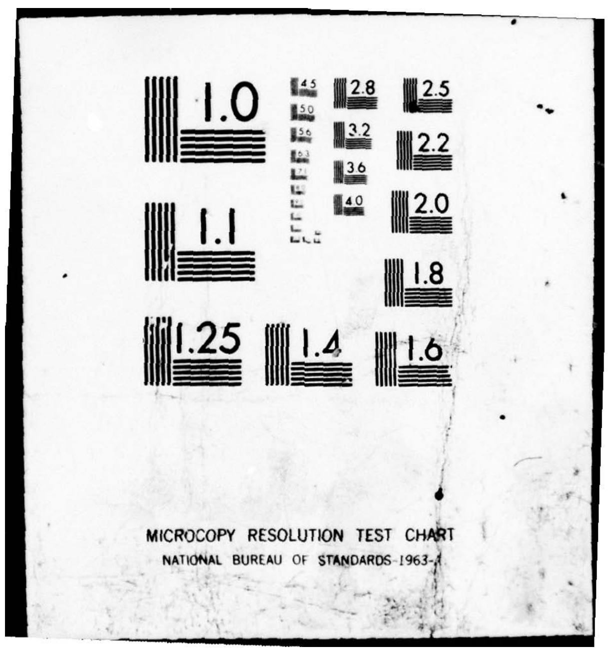
AD-A066 607 WEIDLINGER ASSOCIATES MENLO PARK CALIF F/G 13/13
TRANSIENT ANALYSIS OF EQUIPMENT-STRUCTURE INTERACTION AT HIGH F--ETC(U)
MAY 78 J L SACKMAN, J M KELLY DNA001-77-C-0233

UNCLASSIFIED R-7828 DNA-4675F NL

| OF |

AD
A066607

END
DATE
FILMED
5--79
DDC


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL

DNA 4675F

TRANSIENT ANALYSIS OF EQUIPMENT-STRUCTURE INTERACTION AT HIGH FREQUENCIES

J.L. Sackman
J.M. Kelly
Weidlinger Associates
3000 Sand Hill Road
Menlo Park, California 94025

May 1978

Final Report for Period September 1977—May 1978

CONTRACT No. DNA 001-77-C-0233

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

THIS WORK SPONSORED BY THE DEFENSE NUCLEAR AGENCY UNDER RDT&E RMSS CODE B344077464 Y99QAXSC06152 H2590D.

Prepared for

Director

DEFENSE NUCLEAR AGENCY

Washington, D. C. 20305

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER DNA 4675F	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) TRANSIENT ANALYSIS OF EQUIPMENT-STRUCTURE INTERACTION AT HIGH FREQUENCIES.		5. TYPE OF REPORT & PERIOD COVERED Final Report for Period Sep[redacted]977—May [redacted]78
		6. PERFORMING ORG. REPORT NUMBER R-7828
7. AUTHOR(s) J. L. Sackman J.M. Kelly		8. CONTRACT OR GRANT NUMBER(s) DNA 001-77-C-0233
9. PERFORMING ORGANIZATION NAME AND ADDRESS Weidlinger Associates 3000 Sand Hill Road Menlo Park, California 94025		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NWED Subtask Y99QAXSC061-52
11. CONTROLLING OFFICE NAME AND ADDRESS Director Defense Nuclear Agency Washington, D.C. 20305		12. REPORT DATE May [redacted]78
		13. NUMBER OF PAGES 64
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This work sponsored by the Defense Nuclear Agency under RDT&E RMSS Code B344077464 Y99QAXSC06152 H2590D.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Dynamics
Transient Analysis
Equipment-Structure Interaction

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An analytical method is developed which yields a simple estimate of the maximum dynamic response of light equipment attached to a structure subjected to ground motions. The natural frequency of the equipment, modeled as a single-degree-of-freedom system, is considered to be close, or equal, to one of the natural frequencies of the N-degree-of-freedom structure. This bound provides a convenient rational method for the structural design of the equipment and its installation. → next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

392 150

20. ABSTRACT (Continued)

The approach is based on the transient analysis of lightly damped, tuned or detuned equipment-structure systems in which the mass of the equipment is small in comparison with that of the structure. It is assumed that the information available to the designer is a design spectrum for the ground motion and fixed-base modal properties of the structure alone and of the equipment alone. The results obtained are estimates of the maximum acceleration and displacement of the equipment in terms of this information. The method can also be used to treat closely-spaced modes in structural systems, where the conventional square root of the sum of squares procedure is known to be invalid.

This analytical method has also been applied to equipment-structure systems which are completely untuned and it has been established mathematically that the conventional floor spectrum method is valid for this case. A closed-form solution is obtained which permits the construction of an estimate of the maximum equipment response. This simple estimate furnishes a convenient method for the rational design of equipment and its mounting without the necessity of computing time histories, as required for the conventional floor spectrum method.

TABLE OF CONTENTS

LIST OF FIGURES

SECTION 1

INTRODUCTION

The design of equipment to withstand dynamic loading of its parent structure has been addressed for a wide range of structures by means of theory and experiment. Shock effects on equipment in hardened, protective structures have been estimated by constructing envelopes to the spectral response at the attachment points, Reference 1. It is usual to avoid mounting sensitive equipment directly on the structure if the fixed base frequency of the equipment falls near a natural frequency of the structure; potentially damaging overlap is avoided by redesigning the equipment or by using shock isolation systems, References 1 and 2. In order to study the effectiveness of shock isolation systems in complex structure over a wide frequency range, impedance testing procedures have been developed, References 3 and 4.

Problems involving equipment-structure interaction have been addressed in submarines, where the equipment is massive, and may be heavier and have lower natural frequencies than the structure, References 5 and 6. Special analysis and design procedures have been developed to take advantage of the unique characteristics of such systems. The spectral response at the attachment point of massive equipment exhibits a minimum at its fixed base frequencies. Therefore the envelope to the spectral response is unnecessarily conservative and the maximum stresses in the equipment will most probably occur at the minima in the shock spectra; this is taken into account in design, Reference 7.

Earthquake response of structures and their contents is another area where equipment-structure interaction has been addressed. One common approach is the floor spectrum method, in which interaction is completely neglected. The equipment is treated as a single-degree-of-freedom system subject to the base motion that the structure would experience at the attachment point in the absence of the equipment. In addition to neglecting interaction, time history analysis of the structure to determine that motion must be carried out. Approximate techniques have been proposed to develop floor response spectra from ground spectra that bypass the computational problems associated with this, but these are ad hoc methods and no estimate of their accuracy can be made (Reference 8). An alternative approach is to consider a N + 1 degree-of-freedom model for the equipment-structure system and subject it to time history analyses for a variety of specified ground motion inputs. This can be cumbersome and expensive for design. Furthermore, conventional methods of dynamic analysis are accurate in the computation of the response for the lower modes, and they frequently possess some form of numerical dissipation to damp out spurious participation from the higher modes. In the cases where structure-equipment interaction is important, at intermediate and high frequencies, the use of such codes can mask significant equipment response. Finally, response spectra methods for N + 1 degree-of-freedom systems suffer from uncertainty in how to combine the peak modal values. The approach using the square root of the sum of the squares, References 9 and 10, has been shown to be inaccurate in the case of closely-spaced modal frequencies and light equipment having a natural frequency equal to one of the natural frequencies of the structure. Penzien and Chopra (Reference 11) and Penzien (Reference 12) have studied this problem for a simple tuned system and have proposed a method in which the response spectra of two-degree-of-freedom systems are used. In this approach, the N modes of of the structure are determined and to each mode considered as a single-degree-of-freedom system, the equipment is attached to give a two-degree-of-freedom system. The time history of the response of each of these N two degree of freedom systems is computed numerically and the maximum response of the entire system evaluated using the square root of the sum of the squares of the maximum value for each. This method is shown by comparison with a direct computation of the time history of the response of the total system to be fairly accurate. It is shown below that the method is unnecessarily complicated, in principle incorrect, and that its accuracy, established in Reference 11 only for the particular low order system studied, is in a sense fortuitous.

The present work is concerned with a rational approach to one aspect of this problem, that of light equipment whose frequency is close to one of the natural frequencies of the structure, a situation described as "tuning." To gain insight into this problem area, idealized models are considered which incorporate the characteristics of a structure and internal equipment, the equipment having natural frequencies which are higher than the fundamental frequency of the structure. The combined structure-equipment system considered is one in which the equipment is relatively light. In previous work (Reference 13), we have described the response of this system to steady-state ground shaking. It was shown that significant interaction effects occur only in the case characterized as tuning, or near-tuning; namely, the situation where the frequency of the equipment considered as a single-degree-of-freedom system is the same as, or close to, one of the natural frequencies of the structure. If the equipment frequency is not tuned, or nearly tuned, to a structural frequency, the response curve is roughly the superposition of the structural response and the equipment response with little interaction. This indicates that the conventional floor spectrum method is valid for the solution of transient problems.

If, on the other hand, the equipment frequency is tuned to a structural frequency, it was found that for the combined system there were two closely spaced frequencies on either side of the tuning frequency and that a band of high amplification appears around the tuning frequency which offers a substantial target for sympathetic oscillation. Thus there is a significant interaction in this situation between the structure and the equipment and suggests that the conventional floor spectrum method, which ignores such interaction, will not be valid for the transient analysis of tuned, or nearly tuned, systems.

A typical result for the steady state response of a structure-equipment system is shown in Figure 1 which is taken from Reference 13. In this case, the equipment was tuned to the third natural frequency of the structure, and the curve shown is the ratio of the equipment acceleration to the input ground acceleration considered as a function of the frequency of the input normalized with respect to the natural frequency of the equipment. The mass of the structure in this example was a thousand times that of the equipment.

In this report, we will describe a method valid for the transient analysis of tuned or nearly tuned structure-equipment interaction. This method is considered as a replacement for the floor spectrum analysis. It utilizes the shock spectra for the specified input to the structure and directly provides the shock spectra for the equipment. In this sense, it is simpler and easier to apply to design than the floor spectrum method which requires the time history of the structure response to be computed in order to obtain the equipment spectrum. The simplicity of these results is due to the fact that it is possible to take advantage of the mathematical structure of the equations of the tuned or nearly tuned system and also to

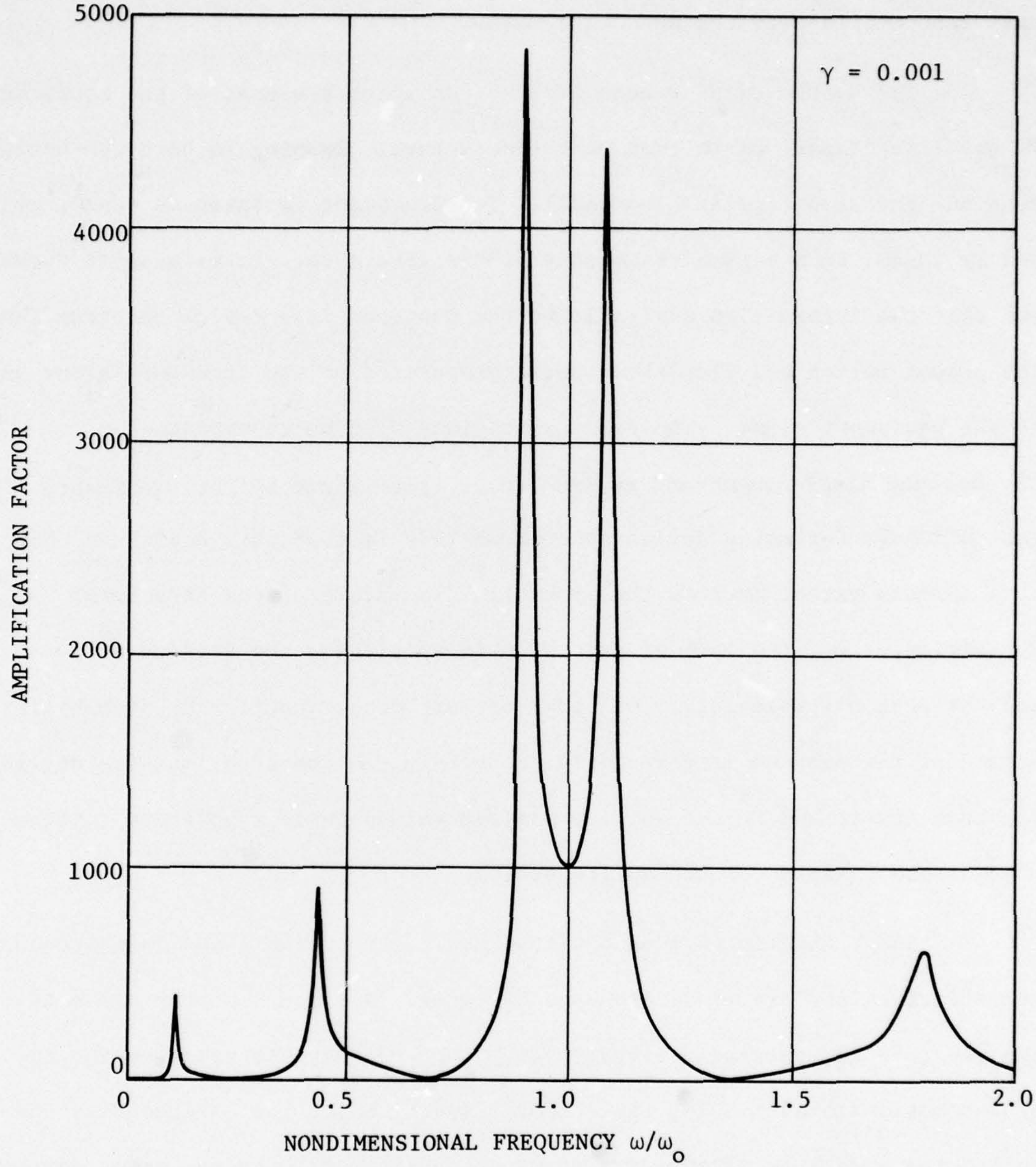


Figure 1. Amplification Factor for Equipment Acceleration as a Function of Frequency: Equipment Tuned to Third Structure Frequency.

make use of asymptotic methods made possible by the smallness of the equipment mass in comparison to the structure mass.

The system to be studied is one for which the mass of the equipment is small in comparison to that of the structure. Damping in both the equipment and the structure is also small. The equipment is taken as tuned, or nearly tuned, to a natural frequency of the structure. It is assumed throughout that the information available to the designer is a design spectrum for the ground motion and fixed-base modal properties of the structure alone and of the equipment alone. The results obtained will be an estimate of the maximum displacement and maximum force experienced by the equipment. The rationale for using design spectra methods is that they are cheap and to a certain extent include the probabilistic nature of the structure definition, i.e., the lack of precision in specifying the structural parameters and the probabilistic nature of shock or earthquake input. The probabilistic nature of the problem is taken into account in the construction of a design spectrum itself and in the way the maximum values in each mode are combined to predict the maximum for the entire system.

