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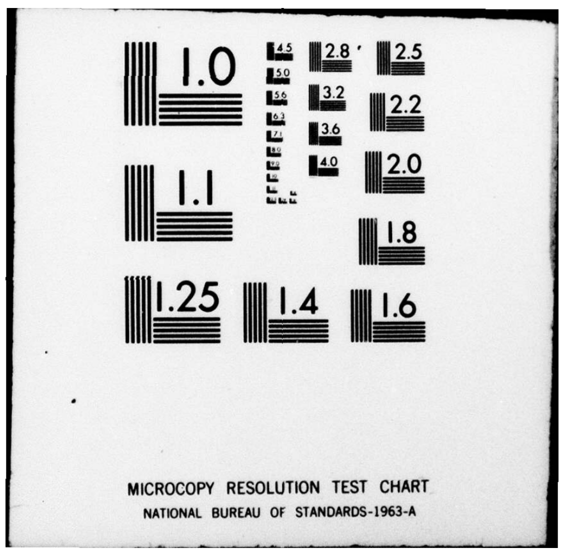
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(SELECTED ARTICLES)

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FTD-ID(RS)T-1685-78

EDITED TRANSLATION

FTD-ID(RS)T-1685-78 13 October 1978

MICROFICHE NR: FTD-78-C-001395

TRANSACTIONS OF THE CENTRAL AERO-HYDRODYNAMICS INSTITUTE (SELECTED ARTICLES)

English pages: 67

Source: Trudy Tsentral'nogo Aero-Gidrodinamicheskogo Instituta im. N. E. Zhukovskogo, Issue 1430, Moscow, 1972, pp. 1-37

Country of Origin: USSR
Translated by: Joseph E. Pearson
Requester: FTD/TQTA
Approved for public release; distribution unlimited.

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

Transactions
Of the Central Aero-Hydrodynamics Institute im. Prof. N. Ye. Zhukovskiy

Issue 1430

The Finite Element Method with Iterations for Calculating the Shapes and Frequencies of the Free Oscillations of Naturally Twisted Propeller Blades

by Z. Ye. Shnurov

Calculating the Flutter of a Helicopter Rotor in Flight

by V. V. Nazarov

The Publishing Department of TsAGI
Moscow 1972

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

The Finite Element Method with Iterations for Calculating the Shapes and the Frequencies of the Free Oscillations of Naturally Twisted Propeller Blades

by Z. Ye. Shnurov

SUMMARY

The Method of Calculating the frequencies and the forms of the free oscillations of a twisted rotating propeller blade with the various methods of attaching a blade, encountered in real designs, was worked out by the Finite Element Method with Iterations. Calculation by this method gives quick agreement, has high resolving power and makes it possible to obtain the solution in cases when it would not be possible to obtain it by other methods or it would be accomplished with insufficient accuracy.

Introduction

The emergence of vertical takeoff aircraft with rotating propellers and of helicopters with rotors having rigid attachment of the blades to hub has once again concentrated attention on the questions of the oscillations of twisted blades.

The increase in the flying speed of helicopters and the necessity of the operation of propellers (including aircraft propellers) under conditions very remote from axial flow requires the creation of such practical methods of calculating the frequencies of natural oscillations and stresses in these blades, which make it possible with a greater degree of accuracy to determine magnitudes which are of interest to the designers. From the early works it is necessary to note [1,2]. In work [1] an approximate method of calculation was developed, in which a twisted blade was simulated by a twisted rod: the plane of its greatest rigidity was located at a certain angle to the plane of rotation, which corresponds, as a rule, to an angle on a relative radius $\bar{r}=0.75$. In work [2] based on the theory of thin rods of Kirchhoff and Clebsch accurate differential equations of the oscillations of blades were obtained and approximate methods of solving these equations for a nonrotating blade were demonstrated.

The methods of calculating developed in works[3,4] have high accuracy in solving the blade model in question, however, its twistedness is not taken into consideration, as a result of which the connection of the oscillations in the planes of the greatest and least rigidity is absent. These works also cannot be used in the calculations of plane blades in the case when inclination of the plane of the greatest rigidity in the plane of rotation occurs, since the connection arising in this case between the oscillations in the planes of the greatest and least rigidity is not taken into account. It is necessary to note, that in certain cases this connection (attachment of the blade to the hub with the aid of a horizontal and vertical hinge) has an insignificant effect on the frequencies of the natural oscillations, in particular, in those cases when the oscillations in the plane of the least rigidity predominate in the actual form of oscillations.

Of the works, dedicated to the calculation of bending oscillations of slightly twisted baldes, it is necessary to not work [5], in which a single-parameter integral equation was developed for determining the forms and frequencies of bending oscillations of a rotating blade, which ensures convergence of the method of successive approximations at any angular velocities of rotation. But the case was examined in it, when one of the main magnitudes of rigidity becomes infinite and, as a consequence of the slight twistedness, the connection of the oscillations in the planes of the greatest and least rigidity disappears. An equation was obtained in the work, which takes into consideration the twistedness of the blade, but the programmed equation did not take this circumstance into account.

The works developed in connection with the calculation of the frequencies of the natural oscillations of compressor and turbine blades can be used for calculating propeller blades, however, the different assumptions in deriving the equations, completely acceptable for the purposes of designing short and broad turbine blades, do not make it possible to consider them sufficiently satisfactory for designing propeller blades. Thus, for example, in work [6] it was assumed, that the curvature in the direction of the greatest rigidity of the cross-section was equal to zero. This assumption is completely satisfactory for those cases, whan for a plane blade, which differs from the one being investigated only by the fact, thatthe twistedness of the cross sections relative to each other is absent and the frequencies of the natural oscillations are greatly different. Such is the situation for turbine blades. For propeller blades (for example, see Fig. 11) these frequencies are sufficiently close and a similar assumption would lead to a noticeable error.

In work [7] a system of differential equations was obtained, which describe the oscillations of a blade taking into consideration the connections, which arise with simultaneous bending in the planes of the greatest and least rigidity and torsion. However, the method of initial parameters employed in this work in the presence of a field of centrifugal forces requires very high accuracy and does not give certainty that the system can be solved at all possible relationships of the lowest frequenciesof natural oscillations and angular velocity of rotation of the rotor.

It is necessary to mention one more important feature, which should be taken into account in developing ....(two sentences and part of another are illegible)... with aid of one so-called horizontal hinge, and its axis cannot coincide with the plane of the greater rigidity of the blade both at the site of the location of the hinge and at any other cross section. This noncoincidence leads to additional connections in the oscillations of a blade in the direction of its least and greatest rigidity. Works [1-7] do not take such a possibility into consideration.

In the present work a method has been worked out, which makes it possible to calculate, for a naturally twisted blade, the frequencies and forms of the natural oscillations simultaneously in the planes of the greatest and least rigidity with various combination methods of attachment to the hub.

Derivation of the Equations

The method expounded below was obtained by employing the method of three moments (well known in strength of materials), used earlier for calculating the deformations of a blade in one plane in a field of centrifugal forces [8] and revised by A. V. Nekrasov [3] for calculating the frequencies and forms of natural oscillations also in a field of centrifugal forces of a plane untwisted blade, performing oscillations only in the plane of the greatest and least rigidity.

Fig. 1

In accordance with the method the blade is represented as a beam, in which the length greatly exceeds the dimensions in the two other directions; it is assumed, that the points of intersection of the main central axes of rigidity of the cross section lie one one straight line and the connection between bending and torsion is absent. The blade in accordance with work [3] is represented in the from of a weightless beam, in each cross section of which are known the directions of the main central axes of inertia and accordingly of the greatest values of bending rigidity. The blade is broken down into sections. At the junction of the neighboring sections are located concentrated masses, which correspond in magnitude to the mass of the halves of the neighboring sections adjacent to the given point. The magnitude of the rigidities and the angles of inclination of the planes of the greatest and least rigidity within the limits of one section are constant (Fig. 1). Thus, the centrifugal force being applied to the concentrated masses, remains constant within the limits of one section and changes abruptly at the boundary with the neighboring section. A zero mass is selected from the conditions determined by the attachment of the blade to the hub. For obtaining a fixed (jammed) or hinge-wise supported blade the zero mass is selected necessarily large. If the blade is attached to an absolutely rigid real body, then the zero mass can be selected in accordance with the mass of this real body. If the blade is attached to an elastic real body, the frequencies of the natural oscillations of which have the same order as the frequencies of the natural oscillations of the blade, then the zero mass can be determined from the condition of equality to zero of the sum of the dynamic rigidities of the blade in the corresponding direction at the zero point (Fig. 1) and of the dynamic rigidity in this same direction of the attached body. Let us point out here, that for one and the same value of frequency the values of the zero mass in the examination of the oscillations of the blade in plane xOz and in plane yOz will be different.

Let us examine the two neighboring sections of the blade i-j and j-k (Fig. 2). Here φ_{ij} is the angle of inclination of the plane of the greatest rigidity towards the plane of rotation in section i-j; φ_{jk} is the angle of inclination of the plane of the greatest rigidity towards the plane of rotation in section j-k.

Plane xOz coincides with the plane of rotation; planes ξ_{ij} Oz; ξ_{jk} Oz; η_{ij} Oz; η_{jk} Oz are respectively the planes of the greatest and the least rigidities in sections i-j and j-k. The concentrated masses are located at points (ij) and (jk). Thus, only one mass is located within the limits of one section.

