

AD-A066 443

STANFORD UNIV CALIF SYSTEMS OPTIMIZATION LAB

F/G 12/1

DUALY EQUIVALENT DECOMPOSITION ALGORITHMS WITH APPICATION TO S--ETC(U)

DEC 78 L NAZARETH

N00014-75-C-0267

UNCLASSIFIED

SOL-78-30

NL

| OF |

AD
A066443



END
DATE
FILMED
5-79
DDC

AD A0 66443



LEVEL II
Systems
Optimization
Laboratory

12



FILE COPY

DDC
RECEIVED
MAR 27 1979
D

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

Department of Operations Research
Stanford University
Stanford, CA 94305

79 03 23 023

LEVEL II

12

ACCESSION NO	
DTIC	White Section <input checked="" type="checkbox"/>
DDC	Dist Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION.....	
BY.....	
DISTRIBUTION/AVAILABILITY CODES	
Dist.	AVAIL. and/or SPECIAL
A	

SYSTEMS OPTIMIZATION LABORATORY
 DEPARTMENT OF OPERATIONS RESEARCH
 Stanford University
 Stanford, California
 94305

AD A0 66443

6
 DUALY EQUIVALENT DECOMPOSITION ALGORITHMS
 WITH APPLICATION TO SOLVING STAIRCASE STRUCTURES.

10 by L. Nazareth
 9 TECHNICAL REPORT, SOL-78-30
 December 1978

11 Dec 78

12 36p.

15

Research and reproduction of this report were partially supported by the Office of Naval Research Contract N00014-75-C-0267, the National Science Foundation Grants MCS76-20019 A01 and ENG77-06761 A01; and the Department of Energy Contract EY-76-S-03-0326 PA #18.

Reproduction in whole or in part is permitted for any purposes of the United States Government. This document has been approved for public release and sale; its distribution is unlimited.

DDC FILE COPY

408 765

LB

DDC
 RECEIVED
 MAR 27 1979
 REGITLED
 D

Abstract

We briefly go over the well known dual relationship between Dantzig-Wolfe Decomposition and Benders Decomposition, in order to develop suitable notation, and then elaborate upon the dual relationship between nested versions of Dantzig-Wolfe and Benders Decomposition. Next we develop a new pair of dually related decompositions termed symmetric Dantzig-Wolfe and symmetric Benders Decomposition. Finally we discuss the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular to staircase structures.

Acknowledgment

My grateful appreciation to Professor G.B. Dantzig and the many others who helped make my visit to the Systems Optimization Laboratory an interesting and valuable learning experience.

My thanks also to the Applied Mathematics Division, Argonne National Laboratory, who jointly with the Systems Optimization Laboratory supported my appointment at Stanford University.

DUALLY EQUIVALENT DECOMPOSITION ALGORITHMS
WITH APPLICATION TO SOLVING STAIRCASE STRUCTURES

by

L. Nazareth

1. Introduction

Solving an LP problem say P by the dual simplex method (Lemke [1]) is equivalent to solving the dual of P by the simplex method (Dantzig [2]). We shall use the term "dually equivalent" to describe a relationship such as this, between two linear programming algorithms.

It is well known that Benders decomposition algorithm [3] is dually equivalent to the Dantzig-Wolfe Decomposition algorithm [4], see e.g., Lasdon [5]. Similarly a nested version of Benders Decomposition algorithm (see e.g., Kallio [6]) is dually equivalent to the nested Dantzig-Wolfe Decomposition algorithm (see Glassey [7], Ho and Manne [8]). This paper is concerned with another pair of dually equivalent algorithms which we believe may a) lead to new and interesting ways to solve staircase systems, b) indicate how different optimization models may be combined. Our paper is organized as follows:

1.1. Overview

In Section 2 we discuss fundamental results. First, we discuss the relationship between Benders and Dantzig-Wolfe Decomposition, primarily to develop notation and lay the basis for the new material.

Nested versions of Dantzig-Wolfe Decomposition have been developed by Glassey [7] and Ho and Manne [8]. By establishing a dual equivalence with nested decomposition, Kallio [6] develops a nested version of Benders algorithm for blocking triangular matrices, but does not go into any detail. Therefore we elaborate a little on this. Next we develop what we call, for want of a better name, symmetric Benders decomposition and symmetric Dantzig-Wolfe decomposition. The symmetric Benders algorithm is related to the tangential approximation method of Geoffrion [9].

Finally, in Section 3, we discuss solution strategies and the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular those with staircase structures.

Our aim in this paper is to discuss some basic approaches to solving structured LP problems. Implementational details will be discussed at a later date.

2. Basic Results

2.1.

We consider the primal LP system

$$\begin{aligned} & \text{minimize} && \sum_{j=1}^n \underline{c}_j^T \underline{x}_j \\ & \text{such that} && \sum_{j=1}^n A_j \underline{x}_j \geq \underline{b} \\ & && \underline{x}_j \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{minimize} \\ & \text{such that} \\ & \underline{x}_j \geq 0 \end{aligned}} \right\} (2.1)\text{-P}$$

where

\underline{c}_j and \underline{x}_j are n_j dimensional vectors

A_j is an $(m \times n_j)$ dimensional matrix

and

\underline{b} is an m dimensional vector.

The dual of (2.1)-P is

$$\begin{aligned} & \text{maximize} && \underline{\pi}^T \underline{b} \\ & \text{such that} && A_j^T \underline{\pi} \leq \underline{c}_j \quad j = 1, 2, \dots, n \\ & && \underline{\pi} \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{maximize} \\ & \text{such that} \\ & \underline{\pi} \geq 0 \end{aligned}} \right\} (2.2)\text{-D}$$

where

$\underline{\pi}$ is an m -dimensional vector

Difficulties of notation often needlessly complicate the description of decomposition algorithms, particularly their nested versions. Therefore we shall confine ourselves in this section to the LP systems (2.1)-P and (2.2)-D with $n = 2$ or $n = 3$, which is quite adequate for our purpose.

A summary of the development of material in this section was given in Section 1.1. Lasdon [p. 382, 5], has described the dual equivalence between Bender's and Dantzig-Wolfe decomposition. In our brief development of nested versions, we shall parallel the development in Lasdon and appeal frequently to his results and notation. Note that nested algorithms have been introduced for specially structured LP's -- staircase structures in Ho and Manne [8], and block triangular structures in Kallio [6]. These are the settings within which the algorithms can be expected to be useful, but they are not restricted in principle to these structures. Since our aim is exposition, we shall let A_j be full matrices. Later, in Section 3, we shall consider special structures.

