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A PARAMETRIC LINEAR COMPLEMENTARITY TECHNIQUE FOR OPTIMAL PORTF--ETC(U)
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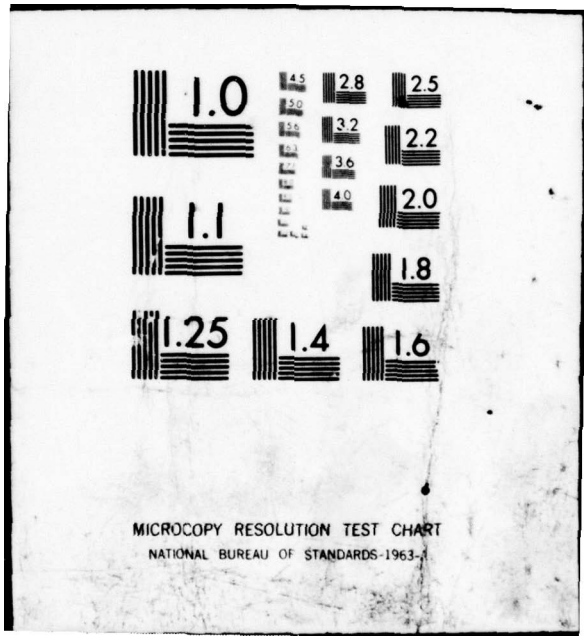
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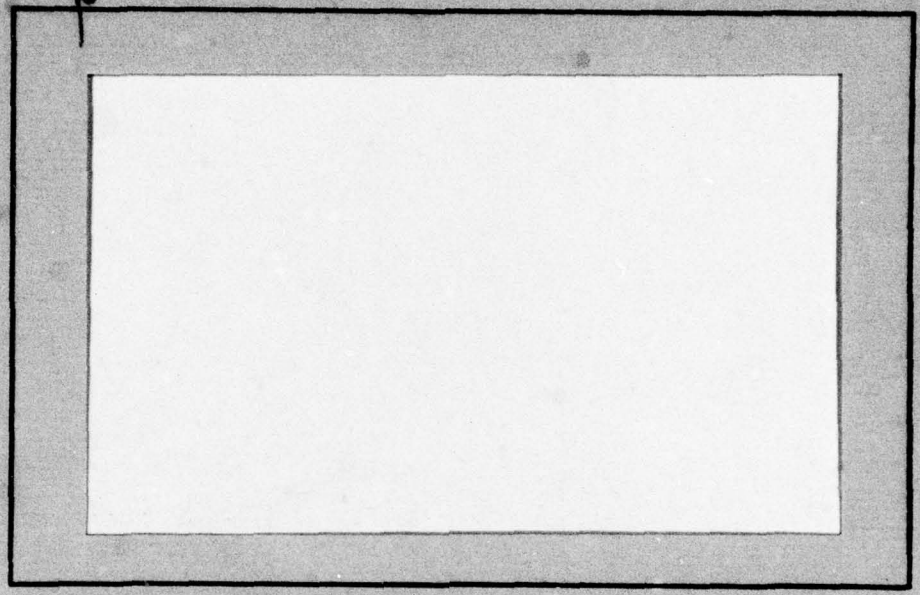

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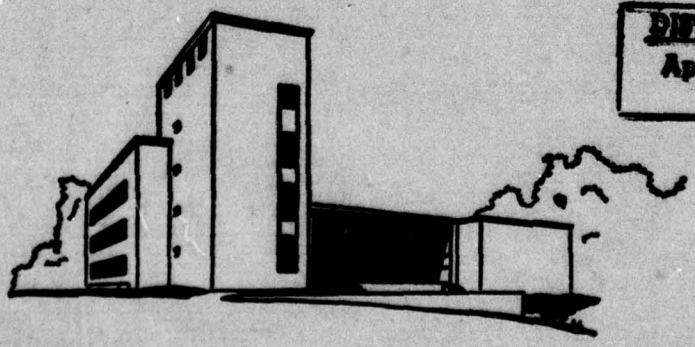

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Management Sciences Research Report No. 428

A PARAMETRIC LINEAR COMPLEMENTARITY TECHNIQUE FOR OPTIMAL PORTFOLIO SELECTION WITH A RISK-FREE ASSET

by

Jong-Shi Pang

January 1979

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The general single-period optimal portfolio selection problem is the following: An investor wishes to invest his wealth in certain risky assets, each of which has a constant scale of return that is a random variable. He could also borrow or lend and the interest rates for borrowing and lending are assumed to be the same and fixed. The latter is referred to as the risk-free (or safe) asset. The objective of the investor is to maximize his expected utility of wealth subject to his budget constraint and certain federal or other personal restrictions.

In Lintner [13] and Ziemba et al. [18], it was shown that the problem can be solved by a two-stage procedure, provided that the investor has a concave utility function and that the asset returns have a multinormal distribution. In stage 1, one solves a certain fractional program having as variables the proportions invested in the risky assets. In stage 2, one uses the optimal solution obtained in stage 1 to solve a certain stochastic program having one single variable which represents the proportion invested in the safe asset.