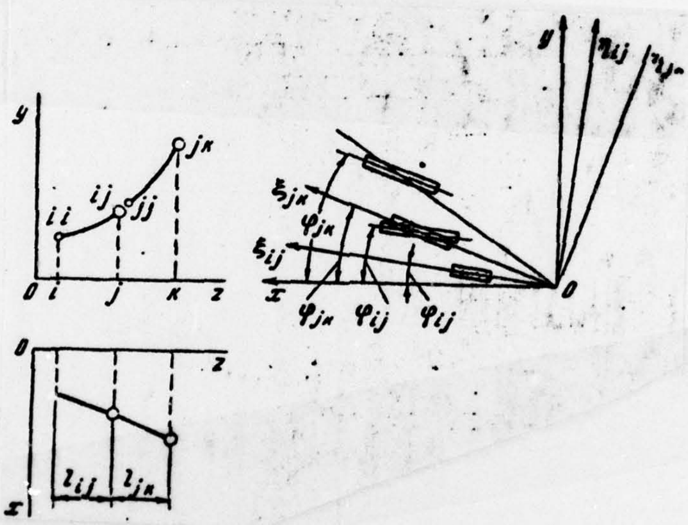

Fig. 2

Let us explain more precisely, that the coordinate system xyz rotates together with the rotor, in this case axis y is directed along the rotor shaft in the direction of the lift, axis z is directed along the blade axis, and axis x is perpendicular to the first two and is oriented in the direction of the rotation of the rotor. This is a so-called general system of coordinates. Moreover, the so-called local coordinate system $\xi\eta\zeta$ is located in the investigation; the axes and are located in the planes of the cross sections of the blade perpendicular to axis z, are are directed respectively along the main central axes of inertia of the cross sections.

Let us write out the equations of equilibrium for sections i-j and j-k in the general coordinate system xyz:

$$\left.\begin{aligned} M_{xji} &= M_{xij} - Q_{yij} l_{ij} - N_{ij}(U_{yij} - U_{yji}); \\ M_{yji} &= M_{yij} - Q_{xij} l_{ij} - N_{ij}(U_{xij} - U_{xji}); \end{aligned}\right| \qquad (1a)$$

$$\left.\begin{aligned} M_{xij} &= M_{xjk} + Q_{yjk} l_{jk} - N_{jk}(U_{yjk} - U_{yij}); \\ M_{yij} &= M_{yjk} + Q_{xjk} l_{jk} - N_{jk}(U_{xjk} - U_{xij}). \end{aligned}\right| \qquad (1b)$$

Although

$$\left.\begin{aligned} U_{xij} = U_{xjj} = U_{xj};\quad M_{yij} = M_{yjj} = M_{yj}; \\ U_{yij} = U_{yjj} = U_{yj};\quad M_{xij} = M_{xjj} = M_{xj}, \end{aligned}\right\} \qquad (2)$$

let us retain for a certain time the double indexing system, since

$$\left.\begin{aligned} U_{xij} \neq U_{xjj};\quad M_{yij} \neq M_{yjj}; \\ U_{yij} \neq U_{yjj};\quad M_{xij} \neq M_{xjj}, \end{aligned}\right| \qquad (3)$$

and subsequently it will be necessary to carry out transformation employing (1)-(2).

Let us note here, that

$$\left.\begin{aligned} Q_{yij} &= -\sum_{j}^{k} m_i \ddot{U}_{yi}, \\ Q_{xij} &= -\sum_{j}^{k} m_i (\ddot{U}_{xi} - \omega^2 U_{xi}) \text{ etc.} \end{aligned}\right\} \qquad (4)$$

Designated in the equations are:

U_y; U_x is the displacement in the direction of the axes of the general coordinate system;

U_v; U_t is the displacement in the direction of the local coordinate system;

M_y; M_x are the flexural moments, the vectors of which are respectively parallel to axes y and x;

Q_y; Q_x are shearing forces, the vectors of which are also parallel to the axes of the general coordinate system y and x;

N is centrifugal force, the vector of which is parallel to axis z.

Having divided each of the expressions (1a) and (1b) respectively by $N_{ij}l_{ij}$ and $N_{jk}l_{jk}$, introducing the designations

$$\left.\begin{aligned} f_{ij} &= \frac{1}{N_{ij}l_{ij}}; \quad b_{ij} = \frac{1}{l_{ij}}; \\ f_{jk} &= \frac{1}{N_{jk}l_{jk}}; \quad b_{jk} = \frac{1}{l_{jk}}, \end{aligned}\right| \tag{5}$$

we will obtain

$$\left.\begin{aligned} f_{ij}M_{xij} - f_{ij}M_{xii} - b_{ij}(U_{yij} - U_{yii}) + \frac{Q_{yij}}{N_{ij}} &= 0, \\ f_{jk}M_{xjk} - f_{jk}M_{xjj} - b_{jk}(U_{yjk} - U_{yjj}) + \frac{Q_{yjk}}{N_{jk}} &= 0; \end{aligned}\right| \tag{6}$$

$$\left.\begin{aligned} f_{ij}M_{yij} - f_{ij}M_{yii} - b_{ij}(U_{xij} - U_{xii}) + \frac{Q_{xij}}{N_{ij}} &= 0. \\ f_{jk}M_{yjk} - f_{jk}M_{yjj} - b_{jk}(U_{xjk} - U_{xjj}) + \frac{Q_{xjk}}{N_{jk}} &= 0. \end{aligned}\right| \tag{7}$$

Subtracting in pairs (7) from (6), we will obtain after certain transformations

$$[illegible]\ f_{j}M\ [illegible] + \frac{Q}{N_{j}} - \frac{Q}{N}\ [illegible] \tag{8a}$$

$$[illegible]\ U_{k} + a_{j} U_{kj} + b_{jk} U_{k}\ [illegible] = f\ [illegible]\ M\ [illegible] - f_{jk} M_{j} - \frac{Q_{k}}{N_{jk}} - \frac{Q}{N_{ij}}. \tag{8b}$$

$$a_{j} = -b_{[illegible]} - b_{[illegible]};\quad c\ [illegible] = -f_{jk} - f_{[illegible]} \tag{9}$$

Then we will examine the deformations of sections i-j and j-k. We will seek the deformations in the planes of the least and greatest rigidity. The differential equations of equilibrium in the sections respectively in the planes ηOz and ξOz are:

$$(EI_{\xi} U_{\eta}'')'' - (NU_{\eta}')'' = 0;\quad (EI_{\eta} U_{\xi}'')'' - (NU_{\xi}')' = 0. \tag{10}$$

It is possible to write the solution of the equations of (10) in the following manner:

$$M_{\xi} = A_{\xi} \operatorname{sh} \mu_{\xi} z - B_{\xi} \operatorname{ch} \mu_{\xi} z;\quad M_{\eta} = A\ [illegible]\ z, \tag{11}$$

where (with the retaining of double indexing)

$$A_{\xi ij} = \frac{M_{\xi ii}}{\operatorname{sh} \alpha_{\xi ij}} - \frac{M_{\xi ii}}{\operatorname{th} \alpha_{\xi ij}};\quad A_{\eta ij} = \frac{M_{\eta ij}}{\operatorname{sh} \alpha_{\eta ij}} - \frac{M_{\eta ii}}{\operatorname{th} \alpha_{\eta ij}}; \tag{12}$$

$$B_{\xi ij} = M_{\xi ii};\quad B_{\eta ij} = M_{\eta ii};$$

$$A_{\xi jk} = \frac{M_{\xi jk}}{\operatorname{sh} \alpha_{\xi jk}} - \frac{M_{\xi jj}}{\operatorname{th} \alpha_{\xi jk}};\quad A_{\eta jk} = \frac{M_{\eta jk}}{\operatorname{sh} \alpha_{\eta jk}} - \frac{M_{\eta jj}}{\operatorname{th} \alpha_{\eta jk}};$$

$$B_{\xi jk} = M_{\xi jj};\quad B_{\eta jk} = M_{\eta jj};$$

$$M_{\xi} = EI_{\xi} U_{\eta}'';\quad M_{\eta} = EI_{\eta} U_{\xi}''; \tag{13}$$

$$\alpha_{\eta ij(jk)} = \mu_{\eta ij(jk)} l_{ij(jk)};\quad \alpha_{\xi ij(jk)} = \mu_{\xi ij(jk)} l_{ij(jk)}. \tag{14}$$

Haing substituted expression (12) in (11) and having twice integrated the corresponding expressions, we will obtain for section i-j the equations connecting the deformations in the plane of the least rigidity with the flexural moment, acting in the same plane:

or

$$\left.\begin{aligned} b_{ij}(U_{\eta ij} - U_{\eta ji}) &= d^{\eta}_{ij} M_{\xi ij} + l^{\eta}_{ij} M_{\xi ji} + \beta^{\eta}_{ji} \\ b_{ij}(U_{\eta ij} - U_{\eta ji}) &= - l^{\eta}_{ij} M_{\xi ij} - d^{\eta}_{ij} M_{\xi ji} + \beta^{\eta}_{ij}. \end{aligned}\right\} \quad (15)$$

The expression for section j-k in the plane of the least rigidity and the corresponding expressions for the plane of the greatest rigidity are analogous. For example, for section j-k

$$\left.\begin{aligned} b_{jk}(U_{\xi jk} - U_{\xi jj}) &= d^{\xi}_{jk} M_{\eta jk} + l^{\xi}_{jk} M_{\eta jj} + \beta^{\xi}_{jj}; \\ b_{jk}(U_{\xi jk} - U_{\xi jj}) &= - l^{\xi}_{jk} M_{\eta jk} - d^{\xi}_{jk} M_{\eta jj} + \beta^{\xi}_{jk}. \end{aligned}\right\} \quad (16)$$

in this case

$$\alpha^{\xi(\eta)}_{ij(jk)} = \sqrt{\frac{N_{ij(jk)} l^2_{ij(jk)}}{EJ^{ij(jk)}_{\xi(\eta)}}}; \quad (17)$$

[illegible] (18)

[illegible] is the tangent of the angle between the projection of the tangent to the elastic line in plane [illegible] Oz and axis Oz with its apex at point ij;

[illegible] is the tangent of the angle between the projection of the tangent to the elastic line in plane [illegible] Oz and axis Oz with its apex at point ii;

[illegible] is the tangent of the angle between the projection of the tangent to the elastic line in plane [illegible] Oz and axis Oz with its apex at point jk.