We shall also employ the notation

$$\left. \begin{aligned} \tilde{A}_j &= (A_1, \dots, A_j) \\ \tilde{x}_j &= (x_1, \dots, x_j) \\ \tilde{c}_j &= (c_1, \dots, c_j) \end{aligned} \right\} (2.3)-N$$

2.2. Basic Decomposition Algorithms

2.2.1. Benders decomposition algorithm

We first briefly describe Benders algorithm, following Lasdon [5], p. 370]. Consider the problem (2.1)-P with $n = 2$.

$$\begin{aligned} & \text{minimize } c_1^T x_1 + c_2^T x_2 \\ & \text{such that } A_1 x_1 + A_2 x_2 \geq b \\ & \qquad \qquad \qquad x_1, x_2 \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{minimize } c_1^T x_1 + c_2^T x_2 \\ & \text{such that } A_1 x_1 + A_2 x_2 \geq b \\ & \qquad \qquad \qquad x_1, x_2 \geq 0 \end{aligned}} \right\} (2.4)\text{-P2}$$

The method, in essence, fixes x_1 at some value and solves the resulting LP for x_2 , then adjusts the value of x_1 and repeats, a procedure known formally as Projection, see Geoffrion [10].

In order to ensure feasibility of the projected LP, x_1 is restricted to lie in the set (assumed non-empty)

$$R_1 \triangleq \{x_1 \geq 0 \mid \exists x_2 \geq 0$$

$$\text{such that } A_2 x_2 \geq b - A_1 x_1\}$$

Then (2.4)-P2 is equivalent to

$$\min_{x_1 \in R_1} \left\{ c_1^T x_1 + \min_{x_2} [c_2^T x_2 \mid A_2 x_2 \geq b - A_1 x_1, x_2 \geq 0] \right\} \quad (2.5)$$

The inner minimization, if dualized, becomes:

BENDERS SUBPROBLEM

$$\left. \begin{aligned} & \text{maximize } (\underline{b} - A_1 \underline{x}_1)^T \underline{u} \\ & \text{such that } A_2^T \underline{u} \leq \underline{c}_2 \\ & \underline{u} \geq 0 \end{aligned} \right\} (2.6)\text{-BSP}$$

and thus (2.5) becomes:

$$\min_{\underline{x}_1 \in R_1} \left\{ \underline{c}_1^T \underline{x}_1 + \max_{\underline{u}} [(\underline{b} - A_1 \underline{x}_1)^T \underline{u}, \underline{u} \in S] \right\} \quad (2.7)$$

where

$$S \triangleq \{ \underline{u} | A_2^T \underline{u} \leq \underline{c}_2, \underline{u} \geq 0 \} .$$

Let us denote the extreme points of S by $\underline{u}_i^p, i = 1, \dots, n_p$ and the extreme rays of S by $\underline{u}_i^r, i = 1, \dots, n_r$. Lasdon [5] establishes two facts

Fact 1: The set R_1 is given by

$$R_1 = \{ \underline{x}_1 | (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^r \leq 0, i = 1, \dots, n_r, \underline{x}_1 \geq 0 \}$$

where

\underline{u}_i^r are the extreme rays of the set $C \triangleq \{ \underline{u} | A_2^T \underline{u} \leq 0, \underline{u} \geq 0 \}$.

Fact 2: If the inner maximization in (2.7) is unbounded, then the inner minimization in (2.5) would be infeasible, contradicting the assumption that $\underline{x}_1 \in R_1$. Thus in the problem (2.7) we need only consider extreme points of S in the inner maximization.

Thus problem (2.7) can be written as

$$\min_{\underline{x}_1 \in R_1} \left\{ c_1^T \underline{x}_1 + \max[(\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^p, i = 1, 2, \dots, n_p] \right\} \quad (2.8)$$

and this is equivalent to

minimize z

$$\text{such that } z \geq c_1^T \underline{x}_1 + (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^p, \quad i = 1, 2, \dots, n_p$$

$$0 \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^r, \quad i = 1, 2, \dots, n_r$$

$$\underline{x}_1 \geq 0.$$

Writing $v \triangleq z - c_1^T \underline{x}_1$ this becomes

BENDERS MASTER

$$\text{minimize } c_1^T \underline{x}_1 + v$$

$$\text{such that } v \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^p \quad i = 1, 2, \dots, n_p$$

$$0 \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^r \quad i = 1, 2, \dots, n_r$$

$$\underline{x}_1 \geq 0$$

} (2.9)-BM

In practice, a relaxation strategy is used, so that only a small fraction of the constraints are employed in a restricted master. Again see Lasdon [5, p. 375], for details.

2.2.2. Dantzig-Wolfe decomposition

We now work with the dual of (2.4)-P2

$$\text{maximize } \underline{b}^T \underline{\pi}$$

$$\text{such that } A_1^T \underline{\pi} \leq \underline{c}_1$$

$$A_2^T \underline{\pi} \leq \underline{c}_2$$

$$\underline{\pi} \geq 0$$

} (2.10)-D2

Treating the constraints $A_2^T \underline{\pi} \leq \underline{c}_2$, $\underline{\pi} \geq 0$ as the subproblem, we have

$$\underline{\pi} = \sum_{j=1}^{n_p} \lambda_j \underline{u}_j^p + \sum_{j=1}^{n_r} \mu_j \underline{u}_j^r,$$

with

$$\sum_{j=1}^{n_p} \lambda_j = 1.$$

The Dantzig-Wolfe [4] master problem becomes

DANTZIG-WOLFE MASTER

$$\begin{aligned}
 & \text{maximize} && \sum_{j=1}^{n_p} (\underline{b}^T \underline{u}_j^p) \lambda_j + \sum_{j=1}^{n_r} (\underline{b}^T \underline{u}_j^r) \mu_j \\
 & \text{such that} && \sum_{j=1}^{n_p} (A_1^T \underline{u}_j^p) \lambda_j + \sum_{j=1}^{n_r} (A_1^T \underline{u}_j^r) \mu_j \leq \underline{c}_1 \\
 & && \sum_{j=1}^{n_p} \lambda_j = 1, \quad \lambda_j, \mu_j \geq 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \text{maximize} \\ \text{such that} \end{aligned}} \right\} (2.11)\text{-DWM}$$