Under the limitations outlined above, the results obtained are easy and efficient for practical use by a designer. A feature of the present analysis, is the extremely simple result that if response spectrum for the ground motion is available, the response spectrum for the equipment can be calculated merely by multiplying it by an amplification factor which depends on the mass ratio, the degree of detuning and the damping.

The reason for the simplicity of the result can be indicated on physical grounds, if it is recognized that the major portion of the response is a classical beat phenomenon. It is well known that in weakly coupled systems with the same frequency, the response of the system involves a perfect energy exchange between each component. The same phenomenon appears here. The weak coupling is achieved through the small mass ratio of equipment to structure. For simplicity of explanation, consider an undamped tuned equipment-structure system. Were the structure alone subjected to a ground motion, the velocity imparted to it would be independent of the mass and determined only by the ground motion. If the same ground motion were to be applied directly to the equipment alone, the same velocity would be imparted to it in the case of tuning. The kinetic energy, on the other hand, will be proportional to the mass of the system excited. In the case of the equipment, this would be much smaller than that of the structure. However, if the equipment is attached to the structure and the structure is subjected to ground motion, the kinetic energy imparted to the structure is transmitted in its entirety to the equipment if tuned, and the velocity imparted is amplified by the reciprocal of the square root of the mass ratio. This simple result recurs for all other response spectra of the equipment.

It is clear that damping will play an important role in this process, since the energy transfer takes many cycles and much of the kinetic energy in a damped system could be dissipated before being transferred. This is accounted for in the analysis when damping is included. We have also studied the case when the equipment frequency is not perfectly tuned to a natural frequency of the structure. If the detuning is small, then two closely spaced frequencies again occur and the determination of the resulting peak response must be calculated with care, as for the tuned case. The final result is analogous to the tuned situation, but the amplification factor is modified by the inclusion of a parameter describing the degree of detuning. When the equipment frequency is grossly detuned from that of all natural frequencies of the structure, then the floor spectrum method is valid.

However, we have derived a much simpler method which gives the peak response of the equipment in terms of the given ground spectra multiplied by amplification factors depending upon the fixed-base modal properties of the structure alone and of the equipment alone.

SECTION 2

SUMMARY OF RESULTS

To assist the reader in understanding the theoretical development and its applications, the final result is presented in this section in terms of the free-field spectral acceleration, S_A, which is a function of frequency and damping, and a modification factor, which depends on properties of the equipment and the structure. Although the theory is developed for the special cases of systems which are (a) damped, tuned; (b) undamped, slightly detuned; and (c) damped, slightly detuned, the result is given by one general expression as follows:

$$|\ddot{u}|_{max} = \frac{S_A\left((1+\frac{\xi}{2})\omega,\ \frac{\beta+B}{2}\right)}{(\gamma+\xi^2+4\beta B)^{\frac{1}{2}}} \qquad (2\text{-}1)$$

where

S_A = free-field spectral acceleration evaluated at the frequency $(1 + \frac{\xi}{2})\omega$, with spectral damping ratio $(\beta+B)/2$.

ξ = detuning parameter, expressing the degree of detuning between the structure (mode of interest has frequency Ω) and the equipment (frequency ω). $\xi = (\Omega - \omega)/\omega$.

B, β = damping ratio for structure, equipment, respectively

$B = C/2\Omega M$

$\beta = c/2\omega m$

C, c = viscous damping coefficients for structure and equipment, respectively.

M, m = masses of structure and equipment, respectively

$\gamma = m/M$

In order to apply Equation (2-1), the free-field spectral acceleration S_A is evaluated at the shifted frequency $(1 + \frac{\xi}{2})\omega$ and at the average damping ratio $(\frac{\beta + B}{2})$. Then the modification factor is evaluated which depends on the detuning ξ, the mass ratio γ, and the damping ratios β and B.

In the event that the mass ratio $\gamma << \xi^2 + 4\beta B$ and $\xi << 1$, corresponding to light, nearly tuned equipment, Equation (2-1) becomes equivalent to the conventional floor spectrum approach and has the further advantage of being applicable when the only available information on ground motion is a free-field design spectrum.

SECTION 3

MODAL ANALYSIS OF STRUCTURE-EQUIPMENT SYSTEM

The main characteristic of the interaction between the structure and the tuned equipment is in the equivalent two-degree-of-freedom system comprising the equipment and the particular mode of the structure to which it is tuned. For comparison with the modal analysis of the structure the appropriate equations of a two-degree-of-freedom system are given.

The equivalent two degree of freedom system is shown in Fig. 2 in which lower case letters refer to the equipment and upper case letters to the suitable modal properties of the structure. The equivalent or effective ground motion is denoted by $u_g^{eff}(t)$ and the equations of motion are:

$$\begin{aligned} M\ddot{U} + C(\dot{U} - \dot{u}_g^{eff}) + K(U - u_g^{eff}) &= F(t) \\ F(t) = -m\ddot{u} &= c(\dot{u} - \dot{U}) + k(u - U) \end{aligned} \qquad (3\text{-}1)$$

where F(t) is the interaction force between the two systems and where m and M are equipment and structure modal masses, k and K are the respective stiffnesses and c and C are the damping coefficients. It is convenient to introduce the variables

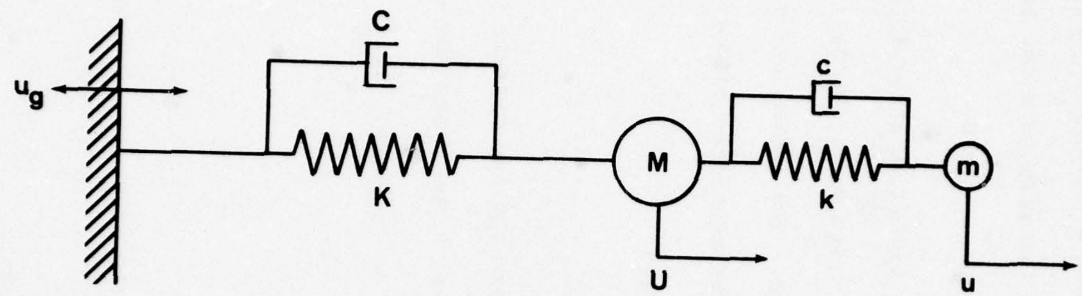
$$\omega = \sqrt{k/m} \quad , \quad \Omega = \sqrt{K/M}$$

$$\beta = c/2\omega m \quad , \quad B = C/2\Omega M$$

the detuning parameter $\zeta = (\Omega-\omega)/\omega$ and the mass ratio $\gamma = m/M$ in terms of which the equations become:

$$\begin{aligned} \ddot{U} + 2B\Omega\dot{U} + \Omega^2 U &= 2B\Omega\dot{u}_g^{eff} + \Omega^2 u_g^{eff} + F/M \\ F(t)/m = -\ddot{u} &= 2\beta\omega(\dot{u}-\dot{U}) + \omega^2(u-U) \end{aligned} \qquad (3\text{-}2)$$

Applying the Laplace transform to these equations and eliminating the structure response gives a single equation for the transform of the equipment



$$\Omega^2 = \frac{K}{M}, \quad \omega^2 = \frac{k}{m}$$

$$2\beta\omega = c; \quad 2B\Omega = C$$

Figure 2. Two-Degree-of-Freedom System Giving System Parameters.

response in the form

$$\bar{u}[(p^2 + 2\beta\omega p + \omega^2) + \frac{\gamma p^2(2\beta\omega p + p^2)}{p^2 + 2B\Omega p + \Omega^2}] = \frac{(2\beta\omega p + \omega^2)(2B\Omega p + \Omega^2)}{p^2 + 2B\Omega p + \Omega^2} \bar{u}_g^{eff} \tag{3-3}$$

The equations of motion of the N degree of freedom structural system take the form

$$\sum_{j=1}^{N} M_{ij}\ddot{U}_j + \sum_{j=1}^{N} C_{ij}\dot{U}_j + \sum_{j=1}^{N} K_{ij}U_j = \sum_{j=1}^{N} C_{ij}r_j\dot{u}_g + \sum_{j=1}^{N} K_{ij}r_ju_g + F(t)e_i \tag{3-4}$$

where M_{ij} is the mass matrix, C_{ij}, the damping matrix and K_{ij} the stiffness matrix. The vector r_i is a vector of influence coefficents introduced to couple the actual ground motion, $u_g(t)$ to the structure and e_i a vector which is zero at every degree of freedom except that one denoted by the index r, to which the equipment is attached where it takes unit value. F is the interaction force between the equipment and the structure.

The natural frequencies Ω_n and modes ϕ_i^n are given by the equation

$$\Omega_n^2 \sum_{j=1}^{N} M_{ij}\phi_j^n = \sum_{j=1}^{N} K_{ij}\phi_j^n \tag{3-5}$$

Assuming that the damping is sufficiently small that it does not introduce coupling between the modes the equations in modal coordinates become

$$\ddot{q}_k + 2B_k\Omega_k\dot{q}_k + \Omega_k^2 q_k = \sum_{i=1}^{N} \phi_i^k F_i/M_k$$

where

$$M_k = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_i^k\phi_j^k M_{ij} \text{ , } 2B_k\Omega_k = \sum_{i=1}^{N}\sum_{j=1}^{N} \phi_i^k \phi_j^k C_{ij}/M_k \tag{3-6}$$

and

$$F_i = \sum_{j=1}^{N} C_{ij}r_j\dot{u}_g + \sum_{j=1}^{N} K_{ij}r_ju_g + Fe_i$$

The Laplace transform of the structure response $U_j(t)$ is given by

$$\bar{U}_j = \sum_{k=1}^{N} \frac{\sum_{i=1}^{N} \Phi_i^k \Phi_j^k \bar{F}_i(p)}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)}$$

with (3.7)

$$\bar{F}_i(p) = \sum_{\ell=1}^{N} (C_{i\ell} r_\ell p + K_{i\ell} r_\ell)\bar{u}_g + \bar{F}e_i$$

The corresponding equation of motion for the equipment displacement u is

$$-m\ddot{u} = F(t) = c(\dot{u} - \dot{U}_r) + K(u - U_r) \tag{3.8}$$

or in terms of Laplace transforms

$$-p^2\bar{u} = \bar{F}(p)/m = (2\beta\omega p + \omega^2)(\bar{u} - \bar{U}_r) \tag{3.9}$$

From Eq. 3.9 a relationship between u and U_r is obtained in the form

$$(p^2 + 2\beta\omega p + \omega^2)\bar{u} = (2\beta\omega p + \omega^2)\bar{U}_r \tag{3.10}$$

and from Eq. 3.7 this can be written as

$$\bar{u}(p^2 + 2\beta\omega p + \omega^2) = (2\beta\omega p + \omega^2) \sum_{k=1}^{N} \frac{\sum_{i=1}^{N} \Phi_i^k \Phi_r^k [\bar{F}e_i + \sum_{\ell=1}^{N} (C_{i\ell} r_\ell p + K_{i\ell} r_\ell)\bar{u}_g]}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)}$$

But we also have $\bar{F} = -mp^2\bar{u}$. Using this to eliminate $\bar{F}$ in the above, leads to the final transformed equation for the equipment response.

$$\bar{u}\left[(p^2 + 2\beta\omega p + \omega^2) + \sum_{k=1}^{N} \frac{mp^2(2\beta\omega p + \omega^2)\Phi_r^k\Phi_r^k}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)}\right]$$

$$= (2\beta\omega + \omega^2) \sum_{k=1}^{N} \frac{\Phi_r^k \sum_{i=1}^{N} \Phi_i^k \sum_{\ell=1}^{N} (C_{i\ell} r_\ell p + K_{i\ell} r_\ell)}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)} \bar{u}_g \tag{3.11}$$

We note that $\sum_{i=1}^{N} K_{i\ell}\Phi_i^k$ can be written as $\Omega_k^2 \sum_{i=1}^{N} M_{i\ell}\Phi_i^k$ and the assumption of small damping allows the representation of $\sum_{i=1}^{N} C_{i\ell}\Phi_i^k$ as $2B_k\Omega_k \sum_{i=1}^{N} M_{i\ell}\Phi_i^k$. Thus the solution for $\bar{u}$ for the multidegree of freedom system takes the form

$$\bar{u}[(p^2 + 2\beta\omega p + \omega^2) + p^2 \sum_{k=1}^{N} \frac{m\Phi_r^{k^2}(2\beta\omega p + \omega^2)}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)}]$$

$$= \sum_{k=1}^{N} \frac{\Phi_r^k \sum_{i=1}^{N} \Phi_i^k \sum_{\ell=1}^{N} M_{i\ell} r_\ell (2B_k\Omega_k p + \Omega_k^2)(2\beta\omega p + \omega^2)}{M_k(p^2 + 2B_k\Omega_k p + \Omega_k^2)} \bar{u}_g \tag{3.12}$$

In performing the inversion of the Laplace transform by the use of residue theory, we are interested in the zeros of the term in brackets on the left hand side. These zeros are the poles of the transfer function. Here we are restricting attention to the case where the equipment frequency is close to a structural frequency, Ω_n, say. This is indicated in Fig. 3. In this figure the two expressions in the brackets on the left hand side of Eq. 3.12 have been plotted separately. These plots were obtained by replacing p by $i\Omega$ and then drawing the graph of each of the two resulting functions in the bracketed expression. For simplicity of illustration we have plotted the figure for the completely undamped case ($\beta = B_1 = \ldots = B_n = 0$). The negative of the first function plots as a simple quadratic in Ω, becoming zero at $\Omega = \omega$, the natural frequency of the equipment. The function involving the summation plots as the complicated curve which goes to $\pm\infty$ at $\Omega = \Omega_k$, k = 1, 2....N, the natural frequencies of the structure. Two such curves have been plotted, one when the equipment mass is small and one when it is not.

