AD-A066 066 STANFORD UNIV CALIF SYSTEMS OPTIMIZATION LAB F/G 5/3

PRICING UNDEREMPLOYED CAPACITY IN A LINEAR ECONOMIC MODEL.(U)

FEB 79 G B DANTZIG, P L JACKSON N00014-75-C-0267

UNCLASSIFIED SOL-79-2 NL

1 OF 1

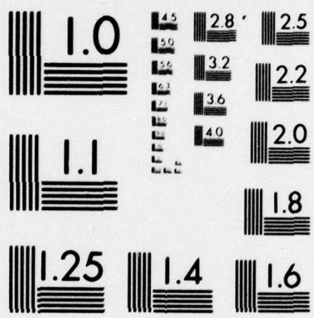
AD A066066

END

DATE FILMED

5-79

DDC


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

LEVEL II

Systems
Optimization
Laboratory

(14)

AD A0 66066

DDC FILE COPY

a
PILOT
publication

Department of Operations Research
Stanford University
Stanford, CA 94305

SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
Stanford University
Stanford, California
94305

PRICING UNDEREMPLOYED CAPACITY IN A LINEAR ECONOMIC MODEL

by

George B. Dantzig and Peter L. Jackson

TECHNICAL REPORT SOL 79-2
February 1979

Research and reproduction of this report were partially supported by the Office of Naval Research Contract N00014-75-C-0267; the National Science Foundation Grants MCS76-81259 A01 and ENG77-06761; the Electric Power Research Institute Contract RP 652-1; and the Institute for Energy Studies at Stanford University.

Abstract

One of the difficulties in relating the shadow prices of a linear economic model to their counterparts in the real economy being modelled is the assumption of perfect competition. Under this assumption competition would force the price of any resource in excess supply down to zero. In real economies, however, owners of capacity routinely receive a return even when that capacity is underemployed, precisely because competition is imperfect. We present a method for determining a stable system of shadow prices consistent with an absence of competition among the owners of slack capacity and show that this implies non-zero prices on all resources, regardless of excess supply.

Pricing Underemployed Capacity in a Linear Economic Model

George B. Dantzig and Peter L. Jackson

Introduction

This paper presents a method of associating a new set of dual variables with the optimal primal solution of a linear program so that the shadow price associated with an item in excess supply need not be zero. The general motivation for the method is the observation that, in real economies, resource prices must be positive if the owners are to have sufficient incentive to supply their resources to the market. A solution to the system in which a resource in excess supply had non-zero price clearly represents a dis-equilibrium situation which would give rise to competition among the owners that would presumably drive the price down toward zero. However, for technical or institutional reasons this competition may not exist, at least in the short run. Barriers to competition can take the form of market entry costs, resource differentiation, conversion costs, and the existence of large bargaining units, all of which effects may not be captured or even represented in the linear economic model. In the absence of competition there may be a range of prices which "work" and we propose to allow a small amount (actually an infinitesimally small amount) of substitution among the resources and capacities that are actually employed and to select a price system which reflects the marginal substitution possibilities within this amount.

<u>PLEASE NOTE:</u> *All α and β are subscripts.*

The Method

We are given the primary optimization problem: maximize Z subject to $X \geq 0$, $Y \geq 0$, $s \geq 0$ and

		Dual Variables	
$a\theta + BX + IY$	$= K$	$\sigma \geq 0$	(1)
$-gY + s$	$= 0$	$\tau \geq 0$	(2)
1θ	$= Z$ (max)		(3)

The objective is to maximize a vector output from the system, the bill-of-goods vector $a\theta$. The vector X may represent both production activities and certain exogenous consumption activities. Similarly, K may represent both resource capacities (upper bounds) and required outputs (the negative of lower bounds) of the system. The vector Y measures excess supply of the resources/commodities of the system and the scalar variable s is a weighted total of this excess. Equations (1) and (3) form a very general representation of a single period linear economic system.

Equation (2) defines a variable s as a composite measure of the slack capacity of the system. In general, the zero components of g correspond to (end-use) commodities and the positive components correspond to resources. We will refer to s as a measure of the availability of "generalized capacity".