Extreme points are, of course, developed as needed by a restricted master. Let \underline{x}_1 be the dual variables for the first set of constraints, and v the dual variable for the convexity constraint. Then the Dantzig-Wolfe subproblem becomes

$$\begin{aligned}
 & \text{maximize} && (\underline{b} - A_1 \underline{x}_1)^T \underline{u} \\
 & \text{such that} && A_2^T \underline{u} \leq \underline{c}_2 \\
 & && \underline{u} \geq 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \text{maximize} \\ \text{such that} \end{aligned}} \right\} (2.12)\text{-DWSP}$$

2.2.3. Dual equivalence of Benders and Dantzig-Wolfe decomposition

This is easily seen by comparing (2.6)-BSP and (2.12)-DWSP, these being the subproblems corresponding to the two decompositions, and noting that (2.9)-BM is the dual of (2.11)-DWM. It can also be shown that both algorithms employ the same test for optimality.

2.3. Nested Decomposition Algorithms

2.3.1. Nested Bender's decomposition

Let us now consider the system (2.1)-P with $n = 3$. Using the notation (2.3)-N, given at the end of Section 2.1, this can be written in the form

$$\begin{aligned} \text{minimize } & \tilde{c}_2^T \tilde{x}_2 + c_3^T x_3 \\ \text{such that } & \tilde{A}_2 \tilde{x}_2 + A_3 x_3 \geq \underline{b} \\ & \tilde{x}_2, x_3 \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{minimize } \\ \text{such that } \\ & \tilde{x}_2, x_3 \geq 0 \end{aligned}} \right\} (2.13)\text{-P3}$$

Then following the development of Section 2.2.1 we can apply Benders Decomposition to (2.13)-P3 and we have the subproblem

NESTED BENDER SUBPROBLEM 1

$$\begin{aligned} \text{maximize } & (\underline{b} - \tilde{A}_2 \tilde{x}_2)^T \underline{u}_1 \\ \text{such that } & A_3^T \underline{u}_1 \leq c_3 \\ & \underline{u}_1 \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{maximize } \\ \text{such that } \\ & \underline{u}_1 \geq 0 \end{aligned}} \right\} (2.14)\text{-NBSP1}$$

where a subscript on \underline{u} is introduced to distinguish variables at the first nested level.

Analogously to (2.9)-BM of Section 2.2.1, the first level
 Benders Master is

NESTED BENDER MASTER 1

$$\begin{aligned}
 & \text{minimize } \bar{c}_2 \bar{x}_2 + v_1 \\
 & \text{such that } v_1 \geq (\underline{b} - \tilde{A}_2 \bar{x}_2)^T \underline{u}_{1i}^p \quad i = 1, 2, \dots, n_{1p} \\
 & \quad \quad \quad 0 \geq (\underline{b} - \tilde{A}_2 \bar{x}_2)^T \underline{u}_{1i}^r \quad i = 1, 2, \dots, n_{1r} \\
 & \quad \quad \quad \bar{x}_2 \geq 0
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} & \text{minimize } \bar{c}_2 \bar{x}_2 + v_1 \\ & \text{such that } v_1 \geq (\underline{b} - \tilde{A}_2 \bar{x}_2)^T \underline{u}_{1i}^p \quad i = 1, 2, \dots, n_{1p} \\ & \quad \quad \quad 0 \geq (\underline{b} - \tilde{A}_2 \bar{x}_2)^T \underline{u}_{1i}^r \quad i = 1, 2, \dots, n_{1r} \\ & \quad \quad \quad \bar{x}_2 \geq 0 \end{aligned}} \right\} (2.15)\text{-NBMI}$$

where $\underline{u}_{1i}^p, \underline{u}_{1i}^r$ are defined in an analogous way to the extreme points
 in (2.9)-BM.

Then if we let

$$U_1^p = [\underline{u}_{11}^p, \underline{u}_{12}^p, \dots, \underline{u}_{1n_{1p}}^p]$$

$$U_1^r = [\underline{u}_{11}^r, \underline{u}_{12}^r, \dots, \underline{u}_{1n_{1r}}^r]$$

and recalling the definitions of \bar{c}_2, \tilde{A}_2 and \bar{x}_2 we can write
 (2.15)-NBMI as

$$\begin{aligned}
& \text{minimize} && c_1^T x_1 + \boxed{c_2^T x_2 + v_1} \\
& \text{such that} && (U_1^P)^T A_1 x_1 + (U_1^P)^T A_2 x_2 + e v_1 \geq U_1^P b \\
& && (U_1^R)^T A_1 x_1 + (U_1^R)^T A_2 x_2 \geq U_1^R b \\
& && x_1, x_2 \geq 0
\end{aligned}$$

where $e^T = (1, 1, \dots, 1)$.

If we now apply Benders Decomposition to this system, and let the subproblem correspond to the variables (x_2, v_1) , we have the second level Benders master given by

NESTED BENDER MASTER 2

$$\text{minimize } c_1^T x_1 + v_2$$

$$\text{such that } v_2 \geq \left[\begin{array}{c} \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \underline{b} \\ \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \underline{b} \end{array} \right] - \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \begin{array}{c} A_1 \\ A_1 \end{array} x_1 \right]^T \underline{u}_{21}^P, \quad i=1,2,\dots,n_{2p}$$

$$0 \geq \left[\begin{array}{c} \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \underline{b} \\ \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \underline{b} \end{array} \right] - \left(\begin{array}{c} U_1^P \\ U_1^R \end{array} \right)^T \begin{array}{c} A_1 \\ A_1 \end{array} x_1 \right]^T \underline{u}_{21}^R, \quad i=1,2,\dots,n_{2r}$$

$$x_1 \geq 0$$

(2.16)-
NBM2

and the subproblem is given by

$$\text{maximize } \left[\begin{pmatrix} U_1^P & \underline{b} \\ U_1^R & \underline{b} \end{pmatrix} - \begin{pmatrix} U_1^P & A_1 \\ U_1^R & A_1 \end{pmatrix} \underline{x}_1 \right]^T \underline{u}_2$$

