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DISPERSION AND CUT-OFF PHENOMENA
IN RODS AND BEAMS

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and
John E. Cole, III

October 1978

Final Report U-573-260
(Phase I - 1 January - 30 September 1978)
Prepared Under Contract N00014-78-C-0210 for
Office of Naval Research, Material Sciences Division
Attention: Dr. Nicholas L. Basdekas, Code 474

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ABSTRACT

Dispersion and impedance properties in various structural waveguides, including non-uniform rods and beams were examined. Here are some conclusions: (1) Even though both a dilatational and a shear potential are required to express longitudinal wave motion in 3-dimensional cylinders, the axial modal impedance equals ρc_z, where ρ is density and c_z the modal phase velocity. (2) At cut-off the axial modal and radial modal impedances respectively diverge and vanish. (3) Except for torsional waveguides, whose wave motion is expressible in terms of a single potential the drivepoint impedance can not be calculated as an eigenmode series, as these do not form a complete orthogonal set. (4) In non-uniform longitudinal waveguides, increasing density and Young's modulus give rise to attenuation by backscattering. Whether these parameters vary exponentially or as a power of the axial coordinate, their contribution to the attenuation is precisely half the spreading loss due to a similarly varying cross section. (5) Cut-off behavior of the fundamental longitudinal mode exists for exponentially varying parameters if Young's modulus and density are identical functions of the axial coordinate; when these parameters vary as powers of the coordinate, certain powers lead to non-propagating deflections or exponentially decaying waves; wedges or cones undergoing flexural vibrations display a cut-off frequency when supported on continuous springs.

LIST OF SYMBOLS

r, z	cylindrical coordinates
γ	axial wavenumber
c, c_z	phase velocity
λ, μ	Lamé's constants
c_d, c_t, c_o	dilatational, shear, and bar velocity, respectively
k_o	longitudinal wavenumber ω/c_o
ρ	density
E	Young's modulus
S	cross sectional area
I	cross sectional moment of inertia
r_g	cross sectional radius of gyration
u, w	axial and transverse displacements of the fundamental mode of non-uniform waveguides
$A, B, C, \alpha, \varepsilon, \delta$	coefficients describing the variation of parameters in non-uniform waveguides (p. 22)

I. DISPERSION AND CUT-OFF PHENOMENA IN RODS AND BEAMS

A. Introduction and Principal Results of this Study

This study was proposed to ONR Structural Mechanics Program as a result of in-house research which led to the conclusion that uniform structural waveguides of small transverse dimensions (1) display a vanishing impedance at every modal cut-off frequency; (2) that each cut-off frequency coincides with the natural frequency of the corresponding standing-wave mode of the cross section, the modal configuration being independent of the z-coordinate defined here as the coordinate along the waveguide axis. The preliminary study was restricted to homogeneous waveguides whose cross-section and material properties are z-independent. Furthermore, as already stated the analysis was limited to waveguides whose transverse dimension was small enough to eliminate thickness vibrations, e.g. Euler beams on distributed springs, simply-supported strips, and thin cylindrical shells. In other words, three-dimensional elasticity theory was not required. In the next section, we shall review the current status of the theory of uniform three-dimensional solid elastic waveguides. In Section 3, new theoretical results generated from the present study will be stated for three-dimensional semi-infinite z-independent waveguides. In Section 4, the properties of the fundamental mode of propagation in semi-infinite waveguides with z-dependent parameters are discussed. Starting with a review of the existing literature, we proceed to analyze non-uniform waveguides which have apparently not been analyzed, particularly waveguides with rapidly varying physical constants and cross sections.

Results generated in this study and solutions to problems which have to our knowledge not been published are:

1. In three-dimensional elastic cylinders conducting longitudinal axi-symmetric waves, the axial modal impedance defined as the ratio of the axial stress averaged over the cross section, divided by the similarly averaged axial velocity equals ρc_z, where c_z is the modal phase velocity. The simplicity of this result, reminiscent of acoustic waveguides, is surprising, because both a dilatational and a shear potential are required to describe the wave motion.

2. For the same waveguide, an expression was obtained for the radial modal impedance, defined as the ratio of the radial shear stress averaged over the cross section divided by the radial velocity of the cylindrical surface.

3. At cut-off, the radial modal impedance, and consequently the resultant radial drive-point impedance vanishes. Simultaneously, the axial modal impedance diverges.

4. In non-uniform waveguides located in the region $z > 0$, whose properties change either exponentially or as a power of z, the effect, on the decay of the fundamental mode with increasing z, of a change in cross section has precisely twice the magnitude of a change in density or in the Young's modulus (see Table 1).*

5. The phase velocity of the fundamental mode equals the local "bar" velocity.

6. Exceptionally, viz. for a constant phase velocity, the fundamental mode of non-uniform waveguides with exponentially varying parameters displays a cut-off frequency. This is a generalization of the well known property of acoustic horns.

For the non-uniform waveguides conducting longitudinal waves, analytical solutions have been constructed in the long wavelength limit, for the following situations (see Table 1):

a. The density, Young's modulus, and cross sectional area vary exponentially with the axial coordinate z at different rates, no restriction being placed on the rate of change with z except in so far as radial displacement are not accounted for. This applies to subsequent configurations as well.

b. The density, Young's modulus, and cross sectional area each vary as a different arbitrary power of z.