In a recent paper, by making certain special assumptions about the covariances of the risky assets, Elton, Gruber and Padberg [7] have derived some procedures for the solution of the fractional program. Unfortunately, their derivation was based on a wrong set of necessary and sufficient conditions for the program.

Our objective in this paper is to develop an efficient method for the numerical solution of the fractional program arising in the first stage of the portfolio problem. The second stage is often very easy to solve. See [18].

The organization of the paper is the following: We first show

that the fractional program can be reformulated as a certain equivalent linear complementarity problem which forms the Kuhn-Tucker necessary and sufficient conditions of optimality for a certain (strictly) convex quadratic program. Then, by establishing a theorem which shows how the complementarity problem can be effectively solved by a simplified version of the parametric principal pivoting algorithm as described in Pang [14], we derive an efficient algorithm for solving the problem. We shall also outline how the proposed algorithm can be profitably applied to a specific model with upper bounds. Finally we shall report some computational results including a brief comparison of our proposed algorithm and Lemke's (see Lemke [12]).

1. THE LINEAR COMPLEMENTARITY EQUIVALENCE

As the case where short selling is allowed is computationally much easier to handle, this paper treats only the case where short selling is prohibited.

The fractional program under consideration can then be stated as:

$$\text{maximize} \quad \mu^T x' \; / \; (x'^T V x')^{\frac{1}{2}} \tag{1}$$

$$\text{subject to} \quad e^T x' = 1 \quad , \quad Cx' \leq d \quad \text{and} \quad x' \geq 0 \; .$$

Here V is the n x n symmetric covariance matrix of the (stochastic) returns of the risky assets and is assumed to be nonsingular (or equivalently, positive definite), μ is the n-vector of expected asset returns in excess of the risk-free return, x' is the n-vector of proportions of wealth invested in each asset, e is the vector of 1's and n is the total number of risky assets under consideration. The matrix C is m by n.

The theorem below shows that the program is equivalent to a linear complementarity problem. It is an extension of a result established in [18] for the special case where the constraints $Cx' \leq d$ are not present.

Theorem 1. Suppose that problem (1) is feasible and that $\mu^T x' > 0$ for some feasible x'. Then problem (1) is equivalent to the linear complementarity problem

$$u = -\mu + Vx + (C^T - ed^T)y \geq 0 \qquad x \geq 0 \qquad (2)$$

$$v = -(C - de^T)x \qquad \geq 0 \qquad y \geq 0$$

$$u^T x = v^T y = 0$$

in the sense that there is a one-to-one correspondence between the optimal solutions to (1) and the complementary solutions to (2).

Remark 1. The assumption ($\mu^T x' > 0$) has the interpretation that the total expected return is positive for some feasible portfolios x'.

Remark 2. The feasibility of (1) is important for the equivalence to hold. In fact, simple examples can be constructed so that (2) has a complementary solution but (1) is infeasible.

Proof of the theorem. Since the objective function in (1) is homogeneous in x', the problem is equivalent to

$$\text{maximize} \quad \mu^T x'' \,/\, (x''^T V x'')^{\frac{1}{2}} \qquad (1')$$

$$\text{subject to} \quad Cx'' \leq de^T x'' \,, \; x'' \geq 0 \quad \text{and} \quad x'' \neq 0$$

under the transformation $x' = x''/ e^T x''$. We show that this latter program is equivalent to the convex quadratic program below

$$\text{minimize } \tfrac{1}{2}x^T Vx - \mu^T x \quad \text{subject} \quad Cx \leq de^T x \text{ and } x \geq 0 . \qquad (2')$$

Indeed, let x'' be an optimal solution to (1'). We observe that $\mu^T x'' > 0$ so that the vector $\bar{x} = (\mu^T x'' / x''^T Vx'')x''$ is feasible for (2'). Moreover, we have

$$\tfrac{1}{2}\bar{x}^T V\bar{x} - \mu^T\bar{x} = - (\mu^T x'')^2 / x''^T Vx'' < 0 .$$

Let y be a feasible solution to (2') with $\frac{1}{2}y^T Vy - \mu^T y < 0$. Then $y \neq 0$ is feasible for (1') and we have $\mu^T y > \frac{1}{2}y^T Vy > 0$. Thus

$$\tfrac{1}{2}y^T Vy - \mu^T y \geq - \tfrac{1}{2}(\mu^T y)^2 / y^T Vy \geq - \tfrac{1}{2}(\mu^T x'')^2 / x''^T Vx'' = \tfrac{1}{2}\bar{x}^T V\bar{x} - \mu^T\bar{x} .$$