$$\text{such that } \left[\begin{array}{cc|c} U_1^P & A_2 & \underline{e} \\ \hline U_1^R & A_2 & 0 \end{array} \right]^T \underline{u}_2 \leq \begin{pmatrix} \underline{c}_2 \\ \underline{1} \end{pmatrix}$$

$$\underline{u}_2 \geq 0$$

and this can be rewritten as

$$\text{maximize } [(\underline{b}^T U_1^P \quad \underline{b}^T U_1^R) - \underline{x}_1^T (A_1^T U_1^P \quad A_1^T U_1^R)] \underline{u}_2$$

$$\text{such that } \left[\begin{array}{cc|c} A_2^T U_1^P & A_2^T U_1^R \\ \hline \underline{e}^T & \underline{0}^T \end{array} \right] \underline{u}_2 \leq \begin{pmatrix} \underline{c}_2 \\ \underline{1} \end{pmatrix}$$

$$\underline{u}_2 \geq 0$$

(2.17)-NBSP2

2.3.2. Nested Dantzig-Wolfe decomposition

Now consider the dual of (2.13)-P3

$$\text{maximize } \underline{b}^T \underline{\pi}$$

$$\text{such that } \tilde{A}_2^T \underline{\pi} \leq \underline{c}_2$$

$$A_3^T \underline{\pi} \leq \underline{c}_3$$

$$\underline{\pi} \geq 0$$

(2.18)-D3

Then carry out the standard Dantzig-Wolfe decomposition of §2.2.2
we have the master given, analogously to (2.11)-DWM by

NESTED DANTZIG-WOLFE MASTER 1

$$\text{maximize } \sum_{i=1}^{n_{1p}} (\underline{b}^T \underline{u}_{1i}^p) \lambda_{1i} + \sum_{i=1}^{n_{1r}} (\underline{b}^T \underline{u}_{1i}^r) \mu_{1i}$$

$$\text{such that } \sum_{i=1}^{n_{1p}} (\tilde{A}_2^T \underline{u}_{1i}^p) \lambda_{1i} + \sum_{i=1}^{n_{1r}} (\tilde{A}_2^T \underline{u}_{1i}^r) \mu_{1i} \geq \underline{c}_2$$

$$\sum_{i=1}^{n_{1p}} \lambda_{1i} = 1$$

$$\lambda_{1i}, \mu_{1i} \geq 0$$

(2.19)-NDWM1

where $\underline{u}_{1i}^p, i = 1, \dots, n_{1p}$ and $\underline{u}_{1i}^r, i = 1, \dots, n_{1r}$ are the extreme points and extreme rays of the subproblem which is given by

NESTED DANTZIG-WOLFE SUBPROBLEM 1

$$\text{maximize } (\underline{b} - \tilde{A}_2 \underline{x}_2)^T \underline{\pi}$$

$$\text{such that } A_3^T \underline{\pi} \leq c_3 \quad (2.20)\text{-NDWSP1}$$

$$\underline{\pi} \geq 0$$

where \underline{x}_2 are the dual variables of rows other than the convexity row in (2.19). If we substitute for \tilde{A}_2 and \tilde{c}_2 in (2.19)-NDWSP1 we get

$$\text{maximize } \sum_i (\underline{b}^T \underline{u}_{1i}^p) \lambda_{1i} + \sum_i (\underline{b}^T \underline{u}_{1i}^r) \mu_{1i}$$

$$\text{such that } \sum_i (A_1^T \underline{u}_{1i}^p) \lambda_{1i} + \sum_i (A_1^T \underline{u}_{1i}^r) \mu_{1i} \leq c_2$$

$$\sum_i (A_2^T \underline{u}_{1i}^p) \lambda_{1i} + \sum_i (A_2^T \underline{u}_{1i}^r) \mu_{1i} \leq c_2$$

$$\sum_i \lambda_{1i} \quad \quad \quad = 1$$

$$\lambda_{1i}, \mu_{1i} \geq 0.$$

Writing

$$U_1^P = (u_{11}^P, \dots, u_{1n_1p}^P)$$

$$U_1^R = (u_{11}^R, \dots, u_{1n_1r}^R)$$

and

$$\lambda_1^T = (\lambda_{11}, \dots, \lambda_{1n_1p})^T$$

and

$$\mu_1^T = (\mu_{11}, \dots, \mu_{1n_1r})^T$$

this becomes

NESTED DANTZIG-WOLFE MASTER 1

$$\text{maximize } (b^T U_1^P) \lambda_1 + (b^T U_1^R) \mu_1$$

$$\text{such that } (A_1^T U_1^P) \lambda_1 + (A_1^T U_1^R) \mu_1 \leq c_1$$

$$(A_2^T U_1^P) \lambda_1 + (A_2^T U_1^R) \mu_1 \leq c_2$$

$$e^T \lambda_1 = 1$$

$$\lambda_1, \mu_1 \geq 0$$

(2.21)-NDWM1

Now if we repeat the Dantzig-Wolfe decomposition, letting the subproblem correspond to the boxed set of constraints, we obtain the second level master as

NESTED DANTZIG-WOLFE MASTER 2

$$\begin{aligned}
 & \text{maximize } \sum_{i=1}^{n_{2p}} [(b_1^T U_1^P | b_1^T U_1^R) u_{21}^P] \lambda_{21} + \sum_{i=1}^{n_{2r}} [(b_1^T U_1^P | b_1^T U_1^R) u_{21}^R] \mu_{21} \\
 & \text{such that } \sum_{i=1}^{n_{2p}} [(A_1^T U_1^P | A_1^T U_1^R) u_{21}^P] \lambda_{21} + \sum_{i=1}^{n_{2r}} [(A_1^T U_1^P | A_1^T U_1^R) u_{21}^R] \mu_{21} \leq c_1 \\
 & \sum_{i=1}^{n_{2p}} \lambda_{21} = 1 \\
 & \lambda_{21}, \mu_{21} \geq 0 .
 \end{aligned} \tag{2.22}-NDWM2$$