The values of Ω at the intersections of these two curves give the locations of the zeros of the bracketed expression on the left hand side of Eq. 3.12, which are the poles of the transfer function for the equipment

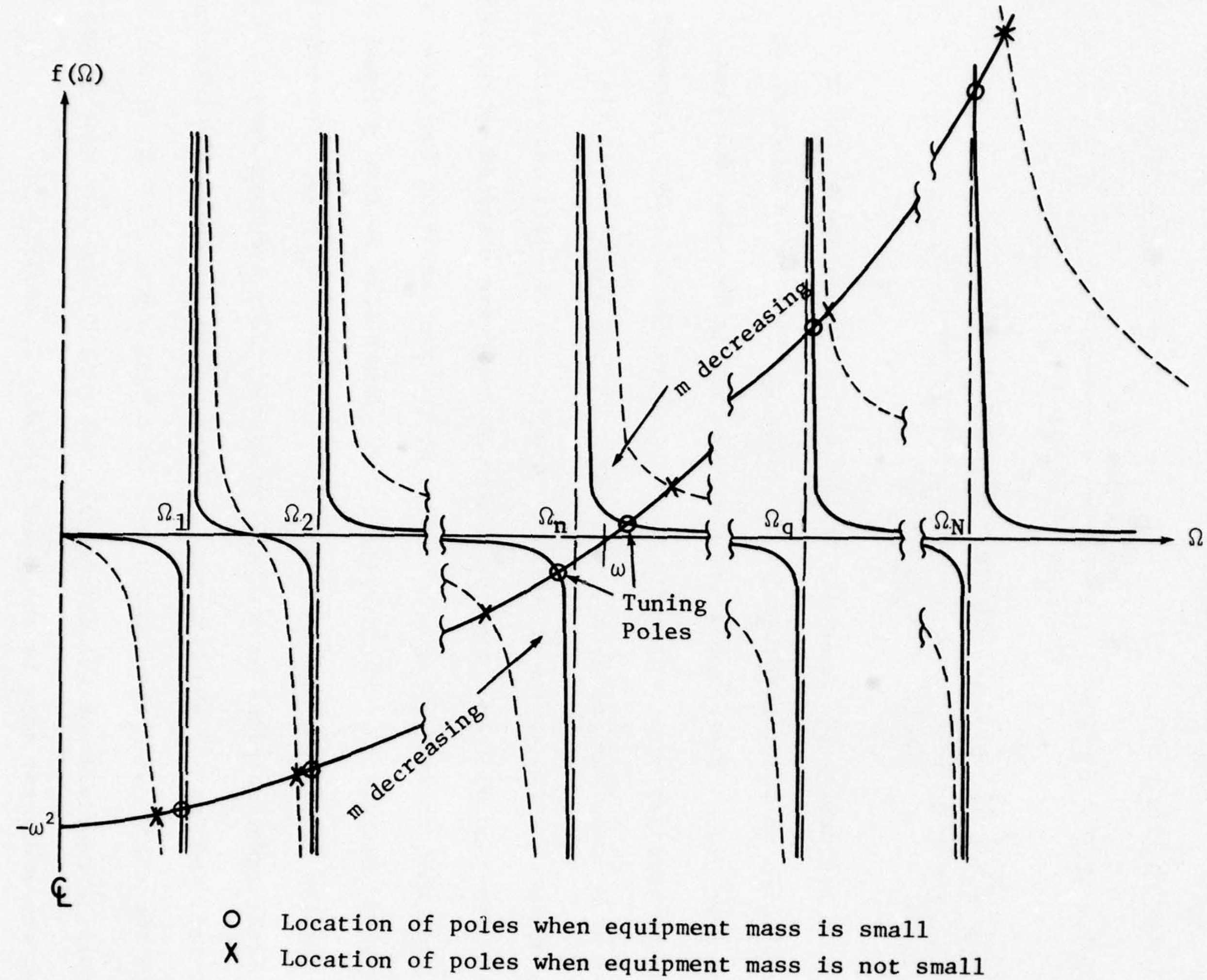


Figure 3. Location of Poles of Equipment Response Transfer Function for Nearly Tuned NDOF Structure.

response, taking into account equipment-structure interaction. It is seen that in the case of small equipment mass these poles, all of which are simple, occur near the natural frequencies of the structure. It is important to note that two closely spaced poles, which we call the tuning poles, are located near the equipment frequency, one below it and the other above it as shown in the figure. These two closely spaced poles coalesce into a double pole when $\omega = \Omega_n$ and $m \to 0$. Thus the contribution to the sum of the residues at all of the poles is dominated by those which are associated with the two tuning poles. In computing the residues at the tuning poles, it is clear that the contribution from the summation expression is dominated by the term where k = n because then the denominator of that term is nearly zero. Hence in the region of $p = i\omega$, Eq. 3.12 can be approximated by

$$\bar{u}[(p^2 + 2\beta\omega p + \omega^2) + p^2 \frac{m\Phi_r^{n^2}}{M_n} \frac{2\beta\omega p + \omega^2}{p^2 + B_n\Omega_n p + \Omega_n^2}]$$

$$= \frac{(2B_n\Omega_n p + \Omega_n^2)(2\beta\omega p + \omega^2)}{p^2 + 2B_n\Omega_n + \Omega_n^2} \left(\sum_{i=1}^{N} \sum_{\ell=1}^{N} \Phi_r^n \Phi_i^n M_{i\ell} r_\ell / M_n \right) \bar{u}_g \qquad (3.13)$$

Comparing this expression to that for the two-degree-of-freedom system, Eq. 3.3, we see that the effective mass ratio is

$$\gamma^{eff} = m\Phi_r^{n^2} / M_n \qquad (3.14)$$

and the equivalent ground motion is

$$u_g^{eff} = C_r^n u_g$$

where

$$C_r^n = \Phi_r^n \sum_{i=1}^{N} \sum_{j=1}^{N} \Phi_i^n M_{ij} r_j / M_n \qquad (3.15)$$

The solution technique used here will be to obtain the contribution in the region near $p = i\omega$ by considering the equivalent two-degree-of-freedom system defined by the above equations. The contribution to the solution from the tuning poles requires special treatement, and this will be done in the context of the equivalent two-degree-of-freedom system. The contributions at the other (N-1) poles is straightforward and will be included after the two-degree-of-freedom analysis has been completed. It should be emphasized that the use of an equivalent two-degree-of-freedom system is not essential, but is only conceptual and introduces no further approximations beyond that made in passing from Eq. 3.12 to Eq. 3.13.

SECTION 4

ANALYSIS OF TRANSFER FUNCTION FOR TWO-DEGREE-OF-FREEDOM NEARLY TUNED SYSTEM

In the previous section the parameters which appear in the equivalent two-degree-of-freedom system have been derived. Returning to the Laplace transform of the equation of motion of the two-degree-of-freedom, Eq. 3.3, to simplify the subsequent notation, the terms B and Ω are used to represent the structural parameters B_n, Ω_n of the tuned mode and γ and u_g should be interpreted as the effective mass ratio and ground motion as given by Eqs. 3.14 and 3.15. In terms of these parameters the transformed equipment acceleration u(p) takes the form

$$\bar{\ddot{u}} = [N(p)/D(p)]\bar{\ddot{u}}_g \tag{4.1}$$

where

$$N(p) = (2\beta\omega p + \omega^2)(2B\omega(1+\xi)p + (1+\xi)^2\omega^2) \tag{4.2}$$

and

$$D(p) = p^4 + \omega p^3(2\beta(1+\xi) + 2B(1+\xi)) + \omega^2 p^2(2 + \gamma + 2\xi + \xi^2 + 2\beta 2B(1+\xi)) + \omega^3 p(2\beta(1+\xi)^2 + 2B(1+\xi)) + \omega^4(1+\xi)^2 \tag{4.3}$$

where $\xi = \dfrac{\Omega - \omega}{\omega}$ is the detuning parameter. In what follows attention will be concentrated on the equipment acceleration. Completely parallel results can be easily developed for the equipment displacement.

The nature of the solution depends essentially on the zeroes of the denominator D(p). Since γ, β, B and ξ are small parameters, the roots of D(p) will be close to those of the system with γ, β, B and ξ taken to be zero; namely

$$p = \pm i\omega$$

To determine the location of the poles of D(p) we replace p in Eq. 4.3 by

$$p = i\omega(1+\delta) \tag{4.4}$$

where δ is a small quantity. Only the plus sign is taken since the roots will appear as complex conjugates. In terms of δ the equation D(p) = 0 takes the form

$$\begin{aligned}&\delta^4 + [4 - i[2\beta(1+\gamma) + 2B(1+\xi))]\delta^3 \\ &+ [4 - \gamma - 2\xi - \xi^2 - 4\beta B(1+\xi) - 3i(2\beta(1+\gamma) + 2B(1+\xi))]\delta^2 \\ &+ [-2\gamma - 4\xi - 2\xi^2 - 8\beta B(1+\xi) - i(2\beta(2+3\gamma - 2\xi - \xi^2) + 4B(1+\xi))]\delta \\ &+ [-\gamma - 4\beta B(1+\xi) - i2\beta(\gamma - 2\xi - \xi^2)] = 0\end{aligned} \tag{4.5}$$

Solutions for this equation are easy to obtain when $\beta = 0$, $B = 0$ and $\gamma \neq 0$, $\xi \neq 0$ and are

$$\delta = \left[1 + \xi + \frac{\gamma}{2} + \frac{\xi^2}{2} \mp \left(\gamma + \xi^2 + \xi^3 + \gamma\xi + \frac{\gamma\xi^2}{4} + \frac{\gamma^2}{4} + \frac{\xi^4}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} - 1$$

$$\simeq \frac{1}{2}\left[\xi \mp (\xi^2 + \gamma)^{\frac{1}{2}}\right]$$

Also if $\gamma = 0$, $\xi = 0$ and $\beta \neq 0$, $B \neq 0$, we have

$$\delta = \sqrt{1 - \beta^2} - 1 + i\beta \ , \ \sqrt{1 - B^2} - 1 + iB \simeq i\beta, \ iB \ .$$

Thus, throughout the analysis it will be assumed that β, B, ξ and $\gamma^{\frac{1}{2}}$ are all of the same order, say ε, and the various approximations for δ will be based on assuming δ of order $\varepsilon << 1$. When the parameters are not of the same order the modifications required are obvious.

The solution of Eq. 4.5 retaining terms of order ε^2 is

$$\delta = \frac{\xi}{2} \pm \frac{\lambda}{2} + i\left(\frac{\beta + B}{2} \pm \frac{\mu}{2}\right) \tag{4.6}$$

where here and throughout the remainder of the analysis the upper signs are taken together to give one root and the lower the other. The quantities λ and μ are given by

$$\lambda = \frac{1}{\sqrt{2}} \left[\left\{ (\xi^2 + \gamma - (\beta-B)^2)^2 + 4\xi^2(\beta-B)^2 \right\}^{\frac{1}{2}} + (\xi^2 + \gamma - (\beta-B)^2) \right]^{\frac{1}{2}} \quad (4.7)$$

$$\mu = \frac{1}{\sqrt{2}} \left[\left\{ (\xi^2 + \gamma - (\beta-B)^2)^2 + 4\xi^2(\beta-B)^2 \right\}^{\frac{1}{2}} \pm (\xi^2 + \gamma - (\beta-B)^2) \right]^{\frac{1}{2}} \quad (4.8)$$

It is easy to show that for $\beta \neq 0$ and/or $B \neq 0$ the imaginary part is always positive, thus leading to damped oscillations. In view of the large number of parameters in this solution there are many special cases and in the following sections we consider some of these of particular interest in further detail.

4.1 UNDAMPED TUNED TWO-DEGREE-OF-FREEDOM SYSTEM

In the case $\beta = 0$, $B = 0$ and $\xi = 0$ the solution of Eq. 4-5 retaining terms of order ε^3 is

$$\delta = \pm \frac{\gamma^{\frac{1}{2}}}{2} \left(1 \pm \frac{\gamma^{\frac{1}{2}}}{4}\right) \quad (4.9)$$

It is useful in this case to retain the higher order terms in δ since these terms are necessary in deriving certain later results on the floor spectrum method. In terms of the transform parameter p the roots are

$$p = i\omega \pm i\omega \frac{\gamma^{\frac{1}{2}}}{2} \left(1 \pm \frac{\gamma^{\frac{1}{2}}}{4}\right) \quad (4.10)$$

These are indicated in the root locus diagram Fig. 4, with the corresponding complex conjugate roots. It is clear that the roots remain on the imaginary axis with the small spread between them equal to $\gamma^{\frac{1}{2}} \omega$. These will lead to an undamped oscillating solution.

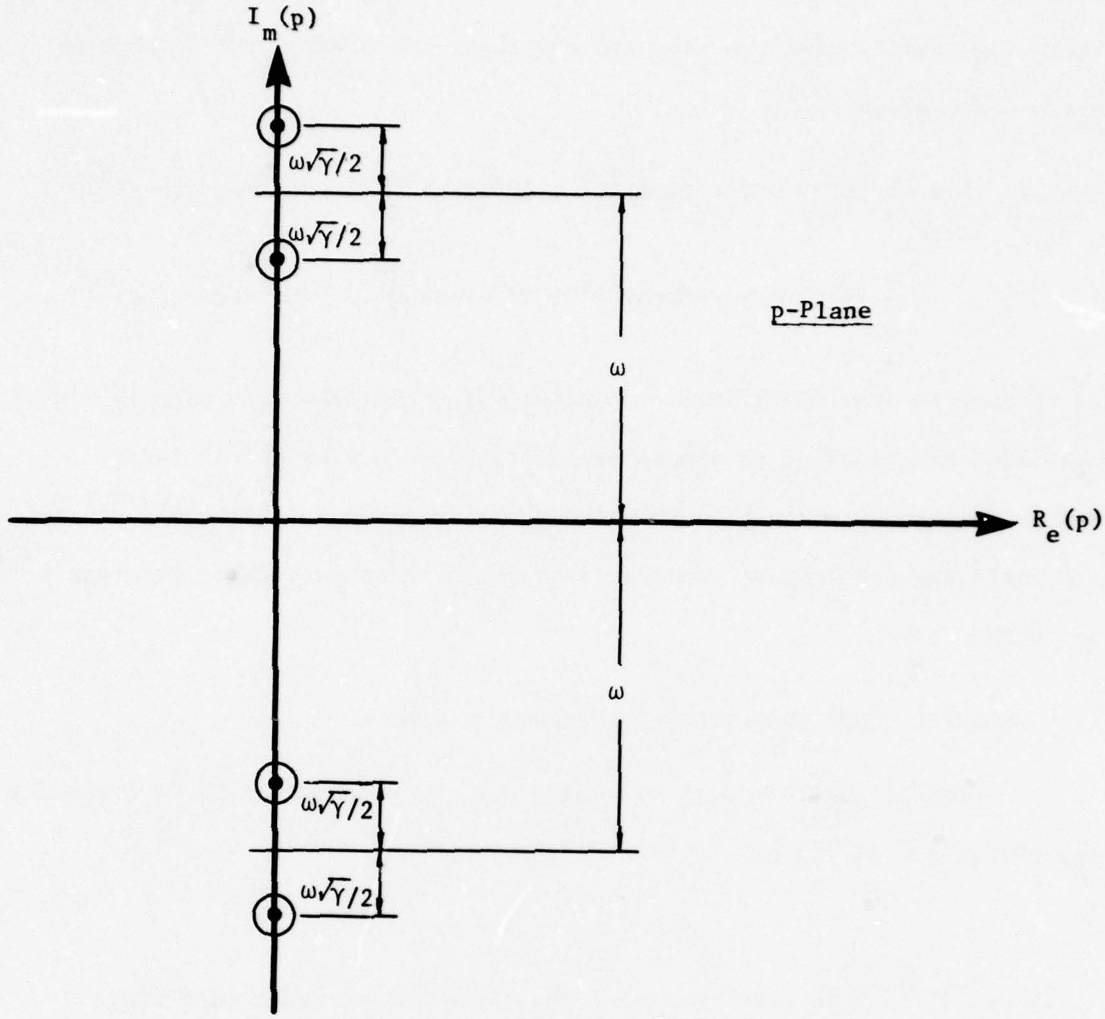


Figure 4. Root Locus Diagram for Undamped, Tuned 2DOF System.