Therefore $\bar{x}$ is optimal for (2'). Conversely, let $\bar{x}$ be an optimal solution to (2'). We claim $\mu^T\bar{x} > 0$. Indeed, if $\mu^T\bar{x} \leq 0$ and if x is a feasible solution for (2') such that $\mu^T x > 0$, then since the vector $y = (\mu^T x / x^T Vx)x$ is also feasible for (2'), we have

$$0 \leq \tfrac{1}{2}\bar{x}^T V\bar{x} - \mu^T\bar{x} \leq \tfrac{1}{2}y^T Vy - \mu^T y = - \tfrac{1}{2}(\mu^T x)^2 / x^T Vx < 0$$

which is impossible. Therefore $\mu^T\bar{x} > 0$. Similarly, we may deduce that $\mu^T\bar{x} = \bar{x}^T V\bar{x}$. Now let x'' be a feasible solution to (1') with $\mu^T x'' > 0$. Then the vector $x = (\mu^T x'' / x''^T Vx'')x''$ is feasible to (2'). Hence, we have

$$- \tfrac{1}{2}(\mu^T\bar{x})^2 / \bar{x}^T V\bar{x} = \tfrac{1}{2}\bar{x}^T V\bar{x} - \mu^T\bar{x} \leq \tfrac{1}{2}x^T Vx - \mu^T x = - \tfrac{1}{2}(\mu^T x'')^2 / x''^T Vx''$$

which implies

$$\mu^T\bar{x} / (\bar{x}^T V\bar{x})^{\frac{1}{2}} \geq \mu^T x'' / (x''^T Vx'')^{\frac{1}{2}} .$$

Therefore $\bar{x}$ is optimal for (1'). The proof can now be completed by noting that the problem (2) is precisely the set of Kuhn-Tucker necessary and sufficient conditions of optimality for the quadratic program (2').

2. THE PARAMETRIC APPROACH

Among the various algorithms which can be used to solve the linear complementarity problem (2) is the parametric version of Graves' principal pivoting algorithm (see Graves[9] for the original non-parametric version and Cottle [2] for the parametric version). A typical pivot step of the algorithm is outlined as follows: Given a parametric linear complementarity problem in the canonical form

$$w = r + \lambda s + Mz$$

where λ is currently of some positive value $\bar{\lambda}$ such that $r + \bar{\lambda}s \geq 0$ and $r_k + \bar{\lambda}s_k = 0$ for some k, we first pivot z_k into the basis and w_k out of the basis if $m_{kk} \neq 0$ (known as a 1 x 1 diagonal pivot). Else we increase z_k to a value until some $r_\ell + \bar{\lambda}s_\ell$ becomes zero, in which case we pivot on $m_{k\ell}$ and $m_{\ell k}$ (a 2 x 2 block pivot).

It is a rather well-known fact that if the diagonal pivot entry is positive in each step and if each pivot is nondegenerate (see [9] for the handling of degeneracy), the algorithm always terminates with a solution to the linear complementarity problem

$$w = r + Mz \geq 0 \ , \ z \geq 0 \ \text{ and } \ z^T w = 0$$

in a finite number of steps (when λ reaches zero). A sufficient condition

for the diagonal pivot entries to be positive is that the original matrix M should have all principal minors positive.

The matrix M in problem (2) which is given by

$$M = \begin{pmatrix} V & A^T \\ -A & 0 \end{pmatrix} \tag{3}$$

where $A = C - de^T$, certainly does not satisfy this sufficient condition. Nevertheless, as the next theorem shows, the same assertion about the diagonal pivot entries remains valid for the problem. The proof of the theorem is given in the Appendix.

Theorem 2. Consider the solution of the linear complementarity problem (2) by the parametric algorithm described above, where the parametric vector s is chosen as $s = \begin{pmatrix} e \\ 0 \end{pmatrix}$. Then the diagonal pivot entry in each pivot step is positive. In other words, the problem can be solved by performing the 1 x 1 diagonal pivots exclusively.

Remark. The positive definiteness of V is important for the theorem to hold. In fact Graves [9] showed that all the pivots are 2 x 2 if V is the zero matrix.

As explained in [14], the only information one needs to have in order to execute the 1 x 1 diagonal pivots consists of (i) the index set of the currently basic z-variables, (ii) the current constant (r -) column and (iii) the current parametric (s -) column. The update of the matrix M is entirely unnecessary.

Applying this idea to the problem (2), we may formulate the algorithm

below. (See the Appendix for the explanation of the notations and recall that $A = C - de^T$ with C being m by n.)

Algorithm for solving problem (2).

Step 0 (Initialization) Set $\beta = \delta = \emptyset$, $\alpha = \{1,\ldots,n\}$ and $\gamma = \{1,\ldots,m\}$.

Step 1 (Main computation) Solve the system of linear equations for $\begin{pmatrix} \tilde{\mu}_\beta & \tilde{e}_\beta \\ b^1_\delta & b^2_\delta \end{pmatrix}$

$$\begin{pmatrix} V_{\beta\beta} & (A_{\delta\beta})^T \\ -A_{\delta\beta} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\mu}_\beta & \tilde{e}_\beta \\ b^1_\delta & b^2_\delta \end{pmatrix} = \begin{pmatrix} \mu_\beta & e_\beta \\ 0 & 0 \end{pmatrix} \tag{4a}$$

and compute

$$\begin{pmatrix} \tilde{\mu}_\alpha & \tilde{e}_\alpha \\ b^1_\gamma & b^2_\gamma \end{pmatrix} = \begin{pmatrix} \mu_\alpha & e_\alpha \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} V_{\alpha\beta} & (A_{\delta\alpha})^T \\ -A_{\gamma\beta} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\mu}_\beta & \tilde{e}_\beta \\ b^1_\delta & b^2_\delta \end{pmatrix} \tag{4b}$$

Step 2 (Ratio test) Determine

$$\lambda = \max\{\max\{\tilde{\mu}_i/\tilde{e}_i : \tilde{e}_i > 0 \quad i \in \alpha\}, \max\{\tilde{\mu}_j/\tilde{e}_j : \tilde{e}_j < 0,\ j \in \beta\},$$

$$\max\{b^1_i/b^2_i : b^2_i > i \in \gamma\}, \max\{b^1_j/b^2_j : b^2_j < 0,\ j \in \delta\}\}.$$

If $\lambda \leq 0$, terminate with the solution

$$\begin{pmatrix} x_\beta \\ y_\delta \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_\beta \\ b^1_\delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_\alpha \\ y_\gamma \end{pmatrix} = 0 \ .$$

Otherwise, let k be a maximizing index and continue.