If we denote the dual variables corresponding to the first set of constraints, by x_1^T , our subproblem is then given by

$$\begin{aligned}
 & \text{maximize } [(b_1^T U_1^P | b_1^T U_1^R) - x_1^T (A_1^T U_1^P | A_1^T U_1^R)] u_2 \\
 & \text{such that } \left[\begin{array}{c|c} A_2^T U_1^P & A_2^T U_1^R \\ \hline e^T & 0^T \end{array} \right] u_2 \leq \begin{pmatrix} c_2 \\ 1 \end{pmatrix}
 \end{aligned} \tag{2.23)-NDWSP2$$

2.3.3. Dual equivalence of nested Benders decomposition and nested Dantzig-Wolfe decomposition

If we denote the dual variables of (2.19)-NDWM1 by \underline{x}_2 and v_1 then we can see that (2.15)-NBM1 is the dual of (2.19)-NDWM1. Also (2.14)-NBSP1 is the same as (2.20)-NDWSP1. Again, if we compare (2.16)-NBM2 with (2.22)-NDWM2 and let \underline{x}_1 and v_2 be the dual variables of (2.22)-NDWM2, we see that (2.22)-NDWM2 and (2.16)-NBM2 are duals. Also (2.17)-NBSP2 is the same as (2.23)-NDWSP2. Note again that we will deal in practice with restricted master problems.

2.4. Symmetric Decomposition Algorithms

2.4.1. Symmetric Benders decomposition

Let us return to problem (2.4)-P2 of Section 1.1

$$\begin{aligned} & \text{minimize } \underline{c}_1^T \underline{x}_1 + \underline{c}_2^T \underline{x}_2 \\ & \text{such that } A_1 \underline{x}_1 + A_2 \underline{x}_2 \geq \underline{b} \\ & \underline{x}_1, \underline{x}_2 \geq 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{minimize } \underline{c}_1^T \underline{x}_1 + \underline{c}_2^T \underline{x}_2 \\ & \text{such that } A_1 \underline{x}_1 + A_2 \underline{x}_2 \geq \underline{b} \\ & \underline{x}_1, \underline{x}_2 \geq 0 \end{aligned}} \right\} (2.24)\text{-P2}$$

We shall employ the following notation of Geoffrion [9]

$$Y_i \triangleq \{y_i \in \mathbb{R}^m \mid \exists \underline{x}_i \text{ such that } A_i \underline{x}_i \geq y_i, \underline{x}_i \geq 0\}$$

for $i = 1, 2$.

$$v_1(y_1) \triangleq \min_{x_1 \geq 0} \{c_1^T x_1 \mid A_1 x_1 \geq y_1\} .$$

If the set is infeasible for the given y_1 , then $v_1(y_1) \triangleq +\infty$.

Geoffrion [9] shows that (2.24)-P2 is equivalent to

$$\begin{aligned} & \text{minimize } v_1(y_1) + v_2(y_2) \\ & \text{such that } y_1 + y_2 \geq b \\ & y_1 \in Y_1, \quad i = 1, 2 . \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{minimize } v_1(y_1) + v_2(y_2) \\ & \text{such that } y_1 + y_2 \geq b \\ & y_1 \in Y_1, \quad i = 1, 2 . \end{aligned}} \right\} (2.25)\text{-P2}$$

Fact 3: In an essentially identical proof to that used by Lasdon [3] to establish Fact 1, Section 2.2.1, we have

$$\{y_1 \in Y_1 \leftrightarrow y_1^T u_{1j}^r \leq 0, \quad j = 1, 2, \dots, n_{1r}\}$$

for $i = 1, 2$, where u_{1j}^r , $j = 1, 2, \dots, n_{1r}$ are the extreme rays of the sets $C_1 \triangleq \{u \mid A_1^T u \leq 0, u \geq 0\}$, $i = 1, 2$. □

Consider now the objective function of (2.25)-P2

$$v_1(y_1) + v_2(y_2) = \min_{x_1 \geq 0} \{c_1^T x_1 \mid A_1 x_1 \geq y_1\} + \min_{x_2 \geq 0} \{c_2^T x_2 \mid A_2 x_2 \geq y_2\}$$

If we dualize each of these minimizations, we can write this as

$$v_1(y_1) + v_2(y_2) = \max_{u_1 \geq 0} \{y_1^T u_1 \mid A_1^T u_1 \leq c_1\} + \max_{u_2 \geq 0} \{y_2^T u_2 \mid A_2^T u_2 \leq c_2\}$$

Fact 4: Again, analogously to Fact 2 of Section 2.2.1, if either maximization problem above went to $+\infty$ along an extreme ray, the corresponding dual minimization problem would be *infeasible*. This would contradict the constraint $y_i \in Y_i$. Therefore we need only consider extreme points of the feasible sets

$$\{u_i \mid A_i^T u_i \leq c_i, u_i \geq 0\}, \quad i = 1, 2 \dots$$

Let us denote these extreme points by u_{ij}^p , $j = 1, 2, \dots, n_{ip}$ for $i = 1, 2$.

Then (2.25)-P2 can be written as

$$\min_{y_1, y_2} [\max\{y_1^T u_{1j}^p, j = 1, \dots, n_{1p}\} + \max\{y_2^T u_{2j}^p, j = 1, 2, \dots, n_{2p}\}]$$

$$\text{such that } y_1 + y_2 \geq b$$

$$y_i \in Y_i, \quad i = 1, 2$$

Thus, finally, this can be written as the symmetric Bender Master problem

SYMMETRIC BENDER MASTER

$$\text{minimize } z_1 + z_2$$

$$\text{such that } z_1 \geq \gamma_1^T u_{1j}^p, \quad j = 1, \dots, n_{1p}$$

$$0 \geq \gamma_1^T u_{1j}^r, \quad j = 1, \dots, n_{1r}$$

$$z_2 \geq \gamma_2^T u_{2j}^p, \quad j = 1, \dots, n_{2p}$$

$$0 \geq \gamma_2^T u_{2j}^r, \quad j = 1, \dots, n_{2r}$$

$$b \leq \gamma_1 + \gamma_2.$$

(2.26)-SBM

and the symmetric Bender subproblems are given by

SYMMETRIC BENDER SUBPROBLEM i

$$\text{maximize } \gamma_i^T u_i$$

$$\text{such that } A_i^T u_i \leq c_i$$

$$u_i \geq 0$$

(2.27)-SBSP1

for $i = 1, 2$.