Let the solution to this problem be denoted θ^0, X^0, Y^0, and s^0 with dual variables σ^0, and τ^0. In general, this solution will satisfy:

$$s^0 \geq 0 \tag{4}$$

$$\tau^0 = 0 \tag{5}$$

$$\sigma^0 a = 1 \tag{6}$$

$$\sigma^0 K = \theta^0 \tag{7}$$

Furthermore, the dual variables partition the matrices B and I, and the corresponding vectors X and Y, into two groups, indexed by subscripts α and β, according as the columns of B and I price out positive or zero, respectively:

$$\sigma^0 B_\alpha > 0, \text{ and } \sigma^0 I_\alpha > 0 \tag{8}$$

$$\sigma^0 B_\beta = 0, \text{ and } \sigma^0 I_\beta = 0 \tag{9}$$

Note that $(X_\alpha{}^0, Y_\alpha{}^0) = 0$.

We require that the system be efficient with regard to the availability of generalized capacity, by which we mean that it is not possible to attain max $(Z) = \theta^0$ with $s < s^0$. Such a solution can be obtained by a secondary optimization: maximize W subject to $\theta \geq 0$, $X_\beta \geq 0$, $Y_\beta \geq 0$ and

		Dual Variables	
$a\theta + B_\beta X_\beta + I_\beta Y_\beta$	$= K$	$\sigma \geq 0$	(10)
$- g_\beta Y_\beta + s$	$= 0$	$\tau \geq 0$	(11)
$1s$	$= W$ (max)		(12)

Note that the original optimal solution θ^0, $X\beta^0$, $Y\beta^0$, and s^0 also satisfies (10) and (11) so that this system is feasible. Moreover, multiplying (10) by σ^0 yields:

$$\sigma^0 a\theta + \sigma^0 B\beta X\beta + \sigma^0 I\beta Y\beta = \sigma^0 K$$

which upon substituting (6), (7) and (9) yields $\theta = \theta^0$ so that the solution obtained also optimizes the original problem.

Let the solution to this secondary problem be denoted θ^1, $X\beta^1$, $Y\beta^1$, s^1 with dual variables σ^1, and τ^1. This solution will satisfy:

$$\tau^1 = 1 \quad (13)$$

$$\sigma^1 a \geq 0 \quad (14)$$

$$\sigma^1 B\beta \geq 0 \quad (15)$$

$$\sigma^1 I\beta \geq g\beta \quad (16)$$

$$\sigma^1 K = s^1 \quad (17)$$

We choose as basis the latter, which is optimal for both the primary and secondary optimizations. Note that the previously obtained prices (σ^0,τ^0) hold in the original problem for this basis so that no revision of these prices is necessary.

We next assume that in the short run, for technical or institutional reasons, the system is "sticky" with respect to generalized capacity. By this, we mean that if the data a, B or K should change by small amounts during the period for which the model is defined the availablility of generalized capacity, as measured by s, would be unable (in the short run) to adjust for these changes. So, given s^1, the real system is "better" represented by the problem: maximize Z subject to $X_\alpha \geq 0$, $X_\beta \geq 0$, $Y_\alpha \geq 0$, $Y_\beta \geq 0$, $t \geq 0$ and

		Dual Variables	
$a\theta + B_\alpha X_\alpha + B_\beta X_\beta + I_\alpha Y_\alpha + I_\beta Y_\beta$	$= K$	σ	(18)
$- g_\alpha Y_\alpha - g_\beta Y_\beta + t$	$= -s^1$	τ	(19)
1θ	$= Z$ (max)		(20)

where we have set $t = s - s^1$. Because of our choice of s^1, max (Z) yields $t = 0$ and (19) is now an active constraint. With $t = 0$ this constraint may be written:

$$g(Y - Y^0) = 0$$

This relationship implies that even though a resouce may be in technical excess supply an increase in its use can only be accompanied by a decrease in the use of one or more other resources, where the substitution occurs according to the vector g. Effectively, what this constraint has done is remove all the slack capacity from the system but permit substitution among the capacities actually in use[1].

----

[1]No substitution actually takes place in this problem since the original primal solution, with $Y = Y^0$, is still optimal. Our device for generating a new system of prices consists of forcing an infinitesimally small amount of substitution to take place.

We do not propose to solve this problem directly because there can be numerical difficulties during computation due to round-off of s^1.

By construction, the original basis (with t taking the place of s in the basis) is optimal for both the primary and secondary optimizations and so will be optimal for this modified problem. Since at an optimum t = 0, the basic solution is degenerate and the optimal prices are not necessarily unique. The set of the optimal values for the dual variables can be easily parametrized by a single parameter, λ. We will resolve the price ambiguity by proposing a stability condition which, in turn, will be seen to imply a particular value of the parameter λ, easy to compute.