4.2 UNDAMPED SLIGHTLY DETUNED TWO-DEGREE-OF-FREEDOM SYSTEM

In this case the roots are

$$\delta = \frac{\xi}{2} \pm \frac{1}{2} (\gamma + \xi^2)^{\frac{1}{2}} \qquad (4.11)$$

which in terms of p is

$$p = i\omega \left(1 + \frac{\xi}{2}\right) \pm i\omega(\gamma + \xi^2)^{\frac{1}{2}}/2 \qquad (4.12)$$

(We note that when $\gamma \to 0$ these become $p = \pm\, i\omega,\ \pm\, i\Omega$, .) These roots and the corresponding complex conjugate roots are indicated in Fig. 5. Again an undamped oscillatory solution results and the spread between the closely spaced roots is now given by $\omega(\gamma + \xi^2)^{\frac{1}{2}}$.

4.3 DAMPED TUNED TWO-DEGREE-OF-FREEDOM SYSTEM

We note first that in the case $\xi = 0$ there is the possibility of a double root of the equation $D(p) = 0$. For this to be so, certain relationships must exist between the coefficients of the various powers of p in the expression for D(p) given by Eq. 18 when $\xi = 0$. It is easy to show that these conditions are:

$$\gamma\beta = 0 \qquad (4.13)$$

and

$$\gamma + 2\beta B = B^2 + (1 + \gamma)^2 \beta^2 \qquad (4.14)$$

for nonzero γ it is clear from Eq. 4.13 that β must be zero and from Eq. 4.14 we must have $\gamma = B^2$.

The solution of Eq. 4.5, when $\xi = 0$, retaining terms of order ε^2 is:

$$\delta = i\,\frac{\beta + B}{2} \pm \frac{1}{2} \left(\gamma - (\beta - B)^2\right)^{\frac{1}{2}} \qquad (4.15)$$

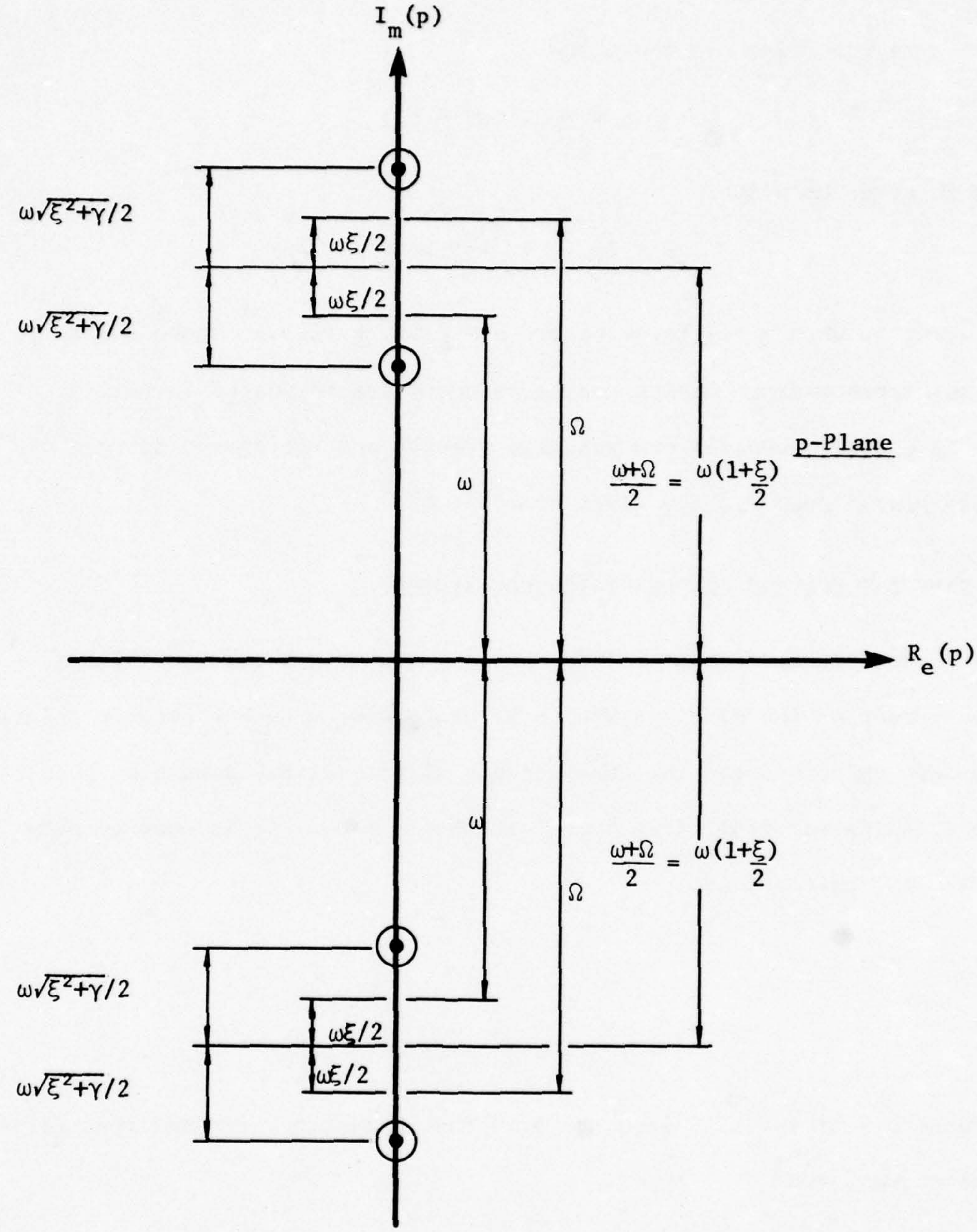


Figure 5. Root Locus Diagram for Undamped, Slightly Detuned 2DOF System.

Utilizing this as a first approximation and using an iterative method the solution of Eq. 4.5 retaining terms of order ε^3 is obtained in the form:

$$\delta = i\,\frac{\beta + B}{2} \pm \frac{1}{2}\{\gamma - (\beta-B)^2 \mp (\gamma-(\beta-B)^2)^{\frac{1}{2}}(\beta^2+B^2-\gamma/2) + i\,(2\beta\gamma+(\beta+B)(\gamma-(\beta-B)^2))\}^{\frac{1}{2}} \tag{4.16}$$

When the term $\gamma - (\beta - B)^2$ is of order ε^2 it dominates in the radical, and it is enough to use Eq. 4.15 for the roots δ .

We note that when $\gamma \to 0$ this gives the two roots

$$\delta = i\beta \quad \text{and} \quad \delta = iB$$

which represents the floor spectrum solution for the damped system. The solution (Eq. 4.15) suggests a double root when $\gamma = (\beta - B)^2$, but it has already been shown that a double root will only occur if $\beta = 0$; thus when $\gamma - (\beta - B)^2$ is of order ε^3 or higher, the complete expression must be used. In the particular case $\gamma = (\beta - B)^2$ the roots are:

$$\delta = i\,\frac{\beta + B \pm \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}{2} \pm \frac{\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}{2} \tag{4.17}$$

When $\beta = 0$, the roots become

$$\delta = i\,\frac{B}{2} \pm \frac{1}{2}\,\{(\gamma-B^2) \mp (\gamma-B^2)^{\frac{1}{2}}(B^2-\frac{\gamma}{2}) + iB(\gamma-B^2)\}^{\frac{1}{2}} \tag{4.18}$$

For fixed B the roots go from

$$\delta = i\,\frac{B}{2} \pm \frac{1}{2}\,\gamma^{\frac{1}{2}} \quad \text{for } \gamma \gg B^2 \quad \text{to} \quad \delta = iB - \frac{B^2}{2}\ ,\ 0 \text{ for } \gamma \to 0\ .$$

When $\gamma = B^2$, Eq. 4.18 indicates a double root at

$$\delta = iB/2$$

However, by consideration of the complete Eq. 4.5 and substitution of $\beta = 0$, $\gamma = B^2$ it is easy to show that the double root is given by:

$$\delta = i\,\frac{B}{2} - \frac{B^2}{8} \qquad (4.19)$$

The pattern of the roots in the p plane for $\beta = 0$ is shown in Fig. 6.

When $\gamma \neq (\beta - B)^2$ the nature of the solution depends on whether $\gamma > (\beta - B)^2$ or $\gamma < (\beta - B)^2$. For the first the roots are given in Eq. 4.15. For the latter they become, to lowest order

$$\delta = i\,(\beta + B \pm ((\beta - B)^2 - \gamma)^{\frac{1}{2}})/2 \qquad (4.20)$$

The roots in the p plane thus have the same imaginary value $i\omega$, but are equally spaced from $(\beta + B)/2$ on a line parallel to the real axis. It is important to note that both roots lie in the left-hand plane for all nonzero values of β, B and γ so that the transient response of the system always decays.

4.4 DAMPED, SLIGHTLY DETUNED TWO-DEGREE-OF-FREEDOM SYSTEM

In this instance the large number of parameters makes it convenient to illustrate the form of the solution by considering the special case when $\beta = B$. Here

$$\lambda = (\gamma + \xi^2)^{\frac{1}{2}}$$

and $\mu = 0$, giving

$$\delta = \frac{\xi}{2} + \frac{1}{2}(\gamma + \xi^2)^{\frac{1}{2}} + i\,\frac{\beta + B}{2} \qquad (4.21)$$

and

$$p = i\omega\,(1 + \frac{\xi}{2} \pm \frac{1}{2}\,(\gamma + \xi^2)^{\frac{1}{2}}) - \omega\,\frac{\beta + B}{2} \qquad (4.22)$$

These roots are similar to those shown in Fig. 5, except that they are shifted to the left by $\omega(\frac{\beta + B}{2})$.

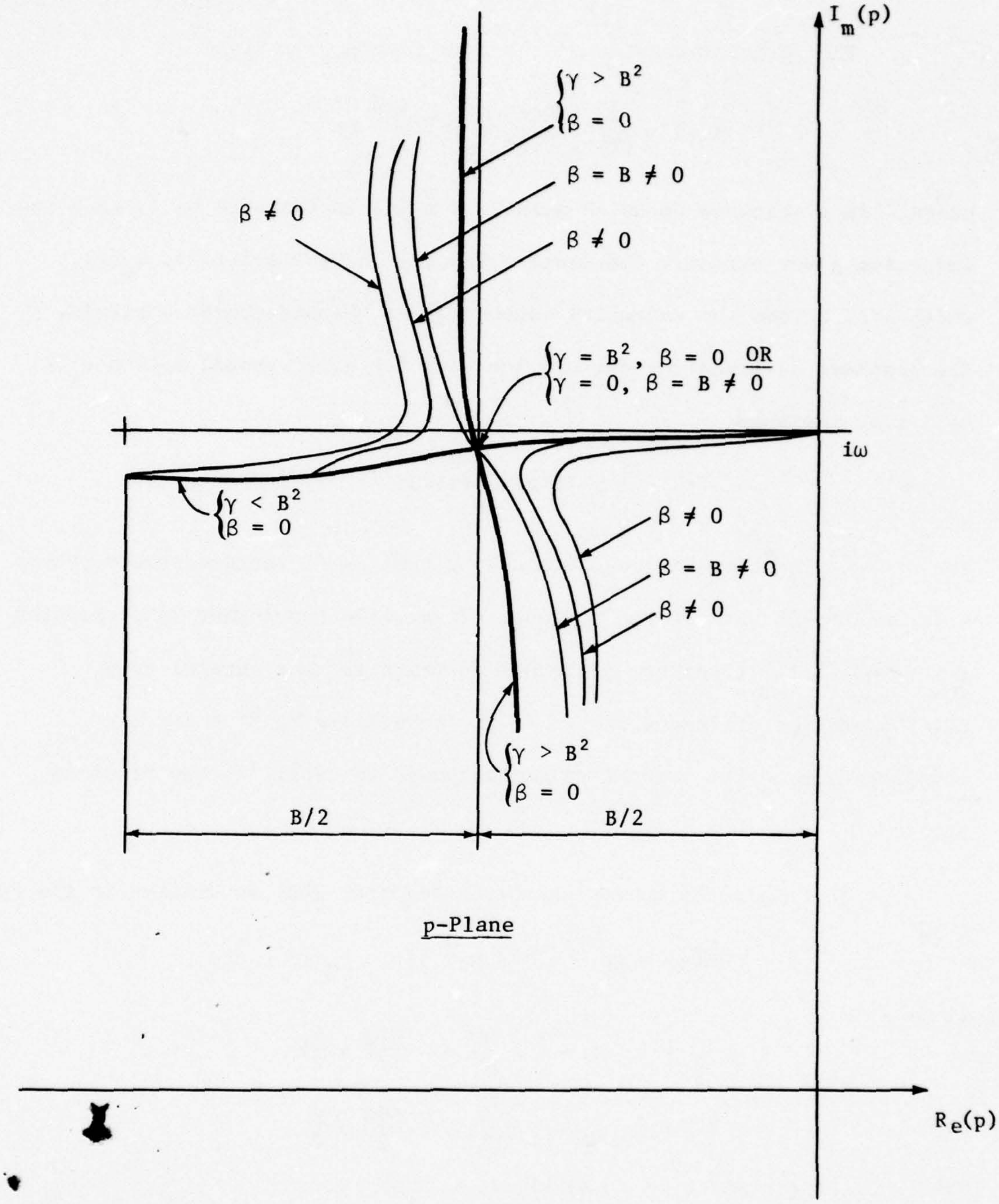


Figure 6. Root Locus for Damped, Tuned 2 DOF System.

SECTION 5

INVERSION OF TRANSFORM SOLUTION FOR TWO-DEGREE-OF-FREEDOM SYSTEM

The formal inversion of the transform Eq. 4.1 is

$$\ddot{u}(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{N(p)}{D(p)} \bar{\ddot{u}}_g(p)\, e^{pt}\, dp \qquad (5.1)$$

where Γ is a suitable Bromwich path. If $\bar{\ddot{u}}_g(p)$ is taken to be 1, then the inversion gives directly the Green's function of the solution, $u_G(t)$, which will become the essential ingredient of the subsequent analysis. The complete solution for the acceleration for given ground motion $u_g(t)$ will take the form

$$\ddot{u}(t) = \int_0^t u_G(t-\tau)\, \ddot{u}_g(\tau)\, d\tau \qquad (5.2)$$

The Green's function will be obtained by the use of residue theory, there being no branch cuts in the p plane. It will be convenient to obtain the inversion of the transformed Green's function for the general case, Eq. 4.6 and for different ranges of the parameters γ, β, B and ξ, corresponding to the special cases discussed in detail in the previous section.