Analogously [illegible] and [illegible] are tangents of the angles between the projection of the tangent to the elastic line in the plane $\xi_{ij}Oz$ or $\eta_{jk}Oz$ and axis Oz with their apices at points ij; ii; jk; and jj or in other words, the components of the first derivative of the displacements in the direction of axes η and ξ.

Since in a general coordinate system the first derivative changes monotonically, i.e., it does not have discontinuities, then for sections ij and jk it is possible to write

$$\beta^x_{ij} = \beta^x_{jj}; \; \beta^y_{ij} = \beta^y_{jj}, \tag{19}$$

where β^x and β^y are respectively components of the first derivative of displacements in the direction of axes x and y.

Let us employ the relation

$$\beta^x_{ij} = \beta^\xi_{ij} \cos \varphi_{ij} - \beta^\eta_{ij} \sin \varphi_{ij}. \tag{20}$$

It follows from (19), that

$$\beta^\xi_{ij} \cos \varphi_{ij} - \beta^\eta_{ij} \sin \varphi_{ij} = \beta^\xi_{jj} \cos \varphi_{jk} - \beta^\eta_{jj} \sin \varphi_{jk}; \tag{21a}$$

$$\beta^\xi_{ij} \sin \varphi_{ij} + \beta^\eta_{ij} \cos \varphi_{ij} = \beta^\xi_{jj} \sin \varphi_{jk} + \beta^\eta_{jj} \cos \varphi_{jk}. \tag{21b}$$

We will obtain from (15) and (16)

$$\left.\begin{aligned}
\beta^\eta_{ij} &= b_{ij}(U_{\eta ij} - U_{\eta ii}) + l^\eta_{ij} M_{\xi ij} + d^\eta_{ij} M_{\xi ii};\\
\beta^\eta_{jj} &= b_{jk}(U_{\eta jk} - U_{\eta jj}) - d^\eta_{jk} M_{\xi jk} - l^\eta_{jk} M_{\xi jj};\\
\beta^\xi_{ij} &= b_{ij}(U_{\xi ij} - U_{\xi ii}) + l^\xi_{ij} M_{\eta ij} + d^\xi_{ij} M_{\eta ii};\\
\beta^\xi_{jj} &= b_{jk}(U_{\xi jk} - U_{\xi jj}) - d^\xi_{jk} M_{\eta jk} - l^\xi_{jk} M_{\eta jj}.
\end{aligned}\right\} \tag{22}$$

Having substituted in turn (22) in (21a) and (21b) and having carried out grouping, we will obtain:

for plane xOz

$$[b_{ij}(U_{\xi ij} - U_{\xi ii})\cos\varphi_{ij} - b_{ij}(U_{\eta ij} - U_{\eta ii})\sin\varphi_{ij}] + [(l^{\xi}_{ij} M_{\eta ij} + d^{\xi}_{ij} M_{\eta ii})\cos\varphi_{ij} - (l^{\eta}_{ij} M_{\xi ij} + d^{\eta}_{ij} M_{\xi ii})\sin\varphi_{ij}] = [b_{jk}(U_{\xi jk} - U_{\xi jj})\cos\varphi_{jk} - b_{jk}(U_{\eta jk} - U_{\eta jj})\sin\varphi_{jk}] + [-(d^{\xi}_{jk} M_{\eta jk} + l^{\xi}_{jk} M_{\eta jj})\cos\varphi_{jk} + (d^{\eta}_{jk} M_{\xi jk} + l^{\eta}_{jk} M_{\xi jj})\sin\varphi_{jk}]; \quad (23)$$

for plane yOz

$$[b_{ij}(U_{\xi ij} - U_{\xi ii})\sin\varphi_{ij} + b_{ij}(U_{\eta ij} - U_{\eta ii})\cos\varphi_{ij}] + [(l^{\xi}_{ij} M_{\eta ij} + d^{\xi}_{ij} M_{\eta ii})\sin\varphi_{ij} + (l^{\eta}_{ij} M_{\xi ij} + d^{\eta}_{ij} M_{\xi ii})\cos\varphi_{ij}] = [b_{jk}(U_{\xi jk} - U_{\xi jj})\sin\varphi_{jk} + b_{jk}(U_{\eta jk} - U_{\eta jj})\cos\varphi_{jk}] + [-(d^{\xi}_{jk} M_{\eta jk} + l^{\xi}_{jk} M_{\eta jj})\sin\varphi_{jk} - (d^{\eta}_{jk} M_{\xi jk} + l^{\eta}_{jk} M_{\xi jj})\cos\varphi_{jk}]. \quad (24)$$

Having used the relationships

$$U_x = U_\xi \cos\varphi - U_\eta \sin\varphi; \quad U_y = U_\xi \sin\varphi - U_\eta \cos\varphi. \quad (25)$$

we will obtain ...(illegible)... transformation for the corresponding sections:

for plane xOz

$$[illegible]$$
$$+ (l^{\xi}_{ij} M_{\eta ij} + d^{\xi}_{ij} M_{\eta ii})\cos\varphi_{ij} - (l^{\eta}_{ij} M_{\xi ij} + d^{\eta}_{ij} M_{\xi ii})\sin\varphi_{ij} + (d^{\xi}_{jk} M_{\eta jk} + l^{\xi}_{jk} M_{\eta jj})\cos\varphi_{jk} - (d^{\eta}_{jk} M_{\xi jk} + l^{\eta}_{jk} M_{\xi jj})\sin\varphi_{jk}; \quad (26a)$$

and analogously for plane yOz

$$b_{ij} U_{yi} - (b_{ij} + b_{jk}) U_{yj} + b_{jk} U_{yk} = (l^{\xi}_{ij} M_{\eta ij} + d^{\xi}_{ij} M_{\eta ii})\sin\varphi_{ij} + (l^{\eta}_{ij} M_{\xi ij} + d^{\eta}_{ij} M_{\xi ii})\cos\varphi_{ij} + (d^{\xi}_{jk} M_{\eta jk} + l^{\xi}_{jk} M_{\eta jj})\sin\varphi_{jk} + (d^{\eta}_{jk} M_{\xi jk} + l^{\eta}_{jk} M_{\xi jj})\cos\varphi_{jk}. \quad (26b)$$

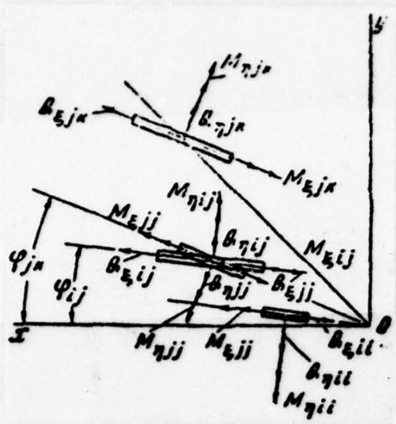

Fig. 3

Expression (26a) connects the displacements in plane xOz with the flexural moments in the planes of the greatest and least rigidity in the four neiboring sections (ii, ij, jj, jk). Expression (26b) connects the displacements in plane yOz with the same flexural moments in the planes of the least and greatest plane in the same four neighboring sections (ii, ij, jj, jk).

These expressions are awkward because they contain values of flexural moments in four sections, whereas the displacements are given in three sections.

However, it is possible to reduce the right sides of (26a) and (26b) to expressions containing flexural moments only in three sections.

Let us note that moments $M_{\xi ij}$, $M_{\xi jj}$, $M_{\eta ij}$ and $M_{\eta jj}$ on the boundary of the neighboring sections are connected with each other by the following relationships (Fig. 3).

$$\left.\begin{aligned} M_{\xi jj} &= M_{\xi ij}\cos(\varphi_{jk}-\varphi_{ij}) - M_{\eta ij}\sin(\varphi_{jk}-\varphi_{ij}); \\ M_{\eta jj} &= M_{\xi ij}\sin(\varphi_{jk}-\varphi_{ij}) + M_{\eta ij}\cos(\varphi_{ij}-\varphi_{ij}), \end{aligned}\right\} \qquad (27a)$$

whence

$$\left.\begin{aligned} M_{\xi ij} &= M_{\xi jj}\cos(\varphi_{jk}-\varphi_{ij}) + M_{\eta jj}\sin(\varphi_{jk}-\varphi_{ij}); \\ M_{\eta ij} &= -M_{\xi jj}\sin(\varphi_{jk}-\varphi_{ij}) + M_{\eta jj}\cos(\varphi_{jk}-\varphi_{ij}). \end{aligned}\right\} \qquad (27b)$$