Step 3 (Updating the index sets) Let

$$\beta = \begin{cases} \beta \cup \{k\} & \text{if ratio occurs at first maximum} \\ \beta \setminus \{k\} & \text{if ratio occurs at second maximum} \\ \beta & \text{otherwise} \end{cases}$$

and $\alpha = \{1,\ldots,n\} \setminus \beta$. Update γ and δ in a similar fashion. Go to Step 1.

We point out that the major computational effort required by the algorithm is contained in Step 1. In general, this step should best be implemented by using an adaptive procedure (such as those described in Gill et al. [8]) to take advantage of the change of the index set $\beta \cup \delta$.

3. A MODEL WITH UPPER BOUNDS

In [7], by making certain assumptions about the covariance matrix V, Elton et al. studied several special cases of the following model

$$\text{maximize } \mu^T x' / (x'^T V x')^{\frac{1}{2}} \quad \text{subject to } e^T x' = 1 \text{ and } 0 \leq x' \leq d \qquad (5)$$

which of course is equivalent to the complementarity problem (2) with C being the identify matrix of order n. We remark here that the linear complementarity formulation used by the authors of the aforementioned reference is not the correct one. Our purpose in this section is to outline how Step 1 of the proposed algorithm can be greatly simplified by taking advantage of the fact that C is the identity matrix. In fact, the entire algorithm can be carried out by operating on the matrix V and the vectors μ and d only.

Referring to the notations of the algorithm, we state the following result.

Theorem 3. Throughout the solution procedure, $\delta \subseteq \beta$.

The idea of the proof is in fact quite simple. By making an inductive

assumption that the assertion is true before a certain pivot and then by deriving an explicit expression for the current constant and parametric columns, it can be observed easily that the maximum ratio (in Step 2 of the algorithm) will not occur at those rows corresponding to the basic variables x_δ and v_α in the current canonical tableau of the complementarity system (cf. (7) in the Appendix). Consequently, the next pivot will not occur in these rows, thereby establishing the claim for this (and thus all subsequent) pivot step(s).

Results similar to this one have appeared in the study of quadratic programs with only upper and lower bounds on the variables (see [14]) and in that of a certain piecewise linear complementarity problem (see Kaneko [11]). Roughly speaking, δ and α consist of those indices whose corresponding variables x'_δ and x'_α are at their upper and lower bounds respectively, and $\beta \setminus \delta$ those that are between the bounds. The theorem then says that a variable which is at one extreme can not immediately reach the other extreme without attaining some intermediate values. Intuitively, this is quite obvious. The theorem also permits one to ignore certain rows in the ratio test. This certainly is a computational saving. For more discussion, see the above two references.

Consider now the solution of the system of linear equations (4a). According to Theorem 3, we may write $\beta = \delta \cup \eta$ with $\eta \cap \delta = \emptyset$. Defining

$$B(\beta,\delta) = \begin{pmatrix} V_{\beta\beta} & (A_{\delta\beta})^T \\ -A_{\delta\beta} & 0 \end{pmatrix},$$

as in the proof of Theorem 2 and recalling $A = I - de^T$, we may write

$$B(\beta,\delta) = \begin{pmatrix} V_{\eta\eta} & V_{\eta\delta} & -e_\eta d_\delta^T \\ V_{\delta\eta} & V_{\delta\delta} & -e_\delta d_\delta^T + I_\delta \\ d_\delta e_\eta^T & d_\delta e_\delta^T - I_\delta & 0 \end{pmatrix}$$

where I_δ denotes the identity matrix of order $|\delta|$. By defining

$$\bar{B}(\beta,\delta) = \begin{pmatrix} V_{\eta\eta} & V_{\eta\delta} & 0 \\ V_{\delta\eta} & V_{\delta\delta} & I_\delta \\ 0 & -I_\delta & 0 \end{pmatrix}, \quad E_1(\beta,\delta) = \begin{pmatrix} 0 & -e_\eta \\ 0 & -e_\delta \\ d_\delta & 0 \end{pmatrix}, \quad E_2(\beta,\delta) = \begin{pmatrix} e_\eta & 0 \\ e_\delta & 0 \\ 0 & d_\delta \end{pmatrix}$$

it is obvious that

$$B(\beta,\delta) = \bar{B}(\beta,\delta) + E_1(\beta,\delta)\, E_2(\beta,\delta)^T$$

which shows that $B(\beta,\delta)$ differs from $\bar{B}(\beta,\delta)$ by a rank-two matrix. By an easy calculation, we may deduce

$$\bar{B}(\beta,\delta)^{-1} = \begin{pmatrix} V_{\eta\eta}^{-1} & 0 & V_{\eta\eta}^{-1}V_{\eta\delta} \\ 0 & 0 & -I_\delta \\ -V_{\delta\eta}V_{\eta\eta}^{-1} & I_\delta & V_{\delta\delta} - V_{\delta\eta}V_{\eta\eta}^{-1}V_{\eta\delta} \end{pmatrix}$$