To obtain the inversion the denominator D(p) is written in the form

$$D(p) = (p - p_1)(p - \bar{p}_1)(p - p_2)(p - \bar{p}_2)$$

where

$$p_1 = i\omega\,(1 + \frac{\xi}{2} + \frac{\lambda}{2}) - \omega\,(\frac{\beta+B}{2} + \frac{\mu}{2})$$

$$p_2 = i\omega\,(1 + \frac{\xi}{2} - \frac{\lambda}{2}) - \omega\,(\frac{\beta+B}{2} - \frac{\mu}{2})$$

and $\bar{p}_1$ and $\bar{p}_2$ are the complex conjugates of p_1 and p_2 . Evaluating the residues at each pole and collecting complex conjugate terms in pairs lead to the result, correct to dominant order,

$$\ddot{u}_G(t) = \frac{\omega}{\lambda^2 + \mu^2} e^{-\omega(\beta+B)t/2} \Big\{ \lambda \sin \frac{\mu}{2} \omega t \cos \omega\frac{\lambda}{2}t \sin \omega(1+\frac{\xi}{2})t \tag{5.3}$$

$$- \lambda \cosh \omega\frac{\mu}{2}t \sin \omega\frac{\lambda}{2}t \cos \omega(1+\frac{\xi}{2})t - \mu \sinh \frac{\mu}{2}\omega t \cos \omega\frac{\lambda}{2}t \cos \omega(1+\frac{\xi}{2})t$$

$$- \mu \cosh \frac{\mu}{2} \omega t \sin \omega\frac{\lambda}{2}t \sin \omega(1+\frac{\xi}{2})t \Big\}$$

The results predicted by this solution are explored in several special cases.

5.1 Undamped, tuned system. When $\beta = 0$, $B = 0$ and $\xi = 0$, the solution retaining terms to the order ε^3 takes the form

$$\ddot{u}_G(t) = \frac{3\omega}{2} \sin \omega t \cos \eta t - \frac{\omega}{\gamma^{\frac{1}{2}}} \cos \omega t \sin \eta t$$

where
$$\eta = \omega\gamma^{\frac{1}{2}}/2 \tag{5.4}$$

5.2 Undamped, slightly detuned two-degree-of-freedom system.

In this case the Green's function for the solution takes the form, retaining here terms up to order ε^2,

$$\ddot{u}_G(t) = - \frac{\omega}{(\xi^2+\gamma)^{\frac{1}{2}}} \cos \omega (1+\frac{\xi}{2})t \sin \eta t \tag{5.5}$$

where now $\eta = (\xi^2+\gamma)^{\frac{1}{2}} \omega/2$.

5.3 Damped, tuned two-degree-of-freedom system.

For the situation where $\xi = 0$ it is convenient to obtain the inversion of the transformed Green's function for three different ranges of the parameters γ, β, B.

i) For $\gamma > (\beta - B)^2$ we can write $D(p)$ in the form

$$D(p) = (p-i\omega+\varepsilon_1\omega)(p+i\omega+\varepsilon_1\omega)(p-i\omega+\varepsilon_2\omega)(p+i\omega+\varepsilon_2\omega)$$

where

$$\varepsilon_1 = \frac{\beta + B}{2} + \frac{i}{2} (\gamma - (\beta-B)^2)^{\frac{1}{2}}$$

$$\varepsilon_1 = \frac{\beta + B}{2} - \frac{i}{2} (\gamma - (\beta-B)^2)^{\frac{1}{2}}$$

Evaluating the residues at each pole and collecting complex conjugate terms in pairs leads to the result, correct to dominant order,

$$\ddot{u}_G(t) = -\frac{\omega\, e^{-(\beta+B)\omega t/2} \cos\omega t \,\sin \sqrt{\gamma-(\beta-B)^2}\,\omega t/2}{\sqrt{\gamma - (\beta-B)^2}} \tag{5.6}$$

This function represents a damped beat type solution, the beat frequency being $\omega \sqrt{\gamma - (\beta-B)^2}/2$ which is much smaller than the tuning frequency ω .

ii) For $\gamma < (\beta-B)^2$ writing D(p) in the same form as before, evaluating the residues at each pole and collecting terms in conjugate pairs leads to a Green's function of the form:

$$\ddot{u}_G(t) = -\frac{\omega\, e^{-(\beta+B)\omega t/2} \cos\omega t \,\sinh \sqrt{(\beta-B)^2 - \gamma}\,\omega t/2}{\sqrt{(\beta-B)^2 - \gamma}} \tag{5.7}$$

Since $(\beta+B)^2 > (\beta-B)^2 - \gamma$ for non-zero β, B and γ, the term $\exp[-(\beta+B)\omega t/2]$ dominates the term $\sinh \sqrt{(\beta-B)^2 - \gamma}\,\omega t/2$. The solution can be interpreted as overdamped beats by analogy with the concept of overdamped vibrations. For large values of ωt the solution has the appearance of an oscillation of frequency ω damped by an exponential with factor

$$-\frac{1}{2}\left(\beta+B - \sqrt{(\beta-B)^2 - \gamma)}\right)\omega t$$

iii) For $\gamma = (\beta-B)^2$ the result in Eq. 4.15 predicts a double pole. However, we have already shown that if $\beta \neq 0$, a double pole will not appear. In fact, the more accurate location of the root gives:

$$p = i\omega - \frac{\beta + B}{2}\,\omega \pm i\,\omega\,\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}/2 \tag{5.8}$$

Proceeding in the same manner as before to evaluate the residues the following result is obtained:

$$\ddot{u}_G(t) = -\frac{\omega\, e^{-(\beta+B)\omega t/2} \cos\omega t \sin \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}\omega t/2}{\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}} \tag{5.9}$$

As in Eq. 5.6 this represents damped beats. The beat period is of order $\varepsilon^{-3/2}$ and is thus very long.

iv) When $\beta = 0$ and $\gamma = B^2$ a genuine double root appears. The Green's function solution in this case takes the form:

$$\ddot{u}_G(t) = -\frac{1}{2}\,\omega^2 t\, e^{-B\omega t/2} \cos \omega t \tag{5.10}$$

This case can be interpreted as critically damped beats.

5.4 Damped, slightly detuned two-degrees-of-freedom system.

In the special case of a detuned damped system with $\beta = B$ the appropriate Green's function solution is

$$\ddot{u}_G(t) = -\frac{\omega}{(\gamma+\xi^2)^{\frac{1}{2}}}\, e^{-\omega(\beta+B)t/2} \sin \eta t \cos \omega(1+\xi/2)t \tag{5.11}$$

where $\eta = (\gamma+\xi^2)^{\frac{1}{2}}\, \omega/2$.

SECTION 6

APPLICATION TO EQUIPMENT MOUNTING DESIGN; UNDAMPED TUNED SYSTEMS

The results obtained in the previous section, for the response of various types of damped, undamped, tuned or untuned systems can be utilized in the design of equipment or equipment mounting. The least complicated forms of these equations are those for the undamped tuned system and it is worthwhile to examine these in detail before proceeding with more general cases. Many of the basic features of the phenomena are more readily apparent for this situation. The methods developed for the damped and tuned or detuned cases are extensions of the method developed for this case.

The results given in Eqs. 5-2 with 5-4 could in principle be used by a designer of equipment or equipment mounting, if a specified ground acceleration history were available to estimate the forces which would be developed in the equipment or its mounting. However, such information is not readily available to a designer and the computation involved in these integrals may also be inconvenient during the design process. It would be more common to begin with a design spectrum which may be specified by a code or determined from averaging several possible inputs as for example in seismic design, Reference 14. We are thus interested in determining to what extent the results in Eqs. 5-2 with 5-4 can be used to provide estimates of maximum acceleration when the information available is the response spectrum of the ground motion $\ddot{u}_g$. In the following sections a number of alternative approaches are explored.

6.1 FLOOR SPECTRUM ANALYSIS

When the Green's function obtained for the undamped tuned system Eq. 5-4 is substituted in Eq. 5-2 the response is given by

$$\ddot{u}(t) = \frac{\omega}{\gamma^{\frac{1}{2}}} \int_0^t \ddot{u}_g(\tau) \left\{ \frac{3}{2} \gamma^{\frac{1}{2}} \cos \eta(t-\tau) \sin \omega(t-\tau) - \sin \eta(t-\tau) \cos \omega(t-\tau) \right\} d\tau \qquad (6\text{-}1)$$

with $\eta = \omega\gamma^{\frac{1}{2}}/2$.

For small values of ηt this reduces to

$$\ddot{u}(t) = \omega \int_0^t \ddot{u}_g(\tau) \left\{ \frac{3}{2} \sin \omega(t-\tau) - \omega(t-\tau) \cos \omega(t-\tau) \right\} d\tau \qquad (6\text{-}2)$$

Note that this result is independent of γ and could be obtained directly by means of the floor spectrum analysis method whereby the input to the structure is used to compute the base motion at the equipment assuming the equipment to be absent and the equipment motion is calculated with this motion as input. It is clear that this approximation is valid for small values of ηt, i.e. $\omega\gamma^{\frac{1}{2}}t/2 << 1$. The essential characteristic of this result is that it neglects interaction between the structure and the equipment.

One other feature of the floor spectrum analysis is that it can be obtained from the basic equations by setting $\gamma = 0$. A double pole will then appear in the tuned case (for an untuned system only simple poles occur). This double pole leads to terms in $t \cos \omega t$ and $t \sin \omega t$ on inversion. Thus the floor spectrum analysis cannot be used to determine maximum displacement or acceleration for undamped tuned systems since it yields responses which grow without limit. It follows that although the floor spectrum analysis is a valuable method for untuned systems it has no meaning for undamped tuned systems.

6.2 MODIFIED GROUND MOTION SPECTRA

When ηt is not much smaller than unity the term in Eq. 6-1 multiplied by $\gamma^{\frac{1}{2}}$ is negligible in comparison to the other and it becomes

$$\ddot{u}(t) = -\omega \int_0^t \ddot{u}_g(\tau) \sin \eta(t-\tau) \cos \omega(t-\tau) d\tau \qquad (6\text{-}3)$$

Expanding the term cos $\omega(t-\tau)$ in Eq. 6-3 allows it to be written in the form

$$\ddot{u}(t) = -\omega \cos (\omega t - \phi) \left\{ \left(\int_0^t [\ddot{u}_g(\tau) \cos \omega\tau] \sin \eta(t-\tau) d\tau \right)^2 + \left(\int_0^t [\ddot{u}_g(\tau) \sin \omega\tau] \sin \eta(t-\tau) d\tau \right)^2 \right\}^{\frac{1}{2}} \qquad (6\text{-}4)$$

where

$$\phi = \tan^{-1} \frac{\int_0^t [\ddot{u}_g(\tau) \sin \omega\tau] \sin \eta(t-\tau) d\tau}{\int_0^t [\ddot{u}_g(\tau) \cos \omega\tau] \sin \eta(t-\tau) d\tau} \qquad (6\text{-}5)$$

The terms

$$\frac{1}{\eta} \int_0^t [\ddot{u}_g(\tau) \cos \omega\tau] \sin \eta(t-\tau) d\tau \qquad (6\text{-}6)$$

$$\frac{1}{\eta} \int_0^t [\ddot{u}_g(\tau) \sin \omega\tau] \sin \eta(t-\tau) d\tau \qquad (6\text{-}7)$$

can be interpreted as the response of a single-degree-of-freedom system with frequency η to the modified ground input accelerations $\ddot{u}_g(t) \cos \omega t$ and $\ddot{u}_g(t) \sin \omega t$. We also note that $\eta \ll \omega$. The term $\cos(\omega t-\phi)$ is a rapidly oscillating function and achieves its maximum many times. The integrals are slowly oscillating functions and represent a slowly varying envelope of the more rapidly oscillating term. The maximum of the product is accordingly very nearly the maximum of the envelope.

Thus, one way to estimate the motion of the equipment is to construct spectra for the modified ground accelerations. From this an estimate of the maximum acceleration is

$$\ddot{u}_{max} = \left\{ S_A^c(\eta)^2 + S_A^s(\eta)^2 \right\}^{1/2} \tag{6-8}$$

A similar expression can be obtained for the displacement. In the above, the terms $S_A^c(\eta)$ and $S_A^s(\eta)$ are undamped acceleration response spectra for the modified ground motions $\ddot{u}_g(t)\cos\omega t$ and $\ddot{u}_g(t)\sin\omega t$, respectively. In principle then it is possible to develop a design technique if the time history $\ddot{u}_g(t)$ is available by constructing the low-frequency response spectra for the modified ground motions. In certain cases the given information may be only the response spectrum of $\ddot{u}_g$ and not the ground motion itself. As far as can be seen at the moment there is no way to compute the spectra for the modified ground motion if the only information available is the spectrum of the actual ground motion.

6.3 AMPLIFIED GROUND MOTION SPECTRUM

In view of the remarks in the previous section, we now develop an alternative approach in which the term $\sin\eta(t-\tau)$ is expanded, leading to

$$\ddot{u}(t) = -\frac{\omega}{\gamma^{1/2}}\sin(\eta t-\theta)\left\{\left(\int_0^t \ddot{u}_g(\tau)\cos\eta\tau\cos\omega(t-\tau)d\tau\right)^2 + \left(\int_0^t \ddot{u}_g(\tau)\sin\eta\tau\cos\omega(t-\tau)d\tau\right)^2\right\}^{1/2} \tag{6-9}$$

where

$$\theta = \tan^{-1}\frac{\int_0^t \ddot{u}_g(\tau)\sin\eta\tau\cos\omega(t-\tau)d\tau}{\int_0^t \ddot{u}_g(\tau)\cos\eta\tau\cos\omega(t-\tau)d\tau} \tag{6-10}$$

We are interested in situations where the ground motion has a prescribed finite duration and for those frequencies ω where the maximum response of a single-degree-of-freedom oscillator, i.e. the response spectra, is achieved late in or after the termination of the ground motion. These frequencies correspond to peaks in the response spectrum of ground motion of the earthquake type. Design spectra, reflecting the probabilistic motion of the input, correspond closely to the peaks of actual spectra and thus presuppose late occuring maxima. In blast-type ground motion which is of short duration it is likely that the maximum values of equipment response will occur at times larger than the duration of the ground motion.