Introducing the designation

$$\Delta\varphi_j = \varphi_{jk} - \varphi_{ij}$$

and having noted that equally

$$M_{\xi jk} = M_{\xi kk}\cos\Delta\varphi_k + M_{\eta kk}\sin\Delta\varphi_k; \quad M_{\eta jk} = -M_{\xi kk}\sin\Delta\varphi_k + M_{\eta kk}\cos\Delta\varphi_k, \qquad (28)$$

let us substitute (27) and (28) in (26a) and we will obtain

$$\begin{aligned} b_{ij}U_{xi} - (b_{ij} - b_{jk})U_{xj} - b_{jk}U_{xk} = M_{\eta ii}d^{\xi}_{ij}\cos\varphi_{ij} + M_{\eta jj}(l^{\xi}_{ij}\cos\Delta\varphi_j\cos\varphi_{ij} - \\ - l^{\eta}_{ij}\sin\Delta\varphi_j\sin\varphi_{ij} + l^{\xi}_{jk}\cos\varphi_{jk}) + M_{\eta kk}(d^{\xi}_{jk}\cos\Delta\varphi_k\cos\varphi_{jk} - d^{\eta}_{jk}\sin\Delta\varphi_k\sin\varphi_{jk}) - \\ - M_{\xi ii}d^{\eta}_{ij}\sin\varphi_{ij} - M_{\xi jj}(l^{\xi}_{ij}\sin\Delta\varphi_j\cos\varphi_{ij} + l^{\eta}_{ij}\cos\Delta\varphi_j\sin\varphi_{ij} + \\ + l^{\eta}_{jk}\sin\varphi_{jk}) - M_{\xi kk}(d^{\xi}_{ij}\sin\Delta\varphi_k\cos\varphi_{jk} + d^{\eta}_{jk}\cos\Delta\varphi_k\sin\varphi_{jk}). \end{aligned} \qquad (29a)$$

Having accomplished the analogous transformations for (26b), we will obtain

$$\begin{aligned} b_{ij}U_{yi} - (b_{ij} - b_{jk})U_{yj} - b_{jk}U_{yk} = M_{\eta ii}d^{\xi}_{ij}\sin\varphi_{ij} + M_{\eta jj}(l^{\xi}_{ij}\cos\Delta\varphi_j\sin\varphi_{ij} + \\ + l^{\eta}_{ij}\sin\Delta\varphi_j\cos\varphi_{ij} + l^{\xi}_{jk}\sin\varphi_{jk}) + M_{\eta kk}(d^{\xi}_{jk}\cos\Delta\varphi_k\sin\varphi_{jk} + d^{\eta}_{jk}\sin\Delta\varphi_k\cos\varphi_{jk}) + \\ + M_{\xi ii}d^{\eta}_{ij}\cos\varphi_{ij} + M_{\xi jj}(-l^{\xi}_{ij}\sin\Delta\varphi_j\sin\varphi_{ij} + l^{\eta}_{ij}\cos\Delta\varphi_j\cos\varphi_{ij} + l^{\eta}_{jk}\cos\varphi_{jk}) + \\ + M_{\xi kk}(-d^{\xi}_{jk}\sin\Delta\varphi_k\sin\varphi_{jk} + d^{\eta}_{jk}\cos\Delta\varphi_k\cos\varphi_{jk}). \end{aligned} \qquad (29b)$$

Thus, the expressions are obtained, which connect the displacements of the three neighboring points with the flexural moments in the planes of the least and greatest rigidity in the three neighboring cross sections.

The first and the last sections are singular sections.

Let us write out the analogous expressions for these sections. The equations of (15) for the first section are:

$$b_{01}(U_{y1} - U_{y01}) = d^{\eta}_{01}M_{\xi 01} + l^{\eta}_{01}M_{\xi 00} + \Re_{01}; \qquad (30a)$$

$$b_{01}(U_{\xi 1} - U_{\xi 0}) = d^{\xi}_{01} M_{\eta 01} + l^{\xi}_{01} M_{\eta 00} + \beta^{\xi}_{00}. \tag{30b}$$

Let us multiply the first equation by the negative value of sin φ_{01}, and the second by cos φ_{01}, then let us add them, having used expressions (20) and (25) for the transformation, and we will obtain

$$b_{01}(U_{x1} - U_{x0}) = M_{\eta 00} l^{\xi}_{01} \cos\varphi_{01} + M_{\eta 01} d^{\xi}_{01} \cos\varphi_{01} - M_{\xi 00} l^{\eta}_{01} \sin\varphi_{01} - \\ - M_{\xi 01} d^{\eta}_{01} \sin\varphi_{01} + \beta_{x0},$$

and having substituted

$$M_{\eta 01} = - M_{\xi 11} \sin\Delta\varphi_1 + M_{\eta 11} \cos\Delta\varphi_1; \quad M_{\xi 01} = M_{\xi 11} \cos\Delta\varphi_1 + M_{\eta 11} \sin\Delta\varphi_1,$$

we will obtain

$$b_{01} = (U_{x1} - U_{x0}) = M_{\eta 00} l^{\xi}_{01} \cos\varphi_{01} + M_{\eta 11}(d^{\xi}_{01} \cos\Delta\varphi_1 \cos\varphi_{01} - \\ - d^{\eta}_{01} \sin\Delta\varphi_1 \sin\varphi_{01}) - M_{\xi 00} l^{\eta}_{01} \sin\varphi_{01} - M_{\xi 11}(d^{\xi}_{01} \sin\Delta\varphi_1 \cos\varphi_{01} + \\ + d^{\eta}_{01} \cos\Delta\varphi_1 \sin\varphi_{01}) + \beta_{x0}. \tag{31a}$$

In an analogous manner, the equation for the displacements in plane yOz can be obtained, if we multiply (30a) by cos φ_{01}, and (30b) by sin φ_{01}:

$$b_{01}(U_{y1} - U_{y0}) = M_{\eta 00} l^{\xi}_{01} \sin\varphi_{01} + M_{\eta 11}(d^{\xi}_{01} \cos\Delta\varphi_1 \sin\varphi_{01} + d^{\xi}_{01} \sin\Delta\varphi_1 \cos\varphi_{01}) + \\ + M_{\xi 00} l^{\eta}_{01} \cos\varphi_{01} + M_{\xi 11}(-d^{\xi}_{01} \sin\Delta\varphi_1 \sin\varphi_{01} + d^{\eta}_{01} \cos\Delta\varphi_1 \cos\varphi_{01}) + \beta_{y0}. \tag{31b}$$

The equations for the displacements of the last two points of the blade can be obtained in the exact same manner. For this it is necessary only in contrast to the first section to employ the expressions of (16) and an analogous expression for plane ηOz.

As a result we will obtain

$$b_{s-1,s}(U_{xs} - U_{x,s-1,s-1}) = - M_{\eta,s-1,s-1} d^{\xi}_{s-1,s} \cos\varphi_{s-1,s} - M_{\eta,s-1,s} l^{\xi}_{s-1,s} \cos\varphi_{s-1,s} + \\ + M_{\xi,s-1,s-1} d^{\eta}_{s-1,s} \sin\varphi_{s-1,s} + M_{\xi,s-1,s} l^{\eta}_{s-1,s} \sin\varphi_{s-1,s} + \beta_{xs}; \tag{32a}$$

$$b_{z-1,z}(U_{yz} - U_{y\,z-1}) = -M_{\eta,z-1,z-1}d^{\xi}_{z-1,z}\sin\varphi_{z-1,z} - M_{\eta,z-1,z}l^{\xi}_{z-1,z}\sin\varphi_{z-1,z} -$$
$$- M_{\xi,z-1,z-1}d^{\eta}_{z-1,z}\cos\varphi_{z-1,z} - M_{\xi,z-1,z}l^{\eta}_{z-1,z}\cos\varphi_{z-1,z} + \beta_{yz}. \qquad (32b)$$

It is necessary to note here, that the systems of equations for the root and terminal(tip) sections differ somewhat from the remaining equations, which is explained by the peculiarities of the extreme sections.

Let us introduce designations, which simplify the notation of equations (29), (31) and (32):

$$d_{ij}^{\xi\xi} = d_{ij}^{\xi}\cos\varphi_{ij};\quad d_{ij}^{\eta\xi} = d_{ij}^{\eta}\sin\varphi_{ij};$$
$$l_j^{\xi\xi} = l_{ij}^{\xi}\cos\Delta\varphi_j\cos\varphi_{ij} - l_{ij}^{\eta}\sin\Delta\varphi_j\sin\varphi_{ij} - l_{jk}^{\xi}\cos\varphi_{jk};$$
$$l_j^{\eta\xi} = l_{ij}^{\xi}\sin\Delta\varphi_j\cos\varphi_{ij} + l_{ij}^{\eta}\cos\Delta\varphi_j\sin\varphi_{ij} + l_{jk}^{\eta}\sin\varphi_{jk};$$
[illegible]
[illegible] $\sin\varphi_{ij};\ d_{ij}^{\eta\eta} = d_{ij}^{\eta}\cos\varphi_{ij};$
$$l_j^{\xi\eta} = l_{ij}^{\xi}\cos\Delta\varphi_j\sin\varphi_{ij} + l_{ij}^{\eta}\sin\Delta\varphi_j\cos\varphi_{ij} - l_{jk}^{\xi}\sin\varphi_{jk};$$
$$l_j^{\eta\eta} = -l_{ij}^{\xi}\sin\Delta\varphi_j\sin\varphi_{ij} + l_{ij}^{\eta}\cos\Delta\varphi_j\cos\varphi_{ij} + l_{jk}^{\eta}\cos\varphi_{jk};$$
$$d_k^{\xi\eta} = d_{jk}^{\xi}\cos\Delta\varphi_k\sin\varphi_{jk} + d_{jk}^{\eta}\sin\Delta\varphi_k\cos\varphi_{jk};$$
$$d_k^{\eta\eta} = -d_{jk}^{\xi}\sin\Delta\varphi_k\sin\varphi_{jk} + d_{jk}^{\eta}\cos\Delta\varphi_k\cos\varphi_{jk}.$$

As a result we will obtain a system of (z+1) pairs written in general form, containing (z+1) pairs of flexural moments acting in the plane of the least and greatest rigidity, and two pairs of components of the first derivative of the elastic line in fixed coordinate system xyz at the zero and last points of the blade.