Hence, according to the Sherman-Morrison-Woodbury formula (see Householder [10, p.124] e.g.) we have

$$B(\beta,\delta)^{-1} = \bar{B}(\beta,\delta)^{-1} - \bar{E}_1(\beta,\delta)G(\beta,\delta)^{-1}\bar{E}_2(\beta,\delta)^T$$

where

$$\bar{E}_1(\beta,\delta) = \bar{B}(\beta,\delta)^{-1}E_1(\beta,\delta) = \begin{pmatrix} V_{\eta\eta}^{-1}V_{\eta\delta}d_\delta & -V_{\eta\eta}^{-1}e_\eta \\ -d_\delta & 0 \\ (V_{\delta\delta}-V_{\delta\eta}V_{\eta\eta}^{-1}V_{\eta\delta})d_\delta & V_{\delta\eta}V_{\eta\eta}^{-1}e_\eta - e_\delta \end{pmatrix}$$

$$\bar{E}_2(\beta,\delta)^T = E_2(\beta,\delta)^T\bar{B}(\beta,\delta)^{-1} = \begin{pmatrix} e_\eta^T V_{\eta\eta}^{-1} & 0 & e_\eta^T V_{\eta\eta}^{-1}V_{\eta\delta} - e_\delta^T \\ -d_\delta^T V_{\delta\eta}V_{\eta\eta}^{-1} & d_\delta^T & d_\delta^T(V_{\delta\delta} - V_{\delta\eta}V_{\eta\eta}^{-1}V_{\eta\delta}) \end{pmatrix}$$

and

$$G(\beta,\delta) = I_2 + E_2(\beta,\delta)^T\bar{B}(\beta,\delta)^{-1}E_1(\beta,\delta)$$

$$= \begin{pmatrix} 1 + (e_\eta^T V_{\eta\eta}^{-1}V_{\eta\delta} - e_\delta^T)d_\delta & -e_\eta^T V_{\eta\eta}^{-1}e_\eta \\ d_\delta^T(V_{\delta\delta} - V_{\delta\eta}V_{\eta\eta}^{-1}V_{\eta\delta})d_\delta & 1 + d_\delta^T(V_{\delta\eta}V_{\eta\eta}^{-1}e_\eta - e_\delta) \end{pmatrix}.$$

Notice that this last matrix $G(\beta,\delta)$ is 2×2. Hence, we obtain

$$\begin{pmatrix} \tilde{\mu}_\beta & \tilde{e}_\beta \\ b_\delta^1 & b_\delta^2 \end{pmatrix} = B(\beta,\delta)^{-1}\begin{pmatrix} \mu_\beta & e_\beta \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\mu}_\eta & \bar{e}_\eta \\ 0 & 0 \\ \bar{\mu}_\delta & \bar{e}_\delta \end{pmatrix} - \bar{E}_1(\beta,\delta)G(\beta,\delta)^{-1}\begin{pmatrix} e_\eta^T\bar{\mu}_\eta & e_\eta^T\bar{e}_\eta \\ d_\delta^T\bar{\mu}_\delta & d_\delta^T\bar{e}_\delta \end{pmatrix}$$

where

$$(\bar{\mu}_\eta\ ,\ \bar{e}_\eta) = V_{\eta\eta}^{-1}\ (\mu_\eta\ ,\ e_\eta)$$

and

$$(\bar{\mu}_\delta , \bar{e}_\delta) = (\mu_\delta , e_\delta) - V_{\delta\eta}(\bar{\mu}_\eta , \bar{e}_\eta) .$$

Consequently, the solution of (4a) can be achieved by the following steps:

1. Solve the system of linear equations for $(\bar{d}_\eta , \bar{\mu}_\eta , \bar{e}_\eta)$

$$V_{\eta\eta}(\bar{d}_\eta , \bar{\mu}_\eta , \bar{e}_\eta) = (V_{\eta\delta} d_\delta , \mu_\eta , e_\eta) \tag{6a}$$

and compute

$$(\bar{d}_\delta , \bar{\mu}_\delta , \bar{e}_\delta) = (V_{\delta\delta} d_\delta , \mu_\delta , e_\delta) - V_{\delta\eta}(\bar{d}_\eta , \bar{\mu}_\eta , \bar{e}_\eta) \tag{6b}$$

2. Solve the 2×2 system of linear equations for $\begin{pmatrix} \bar{g}_1^\mu & \bar{g}_1^e \\ \bar{g}_2^\mu & \bar{g}_2^e \end{pmatrix}$

$$\begin{pmatrix} 1 - d_\delta^T \bar{e}_\delta & - e_\eta^T \bar{e}_\eta \\ d_\delta^T \bar{d}_\delta & 1 - d_\delta^T \bar{e}_\delta \end{pmatrix} \begin{pmatrix} \bar{g}_1^\mu & \bar{g}_1^e \\ \bar{g}_2^\mu & \bar{g}_2^e \end{pmatrix} = \begin{pmatrix} e_\eta^T \bar{\mu}_\eta & e_\eta^T \bar{e}_\eta \\ d_\delta^T \bar{\mu}_\delta & d_\delta^T \bar{e}_\delta \end{pmatrix} .$$