Thus, for values of $\eta t_1 << 1$, where t_1 is the duration of the ground motion, the first integral in Eq. 6-9 can be approximated by

$$\int_0^t \ddot{u}_g(\tau) \cos \omega(t-\tau)d\tau \tag{6-11}$$

and the second neglected since sin ηt will be bounded by $\eta t_1 << 1$. For $\eta t_1 << 1$, then, we have

$$\ddot{u}(t) = - \frac{\omega}{\gamma^{\frac{1}{2}}} \sin \eta t \int_0^t \ddot{u}_g(\tau) \cos \omega(t-\tau)d\tau \tag{6-12}$$

The term in the integral is a function oscillating with frequency ω which is high compared to η and a maximum of that term will nearly coincide with the maximum of sin ηt. An estimate of the maximum value of $\ddot{u}(t)$ is

$$|\ddot{u}|_{max} = \frac{\omega}{\gamma^{\frac{1}{2}}} \max \left| \int_0^t \ddot{u}_g(\tau) \cos \omega(t-\tau)d\tau \right| \tag{6-13}$$

If the displacement, velocity and acceleration response spectra as functions of frequency ω and damping parameter β are denoted by $S_D(\omega, \beta)$, $S_V(\omega, \beta)$ and $S_A(\omega, \beta)$ respectively, we then recognize that

$$\max \left| \int_0^t \ddot{u}_g(\tau) \cos \omega(t-\tau)d\tau \right| \tag{6-14}$$

is the undamped velocity response spectrum $S_V(\omega, 0)$ for a single-degree-of-freedom system with frequency ω. Thus, we have as an estimate of $\ddot{u}_{max}$ the expression

$$|\ddot{u}|_{max} = \frac{\omega}{\gamma^{\frac{1}{2}}} S_V(\omega, 0) \tag{6-15}$$

Since $S_D = S_V/\omega$ and $S_A = \omega S_V$, then

$$|u|_{max} = \frac{S_D(\omega, 0)}{\gamma^{\frac{1}{2}}} \tag{6-16}$$

$$|\ddot{u}|_{max} = \frac{S_A(\omega, 0)}{\gamma^{\frac{1}{2}}} \tag{6-17}$$

It follows that if an engineering designer is given only the response spectrum of the ground motion, the maximum displacement and force in the equipment can be estimated by using these spectra amplified by the factor $\gamma^{\frac{1}{2}}$. These remarks refer of course to the equivalent two-degree-of-freedom system. The results for the general system are obtained by utilizing the factors in Eqs. 3-14 and 3-15.

SECTION 7

APPLICATION TO EQUIPMENT MOUNTING DESIGN; DAMPED, TUNED AND SLIGHTLY UNTUNED SYSTEMS

In the previous section, a number of procedures were developed for undamped tuned systems which could provide information to a designer of equipment or equipment mounting. These procedures were motivated by the fact that in many cases a specified time history of the ground motion applied to the structure would not be available except in the restricted form of a design spectrum. These results enable the designer to utilize the design spectrum for the structure to estimate directly maximum values of the equipment acceleration and displacement.

Three different approaches to the design problem were developed. These were the floor spectrum method, the modified ground motion spectrum method and the amplified ground motion spectrum method. It was shown that for tuned, undamped systems the floor spectrum method was not a valid technique. The modified ground motion spectrum method was valid, but inconvenient and it is clear that the amplified ground motion spectrum was the most convenient to use in estimating the response of light undamped equipment.

For damped tuned systems it is seen from Eq. 4-3 that if $\gamma << 1$, then γ is negligible if $4\beta B >> \gamma$. This means that for such cases the possibility of significant interaction between equipment and structure can be ignored and the floor spectrum method used to determine the response of the equipment. However, the floor spectrum method requires the computation of time histories, and thus, if a design spectrum is the only given information, its use may not be the most convenient.

Since the results of the previous section showed that the amplified response spectrum method was the most convenient to use, similar approximations will be developed for damped and slightly detuned systems.

7.1 UNDAMPED, SLIGHTLY DETUNED SYSTEMS

The acceleration response $\ddot{u}(t)$ to imposed ground acceleration $\ddot{u}_g(t)$ is obtained from Eqs. 5-2 with 5-5 in the form

$$\ddot{u}(t) = -\frac{\omega}{(\xi^2+\gamma)^{\frac{1}{2}}}\int_0^t \ddot{u}_g(\tau)\sin\eta(t-\tau)\cos\omega(1+\frac{\xi}{2})(t-\tau)d\tau \tag{7.1}$$

where

$$\eta = (\xi^2+\gamma)^{\frac{1}{2}}\omega/2$$

Expanding the term sin $\eta(t-\tau)$ as was done in Eq. 6-9 of the previous section and neglecting the analogous terms an estimate of the maximum acceleration is obtained as

$$|\ddot{u}|_{max} = \frac{S_A(\omega(1+\xi/2),0)}{(\gamma+\xi^2)^{\frac{1}{2}}} \tag{7.2}$$

This result is still valid if $\gamma << \xi^2$ providing $\xi << 1$. For such cases the floor spectrum method is applicable, but of course could not be used if the only information on the ground motion is a design spectrum. The above result is clearly more convenient and equally valid. The beating phenomenon which is the physical basis of the result will appear in the floor spectrum solution in the slightly detuned case, the beating being between the two closely spaced frequencies ω and $\omega(1+\xi)$.

7.2 DAMPED, TUNED SYSTEMS

The results obtained in Section 3 for the transfer functions for damped tuned systems. indicate that four sets of the parameters γ, β and B have to be identified.

Case 1: $\gamma > (\beta - B)^2$

The acceleration response $\ddot{u}(t)$ to imposed ground acceleration $\ddot{u}_g(t)$ is given by

$$\ddot{u}(t) = -\frac{\omega}{(\gamma-(\beta-B)^2)^{\frac{1}{2}}} \int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) \sin \eta(t-\tau) d\tau \tag{7.3}$$

where

$$\eta = \frac{\omega}{2} (\gamma-(\beta-B)^2)^{\frac{1}{2}}$$

Expanding the term $\sin \eta(t-\tau)$ leads to

$$\ddot{u}(t) = -\frac{\omega}{(\gamma-(\beta-B)^2)^{\frac{1}{2}}} \cos(\eta t-\theta) \left\{ \left\{ \int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) \cos \eta\tau \, d\tau \right\}^2 + \left\{ \int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) \sin \eta\tau \, d\tau \right\}^2 \right\}^{\frac{1}{2}} \tag{7.4}$$

where

$$\theta = \tan^{-1} \left\{ -\frac{\int_0^t \ddot{u}_g(\tau) \cos \eta\tau \, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) d\tau}{\int_0^t \ddot{u}_g(\tau) \sin \eta\tau \, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) d\tau} \right\} \tag{7.5}$$

We are interested in situations where the ground motion has a prescribed finite duration and frequencies for which the maximum response of a single-degree-of-freedom oscillator is achieved late in or after the termination of the ground motion. Thus, for values of $\eta t_1 << 1$ where t_1 is the duration of the ground motion the first integral in Eq. 7-4 can be approximated by

$$\int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) d\tau$$

and the second neglected since $\sin \eta\tau$ will be bounded by $\eta t_1 << 1$ and $\ddot{u}_g = 0$ for $t > t_1$. Thus we take

$$\ddot{u}(t) = -\frac{\omega \sin \eta t}{(\gamma-(\beta-B)^2)^{\frac{1}{2}}} \int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) d\tau \tag{7.6}$$

When the parameters $\gamma^{\frac{1}{2}}$, β and B are small we may interpret this result in the following way:

For $t > t_1$ the above expression can be written in the form;

$$\ddot{u}(t) = -\frac{\omega^2 \sin \eta t}{2\eta} e^{-(\beta+B)\omega t/2} R \cos(\omega t - \Psi)$$

where

$$R = (A_1^2 + A_2^2)^{\frac{1}{2}} \qquad \text{with}$$

$$A_1 = \int_0^{t_1} \ddot{u}_g(t)\, e^{+(\beta+B)\omega t/2} \cos \omega t\, dt$$

$$A_2 = \int_0^{t_1} \ddot{u}_g(t)\, e^{+(\beta+B)\omega t/2} \sin \omega t\, dt$$

and

$$\Psi = \tan^{-1} A_2/A_1$$

In the above, R and Ψ are constants independent of t for $t > t_1$, and $R \cos(\omega t-\Psi)$ is a rapidly varying function of time. The term

$$\frac{\omega^2}{2\eta} \sin \eta t\, e^{-(\beta+B)\omega t/2}$$

is a slowly varying envelope curve which attains its maximum value at a time t^* given by

$$\tan \eta t^* = 2\eta/(\beta+B)\omega \tag{7.7}$$

The value of $\sin \eta t$ at which the envelope achieves its maximum is

$$\sin \eta t^* = \frac{\eta}{(\eta^2+(\beta+B)^2\omega^2/4)^{\frac{1}{2}}} \tag{7.8}$$

It follows that

$$|\ddot{u}|_{max} = |\ddot{u}(t^*)| = \frac{\omega}{(\gamma-(\beta-B)^2)^{\frac{1}{2}}} |\sin \eta t^*|$$

$$\left| \int_0^{t^*} \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t^*-\tau)/2} \cos \omega(t^*-\tau)\, d\tau \right| \tag{7.9}$$

Note that the term

$$\left| \int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau)\, d\tau \right|_{max}$$

is, to the order of β and B, the velocity response spectrum $S_V(\omega, \frac{\beta+B}{2})$ for a damped single-degree-of-freedom oscillator with damping factor (β+B)/2 and is a bound for the integral in Eq. 7-9. Thus an estimate for the maximum equipment acceleration is

$$|\ddot{u}|_{max} = \frac{\omega |\sin \eta t^*|}{(\gamma-(\beta-B)^2)^{\frac{1}{2}}} S_V(\omega, \frac{\beta+B}{2})$$

Utilizing the value of sin ηt* from Eq. 7-8, the final estimate is

$$|\ddot{u}|_{max} = \frac{\omega\, S_V(\omega, \frac{\beta+B}{2})}{(\gamma+4\beta B)^{\frac{1}{2}}} \qquad (7.10)$$

For the lightly damped systems considered here

$$\omega\, S_V = S_A = \omega^2\, S_D$$

so that the result can be written in the alternative forms,

$$|\ddot{u}(t)|_{max} = S_A(\omega, \frac{\beta+B}{2}) / (\gamma+4\beta B)^{1/2} \qquad (7.11)$$

or

$$|u(t)|_{max} = S_D(\omega, \frac{\beta+B}{2}) / (\gamma+4\beta B)^{1/2} \qquad (7.12)$$

It follows that if a designer is given only response spectra, at various damping values, of the ground motion applied to the structure, the maximum displacement and force in the equipment can be estimated by using the appropriate damped spectra for a damping factor equal to the average of those in the structure and equipment, amplified by the factor $(\gamma+4\beta B)^{-\frac{1}{2}}$. It is to be noted that if β+B is fixed, the maximum value of 4βB is achieved when β = B, yielding the smallest value of the amplification factor. Thus if the total damping is fixed the optimal choice is to have it shared equally by equipment and structure.

Case 2: $\gamma < (\beta - B)^2$

The solution in this case for $\ddot{u}(t)$ in terms of specified ground motion $\ddot{u}_g(t)$ takes the form

$$\ddot{u}(t) = -\frac{\omega}{((\beta-B)^2-\gamma)^{\frac{1}{2}}}\int_0^t \ddot{u}_g(\tau)\, e^{-(\beta+B)\,\omega(t-\tau)/2} \cos \omega(t-\tau) \sinh \eta(t-\tau)d\tau \tag{7.13}$$

where

$$\eta = ((\beta-B)^2 - \gamma)^{\frac{1}{2}}\, \omega/2$$

This can be written in a form analogous to that in Eq. 7-4 with the envelope now in the form $e^{-(\beta+B)\ \omega t/2}\sinh \eta t$. When the envelope is analyzed as before for its maximum value it is found that the time $t = t^*$ is such that

$$\sinh \eta t^* = \eta\ /(\eta^2 + (\beta+B)^2\ \omega^2/4)^{\frac{1}{2}}$$

Following the arguments used to obtain the previous results it is found that as before the amplication factor is $(\gamma+4\beta B)^{\frac{1}{2}}$.

Case 3: $\gamma = (\beta - B)^2$

The solution in this case takes the form

$$\ddot{u}(t) = -\frac{\omega}{\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}\int_0^t \ddot{u}_g(\tau)\ e^{-(\beta+B)\omega(t-\tau)/2} \cos \omega(t-\tau) \sin \eta(t-\tau)d\tau \tag{7.14}$$

where now

$$\eta = \omega\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}/2$$

In this case the envelope takes the form

$$\sin \eta t e^{-(\beta+B)\omega t/2}$$

and the time t^* is given by

$$\tan \eta t^* = 2\eta/(\beta+B)\omega$$

Using this and the appropriate value for η gives as the amplification factor the term

$$\frac{1}{((\beta+B)^2+\beta\gamma)^{\frac{1}{2}}}$$

The term $\beta\gamma$ is of order ε^3 and may be neglected in comparison to $(\beta+B)^2$. Thus the amplification factor is simply $(\beta+B)^{-1}$, but note that since $\gamma = (\beta-B)^2$ that $(\beta+B)^{-1} = (\gamma+4\beta B)^{-\frac{1}{2}}$. Thus, the same result as in the two previous cases applies here.

Case 4: $\gamma = B^2$, $\beta = 0$

This is the situation in which a true double root appears and the solution has the form of a damped floor spectrum result,

$$\ddot{u}(t) = -\frac{\omega^2}{2}\int_0^t \ddot{u}_g(\tau)\, e^{-B\,\omega(t-\tau)/2}\cos\,\omega(t-\tau)\cdot(t-\tau)d\tau \tag{7.15}$$

The envelope is $\omega t\, e^{-B\omega t/2}$ the maximum of which is reached at t^* given by $\omega t^* = 2/B$. This leads to an amplification factor $1/B$. We note, however, that since $\gamma = B^2$ and $\beta = 0$, the amplification factor is again

$$(\gamma+4\beta B)^{-\frac{1}{2}}$$

Thus the Eqs. 7-11 and 7-12 obtained for Case 1 are in fact correct for all combinations of γ, β and B, which is a surprising result when one considers the differences in form of the Green's functions for each case.