The general expression of the j-th pair of equations is:

$$\left.\begin{aligned} b_{ij}U_{xi} + a_jU_{xj} + b_{jk}U_{xk} &= M_{\eta ii}d_{ij}^{\xi\xi} + M_{\eta jj}l_j^{\xi\xi} + \\ &+ M_{\eta kk}d_k^{\xi\xi} - M_{\xi ii}d_{ij}^{\xi\eta} - M_{\xi jj}l_j^{\xi\eta} - M_{\xi kk}d_k^{\xi\eta}; \\ b_{ij}U_{yi} + a_jU_{yj} + b_{jk}U_{yk} &= M_{\eta ii}d_{ij}^{\eta\xi} + M_{\eta jj}l_j^{\eta\xi} + \\ &+ M_{\eta kk}d_k^{\eta\xi} + M_{\xi ii}d_{ij}^{\eta\eta} + M_{\xi jj}l_j^{\eta\eta} + M_{\xi kk}d_k^{\eta\eta}. \end{aligned}\right| \qquad (33)$$

The first pair of equations (the root section) is

$$\left.\begin{aligned} -b_{01}U_{x0}+b_{01}U_{x1} &= M_{\tau,00}l_{01}^{c\xi}+M_{\tau,11}d_{1}^{c\xi}-M_{\xi 00}l_{01}^{s\eta}-M_{\xi 11}d_{1}^{s\eta}+\beta_{x0};\\ -l_{01}U_{y0}+b_{01}U_{y1} &= M_{\tau,00}l_{01}^{s\xi}+M_{\eta 11}d_{1}^{s\xi}+M_{\xi 00}l_{01}^{c\eta}+M_{\xi 11}d_{1}^{c\eta}+\beta_{y0}. \end{aligned}\right\} \qquad (34a)$$

The last pair of equations (the terminal (tip) section) is

$$\left.\begin{aligned} b_{z-1}U_{x;\,z-1}-b_{z-1}U_{xz} &= M_{\tau;\,z-1;\,z-1}\,d_{z-1;\,z}^{c\xi}-M_{\tau;\,z-1;\,z}\,l_{z-1;\,z}^{c\xi}-\\ &\quad -M_{\xi;\,z-1;\,z-1}\,d_{z-1;\,z}^{s\eta}-M_{\xi;\,z-1;\,z}\,l_{z-1;\,z}^{s\eta}+\beta_{xz};\\ b_{z-1}U_{y;\,z-1}\,b_{z-1}U_{yz} &= M_{\tau;\,z-1;\,z-1}\,d_{z-1;\,z}^{s\xi}+M_{\eta;\,z-1;\,z}\,l_{z-1;\,z}^{s\xi}+\\ &\quad +M_{\xi;\,z-1;\,z-1}\,d_{z-1;\,z}^{c\eta}+M_{\xi;\,z-1;\,z}\,l_{z-1;\,z}^{c\eta}-\beta_{yz}. \end{aligned}\right\} \qquad (34b)$$

This system, written in matrix form, has the form of (35) (see the inset). It makes it possible from the known values of the flexural moments and components of the first derivative of the elastic line to find the magnitudes of the displacements.

For solving a problem on natural oscillations by the method of successive approximations one more system of equations is needed, from which it would be possible to obtain the values of the flexural moments and components of the first derivative at the ends of the blade.

For this let us employ system (8a) and (8b).

Let us write equation (8a) somewhat differently:

$$\begin{aligned} b_{ij}U_{xij}+a_{j}U_{xj}+b_{jk}U_{xjk} &= f_{ij}M_{yii}-f_{ij}M_{yij}-\\ &\quad -f_{jk}M_{yjj}-f_{jk}M_{yjk}+\frac{Q_{xjk}}{N_{jk}}-\frac{Q_{xij}}{N_{ij}}, \end{aligned} \qquad (36)$$

since

$$
\left.\begin{aligned}
f_{ij} M'_{y\,ii} &= f_{ij}(-M_{z\,ii}\sin\varphi_{ij} + M_{y\,ii}\cos\varphi_{ij});\\
f_{ij} M'_{y\,ij} &= f_{ij}(-M_{z\,ij}\sin\varphi_{ij} + M_{y\,ij}\cos\varphi_{ij});\\
f_{jk} M'_{y\,jj} &= f_{jk}(-M_{z\,jj}\sin\varphi_{jk} + M_{y\,jj}\cos\varphi_{jk});\\
f_{jk} M'_{y\,jk} &= f_{jk}(-M_{z\,jk}\sin\varphi_{jk} + M_{y\,jk}\cos\varphi_{jk}),
\end{aligned}\right| \qquad (37)
$$

then, having equated the right sides of equalities (26a) and (36), we will obtain, after some transformations

$$
\begin{aligned}
&M_{y\,ii}(d^{\xi}_{ij} - m_{ij})\cos\varphi_{ij} + M_{y\,ij}(l^{\xi}_{ij} + m_{ij})\cos\varphi_{ij} + M_{y\,jj}(l^{\xi}_{jk} + m_{jk})\cos\varphi_{jk} +\\
&+ M_{y\,jk}(d^{\xi}_{jk} - m_{jk})\cos\varphi_{jk} - M_{z\,ii}(d^{\xi}_{ij} - m_{ij})\sin\varphi_{ij} - M_{z\,ij}(l^{\xi}_{ij} + m_{ij})\sin\varphi_{ij} -\\
&- M_{z\,jj}(l^{\xi}_{jk} + m_{jk})\sin\varphi_{jk} - M_{z\,jk}(d^{\xi}_{jk} - m_{jk})\sin\varphi_{jk} = \frac{Q_{x\,jk}}{N_{jk}} - \frac{Q_{x\,ij}}{N_{ij}}.
\end{aligned} \qquad (38)
$$

Let us again use the substitution of (27) and (28), simultaneously introducing the designations

$$
\left.\begin{aligned}
p^{\xi(\tau_0)}_{ij} &= d^{\xi(\tau_0)}_{ij} - m_{ij} = -\frac{\alpha^{\xi(\tau_0)}_{ij}}{\operatorname{sh}\alpha^{\xi(\tau_0)}_{ij}}\, m_{ij}; \quad p^{c\xi(\tau_0)}_{ij} = p^{\xi(\tau_0)}_{ij}\cos\varphi_{ij};\\
p^{\xi(\tau_0)}_{jk} &= d^{\xi(\tau_0)}_{jk} - m_{jk} = -\frac{\alpha^{\xi(\tau_0)}_{jk}}{\operatorname{sh}\alpha^{\xi(\tau_0)}_{jk}}\, m_{jk}; \quad p^{s\xi(\tau_0)}_{ij} = p^{\xi(\tau_0)}_{ij}\sin\varphi_{ij};\\
q^{\xi(\tau_0)}_{ij} &= l^{\xi(\tau_0)}_{ij} + m_{ij} = \frac{\alpha^{\xi(\tau_0)}_{ij}}{\operatorname{th}\alpha^{\xi(\tau_0)}_{ij}}\, m_{ij}; \quad q^{c\xi(\tau_0)}_{ij} = q^{\xi(\tau_0)}_{ij}\cos\varphi_{ij};\\
q^{\xi(\tau_0)}_{jk} &= l^{\xi(\tau_0)}_{jk} + m_{jk} = \frac{\alpha^{\xi(\tau_0)}_{jk}}{\operatorname{th}\alpha^{\xi(\tau_0)}_{jk}}\, m_{jk}; \quad q^{s\xi(\tau_0)}_{ij} = q^{\xi(\tau_0)}_{ij}\sin\varphi_{ij}.
\end{aligned}\right| \qquad (39)
$$

As a result we will obtain for plane xOz

Inset

...	...	$M_{xz-1,z}$	$M_{x\ldots}$	β_{xz}	β_{y0}	M_{y00}	M_{y11}	M_{y22}	...	...	$M_{yz-1,z}$	$M_{yz-1,z-1}$	β_{yz}
						$-l_{01}^{s\eta}$	$-d_1^{s\eta}$						
						$-d_{01}^{s\eta}$	$-l_1^{s\eta}$	$-d_2^{s\eta}$					
$d_3^{c\xi}$							$-d_{12}^{s\eta}$	$-l_2^{s\eta}$	$-d_2^{s\eta}$				
...	...							...	...	...			
$d_{z-3,z-2}^{c\xi}$	$l_{z-2}^{c\xi}$	$d_{z-1}^{c\xi}$							$-d_{z-3,z-2}^{s\eta}$	$-l_{z-2}^{s\eta}$	$-d_{z-1}^{s\eta}$		
	$d_{z-2,z-1}^{c\xi}$	$l_{z-1}^{c\xi}$	$d_{z-1,z}^{c\xi}$							$-d_{z-2,z-1}^{s\eta}$	$-l_{z-1}^{s\eta}$	$-d_z^{s\eta}$	
		$d_{z-1,z}^{c\xi}$	$l_{z-1,z}^{c\xi}$	-1							$-d_{z-1,z}^{s\eta}$	$-l_{z-1,z}^{s\eta}$	
					1	$l_{01}^{c\eta}$	$d_1^{c\eta}$						
						$d_{01}^{c\eta}$	$l_1^{c\eta}$	$d_2^{c\eta}$					
$d_3^{s\xi}$							$d_{12}^{c\eta}$	$l_2^{c\eta}$	$d_3^{c\eta}$				
...	...							...	...	...			
$d_{z-3,z-2}^{s\xi}$	$l_{z-2}^{s\xi}$	$d_{z-1,z}^{s\xi}$							$d_{z-1,z-2}^{c\eta}$	$l_{z-2}^{c\eta}$	$d_{z-1}^{c\eta}$		
	$d_{z-2,z-1}^{s\xi}$	$l_{z-1}^{s\xi}$	$d_{z-1,z}^{s\xi}$							$d_{z-2,z-1}^{c\eta}$	$l_{z-1}^{c\eta}$	$d_{z-1,z}^{c\eta}$	
		$d_{z-1,z}^{s\xi}$	$l_{z-1,z}^{s\xi}$								$d_{z-1,z}^{c\eta}$	$l_{z-1,z}^{c\eta}$	$-1^{c\eta}$