3. Set

$$\begin{pmatrix} \tilde{\mu}_\beta & \tilde{e}_\beta \\ b_\delta^1 & b_\delta^2 \end{pmatrix} = \begin{pmatrix} \bar{\mu}_\eta & \bar{e}_\eta \\ 0 & 0 \\ \bar{\mu}_\delta & \bar{e}_\delta \end{pmatrix} - \begin{pmatrix} \bar{d}_\eta & -\bar{e}_\eta \\ -d_\delta & 0 \\ \bar{d}_\delta & -\bar{e}_\delta \end{pmatrix} \begin{pmatrix} \bar{g}_1^\mu & \bar{g}_1^e \\ \bar{g}_2^\mu & \bar{g}_2^e \end{pmatrix}$$

We point out that the major computational effort required to solve (4a) has now been reduced to the solution of (6a) and the computation of (6b). This reduction is significant because (6) involves only the matrix V whose size is one half that of M (cf. (3)).

Finally, we mention that similar savings can be achieved in computing (4b).

4. COMPUTATIONAL RESULTS

In order to numerically test the proposed parametric approach, we have implemented it for solving some randomly generated problems of the type (5). There are two families of problems being solved. Each of them is characterized by a certain form of the covariance matrix. The reason that we have chosen these particular families in the experimentation is because they are among the most commonly used models in portfolio analysis. In the first family, the matrix has the form

$$V = \Sigma + L\,L^T$$

where Σ is an n x n diagonal matrix with positive diagonal entries and L is an n x m matrix with m much less than n. This structure arises from an m-index model ([14] and Sharpe [17]). The case m = 1 corresponds to the single-index model (Sharpe [16]). The constant correlation coefficient model studied in Elton, Gruber and Padberg [5, 7] also gives rise to a covariance matrix having the above structure with m = 1.

In the second family of problems solved, the covariance matrix is given in partitioned form $V = (V_{ij})$ where for i, j = 1....N,

$$V_{ij} = \begin{cases} c_{ij} f^i (f^j)^T & i \neq j \\ \Sigma^i + c_{ii} f^i (f^i)^T & i = j \end{cases}$$

where $C = (c_{ij})$ is an arbitrary N x N symmetric matrix of scalars, f^i is an N_i-vector and Σ^i is an N_i x N_i diagonal matrix with positive diagonal

entries. For N = 1, this structure reduces to that arising from the single-index model. In general, it assumes that the risky assets are divided into groups so that members in each group satisfy the assumptions of a single-index model. The multi-group model discussed in Elton and Gruber [4] and Elton, Gruber and Padberg [6] gives rise to a covariance matrix having this structure.

In many practical applications, both m (the number of indices) and N (the number of groups) are fairly small compared to n (the number of risky assets). Advantage can be taken of this fact to further reduce the computational effort (and in fact the computer storage as well) required by the proposed approach in solving problems with these structures. To avoid complicated notations, we choose not to present the technical details.

Two sets of experiments were performed on a DEC-20 computer at the computation center in Carnegie-Mellon University. The computations were done in double precision to reduce round off-errors. The computer codes were written in FORTRAN and the timings reported are exclusive of inputs and outputs.

The first set of experiments was concerned with the implementation of the proposed method for treating an m-index model and an N-group model. The objective was to test the capability and efficiency of the method for solving problems of considerably large size. The data were generated as follows: Each component of the vector d was the same and equal to 1.75/n. The number 1.75 was used as a control of the total number of pivots and the total number of variables at their upper bounds so that these numbers would not become too small. For an m-index model, the diagonal entries of Σ were set equal to 2.0. The entries of the matrix L were generated randomly in (-1.0,1.0) and the

components of the vector μ in (0,1). For an N-group model, the components of the vector μ were generated randomly in (0,10) and those of f^i in (-1,1). The diagonal entries of Σ^i were equal to 2 ran + 1 where ran was a random number between 0 and 1. Finally, the matrix C was equal to GG^T where G was an N by N random matrix whose entries were random numbers between -1 and 1. The results are summarized in Tables 1 - 4 below.

m	# of pivots	# of variables between bds.	# of variables at upper bds.	total CPU time (in sec.)	CPU time/pivot (in sec.)
5	410	120	49	21.1867	0.0516
10	402	80	72	30.3223	0.0753
15	414	42	92	41.8303	0.1010
20	432	34	99	56.261	0.1303
25	418	27	100	67.2163	0.161
30	438	26	102	85.852	0.1962

Table 1: Multiple-index model n = 200

m	# of pivots	# of variables between bds.	# of variables at upper bds.	total CPU time (in sec.)	CPU time/pivot (in sec.)
5	1230	380	143	192.4927	0.1565
10	I258	264	209	279.7347	0.2223
15	1259	105	289	364.0687	0.2891
20	1348	56	315	485.9643	0.3604
25	1321	36	326	574.019	0.4344
30	1290	29	330	673.5277	0.522