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* This conclusion is based on the asymptotic large-argument or far-field form of the solutions in Eqs. 29 and 36. It was pointed out to the authors that this asymptotic expression is tantamount to the WKB approximation which does place some restriction on the rate of variation of the waveguide parameters. Our conclusions regarding the respective effects of E, ρ, and S are not restricted to the two analytically tractable classes of waveguides analyzed here but can be generalized to other waveguides to which the WKB approximation is applicable.

7. For flexural waveguides for which the product of bar velocity $(E/\rho)^{1/2}$ times radius of gyration $(I/S)^{1/2}$ varies parabolically or linearly with z, e.g. a cone or a wedge, the long-wavelength solution for the fundamental antisymmetric (i.e. flexural) mode was constructed, both for the free waveguide and the waveguide supported on a distributed spring whose stiffness varies like the cross sectional area. The spring-supported waveguide displays a cut-off frequency at the natural frequency corresponding to rigid-body translational vibration.

B. Dispersion and Cut-off Characteristics of Three-dimensional Elastic Waveguides

1. Torsional Waveguides

The theory of torsional wave motions in cylindrical waveguides, as well as that of compressional and flexural waveguides, is presented in a number of classical texts.[2,3,4] It need therefore not be paraphrased here. The torsional waveguide differs from the latter two in that it requires only one potential to describe its dynamic behavior. In contrast, compressional and flexural waveguides require both a dilatational and a shear potential as will be illustrated in Section C.

The solution of the torsional waveguide is as straight-forward as that of an acoustic waveguide, the single potential required to describe its response being the solution of the wave equation or, for the harmonic conditions assumed, of the Helmholtz equation. It is therefore a simple matter to match an excitation, viz. an oscillatory torque, to the modal series describing the prescribed shear stress distribution over the cross section being excited. Onoe[5] was thus able to construct the driving point impedance or, in his formulation, its reciprocal, the admittance, as a modal series, in the same simple form arrived at for the elementary structural waveguides analyzed in Ref. 1. As in the latter study, Onoe's drive-point admittance, defined as the angular velocity divided by the oscillatory torque, becomes infinite at every cut-off frequency. As explained in Ref. 1, the reason is that the admittance is a modal series whose terms have a denominator proportional to the axial wavenumber of the various modes. By definition, this wavenumber vanishes at the cut-off frequency of a given mode. This is apparent if one examines Onoe's Eq. 29, and notes that ℓ_m is an axial wavenumber. The reason the resultant impedance vanishes (i.e. the resultant admittance becomes infinite) is amply discussed in Ref. 1 and need not be repeated here.

2. Compressional and Flexural Waveguides

For these waveguides dispersion curves have been plotted and impedances have occasionally been computed numerically. This may be an opportune place for correcting an error perpetuated in a familiar monograph,[6] which erroneously states that the characteristic equation of flexural waves in cylindrical waveguides admits only one root corresponding to a single modal dispersion curve.

The reason the impedances (or admittances) of compressional and flexural waveguides have not been formulated rigorously as modal series is that the normal modes of propagation do not yield orthogonality relations suitable for expressing a prescribed end loading as a modal series.[5] Furthermore, the Pochhammer-Chree frequency equation which governs the dispersion curves of cylindrical waveguides and the Rayleigh-Lamb equation which governs the dispersion curves of plates, admit roots in the form of complex wave numbers extending down to vanishing frequencies. Other roots are as for torsional or acoustic waveguides, imaginary below cut-off and real above. The existence of complex roots was first pointed out by Adem[7] for cylindrical waveguides conducting compressional waves. We will see in subsection C.4 that the values of drive-point admittance he computed from a modal series appear to be incorrect because the modes do not form a complete orthogonal set. The existence of complex wavenumbers has been exhaustively studied by Onoe et al. for plates. His results are reproduced in Ref. 4, page 137. This figure shows, in particular, the existence of a large number of higher-order modes which, even though exponentially damped, display a complex wavenumber, i.e. a finite phase velocity at vanishing frequency, incompatible with the cut-off phenomenon. Several other authors, such as Mindlin and

McNiven have obtained similar results both for cylinders and plates, but there is no need here for an exhaustive bibliography. The interested reader is referred to Miklowitz's excellent even though somewhat dated review paper.[8]

Both these characteristics, i.e. the unsuitability of the normal modes for constructing a series expression of the admittance, and the absence of true cut-off frequencies for a number of modes, prevent the analytical formulation of the drive-point impedance (see subsection C.4). This defeats a portion of the original purpose of this study. We can, however, formulate some interesting results concerning modal impedances not formulated by earlier authors by limiting ourselves to modes displaying cut-off frequencies, i.e. imaginary or real rather than complex wave numbers. Even though the drive-point admittance is intractable as a modal series, its behavior at a modal cut-off frequency, being governed by a single mode, is tractable.

C. Axial and Radial Modal Impedances of Longitudinal Wave Modes in Solid Cylinders

1. Statement of the Characteristic Equation, Stresses, and Displacements

The purpose of this section is to derive some novel properties of the modal impedances of the higher modes of propagation of solid structural waveguides conducting compressional waves. For the purpose of this analysis, we have selected the axisymmetric wave motions of a semi-infinite cylindrical waveguide. We shall start from the equations of motion as formulated by Redwood,[3] page 137, and in Davies' monograph.[9] For longitudinal excitation the results will be expressed in the form of two modal impedances. The axial modal impedance is defined as the ratio of the axial stress averaged over the cross section divided by the similarily averaged axial velocity. The radial impedance is the space-averaged axially symmetric, radially oriented shear stress divided by the radial velocity of the cylindrical surface.