7.3 DAMPED, SLIGHTLY DETUNED CASE

The addition of slight detuning considerably modified the form of the response which now becomes

$$\ddot{u}(t) = -\frac{\omega}{(\lambda^2+\mu^2)}\int_0^t e^{-\omega(\beta+B)(t-\tau)/2}\ \ddot{u}_g(\tau)\Big\{-\lambda\sinh\frac{\mu}{2}\omega(t-\tau)\cos\omega\frac{\lambda}{2}(t-\tau)\sin\omega(1+\frac{\xi}{2})(t-\tau)$$

$$+\lambda\cosh\frac{\mu}{2}\omega(t-\tau)\sin\omega\frac{\lambda}{2}(t-\tau)\cos\omega(1+\frac{\xi}{2})(t-\tau)$$

$$+\mu\sinh\frac{\mu}{2}\omega(t-\tau)\cos\omega\frac{\lambda}{2}(t-\tau)\cos\omega(1+\frac{\xi}{2})(t-\tau) \qquad (7.16)$$

$$+\mu\cosh\frac{\mu}{2}\omega(t-\tau)\sin\omega\frac{\lambda}{2}(t-\tau)\sin\omega(1+\frac{\xi}{2})(t-\tau)\Big\}\,d\tau,$$

where λ and μ are defined in terms of ξ, γ, β, B in Eqs. 3-7 and 3-8. To simplify the algebraic manipulations needed for this considerably more complicated expression attention is focused on a single case which will be illustrative of the result. The case selected is that of optimal use of damping. That is, if a fixed total amount of damping $\beta + B$ is specified then the best selection of this damping is the case $\beta = B$. Then Eq. 7-16 takes the form

$$\ddot{u}(t) = -\frac{\omega}{\lambda}\int_0^t e^{-\omega(\beta+B)(t-\tau)/2}\ \ddot{u}_g(\tau)\sin\omega\frac{\lambda}{2}(t-\tau)\cos\omega(1+\frac{\xi}{2})(t-\tau)d\tau \qquad (7.17)$$

where

$$\lambda = (\xi^2+\gamma)^{1/2}$$

Expanding the $\sin\omega\frac{\lambda}{2}(t-\tau)$ term and recalling that for $t >> t_1$, the duration of the ground motion, the term $\sin\frac{\lambda}{2}\omega t$, can be neglected, we have

$$\ddot{u}(t) = -\frac{\omega}{\lambda}e^{-(\beta+B)\omega t/2}\sin\lambda\omega t/2\ R\cos(\omega t-\psi)$$

where here

$$R = (A_1^2+A_2^2)^{1/2} \qquad \text{and } \psi = \tan^{-1} A_2/A_1$$

with

$$A_1 = \int_0^{t_1} \ddot{u}_g(t)\, e^{+(\beta+B)\,\omega t/2} \cos\, \omega(1+\xi/2)t\; dt$$

$$A_2 = \int_0^{t_1} \ddot{u}_g(t)\, e^{+(\beta+B)\,\omega t/2} \sin\, \omega(1+\xi/2)t\; dt$$

The slowly varying envelope function

$$e^{-(\beta+B)\,\omega t/2} \sin \lambda\omega t/2$$

has its maximum at a time t* such that

$$\sin \lambda\omega t^*/2 = \frac{\lambda}{(\lambda^2+(\beta+B)^2)^{\frac{1}{2}}}$$

From this result and using the same reasoning as before

$$|\ddot{u}|_{max} = \frac{S_A\left((1+\frac{\xi}{2})\omega, \frac{\beta+B}{2}\right)}{(\xi^2+\gamma+4\beta B)^{\frac{1}{2}}} \tag{7.18}$$

and

$$|\ddot{u}|_{max} = \frac{S_D\left((1+\frac{\xi}{2})\omega, \frac{\beta+B}{2}\right)}{(\xi^2+\gamma+4\beta B)^{\frac{1}{2}}} \tag{7.19}$$

It is surprising that for all the cases considered a universal result applies: The appropriate response spectrum evaluated at the average damping and the average frequency of structure and equipment is multiplied by the amplification factor

$$(\xi^2+\gamma+4\beta B)^{-\frac{1}{2}}$$

SECTION 8

COMPLETE SOLUTION INCLUDING OTHER POLES

In the previous sections we have determined the contribution to the response from the tuning poles of the equipment-structure system. This is of course the dominant part of the response in the case of light equipment mass, but it is easy to include the contributions from the other poles. To do this recall that the non-tuning poles of Eq. 3-12 are close to their location for the structure alone, as indicated in Fig. 3. For the m^{th} non-tuned mode the poles are

$$p = - B_m \Omega_m \pm i\Omega_m \tag{8-1}$$

Evaluating the residues and dropping the negligible terms, which are those multiplied by the small modal mass ratio, $m\phi_r^{k^2}/M_k \ll 1$, $k = 1$ to N, we obtain, to dominant order, the contribution from the m^{th} poles as

$$\frac{C_r^m}{1-(\Omega_m/\omega)^2}\,\Omega_m\, e^{-B_m\Omega_m t} \sin \Omega_m t \quad ,m \neq n \tag{8-2a}$$

and contributions of the same order from the tuning poles as

$$\sum_{\substack{m=1\\ m\neq n}}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2}\, \omega e^{-\beta\omega t} \sin \omega t \tag{8-2b}$$

where C_r^m is defined in Eq. 3-15. Thus, the complete solution for the response of the equipment takes the form

$$\ddot{u}(t) = \int_0^t \ddot{u}_g(\tau)\left\{ \sum_{\substack{m=1\\ m\neq n}}^{N} \frac{C_r^m}{1-(\Omega_m/\omega)^2}\left(\Omega_m\, e^{-B_m\Omega_m(t-\tau)} \sin \Omega_m(t-\tau)\right) + \sum_{\substack{m=1\\ m\neq n}}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2}\left(\omega e^{-\beta\omega(t-\tau)} \sin \omega(t-\tau)\right) + \ddot{u}_G(t-\tau) \right\} d\tau \tag{8-3}$$

where $\ddot{u}_G(t)$ is the appropriate form of contribution from the tuning poles given in its various forms in Section 5.

In utilizing this complete solution to develop the spectral response in each mode for design purposes, it is important to note that the two parts of the solution in Eq. 8-3 have an entirely different character. The contributions from the non-tuning poles and the nondominant contributions from the tuning poles are conventional and would attain their peaks during the ground excitation or shortly thereafter, while the dominant response from the tuning poles as indicated in Sections 6 and 7 is controlled by the energy transfer from the structure to the equipment through beating, which takes a relatively long time. The maximum in the latter case will be achieved considerably later than the former. It does not make sense to add these in the conventional way such as square root of the sum of squares, or by a similar rule. In fact, they should not be added at all, but treated as separate maxima. The maximum response from the nondominant contributions can be estimated by the conventional method of square root of the sum of squares.

Accordingly, the estimate of the maximum acceleration has two parts, an early peak given by

$$|\ddot{u}|_{max} = \left\{ \sum_{\substack{m=1 \\ m\neq n}}^{N} \left[\frac{C_r^m}{1-(\Omega_m/\omega)^2} S_A(\Omega_m, B_m) \right]^2 + \left[\sum_{\substack{m=1 \\ m\neq n}}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2} S_A(\omega,\beta) \right]^2 \right\}^{\frac{1}{2}} \tag{8-4}$$

and the other, a later peak, from the dominant contribtions of the tuning poles given by

$$|\ddot{u}|_{max} = \frac{C_r^n}{\left(\xi^2 + \gamma^{eff} + 4\beta B_n\right)^{\frac{1}{2}}} S_A\left(\omega(1+\frac{\xi}{2}), \frac{\beta+B_n}{2}\right) \tag{8-5}$$

where $\xi = (\Omega_n - \omega)/\omega$, and γ^{eff} is given by Eq. 3-14. For light equipment mass and lightly damped closely tuned systems the second peak will be the most important.

Although not of immediate interest in this report, it is also possible to utilize the methods developed to obtain estimates of the peak response for systems which are grossly detuned, i.e., where the equipment frequency is spaced between and well away from all of the structural frequencies.

To do this note that for light mass the structure poles are only slightly shifted from their location for the structure alone, namely

$$p = -B_m \Omega_m \pm i\Omega m \tag{8-6}$$

and additional poles at

$$p = -\beta\omega \pm i\omega \tag{8-7}$$

due to the equipment are included as shown in Fig. 7 (illustrating the undamped case). The residues at the structure poles are as before with the contributions from each m = 1 to N poles given by Eq. 8-2a. The residues at the equipment poles, Eq. 8-7, provide a contribution to the Green's function in the form, similar to Eq. 8-2b

$$\left[\sum_{m=1}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2}\right] \omega\, e^{-\beta\omega t} \sin \omega t \tag{8-8}$$

The derivation is completely standard and is similar to the terms from the structure poles. The complete response for the equipment in the grossly untuned case is thus given by

$$\ddot{u}(t) = \int_0^t \ddot{u}_g(\tau) \left\{ \sum_{m=1}^{N} \frac{C_r^m}{1-(\Omega_m/\omega)^2}\, \Omega_m\, e^{-B_m \Omega_m (t-\tau)} \sin \Omega_m (t-\tau) + \left(\sum_{m=1}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2} \right) \omega\, e^{-\beta\omega (t-\tau)} \sin \omega(t-\tau) \right\} d\tau \tag{8-9}$$

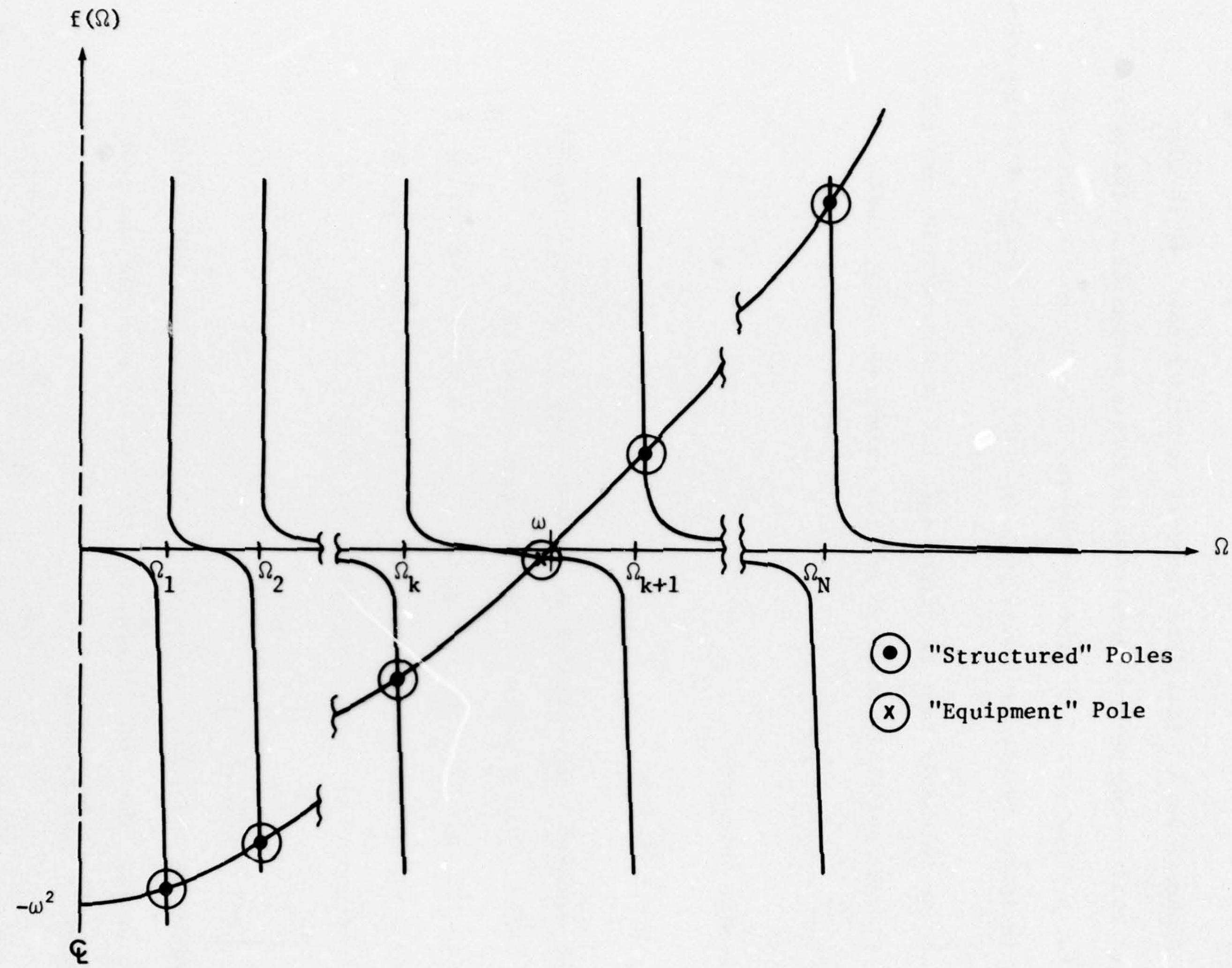


Figure 7. Location of Poles of Equipment Response Transfer Function for Grossly Detuned NDOF Structure.

and the appropriate estimate of the peak response is the conventional one using square root of the sum of squares

$$|\ddot{u}|_{max} = \left\{ \sum_{m=1}^{N} \left[\frac{C_r^m}{1-(\Omega_m/\omega)^2} S_A(\Omega_m, B_m) \right]^2 \right.$$

$$\left. + \left[\sum_{m=1}^{N} \frac{C_r^m}{1-(\omega/\Omega_m)^2} S_A(\omega,\beta) \right]^2 \right\}^{\frac{1}{2}} \qquad (8\text{-}10)$$

This result can be used as an alternative to time history or modal analysis of the composite N + 1 system. It is also an alternative to the standard floor spectrum analysis which requires computation of the time history of the structure alone. This is next applied as input to the equipment and then the time history of the equipment is computed. Note that Eq. 8-9 is completely independent of the modal mass ratios. In fact, it represents the general closed form solution of the floor-spectrum method, and its interpretation directly provides the simple estimate, Eq. 8-10. Indeed, the preceding analysis which led to Eq. 8-9 is the mathematical justification for the use of the floor spectrum method for the grossly detuned system. All of the information needed for Eq. 8-10 is available from the building design, the equipment frequency, and the design spectrum; it should thus be very convenient for the designer to use in practical design applications.

It is worthwhile noting that the methods developed for dealing with the tuned poles can be used to determine the response of systems with closely spaced modes even if no equipment is included. Of course, in this case the approximations used here based on the small mass ratio could not be used, but the treatment of the envelope of the beating response would be entirely similar. It is not to be expected, however, that, in the case of closely spaced modes the maximum response would be much different from that of the other modes, except that it will occur at a much later time. It is the light equipment mass that produces the large amplification, and the dominance of the result in Eq. 8-5 for the late peak over that in Eq. 8-4 for the early peak.

This is the reason that the peculiar ad hoc approach used by Penzien and Chopra in Reference 11 led to a good result. In their approach, they model an N-degree-of-freedom system with a light appendage by considering a set of N-two-degree-of-freedom systems in which one component is one of the N modes of the structure without an appendage and the other is always the appendage and numerically solve the N sets of coupled differential equations. They then add together the peaks from each of the two-degree-of-freedom systems by the square root of the sum of the squares to obtain the maximum response of the appendage. That the result is reasonable arises from the fact that in a tuned system the contribution from the tuning poles, and thus in their case from the particular two-degree-of-freedom system which is tuned, dominates the rest. For this reason, their result is fortuitous. Were this approach used for an untuned system, it could produce erroneous results.

REFERENCES

1. Newmark, N.M. and Haltiwanger, J.D., "Air Force Design Manual. Principles and Practices for Design of Hardened Structures," University of Illinois, Technical Documentary Report No. AFSWC-TDR-62-138. Air Force Special Weapons Center, Kirtland Air Force Base, New Mexico, December 1962.

2. Crawford, R.E., Higgens, C.J., and Bultmann, E.H., "The Air Force Design Manual for Design and Analysis of Hardened Structures," Civil Nuclear Systems Corp., AFWL-TR-74-102, Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, October 1974.