Table 2: Multiple-index model n = 600

# members in each grp.	# of pivots	# variables between bds.	# variables at upper bds.	total CPU time (in sec.)	CPU time/pivot (in sec.)
20	197	79	15	2.839	0.0144
40	428	164	33	11.685	0.0273
60	607	243	47	24,418	0.0402
80	805	321	71	42.842	0.0532
100	1026	410	81	68.307	0.0666

Table 3: Multiple-group model N = 5

# members in each grp.	# of pivots	# variables between bds.	# variables at upper bds.	total CPU time (in sec)	CPU time/pivot (in sec)
20	811	317	73	51.799	0.0639
40	1656	628	141	188.314	0.1137
60	2439	955	216	399.642	0.1228
80	3251	1233	300	721.251	0.2218
100	4163	1579	353	1,088.565	0.2615

Table 4: Multiple-group model N = 20

The objective of the second set of experiments was to compare the proposed method with Lemke's algorithm for solving problems of the above type. The code that we used for the latter algorithm was called LCPBIG and was written at the Systems Optimization Laboratory of the Department of Operations Research at Stanford University. (The author is grateful to Professor R. W. Cottle for making this code available.) The data were generated exactly as above, except that the upper bounds d_i were equal to 1.35/n. The comparison is

summarized in Tables 5 and 6 below.

n	m	# of pivots	# variables between bds.	# variables at upper bds.	total CPU time (in sec.) proposed method	Lemke
20	2	39	11	8	.185	1.107
20	14	40	10	9	.777	1.110
30	2	63	17	13	.384	3.930
30	18	56	6	19	2.025	3.073
30	20	65	7	18	2.352	4.027
40	2	89	27	13	.766	9.877
40	10	88	14	21	1.763	9.850
40	20	98	12	21	4.067	11.364

Table 5: Comparison for an m-index model

N	# members in each grp.	# of pivots	# variables between bds.	# variables at upper bds.	total CPU time (in sec.) proposed method	Lemke
2	5	23	7	3	.053	.275
2	10	40	10	10	.128	1.243
2	20	81	23	16	.417	8.400
3	2	11	3	3	.024	.111
3	10	61	15	14	.311	3.678
4	10	87	23	16	.564	9.080
5	8	89	23	17	.632	9.612

Table 6: Comparison for an N-group model

We point out four remarks.

1. The numbers of pivots shown in the last two tables are the same for both algorithms.

2. In all cases, the numbers m and N are kept fairly small in order to be consistent with the smallest of the ratios m/n and N/n in many practical applications of these models.

3. The data in all the problems solved are extremely dense. In fact, this is an essential reason why we have compared the two algorithms only on small problems. (The code LCPBIG was written for solving linear complementarity problems with matrix having six thousand or less nonzero entries.)

4. The proposed algorithm is rather sensitive on the size of m and N (whereas Lemke's algorithm is not).

From these experiments, we may draw the following two conclusions:

1. In terms of computation times, the proposed algorithm is consistently superior to Lemke's. In most cases (when m/n or N/n is small) the former is several times faster than the latter. The reader is referred to Pang et al. [15] for a brief explanation based on operation counts of the algorithms.

2. The proposed algorithm is capable of solving large problems in a fairly efficient manner.

Finally, we point out that in addition to the superiority in computation times, the proposed algorithm also requires a substantially less amount of computer storage than Lemke's algorithm.

APPENDIX

Here we establish the theorem below. Before doing so, we explain the notations to be used. Let A be an m by n matrix. If α and β are subsets of $\{1,\ldots,m\}$ and $\{1,\ldots,n\}$ respectively, by $A_{\alpha\beta}$ we mean the submatrix of A whose rows and columns are indexed by α and β respectively. If $j \in \{1,\ldots,m\}$, we denote the j-th row of A by A_j. Similar notations are used for vectors.

Theorem. Consider the solution of the parametric linear complementarity problem

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} q \\ b \end{pmatrix} + \lambda \begin{pmatrix} e \\ 0 \end{pmatrix} + \begin{pmatrix} F & A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} \geq 0$$

$$u^T x = v^T y = 0$$

where $F \in R^{n \times n}$ is symmetric and positive definite, $A \in R^{m \times n}$ is arbitrary, e is the vector of ones, q is arbitrary and $b \geq 0$, by the parametric version of Graves' principal pivoting algorithm. Then all the diagonal pivot entries are positive. In particular, in a finite number of steps, the algorithm will terminate with $\lambda = 0$.

We need the following lemma whose proof is not difficult and thus omitted.