The displacements are derived from a dilatational potential ϕ and a shear potential ψ. For axisymmetric wave motions, both potentials are governed by the two-dimensional Helmholtz equation in z and r:

$$\nabla^2\phi + \frac{\omega^2}{c_d^2}\phi = 0$$

$$\nabla^2\psi + \frac{\omega^2}{c_t^2}\psi = 0 \tag{1}$$

Here c_d and c_t are respectively the velocity of dilatational and transverse waves in the elastic medium. If λ and μ are Lamé's constants and ρ the density, the former velocity equals $[(\lambda + 2\mu)/\rho]^{1/2}$ and the latter $(\mu/\rho)^{1/2}$. Selecting solutions of the form

$$\phi = \phi_o(r)\exp(i\gamma z - i\omega t)$$

and similarly for ψ_o, one finds that ϕ_o and ψ_o are Bessel functions

$$\phi(r) = A\, J_o(hr)\exp(i\gamma z-i\omega t),\quad h^2 = \frac{\omega^2}{c_d^2} - \gamma^2$$

$$\psi(r) = C\, J_o(kr)\exp(i\gamma z-i\omega t),\quad k^2 = \frac{\omega^2}{c_t^2} - \gamma^2 \tag{2}$$

Note that ϕ and A have units of length squared, while ψ and C have units of length cubed. The corresponding radial and axial displacements are

$$u_r = \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z}$$

$$u_z = \frac{\partial\phi}{\partial z} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{3}$$

The stresses are then obtained from the generalized form of Hooke's law,

$$\tau_{rr} = \lambda \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_r}{\partial r} \tag{4a}$$

$$\tau_{zz} = \lambda \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) + 2\mu \frac{\partial u_z}{\partial z} \tag{4b}$$

$$\tau_{zr} = \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \tag{4c}$$

The notation used for stresses is Redwood's.[3] The boundary conditions are $\tau_{rr} = \tau_{zr} = 0$ at $r=a$. One thus obtains a characteristic equation whose roots yield the dispersion curves:

$$\gamma^2 \frac{kJ_o(ka)}{J_1(ka)} - \frac{1}{2a} \frac{\omega^2}{c_t^2} + \left(\frac{\omega^2}{2c_t^2} - \gamma^2 \right)^2 \frac{J_o(ha)}{hJ_1(ha)} = 0 \tag{5}$$

The boundary conditions can also be used to express the coefficient A in Eq. 2 in terms of the coefficient C, or <u>vice versa</u>

$$\frac{C}{A} = \frac{-2ih\gamma}{k(\gamma^2-k^2)} \frac{J_1(ha)}{J_1(ka)} \tag{6}$$

This ratio has units of length as anticipated from the statement after Eqs. 2. At cut-off where $\gamma=0$, this ratio, and hence the shear potential vanishes.

2. The Axial Modal Impedance Z_z

Combining Eqs. 6 with 4b, we obtain the axial stress

$$\tau_{zz} = - \rho c_t^2 A \left[\left(\frac{\omega^2}{c_t^2} - 2h^2 \right) J_o(hr) + \frac{4\gamma^2 hk}{\gamma^2-k^2} \frac{J_1(ha)}{J_1(ka)} J_o(kr) \right] \tag{7}$$

The axial velocity is

$$\dot{u}_z = -i\omega u_z = \omega A\gamma [J_o(hr) + \frac{2hk}{\gamma^2-k^2} \frac{J_1(ha)}{J_1(ka)} J_o(kr)]$$

This impedance governs the axial modal response to an axial excitation. It is defined here as

$$Z_z = -\frac{\int_0^a \tau_{zz}(r)\,r\,dr}{\int_0^a \dot{u}_z(r)\,r\,dr} \tag{8}$$

The integrals are readily evaluated by noting that

$$\int_0^a J_o(hr)\,r\,dr = \frac{a}{h} J_1(ha) \tag{9}$$

and similarly for $J_o(kr)$. Consequently, the integral in the numerator yields

$$-\int_0^a \tau_{zz} r\,dr = \frac{A\rho c_t^2 a}{h}\left[\left(\frac{\omega^2}{c_t^2} - 2h^2\right) + \frac{4\gamma^2 h^2}{\gamma^2-k^2}\right] J_1(ha) \tag{10}$$

Multiplying through by c_t^2, reducing the terms in brackets to the same denominator, and noting that, from Eq. 2,

$$c_t^2 = \frac{\omega^2}{k^2+\gamma^2} \tag{10a}$$

Eq. 10 becomes

$$-\int_0^a \tau_{zz} r\,dr = \frac{A\rho\omega^2 a J_1(ha)}{h(\gamma^2-k^2)}\left(\gamma^2-k^2+2h^2\right) \tag{11}$$

The denominator of Eq. 8 similarly yields

$$\int_0^a \dot{u}_z r\,dr = \frac{A\omega\gamma a}{h}\left(1 + \frac{2h^2}{(\gamma^2-k^2)}\right) J_1(ha)$$

$$= \frac{A\omega\gamma a\ J_1(ha)}{h(\gamma^2-k^2)}\ (\gamma^2-k^2+2h^2) \tag{12}$$

Substituting Eqs. 11 and 12 in Eq. 8, the impedance now takes the simple form

$$Z_z = \rho\omega/\gamma \tag{13}$$

Referring to Eq. 2, we note that the axial wavenumber is related to the phase velocity as

$$\gamma \equiv \omega/c_z$$

The impedance thus reduces to

$$Z_z = \rho c_z \tag{14}$$

The axial modal impedance of the compressional waveguide is in the form of the plane-wave impedance. This result is remarkable by its simplicity if one considers that two potentials characterized by two different characteristic velocities c_d and c_t are required to formulate the modal impedance. Even though Davies[9] evaluates this impedance his expression is lengthy and involved, being expressed in terms of auxiliary functions. He therefore did not realize the potential compactness and the physical meaning of this impedance. Since cut-off is characterized by $\gamma=0$, $c_z=\infty$, Eqs. 13 and 14 indicate that the axial modal impedance becomes infinite at cut-off.