To begin, we claim that the optimal dual variables for problem will be of the form:

$$\sigma^2 = \sigma^0 + \lambda\sigma^1 \quad (21)$$

$$\tau^2 = \lambda \quad (22)$$

where λ is chosen from an interval $[0,L]$ for some positive real number, L. This can be seen by combining (6) - (9) and (13) - (17):

$$\sigma^2 a = \sigma^0 a + \lambda\sigma^1 a \geq 0 \quad (23)$$

$$\sigma^2 B\beta = \sigma^0 B\beta + \lambda\sigma^1 B\beta \geq 0 \quad (24)$$

$$\sigma^2 I\beta - \tau^2 g\beta = \sigma^0 I\beta + \lambda(\sigma^1 I\beta - g\beta) \geq 0 \quad (25)$$

$$\sigma^2 K - \tau^2 s^1 = \sigma^0 k + \lambda(\sigma^1 K - s^1) = \theta^0 \quad (26)$$

which hold for all values of $\lambda \geq 0$. In addition for optimality we require

$$\sigma^2 B_\alpha = \sigma^0 B_\alpha + \lambda \sigma^1 B_\alpha \geq 0 \qquad (27)$$

$$\sigma^2 I_\alpha - \tau^2 g_\alpha = \sigma^0 I_\alpha + \lambda(\sigma^1 I_\alpha - g_\alpha) \geq 0 \qquad (28)$$

which hold for at least $\lambda = 0$. Since $\sigma^0 B_\alpha$ and $\sigma^0 I_\alpha$ are both strictly positive there is some $L > 0$ for which the relations (27) and (28) hold for all $\lambda \in [0,L]$. Relations (23), (24), (25), (27), and (28) represent dual feasibility and equation (26) shows that the solution satisfies strong duality. Thus, (σ^2,τ^2) is optimal for all $\lambda \in [0,L]$. The proof that there is a maximal $L < +\infty$ rests on a further assumption, to be made shortly. Let λ^2 equal the maximal such L, assuming it exists. From (27) and (28) it can be seen that:

$$\lambda^2 \equiv \max (L)$$

$$= \min \left[\min_{\substack{i\in\alpha \\ \sigma^1 B_i < 0}} \frac{\sigma^0 B_i}{-\sigma^1 B_i} \, , \, \min_{\substack{i\in\alpha \\ \sigma^1 I_i - g_i < 0}} \frac{\sigma^0 I_i}{-(\sigma^1 I_i - g_i)} \right] \qquad (29)$$

The numerator in each expression within the brackets of (29) is the per unit amount that the objective function for the primary optimization (θ) is reduced by the introduction of a non-basic activity corresponding to $i \in \alpha$. The denominator is the per unit amount that the objective function for the secondary optimization (s) is increased by the introduction of the same activity. To emphasize this tradeoff, which λ^2 optimizes, write:

$$\lambda^2 \equiv \frac{\Delta\theta}{-\Delta s} \qquad (30)$$

To motivate the next step, we propose a stability condition. Imagine that the generalized resource is owned by some monopolistic agent in the economy. Under the original basis with $\tau^0 = 0$, this agent receives nothing. However, since t is basic and zero in the solution these prices may not be unique. By reducing the amount of the generalized capacity which the agent supplies to the economy by some small amount, ϵ, the agent can effect a change in prices resulting in $\tau > 0$. Under the new price system, the agent's return is then maximized by supplying as much of the resource as he now has available (that is, by letting ϵ tend to zero). If there is a price system which is optimal for all ϵ in an interval $[0,\epsilon_1]$ for some sufficiently small ϵ_1 we will define this to be the stable system of prices we seek at $\epsilon = 0$.

The next step, then, is to remove an infinitesimal amount, ϵ ($\epsilon>0$), of the generalized capacity from the system[2]. Rewritten, the problem becomes: maximize Z subject to $X_\alpha \geq 0$, $X_\beta \geq 0$, $Y_\alpha \geq 0$, $Y_\beta \geq 0$, $t \geq 0$ and

		Dual Variables	
$a\theta + B_\alpha X_\alpha + B_\beta X_\beta + I_\alpha Y_\alpha + I_\beta Y_\beta$	$= K$	σ	(31)
$- g_\alpha Y_\alpha - g_\beta Y_\beta + t$	$= -(s^1 + \epsilon)$	τ	(32)
1θ	$= Z$ (max)		(33)

----

[2]An alternative approach would be to "introduce into" rather than "remove from" the system. However, no change in prices would result since t is already in the basis.

