equivalent for the fractional vertex packing problem, but the equivalence obviously holds for the fractional matching problem  $LEM(G)$  as well.

Thus finding a maximum  $f$ -matching in an arbitrary graph  $G$  with  $n$  vertices is reduced to finding a maximum matching in a bipartite graph  $\hat{G}$  with  $2n$  vertices. Hopcroft and Karp [5] give an easy to implement  $O(n^{5/2})$  algorithm for finding a maximum matching in a bipartite graph. An algorithm of the same complexity ( $O(n^{5/2})$ ), but much harder to implement, for maximum matching in an arbitrary graph, has been proposed by Even and Kariv [4]. Thus the worst case behavior of algorithms for maximum matching in a bipartite graph seems to be no better than that of their counterparts for arbitrary graphs. Nevertheless, the worst case is not the only, and perhaps not the most relevant one. The need for handling odd cycles in a matching algorithm for arbitrary graphs either by shrinking or by special labeling rules, make these algorithms considerably more difficult to implement than matching algorithms for bipartite graphs. Thus the idea of using the above described equivalence has some merit, even though it could not improve the worst-case bound. However, working out the details of an algorithm based on this approach is beyond the scope of our paper.

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number  $q$  of vertex-disjoint odd cycles for which a maximum  $f$ -matching has fractional components, equals the maximum number  $s$  of vertex-disjoint odd cycles, separated by a maximum matching; (3) the difference between the cardinality of a maximum  $f$ -matching and that of a maximum matching in  $G$  is one half times  $s$ ; (4) any maximum  $f$ -matching with fractional components for a minimum number  $s$  of vertex-disjoint odd cycles defines a maximum matching obtainable from it in  $s$  steps; and (5) if a maximum  $f$ -matching has fractional components for a set  $Q$  of odd cycles that is not minimum, there exists another maximum  $f$ -matching with fractional components for a minimum-cardinality set  $S$  of odd cycles, such that  $S \subseteq Q$ ,  $|Q/S|$  is even, and the cycles in  $Q/S$  are pairwise connected by alternating paths.

← abs. val  $Q/S$   $Q/S$   
is a subset of

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