3. The Radial Modal Impedance

This impedance is defined here as the radially-oriented shear stress averaged over the cross sectional area divided by the radial velocity $u_r(a)$ on the cylinder surface.

$$Z_r = \frac{2\int_0^a \tau_{zr} r\,dr}{a^2 \dot{u}_r(a)} \tag{15}$$

The radial velocity is readily constructed from Eqs. 2 and 3,

$$\dot{u}_r(r) = i\omega hA \left[\frac{2\gamma^2}{\gamma^2-k^2}\,\frac{J_1(ha)}{J_1(ka)}\,J_1(kr) - J_1(hr)\right] \tag{16}$$

Setting r=a, this reduces to

$$\dot{u}_r(a) = \frac{i\omega hA(\gamma^2+k^2)}{\gamma^2-k^2} \cdot J_1(ha) = \frac{i\omega^3 hA\, J_1(ha)}{c_t^2(\gamma^2-k^2)} \tag{17}$$

where use has been made of Eq. 10a. The shear stress is, from Eq. 4c,

$$\tau_{zr} = -i2\rho c_t^2 \gamma hA \left[J_1(hr) - \frac{J_1(ha)}{J_1(ka)} J_1(kr) \right] \tag{13}$$

The integrals in Eq. 15 can be evaluated by means of the relation[10]

$$\int_0^a J_1(hr)\, r dr = \frac{\pi a}{2h} [J_1(ha)\, \underset{\sim}{H}_0(ha) - \underset{\sim}{H}_1(ha) J_0(ha)]$$

Where $\underset{\sim}{H}_n$ is the Struve function. Substituting this, and the equivalent expression for the integral of $J_1(kr)$, the numerator in Eq. 15 becomes

$$2\int_0^a \tau_{zr} r dr = -i2\pi\rho c_t^2 Aa\gamma \left\{ J_1(ha) \left[\underset{\sim}{H}_0(ha) - \frac{h}{k} \underset{\sim}{H}_0(ka) + \frac{hJ_0(ka)\underset{\sim}{H}_1(ka)}{kJ_1(ka)} \right] - J_0(ha)\underset{\sim}{H}_1(ha) \right\} \tag{19}$$

The radial impedance finally becomes

$$Z_r = \frac{2\pi\rho c_t^4 \gamma(\gamma^2-k^2)}{ah\omega^3} \cdot \left\{ -\underset{\sim}{H}_0(ha) + \frac{h}{k} \left[\underset{\sim}{H}_0(ka) - \frac{J_0(ka)\underset{\sim}{H}_1(ka)}{J_1(ka)} \right] + \frac{J_0(ha)\underset{\sim}{H}_1(ha)}{J_1(ha)} \right\} \tag{20}$$

At cut-off, the modal radial impedance vanishes, since $\gamma=0$. The cut-off frequency is therefore seen once again to represent a resonance of the two-dimensional z-independent standing wave system of the waveguide cross section. It was shown in Ref. 1 that the divergence of a single mode is sufficient to cause the drive-point admittance to diverge.

4. Discussion of the Drive-point Impedance

If the stresses and displacements given above were functions of a single potential, the boundary condition prescribed over the driven cross section, e.g. $\tau_{zz}(r)$ could be simply computed by expanding it as a series of Bessel functions, and using the orthogonality relation of Bessel functions to compute the modal coefficients. More generally, this method can be used when the boundary condition takes the form

$$J_n(ka) + AJ_n'(ka) = 0 \tag{21}$$

where A is a constant and k a wavenumber. The fact that τ_{zz}, Eq. 7 contains two Bessel functions, one for each potential, eliminates this approach even though each potential separately, being a solution of the Sturm-Liouville equation can be expanded in an orthogonal set of eigen functions.[11] Because their stresses are expressible with a single potential satisfying that equation, torsional waveguides are the one 3-dimensional elastic waveguide whose impedance is analytically tractable.[5] The coupling of eigen modes in thick cylindrical shells rigorously analyzed as a problem in three-dimensional elasticity is noted by Armenakas et al.[12] We must be uneasy about the manner in which Adem[7] formulates the forced response of a cylindrical waveguide in terms of its eigen functions: "If we know only some of the roots (of the characteristic equation), then for the other roots we can take $B(\xi_q) = 0$," where $B(\xi_q)$ is the amplitude of the eigen mode of order q. Quite clearly the fact that one can arbitrarily eliminate some eigen functions, shows that they do not form a complete orthogonal set.

The orthogonality of the modes in infinite or semi-infinite places (Lamb waves) was examined by Lyon.[13] He was apparently the first worker to point out that these modes do not form an orthogonal set with respect to the plate thickness. This problem has been receiving some attention in the Russian literature.[14,15] Even though the latter two papers construct some orthogonality relations, they are not suitable for the series expansion of applied loads, i.e. for the construction of the drive-point impedance as a modal series.