Note that the master has block angular form, and that again, in practice, we shall deal with a restricted master problem.

2.4.2. Symmetric Dantzig-Wolfe decomposition

Consider again the problem (2.10)-D2

$$\begin{array}{ll}
 \text{maximize} & \underline{b}^T \pi \\
 \text{such that} & A_1^T \pi \leq \underline{c}_1 \\
 & A_2^T \pi \leq \underline{c}_2 \\
 & \pi \geq 0
 \end{array} \quad \left. \vphantom{\begin{array}{l} \text{maximize} \\ \text{such that} \end{array}} \right\} (2.28)\text{-D2}$$

A feasible solution π_1 of the set $S_1 \triangleq \{\pi | A_1^T \pi \leq \underline{c}_1, \pi \geq 0\}$ satisfies

$$\pi_1 = \sum_{j=1}^{n_{1p}} \lambda_{1j} u_{1j}^p + \sum_{j=1}^{n_{1r}} \mu_{1j} u_{1j}^r$$

with

$$\sum_{j=1}^{n_{1p}} \lambda_{1j} = 1$$

$$\lambda_{1j} \geq 0, \quad \mu_{1j} \geq 0$$

where $u_{1j}^p, j = 1, \dots, n_{1p}$ are the extreme points of S_1 , and $u_{1j}^r, j = 1, \dots, n_{1r}$ are the extreme rays of S_1 .

Likewise a feasible solution π_2 of $S_2 = \{\pi_2 | A_2^T \pi_2 \leq c_2, \pi_2 \geq 0\}$ satisfies

$$\pi_2 = \sum_{j=1}^{n_{2p}} \lambda_{2j} u_{2j}^p + \sum_{j=1}^{n_{2r}} \mu_{2j} u_{2j}^r$$

$$\sum_{j=1}^{n_{2p}} \lambda_{2j} = 1$$

$$\lambda_{2j}, \mu_{2j} \geq 0$$

Then (2.28)-D2 can be written as

SYMMETRIC DANTZIG-WOLFE DECOMPOSITION MASTER

$$\text{maximize } \underline{b}^T \pi$$

$$\text{such that } \pi - \left(\sum_j \lambda_{1j} u_{1j}^p + \sum_j \mu_{1j} u_{1j}^r \right) = 0$$

$$\pi - \left(\sum_j \lambda_{2j} u_{2j}^p + \sum_j \mu_{2j} u_{2j}^r \right) = 0$$

$$\sum_j \lambda_{1j} = 1$$

$$\sum_j \lambda_{2j} = 1$$

$$\pi, \lambda_{1j}, \lambda_{2j}, \mu_{1j}, \mu_{2j} \geq 0.$$

(2.29)-SDWM

If we denote the dual variables corresponding to the first two sets of constraints by \underline{y}_1 and \underline{y}_2 , then the two subproblems for symmetric Dantzig-Wolfe Decomposition are given by

SYMMETRIC DANTZIG-WOLFE SUBPROBLEM 1

$$\text{maximize } \underline{y}_1^T \underline{u}_1$$

$$\text{such that } A_1^T \underline{u}_1 \leq \underline{c}_1 \quad (2.30)\text{-SDWSP1}$$

$$\underline{u}_1 \geq 0$$

2.4.3. Dual equivalence of symmetric Benders and symmetric Dantzig-Wolfe decomposition

This is immediately seen by noting that (2.27)-SBSP1 and (2.30)-SDWSP1 are identical, and that (2.26)-SBM is dual to (2.29)-SDWM. Again, in practice we use restricted master problems.

3. Concluding Remarks on Solution Strategies and Application to Special Structures

In Section 2 we dealt with algorithms for solving (2.1)-P and (2.2)-D, confining the discussion to the case $n = 2$ or 3. It is easy to see how the algorithms extend to arbitrary n . In this section we elaborate upon these algorithms discussing, in particular, their advantages and disadvantages. We shall be primarily concerned with their application to staircase structures. Figure 3.1a) and 3.1b) illustrate two dual staircase LP's, which are special cases of general LP's illustrated in Figures 3.2a) and 3.2b)

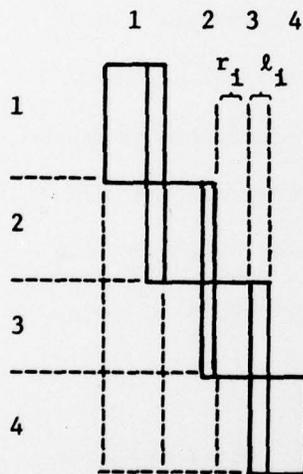


Figure 3.1a)

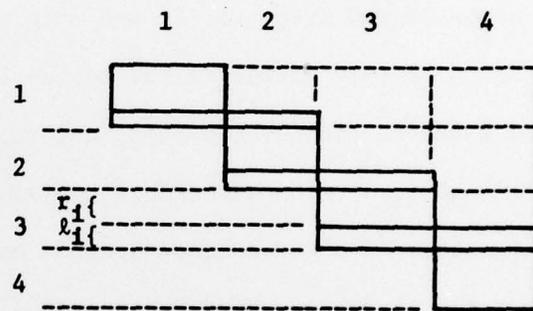


Figure 3.1b)

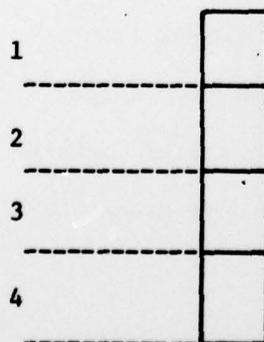


Figure 3.2a)

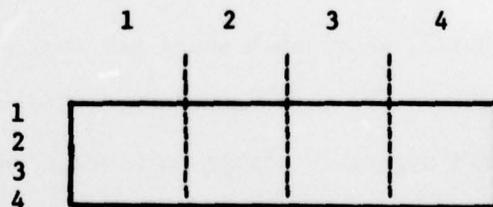


Figure 3.2b)

Note that in staircase structures there are often only a few linking columns in Figure 3.1a) and correspondingly only a few linking rows in the LP dual to it, Figure 3.1b).

3.1. Nested Decomposition Algorithms

The algorithms for Glassey [7] and Ho and Manne [8] were developed for staircase LP's of the form shown in Figure 3.1b). The algorithm of Kallio [6] is a generalization of these to the case when the matrix is block upper triangular.