Assume that this system is feasible for sufficiently small ϵ. Since s^1 is maximal for $\theta = \theta^1$ any solution to (31) - (32) must have $\theta < \theta^1$ and t non-basic. Again, we do not propose to solve this system directly because of numerical problems due to redundancy and round-off errors in representing s^1 exactly and in choosing "small" values of ϵ.

Defining σ^2 and τ^2 as before, we note that there is some $L > 0$ such that (σ^2,τ^2) is a dual feasible solution for all $\lambda \in [0,L]$. That is, relations (27) and (28) hold for all $\lambda \in [0,L]$ and the relations (23) - (25) hold for all $\lambda \geq 0$. With the new right hand side, relation (26) now becomes:

$$\sigma^2 K - \tau^2(s^1+\epsilon) = \sigma^0 K + \lambda(\sigma^1 K - s^1) - \lambda\epsilon$$
$$= \theta^0 - \lambda\epsilon \qquad (34)$$

If there is no upper bound for L then (σ^2,τ^2) is dual feasible for all $\lambda \geq 0$ and equation (34) shows that the dual program is unbounded. Thus, the assumption that (31) - (32) has a feasible solution, and hence a bounded dual, implies that λ^2 as defined in (29) does exist and that L has a finite upper bound.

Proposition: If, using the original basis, all basic variables other than t are strictly positive and if the system (31) - (32) is feasible for small ϵ, then there exists a sufficiently small $\epsilon_1 > 0$ such that the dual variables given by $\sigma^2 = \sigma^0 + \lambda^2\sigma^1$ and $\tau^2 = \lambda^2$, for λ^2 as defined in (29), will be optimal for all ϵ in the interval $[0,\epsilon_1]$.

Proof: Consider the original basis applied to the system (31) - (32). Since t was basic, the solution value of t is given by the inner product of the "t row" of the basis inverse and the right hand side. It is easily verified that the "t row" of the basis inverse must be (σ^1,τ^1) since this basis is optimal for the secondary optimization. Hence the solution value of t is given by:

$$
\begin{aligned}
t &= \sigma^1 K - \tau^1(s^1 + \epsilon) \\
&= \sigma^1 K - s^1 - \epsilon \\
&= - \epsilon
\end{aligned}
$$

using the substitutions (13) and (17). Given the non-negativity constraint on t this basis is not feasible for any positive ϵ.

Let X_i, or possibly Y_i, be the non-basic variable which minimizes the expression for λ^2 (29) and consider the effect of increasing this variable. For convenience let us assume this variable is X_i. Denote the column of coefficients in the problem associated with this variable by P. Note that in general the vector P is either of the form $[B_i',0]'$ or $[I_i',g_i]'$. Premultiplying this column by the basis inverse yields the representation of P in terms of the basis. It may also be interpreted as the column of "substitution factors" (Dantzig [1], p. 268). The substitution factor for the basic variable θ will be the inner product of the "θ row" of the basis inverse and the vector P. Since this basis is optimal for the primary optimization, it is easily verified that (σ^0,τ^0) is the "θ row" of the basis inverse. It follows that the substitution factor for θ, $(\sigma^0,\tau^0)P$, is $\Delta\theta$ as defined in (29) and (30):

$$\theta = \theta^0 - \Delta\theta \cdot X_i \qquad (35)$$

Similarly, the substitution factor for t, $(\sigma^1,\tau^1)P$, is Δs as defined in (29) and (30):

$$t = -\epsilon - \Delta s \cdot X_i \qquad (36)$$

Consequently, we can maintain feasibility ($t \geq 0$) by setting $t = 0$ and $X_i = \epsilon/(-\Delta s) \geq 0$.

Assume first that all basic variables other than t are strictly positive so that there is a range $[0,\underline{X_i}]$ over which X_i can be increased without forcing other basic variables negative. Let $\epsilon_1 = \Delta s \cdot \underline{X_i}$. For any $\epsilon \in [0,\epsilon_1]$ we can maintain primal feasibility by pivoting X_i into the basis at level $\epsilon/(-\Delta s)$ and dropping t from the basis.