D. The Fundamental Mode of Non-uniform Waveguides Conducting Longitudinal and Transverse Waves

1. General Discussion

The motions in axisymmetric waveguides of exponentially varying cross section, i.e. in solid horns, have been studied by means of an approximate theory valid when the transverse dimension is small in terms of wavelengths,[16] thus automatically eliminating three-dimensional cut-off phenomena.

Keller and one of his students constructed an asymptotic WKB-type solution for higher modes including flexural, torsional, and longitudinal wave motions, restricted to solid waveguides with a slowly varying cross section.[17] This restriction results in a solution which predicts local z-dependent cut-off frequencies equal to those of the uniform cylinder whose local cross section equals that of the non-uniform waveguide cross section at z. In view of the difficulties mentioned in the preceeding section, it is not surprising that the authors made no attempt to evaluate drive-point impedances.

We can however, draw certain conclusions from Ref. 17. We have seen that the axial modal impedance becomes infinite at cut-off. Consequently, if we consider acoustic energy propagating in the direction of decreasing cross sections, all modes but the plane-wave mode will gradually be reflected back, thus giving rise to standing waves. A similar conclusion is reached for acoustic waveguides of slowly varying cross section.[18]

We shall examine some specific non-uniform waveguides which admit an analytical solution but which have not yet been treated in the literature. We shall obtain dispersion relations for the fundamental modes of longitudinal and flexural waveguides, but shall not construct the Green's influence functions.

2. Longitudinal Waves in Waveguides with Density, Young's Modulus, and Cross-section Varying Exponentially at Arbitrary Rates

These quantities vary as

$$\rho = \rho_o\, e^{\alpha z}$$

$$E = E_o\, e^{\varepsilon z} \tag{22}$$

$$S = S_o\, e^{\delta z}$$

The constants α, ε, and δ are not restricted in magnitude. For the long-wavelength situation exclusively considered throughout this Section D, radial motion associated with the fundamental mode can be neglected. The steady-state equation of motion of a non-uniform waveguide is therefore an ordinary D.E. formulated in terms of the axial displacement, which need no longer be identified by the subscript z:

$$\frac{d}{dz}\left(ES\frac{du}{dz}\right) + \omega^2 \rho S u = 0 \tag{23}$$

Differentiating, and dividing through by E S, one obtains the equation

$$\frac{d^2u}{dz^2} + \left(\frac{1}{E}\frac{dE}{dz} + \frac{1}{S}\frac{dS}{dz}\right)\frac{du}{dz} + \frac{\omega^2\rho}{E}u = 0 \tag{24}$$

For the parameters described in Eq. 22

$$\frac{d^2u}{dz^2} + (\varepsilon+\delta)\frac{du}{dz} + k_o^2 \exp[(\alpha-\varepsilon)z]\, u = 0 \tag{25}$$

where we have used the bar velocity and wavenumber

$$c_o \equiv (E_o/\rho_o)^{1/2}\ , \quad c(z) = c_o \exp[(\varepsilon-\alpha)z/2] \tag{26}$$

$$k_o \equiv \omega/c_o\ , \quad \gamma(z) = k_o \exp[(\alpha-\varepsilon)z/2]$$

We now make the transformation of variables

$$u = \bar{u} \exp(- \frac{\varepsilon+\delta}{2} z)$$

$$\frac{du}{dz} = \left(\frac{d\bar{u}}{dz} - \frac{\varepsilon+\delta}{2} \bar{u}\right) \exp \left(- \frac{\varepsilon+\delta}{2} z\right) \tag{27}$$

$$\frac{d^2u}{dz^2} = \left[\frac{d^2\bar{u}}{dz^2} - (\varepsilon-\delta) \frac{d\bar{u}}{dz} + \frac{(\varepsilon+\delta)^2}{4} \bar{u}\right] \exp \left(- \frac{\varepsilon+\delta}{2} z\right)$$

Substituting in Eq. 25, one obtains the equation governing $\bar{u}$:

$$\frac{d^2\bar{u}}{dz^2} + \left[k_o^2 \exp(\alpha-\varepsilon) z - \frac{(\varepsilon+\delta)^2}{4}\right] \bar{u} = 0$$

We now perform a transformation in the independent variable

$$z = \frac{2}{\alpha-\varepsilon} \bar{z}$$

The equation now becomes

$$\frac{d^2\bar{u}}{d\bar{z}^2} + \left[\frac{4k_o^2}{(\alpha+\varepsilon)^2} e^{2\bar{z}} - \frac{(\varepsilon-\delta)^2}{(\alpha-\varepsilon)^2}\right] u = 0$$

This in the form of Bessel's equation.[19] The transformed variable $\bar{u}$ is obtained in the form of a cylinder function C of order $(\varepsilon+\delta)/(\alpha-\varepsilon)$ and argument $2k_o e^{\bar{z}}/(\alpha-\varepsilon)$. Transforming back to the original variables, we finally have

$$u(z) = U \exp \left(- \frac{\varepsilon+\delta}{2}\right) C_{(\varepsilon+\delta)/(\alpha-\varepsilon)} \left[\frac{2k_o}{\alpha-\varepsilon} \exp \left(\frac{\alpha-\varepsilon}{2} z\right)\right] \tag{29}$$