In these algorithms, stage 4 would correspond to the first subproblem and stages 1, 2, and 3 to the first level master. Then stage 3 of this master is the second level subproblem and stages 1 and 2 the second level master, and so on. A major contribution of Ho [11] was to develop a workable implementation of the procedure and demonstrate its effectiveness, since many details must be worked out in order to specify a workable nested decomposition method.

We should note the following:

- a) The nested decomposition method of Section 2.3.2 works best for structures of the form 3.1b), since each master will then have relatively few rows. If applied to the staircase structure 3.1a), where each stage has many more rows than columns, each master and thus subsequent subproblems would become very large.
- b) All structure within each stage can be lost in the nested Dantzig-Wolfe Decomposition. Thus, if the constraints of some stage are

implicitly defined, e.g., by a network, we cannot exploit this added structure in the problem.

- c) The procedure lacks symmetry with respect to interchange of stages.
- d) There is an inherent propagation and build up of error in the nested decomposition process. Errors in the extreme points of a subproblem, result in errors in the columns of the master to which these extreme points are transmitted, which in turn result in errors in the next level of the decomposition. For a more detailed discussion see Nazareth [1]2.
- e) A post-optimality procedure (called Phase 3) is needed to recover the solution (see Ho [11]).
- f) Finally, note that the nested Dantzig-Wolfe decomposition can be carried out in other ways. Thus in Figure 3.1b) stages 3 and 4 could be the first level subproblem and stages 1 and 2 the first level master then each of these could be decomposed in turn into a master and subproblem. A serious disadvantage of such a procedure is that a Phase 3 process would be needed at intermediate stages in order to recover an extreme point of a subproblem, for transmission to a master.

By the dual equivalence discussed in Section 2.3.3, we would expect the Nested Benders algorithm to work best on structures of the form 3.1a) and 3.2a). Because we wish to have primal feasibility explicitly in 3.2b) we might consider applying nested Benders decomposition to this structure. However, note that we would then have the

disadvantages of nested Dantzig-Wolfe decomposition applied to 3.2a) i.e., at first sight it appears as though we get master problems with a large number of rows, though there may be ways to get around this by working with the dual.

3.2. Symmetric Decomposition Algorithms

In contrast with the Nested Dantzig-Wolfe decomposition algorithm, the symmetric Dantzig-Wolfe algorithm applies most naturally to structures of the form 3.1a) and 3.2a), since the size of the symmetric Dantzig-Wolfe master is determined by the number of columns. Let ℓ_i denote the number of linking columns for stage i in Figure 3.1a) and r_i the number of remaining columns for stage i (correspondingly ℓ_i and r_i are the number linking rows and remaining rows in Figure 3.1b)). If we decompose into 4 subproblems, one for each stage in Figure 3.1a), the symmetric Dantzig-Wolfe master has $\sum_{i=1}^4 r_i + 2\sum_{i=1}^3 \ell_i + 4$ rows and a structure illustrated in Figure 3.3. Note that the shaded blocks correspond to dense matrices, a potential disadvantage.

The first portion of the matrix has $\sum_{i=1}^4 r_i + \sum_{i=1}^3 \ell_i$ columns. In general, if extreme points of subproblems are not degenerate, and cancellation does not occur, we would expect that variables corresponding to this first part to have non-zero values. Thus we would expect a basic solution of the symmetric master to have approximately $\sum_{i=1}^3 \ell_i + 4$ columns from the dense block diagonal part of the matrix. If ℓ_i is small, these would be relatively few. Operations involving

a basis, for example, LU factorization, in this system can then be handled more efficiently than corresponding operations in the sparse staircase system 3.1b).