(35)

U_{x0}	U_{x1}	U_{x2}	U_{x3}	...	$U_{x,z-1}$	U_{xz}	U_{y0}	U_{y1}	U_{y2}	U_{y3}	...	$U_{y,z-1}$	U_{yz}
	$-(\varphi^2+\omega^2)t_{01}$	$-(\varphi^2+\omega^2)t_{02}$	$-(\varphi^2+\omega^2)t_{03}$		$-(\varphi^2+\omega^2)t_{0,z-1}$	$-(\varphi^2+\omega^2)t_{0z}$							
	$-(\varphi^2+\omega^2)S_1$	$-(\varphi^2+\omega^2)t_{12}$	$-(\varphi^2+\omega^2)t_{13}$		$-(\varphi^2+\omega^2)t_{1,z-1}$	$-(\varphi^2+\omega^2)t_{1z}$							
		$-(\varphi^2+\omega^2)S_2$	$-(\varphi^2+\omega^2)t_{23}$		$-(\varphi^2+\omega^2)t_{2,z-1}$	$-(\varphi^2+\omega^2)t$							
				$-(\varphi^2+\omega^2)S_{z-2}$	$-(\varphi^2+\omega^2)t_{z-2,z-1}$	$-(\varphi^2+\omega^2)t_{z-2,z}$							
					$-(\varphi^2+\omega^2)t_{z-1}$	$-(\varphi^2+\omega^2)t_{z-1,z}$							
						$-(\varphi^2+\omega^2)S_z$							
								$-\rho^2 t_{01}$	$-\rho^2 t_{02}$	$-\rho^2 t_{03}$	...	$-\rho^2 t_{0,z-1}$	$-\rho^2 t_{0z}$
								$-\rho^2 S_1$	$-\rho^2 t_{12}$	$-\rho^2 t_{13}$	...	$-\rho^2 t_{1,z-1}$	$-\rho^2 t_{1z}$
									$-\rho^2 S_2$	$-\rho^2 t_{23}$	...	$-\rho^2 t_{2,z-1}$	$-\rho^2 t_{2z}$
											$-\rho^2 t_{z-2,z}$	$-\rho^2 t_{z-2,z-1}$	$-\rho^2 t_{z-2,z}$
												$-\rho^2 S_{z-1}$	$-\rho^2 t_{z-1,z}$
													$-\rho^2 S_z$

(45)

Inset

[illegible]	U_{x0}	U_{x1}	U_{x2}	U_{x3}	. . .	$U_{x,z-1}$	U_{xz}	U_{y0}	U_{y1}	U_{y2}	U_{y3}	. . .	$U_{y,z-1}$	U_{yz}	β_{x0}	$M_{\eta 00}$	$M_{\eta 11}$	M
1_x	$-b_{01}$	b_{01}													1	$l_{01}^{c\xi}$	$d_1^{c\xi}$	
2_x	b_{01}	a_1	b_{12}													$d_{01}^{c\xi}$	$l_1^{c\xi}$	a
3_x		b_{12}	a_2	b_{23}													$d_{12}^{c\xi}$	l
. . .			b_{23}	a_3	b_{34}												⋮	.
$(Z-1)_x$				. . .	. . .	. . .												
Z_x				$b_{z-2,z-1}$	a_{z-1}	$b_{z-1,z}$												
$(Z+1)_x$						$b_{z-1,z}$	$-b_{z-1,z}$											
1_y								$-b_{01}$	b_{01}							$l^{s\xi}$	$d^{s\xi}$	
2_y								b_{01}	a_1	b_{12}						$d_{01}^{s\xi}$	$l_1^{s\xi}$	a
3_y									b_{12}	a_2	b_{23}						$d_{12}^{s\xi}$	l
. . .										b_{23}	a_3	b_{34}						.
$(Z-1)_y$											. . .	. . .	. . .					
Z_y												$b_{z-2,z-1}$	a_{z-1}	$b_{z-1,z}$				
$(Z+1)_y$													$b_{z-1,z}$	$-b_{z-1,z}$				

Nуравн.	β_{x0}	$M_{\eta 00}$	$M_{\eta 11}$	$M_{\eta 22}$	. . .	. . .	$M_{\eta z-1,z-1}$	$M_{\eta z-1,z}$	β_{xz}	β_{y0}	$M_{\xi 00}$	$M_{\xi 11}$	$M_{\xi 22}$	. . .	. . .	$M_{\xi z-1,z-1}$	$M_{\xi z-1,z}$	β_{yz}
1_x	1	$g_{01}^{c\xi}$	$p_1^{c\xi}$								$-g_{01}^{s\eta}$	$-p_1^{s\eta}$						
2_x		$p_{01}^{c\xi}$	$g_1^{c\xi}$	$p_2^{c\xi}$							$-p_{01}^{s\eta}$	$-g_1^{s\eta}$	$-p_2^{s\eta}$					
3_x			$p_{12}^{c\xi}$	$g_{23}^{c\xi}$	$p_3^{c\xi}$								$-g_2^{s\eta}$	$-p_3^{s\eta}$				
$(Z-1)_x$					$p_{z-3,z-2}^{c\xi}$	$g_{z-1}^{c\xi}$	$p_{z-1}^{c\xi}$							$-p_{z-3}^{s\eta}$	$-g_{z-2}^{s\eta}$	$-p_{z-1}^{s\eta}$		
Z_x						$p_{z-2,z-1}^{c\xi}$	$g_{z-1}^{c\xi}$	$p_{z-1,z}^{c\xi}$							$-p_{z-2,z-1}^{s\eta}$	$-g_{z-2,z-1}^{s\eta}$	$-p_{z-1,z}^{s\eta}$	
$(Z+1)_x$							$p_{z-1,z}^{c\xi}$	$g_{z-1,z}^{c\xi}$	-1							$-p_{z-1,z}^{s\eta}$	$g_{z-1,z}^{s\eta}$	
1_y		[illegible]	[illegible]							1	$g_{01}^{c\eta}$	$p_1^{c\eta}$						
2_y		p_{01}	$g_1^{s\xi}$	$p_2^{s\xi}$							$p_{01}^{c\eta}$	$g_1^{c\eta}$	$p_2^{c\eta}$					
3_y			$p_{12}^{s\xi}$	$g_2^{s\xi}$	$p_3^{s\xi}$							$p_{12}^{c\eta}$	$g^{c\eta}$	$p_3^{c\eta}$				
$(Z-1)_y$					$p_{z-3,z-2}^{s\xi}$	$g_{z-2}^{s\xi}$	$p_{z-1}^{s\xi}$							$p_{z-3,z-2}^{c\eta}$	$g_{z-2}^{c\eta}$	$p_{z-1}^{c\eta}$		
Z_y						$p_{z-2,z-1}^{s\xi}$	$g_{z-1}^{s\xi}$	$p_{z,z-1}^{s\xi}$							$p_{z-2,z-1}^{c\eta}$	$g_{z-1}^{c\eta}$	$p_{z-1,z}^{c\eta}$	
$(Z+1)_y$							$p_{z-1,z}^{s\xi}$	$g_{z-1,z}^{s\xi}$								$p_{z-1,z}^{c\eta}$	$g_{z-1,z}^{c\eta}$	-1

$$
\begin{aligned}
&M_{\eta ii} P^{c\xi}_{ij} + M_{\eta jj}(q^{c\xi}_{jk} + q^{c\xi}_{ij} \cos \Delta\varphi_j - q^{s\eta}_{ij} \sin \Delta\varphi_j) + \\
&+ M_{\eta kk}(P^{c\xi}_{jk} \cos \Delta\varphi_k - P^{s\eta}_{jk} \sin \Delta\varphi_k) - M_{\xi ii} P^{s\eta}_{ij} - \\
&- M_{\xi jj}(q^{s\eta}_{jk} + q^{s\eta}_{ij} \cos \Delta\varphi_j + q^{c\xi}_{ij} \sin \Delta\varphi_j) - M_{\xi kk}(P^{s\eta}_{jk} \cos \Delta\varphi_k + P^{c\xi}_{jk} \sin \Delta\varphi_k) - \\
&= \frac{Q_{xjk}}{N_{jk}} - \frac{Q_{xij}}{N_{ij}}.
\end{aligned}
\tag{40a}
$$

Analogous transformations will give for plane yOz

$$
\begin{aligned}
&M_{\eta ii} P^{s\xi}_{ij} + M_{\eta jj}(q^{s\xi}_{jk} + q^{s\xi}_{ij} \cos \Delta\varphi_j - q^{c\eta}_{ij} \sin \Delta\varphi_j) + \\
&+ M_{\eta kk}(P^{s\xi}_{jk} \cos \Delta\varphi_k + P^{c\eta}_{jk} \sin \Delta\varphi_k) + M_{\xi ii} P^{c\eta}_{ij} + \\
&+ M_{\xi jj}(q^{c\eta}_{jk} + q^{c\eta}_{ij} \cos \Delta\varphi_j - q^{s\xi}_{ij} \sin \Delta\varphi_j) + M_{\xi kk}(P^{c\eta}_{jk} \cos \Delta\varphi_k - P^{s\xi}_{jk} \sin \Delta\varphi_k) = \\
&= \frac{Q_{yjk}}{N_{jk}} - \frac{Q_{yij}}{N_{ij}}.
\end{aligned}
\tag{40b}
$$

Thus, equations were obyained in general form, which connect with each other the flexural moments acting at the three neighboring in the planes of the least and greatest rigidity, with the shearing forces at two neighboring sections of the blade (ij and jk) and directed along the axes of the fixed coordinate system.