The particular cylinder function selected must be well behaved in the region of z relevant to the situation being investigated. We shall construct general solutions but shall not illustrate the matching of linear combinations of cylinder functions to particular excitations. We shall merely concern ourselves with the dispersion characteristics of the phase velocity. For this purpose, we consider waves propagating in the positive z-direction. Assuming,

$\varepsilon + \delta > 0$, $\alpha - \varepsilon > 0$, a suitable solution, i.e. one which converges as $z \to \infty$, is the Hankel function of the first kind. For large argument

$$H^{(1)}_{(\varepsilon+\delta)/(\alpha-\varepsilon)} \approx \left(\frac{\alpha-\varepsilon}{\pi k_o}\right)^{1/2} \exp-\left(\frac{\alpha-\varepsilon}{4} z\right) \exp i\left[\frac{2k_o}{\alpha-\varepsilon} \cdot \exp\left(\frac{\alpha-\varepsilon}{2}\right) z - \frac{\pi}{4}(2\nu+1)\right] \tag{30}$$

where ν is the order of the Hankel function. When this is substituted in Eq. 29, the solution becomes

$$u(z,t) = U\left(\frac{\alpha-\varepsilon}{\pi k_o}\right)^{1/2} \exp\left(-\frac{\alpha+\varepsilon+2\delta}{4} z\right) e^{i\phi} \tag{31}$$

where

$$\phi = \frac{2k_o}{\alpha-\varepsilon} \exp\left(\frac{\alpha-\varepsilon}{2}\right) z - \omega t - \frac{\pi}{4}(2\nu+1) \tag{32}$$

The variation of the amplitude of the solution with increasing z is embodied in the real exponential term. The change in cross sectional has twice the effect of either the density or the Young's modulus for comparable exponential coefficients. Signal attenuation by the increase in cross section is in the nature of a spreading loss. The attenuation associated with the increase in Young's modulus and density embodies gradual backscattering as the signal penetrates into a region of ever increasing characteristic impedance. For negative powers of the exponentials in Eq. 22, the signal increases, as anticipated. The solution displays z-dependence on the coefficient determining the cross sectional area variation that is precisely twice that of either modulus of density coefficients. The phase velocity is obtained from the phase angle

$$c = -\frac{\partial\phi}{\partial t}\frac{\partial z}{\partial\phi} \tag{33}$$

where

$$\frac{\partial\phi}{\partial t} = -\omega \quad ; \quad \frac{\partial\phi}{\partial z} = k_o \exp\left(\frac{\alpha-\varepsilon}{2} z\right) \tag{33}$$

We can solve for a dispersive phase velocity which equals the local bar velocity, Eq. 26.

When $\alpha \to \varepsilon$, i.e. when the bar velocity is z-independent, the order of the Hankel function diverges, as does its argument. Returning to the original differential , Eq. 25, we note that the coefficient of the linear term becomes z-independent. The solution is an exponential, say exp(Nz), where N is a solution of the characteristic equation

$$N^2 + (\varepsilon+\delta)N + k_o^2 = 0$$

This is a quadratic equation with two roots. The axial displacement now becomes

$$u(z) = U \exp\left(\left\{-\frac{(\varepsilon+\delta)}{2} \pm \left[\frac{(\varepsilon+\delta)^2}{4} - k_o^2\right]^{1/2}\right\} z\right) \quad , \quad \alpha = \varepsilon \qquad (34)$$

This is an exponentially damped solution which admits damped propagating waves in the frequency range where the square root is imaginary, i.e. above the cut-off frequency.

$$f_{co} = \frac{(\varepsilon+\delta)c_o}{4\pi} \quad , \qquad \alpha = \varepsilon \qquad (35)$$

If $\alpha = \varepsilon = 0$, this cut-off frequency reduces, as anticipated, to the familiar result obtained for the exponential horn.

3. Longitudinal Waveguide Whose Density, Young's Modulus, and Cross Sectional Area Vary as Arbitrary Powers of z

The density, Young's modulus, and cross sectional area, vary as

$$\rho = Az^n$$

$$E = Bz^m$$

$$S = Cz^r$$

$$c = (B/A)^{1/2}\, z^{(m-n)/2}$$

Eq. 24 becomes

$$\frac{d^2u}{dz^2} + \frac{m+r}{z}\frac{du}{dz} + \frac{\omega^2 A}{B} z^{n-m} u = 0 \tag{35}$$

This is a form of Bessel's equation.[20] Its solution is proportional to a cylinder function of fractional order:

$$u(z) = Uz^{(1-m-r)/2} C_{(1-m-r)/(n-m+2)} \left[\frac{2\omega A^{1/2} z^{(n-m+2)/2}}{(n-m+2)B^{1/2}} \right] \tag{36}$$

As in the discussion of Eq. 29, a Hankel function of the first kind is selected to represent waves propagating in the positive z-direction. The resulting phase angle, when substituted in Eq. 33, leads to the usual conclusion that the phase velocity equals the local bar velocity, Eq. 34. In this same region, the cylinder function converges as $z^{-(n-m+2)/4}$. The amplitude of the plane wave therefore converges as $z^{-(n+m+2r)/4}$. We note that as for exponentially varying waveguide parameters, Eq. 31 an increase in density and in Young's modulus bring about a comparable acceleration in convergence of the solution with increasing z, and that a change in cross section has an effect of precisely twice this magnitude. Negative powers of z in Eq. 34 produce, as anticipated, an enhancement of the signal. The solution in Eq. 36 is expressible in terms of familiar cylinder functions for specified z-dependences of the waveguide parameters. The results are listed on Table 1. The cases which admit a solution in form of Airy functions are related to similar solutions for acoustic waveguides with variable cross sections[18] and for acoustic waveguides with sound velocity gradients and absorptive boundaries.[21] Both these analyses of acoustic waveguides were carried through in sufficient depth to include higher modes and cut-off frequencies. The greater complexity of the solid elastic waveguide requires a laborious analysis probably not justified by the limited practical interest of this waveguide configuration. When m=n=0, the solution reduces to the familiar solution for acoustical horns. The most widely used is the conical horn (r=2), which does not display a cut-off frequency in contrast to the exponential horn mentioned in the preceding section.