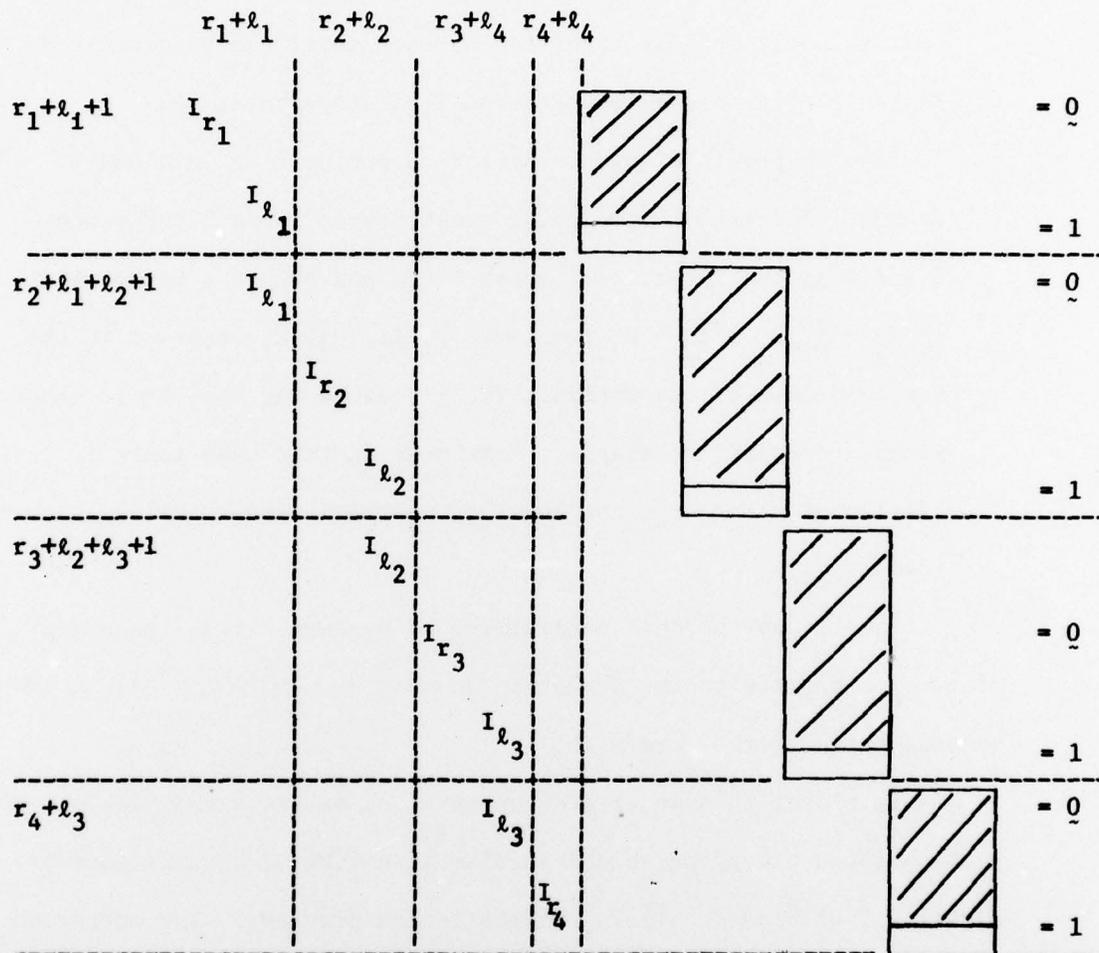


Figure 3.3

Notation for Figure 3.3: I_{l_1} denotes an $l_1 \times l_1$ identity matrix.

In contrast to the nested algorithms note:

- a) No Phase 3 is required to recover the solution.
- b) Special structure within a subproblem is not lost. In particular different models can be integrated into a single model and at the same time the special structure of each model can be exploited.
- c) There is no transmission of error from stage to stage.
- d) It is also possible to do symmetric decomposition in a nested manner. For example, we could treat stages 1 and 2 and stages 3 and 4 as the first level subproblems and obtain a master with $\sum_1^3 \ell_1 + \sum_1^4 r_1 + \ell_2 + 2$ rows and $\sum_1^3 \ell_1 + \sum_1^4 r_1$ columns in the front portion of the matrix. The procedure can then be repeated within each pair of stages. Note that in this case there is transmission of error from our level of decomposition to all subsequent levels.

Finally by the dual equivalence of Section 2.4.3, the above discussion applies to the Symmetric Benders algorithms, which is used on structures of the form 3.1b).

In effect in symmetric decomposition, we are exchanging our LP problem for a block/dual block angular structure of special form of Figure 3.3 with dense blocks and smaller subproblems. The latter do not lose any structures of their own and are amenable to solution by specialized techniques. Symmetric decomposition may have advantages for solving certain sorts of staircase models and for combining different models into a single integrated LP model.

References

- [1] Lemke, C.E., "The Dual Method for Solving the Linear Programming Problem," Nav. Res. Log. Quart., 1, p. 36-47, (1954).
- [2] Dantzig, G.B., Linear Programming and Extensions, Princeton University Press, (1963).
- [3] Benders, J.F., "Partitioning Procedures for Solving Mixed Variables Programming Problems," Numerische Mathematik, 4, p. 238-252, (1962).
- [4] Dantzig, G.B. and Wolfe, P., "Decomposition Principle for Linear Programs," Op. Res. 8, 1, p. 101-111, (1960).
- [5] Lasdon, L., Optimization Theory for Large Systems, MacMillan, (1970).
- [6] Kallio, M.J., "On Large Scale Linear Programming," Systems Optimization Laboratory report SOL 75-7, Department of Operations Research, Stanford University, (1975).
- [7] Glassey, C.R., "Dynamic LP's for Production Scheduling," Op. Res. 19, p. 45-56, (1971).
- [8] Ho, J., and Manne, A., "Nested Decomposition for Dynamic Models," Math. Prog. 6, p. 121-140, (1974).
- [9] Geoffrion, A., "Primal Resource Directive Approaches for Optimizing Non-linear Decomposable Systems," Working Paper 141, OR Division, Western Management Science Institute, University of California, Los Angeles, (1968).
- [10] Geoffrion, A., "Elements of Large Scale Mathematical Programming," Manag. Sci., 16 No. 11, p. 652-691, (1970)
- [11] Ho, J., "Nested Decomposition of Large Scale Linear Programs with the Staircase Structure," Systems Optimization Laboratory report, SOL 74-4, Department of Operations Research, Stanford University, (1974).
- [12] Nazareth, L., "Error Propagation and Solution Reconstruction in Nested Decomposition," Systems Optimization Laboratory Report, SOL 78-27, Department of Operations Research, Stanford University, (1978).

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM						
1. REPORT NUMBER SOL 78-30	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER						
4. TITLE (and Subtitle) DUALY EQUIVALENT DECOMPOSITION ALGORITHMS WITH APPLICATION TO SOLVING STAIRCASE STRUCTURES		5. TYPE OF REPORT & PERIOD COVERED Technical Report						
		6. PERFORMING ORG. REPORT NUMBER SOL 78-30						
7. AUTHOR(s) L. Nazareth		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0267						
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research -- SOL Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-047-143						
11. CONTROLLING OFFICE NAME AND ADDRESS Operations Research Program -- ONR Department of the Navy 800 N. Quincy Street, Arlington, VA 22217		12. REPORT DATE December 1978						
		13. NUMBER OF PAGES 31						
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified						
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE						
16. DISTRIBUTION STATEMENT (of this Report) This report has been approved for public release and sale; its distribution is unlimited.								
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)								
18. SUPPLEMENTARY NOTES								
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)								
<table border="0"> <tr> <td>Nested Decomposition</td> <td>Dantzig-Wolfe Decomposition</td> </tr> <tr> <td>Benders Decomposition</td> <td>Dually Related Decompositions</td> </tr> <tr> <td>Symmetric Decompositions</td> <td>Staircase LP Problems</td> </tr> </table>			Nested Decomposition	Dantzig-Wolfe Decomposition	Benders Decomposition	Dually Related Decompositions	Symmetric Decompositions	Staircase LP Problems
Nested Decomposition	Dantzig-Wolfe Decomposition							
Benders Decomposition	Dually Related Decompositions							
Symmetric Decompositions	Staircase LP Problems							
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We briefly go over the well known dual relationship between Dantzig-Wolfe Decomposition and Benders Decomposition, in order to develop suitable notation and then elaborate upon the dual relationship between nested versions of Dantzig-Wolfe and Benders Decomposition. Next we develop a new pair of dually related decompositions termed symmetric Dantzig-Wolfe and symmetric Benders Decomposition. Finally we discuss the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular to staircase structures.								

DD FORM 1473
1 JAN 73

EDITION OF 1 NOV 68 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)