In examining the special root and tip sections of the blade, it is not difficult to obtain expressions connecting the flexural moments at the two neighboring points with the shearing force acting in the root and tip sections.

Let us introduce designations, which simplify the notation:

$$
\left.
\begin{aligned}
&q_j^{\xi'} = q_{jk}^{\xi'} + q_{ij}^{\xi'} \cos \Delta\varphi_j - q_{ij}^{\eta'} \sin \Delta\varphi_j;\\
&P_k^{\xi'} = P_{jk}^{\xi'} \cos \Delta\varphi_k - P_{jk}^{\eta'} \sin \Delta\varphi_k;\\
&q_j^{\eta'} = q_{jk}^{\eta'} + q_{ij}^{\eta'} \cos \Delta\varphi_j + q_{ij}^{\xi'} \sin \Delta\varphi_j;\\
&P_k^{\eta'} = P_{ij}^{\eta'} \cos \Delta\varphi_k + P_{jk}^{\xi'} \sin \Delta\varphi_k;\\
&q_j^{\xi''} = q_{jk}^{\xi''} + q_{ij}^{\xi''} \cos \Delta\varphi_j + q_{ij}^{\eta''} \sin \Delta\varphi_j;\\
&P_k^{\xi''} = P_{jk}^{\xi''} \cos \Delta\varphi_k + P_{jk}^{\eta''} \sin \Delta\varphi_k;\\
&q_j^{\eta''} = q_{jk}^{\eta''} + q_{ij}^{\eta''} \cos \Delta\varphi_j - q_{ij}^{\xi''} \sin \Delta\varphi_j;\\
&P_k^{\eta''} = P_{jk}^{\eta''} \cos \Delta\varphi_k - P_{jk}^{\xi''} \sin \Delta\varphi_k.
\end{aligned}
\right| \qquad (41)
$$

As a result we obtain a system of (z+1) pairs of euations written in the general form, which contain (z+1) pairs of flexural moments acting in the planes of the greatest and least rigidity, and two pairs of components of the first derivative of the elastic line in the general coordinate system xyz for the zero and the last point of the blade. The equations for the first and the last sections, which contain these components, differ somewhat in structure from the remaining equations and take the boundary conditions into account. These equations are written out below:

the general expression of the j-th pair of equations is

$$
\left.
\begin{aligned}
&M_{\eta ii} P_{ij}^{\xi'} + M_{\eta jj} q_j^{\xi'} + M_{\eta kk} P_k^{\xi'} - M_{\xi ii} P_{ij}^{\eta'} - M_{\xi jj} q_j^{\eta'} - M_{\xi kk} P_k^{\eta'} =\\
&\qquad\qquad - \frac{Q_{xjk}}{N_{jk}} - \frac{Q_{xij}}{N_{ij}};\\
&M_{\eta ii} P_{ij}^{\xi''} + M_{\eta jj} q_j^{\xi''} + M_{\eta kk} P_k^{\xi''} + M_{\xi ii} P_{ij}^{\eta''} + M_{\xi jj} q_j^{\eta''} +\\
&\qquad\qquad + M_{\xi kk} P_k^{\eta''} = \frac{Q_{yjk}}{N_{jk}} - \frac{Q_{yij}}{N_{ij}};
\end{aligned}
\right| \qquad (42a)
$$

the equations for the root section are

$$\left.\begin{aligned}
&\beta_{x0} + M_{\eta 00}\, q_{01}^{c\xi} + M_{\eta 11}\, P_1^{c\eta} - M_{\xi 00}\, q_{01}^{s\eta} - M_{\xi 11}\, P_1^{s\eta} = \frac{Q_{x01}}{N_{01}}\,;\\
&\beta_{y0} - M_{\eta 00}\, q_{01}^{s\xi} + M_{\eta 11}\, P_1^{s\xi} + M_{\xi 00}\, q_{01}^{c\eta} + M_{\xi 11}\, P_1^{c\eta} = \frac{Q_{y01}}{N_{01}}\,;
\end{aligned}\right\} \qquad (42b)$$

the equations for the terminal (tip) section are

$$\left.\begin{aligned}
&-\beta_{xz} + M_{\eta z-1,\,z-1}\, P_{z-1,\,z}^{c\xi} + M_{\eta z-1,\,z}\, q_{z-1,\,z}^{c\xi} - M_{\xi z-1,\,z-1}\, P_{z-1,\,z}^{s\eta} -\\
&\qquad - M_{\xi z-1,\,z}\, q_{z-1,\,z}^{s\eta} = -\frac{Q_{xz-1,\,z}}{N_{z-1,\,z}}\,;\\
&-\beta_{yz} + M_{\eta z-1,\,z-1}\, P_{z-1,\,z}^{s\xi} + M_{\eta z-1,\,z}\, q_{z-1,\,z}^{s\xi} + M_{\xi z-1,\,z-1}\, P_{z-1,\,z}^{c\eta} -\\
&\qquad - M_{\xi z-1,\,z}\, q_{z-1,\,z} = -\frac{Q_{yz-1,\,z}}{N_{z-1,\,z}}\,.
\end{aligned}\right\} \qquad (42c)$$

Since

$$Q_{xij} = -\sum_{j}^{z} m_i\,(\ddot{U}x_i - \omega^2 Ux_i); \quad Q_{yij} = -\sum_{j}^{z} m_i\,\ddot{U}y_i,$$

then these are differential equations. It is simple to convert them into algebraic equations, representing the solution in the form

$$\begin{aligned}
&Ux_i(t) = Ux_i \sin pt; \quad M_{\eta i}(t) = M_{\eta i} \sin pt;\\
&Uy_i(t) = Uy_i \sin pt; \quad M_{\xi i}(t) = M_{\xi i} \sin pt;\\
&[illegible]
\end{aligned}$$

where [illegible] ; and [illegible] are amplitudes of the value of the functions. Having differentiated twice we will obtain

$$Qx_{ij} = (p^2 + \omega^2) \sum_{j}^{z} m_i\, Ux_i; \quad Q_{yij} = p^2 \sum_{j}^{z} m_i\, Uy_i$$

Such a system is written in matrix form (43) (see the inset). It makes it possible from the known values of the shearing forces to find the magnitudes of the flexuralmoments in the planes of the greatest and least rigidity and the components of the first derivative of the elastic line with respect to the ends of the blade in a fixed coordinate system.

The General Sequence of Calculation

The finite element method with iterations developed in the present work is structured to a certain extent analogous to other methods for the calculation of frequencies and forms of free oscillations. However, due to a number of special features it is advisable to state it in the most general form.

The approximations are accomplished in the following order.

The shape of the oscillations for the calculation of the first approximation is assumed determined by the functions $U_{yi}^{(0)}$ and $U_{xi}^{(0)}$. The form is selected in such a manner so that $U_{yz}^{(0)} = U_{xz}^{(0)} = 1$. It is simultaneously assumed that the frequency of the natural oscillations of the firstappoximation is $p_{(1)}^{2(0)} = 1$. Then the shearing forces located on the right side are determined respectively for the planes xOz and yOz as

$$Q_{xij} = (1 + \omega^2)\sum_{j}^{z} m_i U_{xi}; \quad Q_{yij} = 1 \sum_{j}^{z} m_i U_{yi}.$$

The flexural moments in the planes of the greatest and least rigidity and the components of the first derivative of the elastic line in the root (β_{x0} and β_{y0}) are determined from the solution of system (43).

Then the displacements U_{yi} and $U_{xi}^{(1)}$ and the frequency of the natural oscillations of the first approximation are determined from system (35)

$$p_{y\,(1)}^{2\,(1)}=\frac{1}{U_{yz}^{(1)}}\quad \text{or}\quad p_{x\,(1)}^{2\,(1)}=\frac{1}{U_{xz}^{(1)}}-\omega^2.$$

The form of the natural oscillations is then determined after the first approximation:

$$\overline{U}_{yi\,(1)}^{(1)}=U_{yi\,(1)}^{(1)}\,p_{y\,(1)}^{2\,(1)},\quad \text{здесь}\quad \overline{U}_{yz\,(1)}^{(1)}=1;$$

$$\overline{U}_{xi\,(1)}^{(1)}=U_{xi}^{(1)}\,(p_{y\,(1)}^{2\,(1)}+\omega^2),\quad \overline{U}_{xi\,(1)}^{(1)}\neq 1$$

or

$$\overline{U}_{yi\,(1)}^{(1)}=U_{yi}^{(1)}\,p_{x\,(1)}^{2\,(1)},\quad \text{здесь}\quad \overline{U}_{yz\,(1)}^{(1)}\neq 1;$$

$$U_{xi\,(1)}^{(1)}=U_{xi}^{(1)}\,(p_{x\,(1)}^{2\,(1)}+\omega^2),\ \text{причем}\ \overline{U}_{xi\,(1)}^{(1)}=1.$$

Then functions $\overline{U}_{yi}^{(1)}$ and $\overline{U}_{xi}^{(1)}$ are again substituted in system (43) and the cycle of the second approximation is repeated. These cycles are further repeated until the specified accuracy is obtained.