When (m-n)=2, Eq. 36 becomes indeterminate. The differential equation of motion becomes

$$\frac{d^2u}{dz^2} + \frac{m+r}{z}\frac{du}{dz} + \frac{\omega^2 A}{Bz^2}u = 0 \qquad m-n=2$$

The solution is in the form of U/z^M. Substituting in Eq. 37, one constructs the characteristic equation

$$M^2-M(m+r-1) + \omega^2(A/B) = 0 \quad , \qquad m-n=2$$

Solving this quadratic equation, one obtains a solution in the form

$$u(z) = \exp\left(-\left\{\frac{(m+r-1)}{2} \mp \left[\frac{(m+r-1)^2}{4} - \frac{\omega^2 A}{B}\right]^{1/2}\right\}z\right), \quad m-n = 2 \qquad (38)$$

This deformation is in the form of a non-propagating near-field at small frequencies where the exponent is real. The response is a damped propagating wave when the power is complex, i.e., in the frequency range

$$f > \frac{1}{2}\left(\frac{B}{A}\right)^{1/2}\frac{m+r-1}{2}$$

To conclude this section, we turn our attention to the fundamental mode of non-uniform flexural waveguides.

4. Non-uniform Flexural Waveguides

In the low-frequency limit, the Bernoulli-Euler equation which governs the steady-state flexural vibrations of non-uniform beams is

$$EI\frac{d^4w}{dz^4} + \frac{d^2}{dz^2}(EI)\frac{d^2w}{dz^2} - \rho S\omega^2 w = 0 \qquad (39)$$

This takes the more explicit form

$$\frac{d^4w}{dz^4} + \left(\frac{1}{I}\frac{d^2I}{dz^2} + \frac{1}{E}\frac{d^2E}{dz^2} + \frac{2}{EI}\frac{dE}{dz}\frac{dI}{dz}\right)\frac{d^2w}{dz^2} - \frac{\rho S\omega^2 w}{EI} = 0 \qquad (40)$$

For a uniform beam, the coefficient of the second derivative vanishes, and that of the linear term is constant. The solution takes on the familiar form

$$w(z) = W\, e^{i\gamma z} \tag{41}$$

where γ is the flexural wavenumber

$$\gamma = (k_o/r_g)^{1/2}$$

where

$$c_o = (E/\rho)^{1/2}$$

$$r_g = (I/S)^{1/2}$$

Eq. 40 reduces to Bessel's equation if the d^2w/dz^2-term vanishes and if the coefficient of the linear term is proportional to z^{-2}. Both these conditions are satisfied if the product ρS varies as z^{-2}, E and I being constant. This is of course an unrealistic assumption. We can however construct a meaningful mathematical model by taking E, ρ, and hence c_o constant, and

$$r_g = Bz \quad , \quad \gamma = (k_o/Bz)^{1/2} \tag{42}$$

This condition is satisfied by a <u>wedge</u>, for which

$$S = Az, \quad I = AB^2z^3 \tag{43a}$$

and a <u>cone</u> or <u>pyramid</u>, for which

$$S = Az^2, \quad I = AB^2z^4 \tag{43b}$$

Furthermore, to make the D.E. tractable, the coefficient of the second derivative in Eq. 40 must be negligible. The solution thus obtained will be shown to be in the form of a cylinder function. In the large argument limit, this solution yields the second derivative

$$\frac{d^2w}{dz^2} = -4\gamma^2 w \qquad \gamma z >> 1$$

The second derivatives can be ignored if $\gamma^2z^2 << 24$ for the wedge, or $<< 48$ for the cone, conditions obviously not satisfied in the region adjoining the waveguide apex, where z is small in terms of wavelengths.

The approximate differential equation is now formally similar to the flexural equation of motion of uniform beams, even though the coefficient of the linear term varies as z^{-2}:

$$\frac{d^4w}{dz^4} - \gamma^4 w = 0 \quad , \quad \gamma^2 = \frac{k_o}{Bz} \tag{44}$$

This equation admits a solution in the form of a cylinder function[22]

$$w = WzC_2(2iq\gamma z) = WzC_2[2iq(k_o z/B)^{1/2}], \quad q = 0,\ 1,\ 2 \text{ or } 3 \tag{45}$$

It is interesting to note that Kirchhoff solved the problem of the finite wedge in terms of cylinder functions as early as 1879.

Eq. 45 embodies propagating waves in the form of Hankel functions of order 2. For large argument, the function is proportional to exp(iϕ) where

$$\phi = 2(k_o z/B)^{1/2} - \omega t$$

The corresponding phase velocity computed from Eq. 33 is

$$c = (\omega Bz)^{1/2} = (\omega c_o r_g)^{1/2}$$

As usual, the phase velocity at z is the phase velocity which would be observed in a uniform waveguide whose material properties and cross section coincide with those of cross section z.