It is easily verified that the dual variables corresponding to this new basis must be (σ^2,τ^2) with $\tau^2 = \lambda = \lambda^2$. We have already shown that this solution is dual feasible. From (30), (34) and (35) the right hand side prices out:

$$\begin{aligned} \sigma^2K - \tau^2(s^1 + \epsilon) &= \theta^0 - \lambda^2\epsilon \\ &= \theta^0 - \epsilon\Delta\theta/(-\Delta s) \\ &= \theta^0 - \Delta\theta \cdot X_i \\ &= \theta \qquad (37) \end{aligned}$$

demonstrating that strong duality holds. The new basis must be optimal for all $\epsilon \in [0,\epsilon_1]$. Q.E.D.

In general, however, we cannot expect all the primal basic variables to be positive so the possibility remains that this new basis will be optimal for $\epsilon = 0$ but not optimal for any positive ϵ. Since we do not propose to actually perform the optimization of (31), (32), and (33) for some pre-selected ϵ, we will content ourselves with defining the new prices to be the same as those obtained in the non-degenerate case - namely (σ^2,τ^2) with $\tau^2 = \lambda^2$. In either case, since $\tau^2 > 0$, any agent owning generalized capacity has less incentive to withhold the resource than under the original price system $(\tau = 0)$.

Combining relations (25) and (28) reveals that we have achieved our objective:

$$\tau^2 > 0 \text{ ; and,}$$

$$\sigma^2 I_i \geq \tau^2 g_i > 0 \text{ , for all } i \text{ such that } g_i > 0. \quad (38)$$

Under the original price system we had $\sigma^0 I_\beta = 0$ which meant that items in excess supply received a zero price. Under the new price system $(\tau^2 > 0)$ we have that a resource (in general, any item i for which $g_i > 0$) will receive a positive price regardless of technical excess supply. Interpreting τ^2 as the price per unit of generalized capacity and g_i as the physical conversion factor for specific resource i, (38) states that the price of a resource must not be less than its value as generalized capacity. Furthermore, we may interpret $\tau^2 s^1$ as a transfer payment to the owners of slack capacity, suggesting that our device will provide prices more compatible with models using an "institutional arrangements" approach (eg. Dantzig [2]).

In summary, equilibrium prices of a linear economic model can be unstable. Small changes in capacities or resources can induce wide variations in prices. As an alternative, we have looked at an economy where an absence of competition prevents changes in slack capacity from optimal (equilibrium) levels except for some potential substitution among capacities in use. We proposed new prices obtained through a device of forcing an infinitesimally small amount of substitution to take place among the capacities in use. These new prices are given by (21) and (22) with $\lambda = \lambda^2$ as given by (29). These new prices are stable in the sense that they are invariant to small changes in available resources and capacities.

References

[1] Dantzig, George B., Linear Programming and Extensions, Princeton University Press, (Princeton, 1963).

[2] Dantzig, George B., "An Institutionalized Divvy Economy," Journal of Economic Theory, Vol 11, No. 3 (December, 1975).

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) SOL-79-2
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): (6) PRICING UNDEREMPLOYED CAPACITY IN A LINEAR ECONOMIC MODEL
5. TYPE OF REPORT & PERIOD COVERED: (9) Technical Report
6. PERFORMING ORG. REPORT NUMBER: SOL 79-2
7. AUTHOR(s): (10) George B./ Dantzig [redacted] Peter L./Jackson
8. CONTRACT OR GRANT NUMBER(s): (15) N00014-75-C-0267, NSF-MCS76-81259
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Operations Research -- SOL, Stanford University, Stanford, CA 94305
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR-047-143
11. CONTROLLING OFFICE NAME AND ADDRESS: Operations Research Program -- ONR, Department of the Navy, 800 N. Quincy Street, Arlington, VA 22217
12. REPORT DATE: (11) Feb[redacted]979
13. NUMBER OF PAGES: 13 (12) 17p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear Economic Model	Shadow Prices
Capacity	Substitution
Transfer Payments	Equilibrium

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

One of the difficulties in relating the shadow prices of a linear economic model to their counterparts in the real economy being modelled is the assumption of perfect competition. Under this assumption competition would force the price of any resource in excess supply down to zero. In real economies, however, owners of capacity routinely receive a return even when that capacity is underemployed, precisely because competition is imperfect. We present a method for determining a stable system of shadow prices consistent with an absence of competition among the owners of slack capacity and show that this implies non-zero prices on all resources, regardless of excess supply.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

408 765

JOB