Lemma. Let F and A be as in the theorem. Let β and δ be nonempty subsets of $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$ respectively. Then the matrix

$$B(\beta,\delta) = \begin{pmatrix} F_{\beta\beta} & (A_{\delta\beta})^T \\ -A_{\delta\beta} & 0 \end{pmatrix}$$

is nonsingular if and only if the matrix $A_{\delta\beta}$ has full row rank. In this

case, the matrix $A_{\delta\beta}F_{\beta\beta}^{-1}(A_{\delta\beta})^T$ is positive definite and

$$B(\beta,\delta)^{-1} = \begin{pmatrix} H(\beta,\delta) & -F_{\beta\beta}^{-1}(A_{\delta\beta})^T[A_{\delta\beta}F_{\beta\beta}^{-1}(A_{\delta\beta})^T]^{-1} \\ [A_{\delta\beta}F_{\beta\beta}^{-1}(A_{\delta\beta})^T]^{-1}A_{\delta\beta}F_{\beta\beta}^{-1} & [A_{\delta\beta}F_{\beta\beta}^{-1}(A_{\delta\beta})^T]^{-1} \end{pmatrix}$$

where

$$H(\beta,\delta) = F_{\beta\beta}^{-1} - F_{\beta\beta}^{-1}(A_{\delta\beta})^T[A_{\delta\beta}F_{\beta\beta}^{-1}(A_{\delta\beta})^T]^{-1}A_{\delta\beta}F_{\beta\beta}^{-1}$$

is symmetric and positive semi-definite. Moreover, det $B(\beta,\delta)$ is positive.

Proof of the theroem. Consider any pivot step of the algorithm. Let β and δ denote, respectively, the sets of indices of the currently basic x and y variables before the pivot. With no loss of generality, we may assume that both β and δ are nonempty. Let $B(\beta,\delta)$ be the matrix defined in the lemma. It is a rather well-known fact from the theory of pivotal algebra that $B(\beta,\delta)$ is nonsingular. Then the current canonical form of the complementarity system can be written in partitioned form as

		λ	u_β	v_δ	x_α	y_γ
x_β		$-\tilde{e}_\beta$	$B(\beta,\delta)^{-1}$			
y_δ		$-b_\delta^2$				
u_α	= current q	$\tilde{e}_\alpha$				
v_γ		b_γ^2			N	

(7)

where α and γ are respectively the complements of β and δ in $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$.

$$\begin{pmatrix} \tilde{e}_\beta \\ b^2_\delta \end{pmatrix} = B(\beta,\delta)^{-1}\begin{pmatrix} e_\beta \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \tilde{e}_\alpha \\ b^2_\gamma \end{pmatrix} = \begin{pmatrix} e_\alpha \\ 0 \end{pmatrix} - \begin{pmatrix} F_{\alpha\beta} & (A_{\delta\alpha})^T \\ -A_{\gamma\beta} & 0 \end{pmatrix}\begin{pmatrix} \tilde{e}_\beta \\ b^2_\delta \end{pmatrix}$$

and

$$N = \begin{pmatrix} F_{\alpha\alpha} & (A_{\gamma\alpha})^T \\ -A_{\gamma\alpha} & 0 \end{pmatrix} - \begin{pmatrix} F_{\alpha\beta} & (A_{\delta\alpha})^T \\ -A_{\gamma\beta} & 0 \end{pmatrix} B(\beta,\delta)^{-1} \begin{pmatrix} F_{\beta\alpha} & (A_{\gamma\beta})^T \\ -A_{\delta\alpha} & 0 \end{pmatrix}.$$

We divide the proof into four cases.

1. The diagonal pivot entry occurs at a x_β-row. In this case, the entry is $H(\beta,\delta)_{jj}$ where $j \in \beta$ is such that $H(\beta,\delta)_j e_\beta = \tilde{e}_j < 0$. Since $H(\beta,\delta)$ is symmetric and positive semi-definite, the fact that $H(\beta,\delta)_j e_\beta$ is nonzero implies that $H(\beta,\delta)_{jj}$ is positive (see Cottle [1] e.g.).

2. The diagonal pivot entry occurs at a y_δ-row. In this case, the entry is $\{[A_{\delta\beta}F^{-1}_{\beta\beta}(A_{\delta\beta})^T]^{-1}\}_{jj}$ where $j \in \delta$. Since the matrix $[A_{\delta\beta}F^{-1}_{\beta\beta}(A_{\delta\beta})^T]^{-1}$ is positive definite, the desired pivot entry is therefore positive.

3. The diagonal pivot entry occurs at a u_α-row. In this case, the entry is $F_{jj} - (F_{j\beta}\ (A_{\delta j})^T)B(\beta,\delta)^{-1}\begin{pmatrix} F_{\beta j} \\ -A_{\delta j} \end{pmatrix}$ where $j \in \alpha$. It is the Schur complement of $B(\beta,\delta)$ in $B(\beta \cup \{j\}, \delta)$ (see Cottle [3]). This latter matrix $B(\beta \cup \{j\}, \delta)$ is nonsingular by the lemma. Hence the desired pivot entry is positive.

4. The diagonal pivot entry occurs at a v_γ-row. In this case, the entry is $A_{j\beta}H(\beta,\delta)(A_{j\beta})^T$ where $j \in \gamma$ is such that $A_{j\beta}H(\beta,\delta)e_\beta > 0$. If the pivot entry

were zero, then we would have $A_{j\beta}H(\beta,\delta) = 0$ by the symmetry and positive semi-definiteness of $H(\beta,\delta)$(see [1] e.g.). But this is impossible.

Consequently, we conclude that no matter where the next diagonal pivot entry is, it must be positive. This completes the proof of the theorem.

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