To conclude this section we consider a non-uniform flexural waveguide whose fundamental mode displays cut-off behavior. For a waveguide mounted on a distributed spring, the equation of motion becomes

$$\left(EI \frac{d^4}{dz^4} + K - \omega^2 \rho S\right) w = 0$$

or

$$\left[\frac{d^4}{dz^4} - \frac{\omega^2}{c_o^2 r_g^2}\left(1 - \frac{K}{\omega^2 \rho S}\right)\right] w = 0 \tag{46}$$

where K is the spring stiffness per unit length. The waveguide's z-dependence is still governed by Eqs. 43. The spring stiffness is selected to vary linearly with z for a wedge, Eq. 43a,

$$K = Dz$$

and parabolically for a cone or pyramid, Eq. 43b,

$$K = Dz^2$$

For either configuration the ratio $K/\rho S$ is z-independent. It is in fact the natural frequency ω_n of the undeformed beam undergoing translational, i.e. z-independent vibrations. Using Eq. 44 we have

$$\left[\frac{d^4}{dz^4} - \gamma^4 \left(1 - \frac{\omega_n^2}{\omega^2}\right)\right] w = 0 \quad , \qquad \gamma^2 = \frac{k_o}{Bz} \tag{47}$$

The solution of this equation is of the form of Eqs. 43 and 44, but the flexural wave number is multiplied by $(1 - \omega_n^2/\omega^2)^{1/4}$.

The construction of the dispersion curves proceeds as in the preceding subsection:

$$c = (\omega c_o Bz)^{1/2} \left(1 - \frac{\omega_n^2}{\omega^2}\right)^{1/4} \tag{48}$$

With increasing frequency, this phase velocity tends to Eq. 45. It diverges as $\omega \to \omega_n$, and is complex in the frequency range $\omega < \omega_n$. The wave therefore attenuates exponentially as it propagates. The frequency $\omega = \omega_n$ is associated with a zero drive-point impedance, a transverse force exciting the rigid-body resonance. Like the phase velocity, the characteristic impedance of flexural waves is infinite at this frequency.

This cut-off phenomenon is observed in uniform spring-mounted flexural waveguides.[1] The condition for its occurrence in a waveguide with variable mass per unit length is that the spring stiffness and the mass of the beam display the same z-dependence, thus permitting rigid-body translational vibrations uncoupled from any rotational motion.

Table 1 - Effect of Various Parameters on the Propagation of the Fundamental Mode in Non-uniform "Longitudinal" Waveguides

(All Coefficients are Positive; the Waveguide Extends over the Region $z>0$)

z-Dependent Waveguide Parameters	Form of Solution	Phase Velocity	Large-z Convergence of Wave Amplitude
$\rho = \rho_o e^{\alpha z}$ $E = E_o e^{\varepsilon z}$ $S = S_o e^{\delta z}$	Cylinder functions Eqs. 29, 31	$\left(\frac{E}{\rho}\right)^{1/2} e^{(\varepsilon-\alpha)z/2}$	$\exp\left(-\frac{\alpha+\varepsilon+2\delta}{4}\right) z$
Ditto with $\alpha = \varepsilon$	Exponential function, Eq. 34	$(E_o/\rho_o)^{1/2}$; cut-off frequency, Eq. 35	$\exp\left(-\frac{\alpha+\delta}{2}\right) z$
$\rho = Az^n$ $E = Bz^m$ $S = Cz^r$	Cylinder functions Eq. 36	$\left(\frac{B}{A}\right)^{1/2} z^{(m-n)/2}$	$z^{-(n+m+2r)/4}$
Ditto with $n = m - 2$	Polynominal in z, Eq. 38	Non-propagating	Negative power of z, either real or complex
$\rho = Az$	Airy functions	$\left(\frac{E}{A}\right)^{1/2} z^{-1/2}$	$z^{-1/4}$
$\rho = Az^n$ $E = Bz^n$	Cylinder functions for n even; spherical spherical Bessel functions for n an odd integer	$\left(\frac{B}{A}\right)^{1/2}$	$z^{-n/2}$
$\rho = Az^n$ $E = Bz^n$ $S = Cz^n$	Spherical Bessel functions for n integer	$\left(\frac{B}{A}\right)^{1/2}$	z^{-n}

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Dispersion and impedance properties in various structural waveguides, including non-uniform rods and beams were examined. Here are some conclusions: (1) Even though both a dilatational and a shear potential are required to express longitudinal wave motion in 3-dimensional cylinders, the axial modal impedance equals ρc_z, where ρ is density and c_z the modal phase velocity. (2) At cut-off the axial modal and radial modal impedances respectively diverge and vanish. (3) Except for torsional waveguides, whose wave motion

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(20. ABSTRACT - Continued)

is expressible in terms of a single potential the drivepoint impedance can not be calculated as an eigenmode series, as these do not form a complete orthogonal set. (4) In non-uniform longitudinal waveguides, increasing density and Young's modulus give rise to attenuation by backscattering. Whether these parameters vary exponentially or as a power of the axial coordinate, their contribution to the attenuation is precisely half the spreading loss due to a similarly varying cross section. (5) Cut-off behavior of the fundamental longitudinal mode exists for exponentially varying parameters if Young's modulus and density are identical functions of the axial coordinate; when these parameters vary as powers of the coordinate, certain powers lead to non-propagating deflections or exponentially decaying waves; wedges or cones undergoing flexural vibrations display a cut-off frequency when supported on continuous springs.

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