3. Kennedy, T.E. and Safford, F.B., "The Use of Vibration/Impedance Measurements to predict Blast-Induced Structural Vibrations," Fourth International Symposium on the Military Applications of Blast Simulation," Foulness, Southend-on-Sea, England, 9-12 September 1974.

4. Safford, F.B. and Walker, R.E., "Development of Improved Techniques for Measurement of Mechanical Impedance," HNDTR-77-31-ED-SR, Huntsville, Alabama, U.S. Army Corps of Engineers, May 1977.

5. Ranlet, D., DiMaggio, F.L., Bleich, H.H., and Baron, M.L., "Elastic Response of Submerged Shells with Internally Attached Structures to Shock Loading," Journal of Computers and Structures, Vol. 7, No. 3, pp. 355-364, June 1977.

6. Ranlet, D. and DiMaggio, F.L., "Transient Response of Shells with Internally Attached Structures," by Weidlinger Associates for Office of Naval Research, Contract No. N00014-72-C-0119, Technical Report No. 21, August 1977.

7. Harris, C.M. and Crede, C.E., editors, "Shock and Vibration Handbook," Vol. 2, Chapter 26, Shock Testing Machines by Irwin Vigness, pp. 26-1 thru 26-24, McGraw-Hill Book Co., Inc., New York 1961.

8. Peters, K., Schmitz, D., and Wagner, U., "The Problem of Resonance in the Evaluation of Floor Response Spectra", Transactions of the 4th International Conference on Structural Mechanics in Reactor Technology, Vol. K, 4/11, 1977.

9. Skinner, R.I., Skilton, D.W.C., and Laws, D.A. "Unbalanced Buildings and Buildings with Light Towers under Earthquake Forces," Proceedings of the Third World Conference on Earthquake Engineering, Vol. II, New Zealand, 1965.

10. Newmark, N.M., "Earthquake Response Analysis of Reactor Structures," Nuclear Engineering and Design, Vol. 20, 1972, pp. 303-322.

11. Penzien, J., and Chopra, A.K., "Earthquake Response of Appendage on a Multi-Story Building," Proceedings of the Third World Conference on Earthquake Engineering, Vol. II, New Zealand, 1965.

12. Penzien, J., "Earthquake Response of Irregularly Shaped Buildings," Proceedings of the Fourth World Conference on Earthquake Engineering, Vol. II, Chile, 1969.

13. Kelly, J.M. and Sackman, J.L., "Equipment-Structure Interaction at High Frequencies," Report No. DNA 4298T, Weidlinger Associates, Menlo Park, California, 1977.

14. Housner, G.W., "Design Spectrum," Chapter 5 in Earthquake Engineering, Robert L. Wiegel, coordinating editor, Prentice-Hall, Englewood Cliffs, New Jersey, 1970.

NOTATION

A_1, A_2	constants
c	equipment damping coefficient
C	model damping coefficient
C_{ij}	structure damping matrix
C_j^m	modal participation factor
D( )	transform transfer function denominator
e_i	equipment attachment vector
F( )	interaction force between equipment and structure
i, j, k, ℓ, etc.	indices
k	equipment support stiffness
K	model elastic stiffness
K_{ij}	structure stiffness matrix
m	equipment mass
M	model larger mass
M_{ij}	structure mass matrix
M_n	modal mass
N	total number of structural degrees of freedom
N( )	transform transfer function numerator
p	Laplace transform parameter
q	generalized modal coordinate
r_i	influence coefficient vector
t	time
t_1	duration of input ground motion

$S_A(,)$	acceleration response spectrum
$S_V(,)$	velocity response spectrum
$S_D(,)$	displacement response spectrum
$u, \dot{u}, \ddot{u}$	equipment displacement, velocity, acceleration
$U, \dot{U}, \ddot{U}$	model displacement, velocity, acceleration
$U_i, \dot{U}_i, \ddot{U}_i$	structure displacement, velocity, acceleration
$u_g, \ddot{u}_g$	ground displacement, acceleration
$\ddot{u}_g^{eff}$	effective ground acceleration
u_G	Green's function for equipment response
β, B, B_n	damping factors
γ	mass ratio
δ	root locus variable
ε	small parameter
$\varepsilon_1, \varepsilon_2$	roots
λ, μ	roots
η	beat frequency
ψ	phase angle
ω	equipment natural frequency
Ω_n	structural natural frequency

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE

Assistant to the Secretary of Defense
Atomic Energy
ATTN: Executive Assistant

Defense Advanced Rsch. Proj. Agency
ATTN: TIO

Defense Civil Preparedness Agency
Assistant Director for Research
ATTN: Staff Dir. Rsch., G. Sisson
10 cy ATTN: Admin. Officer

Defense Communications Agency
ATTN: CCTC C670, R. Lipp

Defense Documentation Center
12 cy ATTN: DD

Defense Intelligence Agency
ATTN: RDS-3A
ATTN: DB-4E
ATTN: DT-1C
ATTN: DT-2
ATTN: DB-4C, E. O'Farrell

Defense Nuclear Agency
ATTN: DDST
2 cy ATTN: SPSS
4 cy ATTN: TITL

Department of Defense Explo. Safety Board
ATTN: Chairman

Field Command
Defense Nuclear Agency
ATTN: FCTMOF
ATTN: FCPR
ATTN: FCT

Joint Strat. Tgt. Planning Staff
ATTN: XPFS
ATTN: JLTW-2
ATTN: NRI-Stinfo Library
ATTN: DOXT

Livermore Division Fld. Command, DNA
ATTN: FCPRL

Commandant
NATO School (SHAPE)
ATTN: U.S. Documents Officer

Under Secy. of Def. for Rsch. & Engrg.
ATTN: Strategic & Space Systems (OS)

WWMCCS System Engineering Org.
ATTN: T. Neighbors

DEPARTMENT OF THE ARMY

BMD Advanced Technology Center
ATTN: ATC-T
ATTN: 1CRDABH-X

DEPARTMENT OF THE ARMY (Continued)

BMD Program Office
ATTN: DACS-BMT, J. Shea

BMD Systems Command
ATTN: BMDSC-H, N. Hurst

Chief of Engineers
ATTN: DAEN-MCE-D
ATTN: DAEN-RDM

Harry Diamond Laboratories
ATTN: DELHD-N-NP
ATTN: DELHD-N-TI

Redstone Scientific Information Ctr.
U.S. Army R&D Command
ATTN: Chief, Documents

U.S. Army Ballistic Research Labs
ATTN: DRDAR-BLE, J. Keefer
ATTN: DRXBR-BLE, W. Taylor
2 cy ATTN: Technical Library

U.S. Army Communications Command
ATTN: Technical Reference Division

U.S. Army Engineer Div. Huntsville
ATTN: HNDED-SR

U.S. Army Engineer Div. Ohio River
ATTN: ORDAS-L

U.S. Army Engr. Waterways Exper. Station
ATTN: L. Ingram
ATTN: Library
ATTN: W. Flathau

U.S. Army Foreign Science & Tech. Ctr.
ATTN: DRXST-SD

U.S. Army Material & Mechanics Rsch. Ctr.
ATTN: Technical Library
ATTN: R. Shea
ATTN: J. Mescall

U.S. Army Materiel Dev. & Readiness Command
ATTN: DRCDE-D, L. Flynn
ATTN: DRXAM-TL

U.S. Army Missile R&D Command
ATTN: DRDMI-XS

U.S. Army Mobility Equip. R&D Command
ATTN: DRDME-WC

U.S. Army Nuclear & Chemical Agency
ATTN: Library

DEPARTMENT OF THE NAVY

Civil Engineering Laboratory
Naval Construction Battalion Center
ATTN: Code L08A
ATTN: S. Takahashi
ATTN: R. Odello

DEPARTMENT OF THE NAVY (Continued)

David W. Taylor Naval Ship R&D Ctr.
ATTN: Code L42-3

Naval Electronic Systems Command
ATTN: PME 117-21A

Naval Facilities Engineering Command
ATTN: Code 03A
ATTN: Code 09M22C
ATTN: Code 04B

Naval Material Command
ATTN: MAT 08T-22

Naval Ocean Systems Center
ATTN: Code 4471
ATTN: E. Cooper

Naval Postgraduate School
ATTN: Code 0142

Naval Research Laboratory
ATTN: Code 2627
ATTN: Code 8440, F. Rosenthal

Naval Sea Systems Command
ATTN: Code 03511
ATTN: ORD-91313

Naval Ship Engineering Center
ATTN: Code 09G3
ATTN: NSEC 6105

Naval Surface Weapons Center
ATTN: Code F31

Naval Surface Weapons Center
Dahlgren Laboratory
ATTN: Tech. Library & Info. Services Branch

Naval War College
ATTN: Code E-11

Naval Weapons Center
ATTN: Code 533
ATTN: P. Cordle
ATTN: C. Austin

Naval Weapons Evaluation Facility
Kirtland Air Force Base
ATTN: Code 10
ATTN: R. Hughes

Office of Naval Research
ATTN: Code 474, N. Perrone
ATTN: Code 461, J. Warner
ATTN: Code 715

Office of the Chief of Naval Operations
ATTN: OP 981
ATTN: OP 03EG

Strategic Systems Project Office
ATTN: NSP-43
ATTN: NSP-272

DEPARTMENT OF THE AIR FORCE

Air Force Geophysics Laboratory
ATTN: LWW, K. Thompson

DEPARTMENT OF THE AIR FORCE (Continued)

Aerospace Defense Command/XPD
ATTN: XP
ATTN: XPDQQ

Air Force Institute of Technology, Air University
ATTN: Library

Air Force Systems Command
ATTN: DLCAM

Air Force Weapons Laboratory
ATTN: DE, M. Plamondon
ATTN: DE-I
ATTN: DEX
ATTN: SUL

Assistant Chief of Staff
Intelligence
ATTN: IN

Deputy Chief of Staff
Research, Development, & Acq.
ATTN: AFRDPX
ATTN: AFRDQSM

Foreign Technology Division, AFSC
ATTN: NICD Library

Rome Air Development Center, AFSC
ATTN: Documents Library/TSLD

Space & Missile Systems Organization/DE
Air Force Systems Command
ATTN: DEB

Space & Missile Systems Organization/DY
Air Force Systems Command
ATTN: DYS

Space & Missile Systems Organization/MN
Air Force Systems Command
ATTN: MMH
ATTN: MNNH

Strategic Air Command/XPFS
ATTN: NRI-Stinfo Library
ATTN: XPFS

DEPARTMENT OF ENERGY

Department of Energy
Albuquerque Operations Office
ATTN: Doc. Con. for Technical Library

Department of Energy
Nevada Operations Office
ATTN: Doc. Con. for Technical Library

Office of Military Application
ATTN: Doc. Con. for Test Office

OTHER GOVERNMENT AGENCY

Department of the Interior
Bureau of Mines
ATTN: Tech. Library

DEPARTMENT OF DEFENSE CONTRACTORS

Aerospace Corp.
ATTN: Technical Information Services
ATTN: P. Mathur

Agbabian Associates
ATTN: M. Agbabian

Analytic Services, Inc.
ATTN: G. Hesselbacher

Applied Theory, Inc.
2 cy ATTN: J. Trulio

BDM Corp.
ATTN: Corporate Library
ATTN: A. Lavagnino

BDM Corp.
Albuquerque International
ATTN: R. Hensley

Boeing Co.
ATTN: R. Carlson
ATTN: Aerospace Library

Brown Engineering Company, Inc.
Cummings Research Park
ATTN: M. Patel

California Research & Technology, Inc.
ATTN: K. Kreyenhagen
ATTN: S. Shuster
ATTN: Library

Civil/Nuclear Systems Corp.
ATTN: J. Bratton

University of Dayton
Industrial Security Super KL-505
ATTN: H. Swift

EG&G Washington Analytical Services Center, Inc.
ATTN: Library

Eric H. Wang
Civil Engineering Rsch. Fac.
University Station
The University of New Mexico
ATTN: G. Lane

General Electric Company—TEMPO
Center for Advanced Studies
ATTN: DASIAC

General Research Corp.
Santa Barbara Division
ATTN: B. Alexander

IIT Research Institute
ATTN: Documents Library

University of Illinois
Consulting Services
ATTN: N. Newmark

Institute for Defense Analyses
ATTN: Classified Library

DEPARTMENT OF DEFENSE CONTRACTORS (Continued)

Kaman AviDyne
Division of Kaman Sciences Corp.
ATTN: F. Criscione
ATTN: Library

Kaman Sciences Corp.
ATTN: Library
ATTN: F. Shelton

Lawrence Livermore Laboratory
University of California
ATTN: Doc. Con. for Technical Information Department Library

Lockheed Missiles & Space Co., Inc.
ATTN: TIC-Library

Los Alamos Scientific Laboratory
ATTN: Doc. Con. for Reports Library

Martin Marietta Corp.
Orlando Division
ATTN: G. Fotieo

McDonnell Douglas Corp.
ATTN: R. Halprin

Merritt CASES, Inc.
ATTN: Library
ATTN: J. Merritt

Meteorology Research, Inc.
ATTN: W. Green

Pacifica Technology
ATTN: G. Kent

Physics International Co.
ATTN: Technical Library
ATTN: C. Vincent

R&D Associates
ATTN: C. MacDonald
ATTN: Technical Information Center
ATTN: H. Brode
ATTN: J. Lewis
ATTN: R. Port

R&D Associates
ATTN: H. Cooper

Rand Corp.
ATTN: C. Mow

Sandia Laboratories
Livermore Laboratory
ATTN: Doc. Con. for Library & Security Classification Div.

Sandia Laboratories
ATTN: Doc. Con. for 3141
ATTN: Doc. Con. for L. Vortman

Science Applications, Inc.
ATTN: Technical Library

Science Applications, Inc.
ATTN: Technical Library

DEPARTMENT OF DEFENSE CONTRACTORS (Continued)

Science Applications, Inc.
ATTN: D. Bernstein
ATTN: D. Maxwell

Science Applications, Inc.
ATTN: B. Chambers, III

Southwest Research INstitute
ATTN: W. Baker

SRI International
ATTN: G. Abrahamson

Systems, Science & Software, Inc.
ATTN: Library
ATTN: D. Grine

Terra Tek, Inc.
ATTN: Library

TRW Defense & Space Sys. Group
ATTN: Technical Information Center
ATTN: P. Bhutta
2 cy ATTN: P. Dai

DEPARTMENT OF DEFENSE CONTRACTORS (Continued)

TRW Defense & Space Sys. Group
San Bernardino Opeations
ATTN: F. Wong

Universal Analytics, Inc.
ATTN: E. Field

Weidlinger Assoc., Consulting Engineers
ATTN: M. Baron

Weidlinger Assoc., Consulting Engineers
ATTN: J. Sackman
ATTN: J. Kelly
2 cy ATTN: J. Isenberg

Westinghouse Electric Corp.
Marine Divison
ATTN: W. Volz