The method of successive approximations employed here leads to the obtaining of the form of the natural oscillations with the lowest frequency. Let us call it the frequency of the first tone, having designated the final form and frequency in the following manner

[illegible]

The determination of the form and the frequency of the natural oscillations of the second tone is the next step. The process of successive approximations is repeated as previously, with only one difference, that it is necessary to fulfill the condition of orthogonality of the form of the second tone to the form of the first. This condition is written in the following manner:

$$\sum_{i=0}^{i=z} m_i\,\overline{U}_{xi\,(1)}^{(k)}\,x_{i\,(1)}+\sum_{i=0}^{i=z} m_i\,\overline{U}_{yi\,(1)}^{(k)}\,y_{i\,(y)}=0,$$

where $U^{(k)}_{xi(11)}$; $U^{(k)}_{yi(11)}$ are functions of the displacements of the k-th approximation of the second tone. Hence, the functions of the displacements are written as

$$\overline{U}^{(k)}_{xi(11)} = \tilde{U}^{(k)}_{xi(11)} - C^{(k)}_{21} x_{i(1)}; \quad \overline{U}_{yi(11)} = \tilde{U}^{(k)}_{yi(11)} - C^{(k)}_{21} y_{i(1)},$$

where $\tilde{U}^{(k)}_{xi(11)}$ and $\tilde{U}^{(k)}_{yi(11)}$ are functions of the displacements obtained from the solution of system (35);

$$C^{(k)}_{21} = \frac{\sum\limits_{i=0}^{i=s} m_i \tilde{U}^{(k)}_{xi(11)} x_{i(1)} + \sum\limits_{i=0}^{i=s} m_i \tilde{U}^{(k)}_{yi(11)} y_{i(1)}}{\sum\limits_{i=0}^{i=s} m_i x^2_{i(1)} + \sum\limits_{i=0}^{i=s} m_i y^2_{i(1)}}.$$

The constant C_{21} (the constant of the orthagonalization of the second tone to the first) varies with each approximation.

$$p^{2(k)}_{x(11)} = \frac{1}{\tilde{U}^{(k)}_{xs(11)} - C^{(k)}_{21}} - \omega^2 \quad \text{or} \quad p^{2(k)}_{y(11)} = \frac{\tilde{u}^{(k-1)}_{ys(11)}}{\tilde{U}^{(k)}_{ys(11)} - C_{21} y^{(k)}_{s(1)}},$$

if the form of the oscillations of the first tone is such that $\overline{U}_{xs(1)} = 1$, and

$$p^{2(k)}_{x(11)} = \frac{\tilde{U}^{(k-1)}_{xs(11)}}{\tilde{U}^{(k)}_{xs(11)} - C^{(k)}_{21} x_{s(1)}} - \omega^2 \text{ or } p^{2(k)}_{y(11)} = \frac{1}{\tilde{U}^{(k)}_{ys(11)} - C^{(k)}_{21}},$$

if the form of the oscillations of the first tone is such that $\overline{U}_{ys(1)} = 1$.

Generally speaking, the process of successive approximations must be structured in such a manner that the form of the oscillations determined by the smallest of the two values of frequencies obtained after each approximation was established in the next approximation. This is necessary so as not to omit any form of the oscillations, since cases are possible, when

$$\tilde{U}^{(k)}_{xs(11)} = C^{(k)}_{21}$$

or

$$\tilde{U}^{(k)}_{ys(11)} = C^{(k)}_{21}.$$

The form of the natural oscillations of the k-th approximation of the second tone is defined by the expressions

$$\overline{U}^{(k)}_{xi(II)} = (\tilde{U}^{(k)}_{xi(II)} - C^{(k)}_{21} x_{i(I)})(p^{2(k)}_{y(II)} + \omega^2);$$
$$\overline{U}^{(k)}_{yi(II)} = (\tilde{U}^{(k)}_{yi(II)} - C^{(k)}_{21} y_{i(I)})\, p^{2(k)}_{y(II)},$$

where p^2_y means that normalization occurred with respect to the function of the displacements in plane yOz

[illegible]

$$\overline{U}^{(k)}_{yi(II)} = (\tilde{U}^{(k)}_{yi(II)} - C^{(k)}_{21} x_{i(I)})\, p^{2(k)}_{x(II)}.$$

where p^2_x means that normalization was carried out with respect to the function of the displacements in plane xOz.

The form of the natural oscillations of the second tone obtained with the specified accuracy is designated in this manner:

$$U^{(n)}_{xi(II)} = x_{i(II)},$$
$$U^{(n)}_{yi(II)} = y_{i(II)}.$$

Then the frequencies and the forms of the natural oscillations of the next tones are determined.

The alternation of operations remains the same as in the calculation of the second tone. The difference is, that for the N-th tone the function of the displacement for the k-th approximation is written as

$$\overline{U}^{(k)}_{xi(N)} = \left[\tilde{U}^{(k)}_{xi(N)} - \sum_{M=1}^{M=N-1} C^{(k)}_{NM}\, x_{i(M)}\right](p^{2(k)}_{y(N)} + \omega^2);$$
$$\overline{U}^{(k)}_{yi(N)} = \left[\tilde{U}^{(k)}_{yi(N)} - \sum_{M=1}^{M=N-1} C^{(k)}_{NM}\, y_{i(M)}\right] p^{2(k)}_{y(N)};$$

the constant of orthogonalization is

$$C^{(k)}_{NM} = \frac{\sum_{i=0}^{i=s} m_i \tilde{U}^{(k)}_{x(N)} x_{i(M)} + \sum_{i=0}^{i=s} m_i \tilde{U}^{(k)}_{y(N)} y_{i(M)}}{\sum_{i=0}^{i=s} m_i x^2_{i(M)} + \sum_{i=0}^{i=s} m_i y^2_{i(M)}};$$

the frequency of the natural frequencies is

$$p_{x\,(N)}^{2\,(k)}=\frac{1}{\tilde{U}_{xz\,(N)}^{(k)}-\sum\limits_{M=1}^{M=N-1}C_{MN}^{(k)}}-\omega^2 \text{ or } \quad p_{y\,(N)}^{2\,(k)}=\frac{\overline{U}_{yz\,(N)}^{(k-1)}}{\tilde{U}_{yz\,(N)}^{(k)}-\sum\limits_{M=1}^{M=N-1}C_{NM}^{(k)}y_{z\,(M)}},$$

if the forms of the oscillations are such that $U_{xz\,(N)}=1$, or

$$p_{x\,(N)}^{2\,(k)}=\frac{\overline{U}_{xz\,(N)}^{(k-1)}}{\tilde{U}_{xz\,(N)}^{(k)}-\sum\limits_{M=1}^{M=N-1}C_{NM}^{(k)}x_{z\,(M)}}-\omega^2;$$

$$p_{y\,(N)}^{2\,(k)}=\frac{1}{\tilde{U}_{yz\,(N)}^{(k)}-\sum\limits_{M=1}^{M=N-1}C_{NM}^{(k)}},$$

if the forms of the oscillations are such that $U_{yz\,(N)}$ = 1.

We will point out below certain of the details in carrying out the operations in determining the flexural moments and values of [illegible] and [illegible].

Determining the Flexural Moments and Components of the First Derivative of the Elastic line

..........(First two lines are illegible).........................

1) rigid fastening with flexure in two mutually perpendicular planes:

2) rigid fastening in the plane of rotation and hinged support in the perpendicular plane:

3) hinged attachment in both planes (in this case the hinges can be located at different distances from the site of the attachment of the blade).

By analogy with a helicopter rotor we will call these hinges horizontal hinges and vertical hinges.

Naturally, the different conditions of attachment introduce differences into system (43), which is written in general form so that it would be possible to examine the case of simultaneous attachment in two planes or simultaneous hinged attachment in these same planes.

Let us first examine the simpler and more general case of a blade completely fastened at the root.

The boundary conditions in the root are: $y_0^x = 0;\ y_0^y = 0.$

$$\sum_{i=0}^{i=s} m_i U_{xi} = 0;\quad \sum_{i=0}^{i=s} m_i U_{yi} = 0.$$

Generally, system (43) represents on the left side four three-band submatrices, joined by common unknowns. For the solution we employed the Gauss method used by A. V. Nekrasov in [3] for solving a system representing one of the indicated submatrices.

The algorithm is shown below, by which the matrix is resolved, with the necessary explanations prefaced to it.

Let us examine system of equations 1_x and 1_y [see (43) in the inset]. Since the blade is fastened ($\beta_{x0} = \beta_{y0} = 0$), it is possible to write

$$\left.\begin{aligned} &M_{\eta,00} g_{0i}^{cx} + M_{\eta 11} P_1^{cx} - M_{\xi 00} g_{0i}^{cy} - M_{\xi 11} P_1^{cy} = F_0^x; \\ &M_{\eta,00} g_{0i}^{cx} + M_{\eta 11} P_1^{cx} + M_{\xi 00} g_{0i}^{cy} + M_{\xi 11} P_1^{cy} = F_0^y, \end{aligned}\right\} \qquad (44)$$

where

$$F_0^x = -(p^2+\omega^2)\sum_{i=1}^{i=z} t_{0i} U_{xi}; \quad t_{0i} = m_i\left(-\frac{1}{N_{0i}}\right);$$

$$F_0^y = -p^2 \sum_{i=1}^{i=z} t_{0i} U_{yi}.$$

Let us represent the flexural moments in the following manner:

$$\left.\begin{aligned} M_{\nu,00} &= \gamma_{\nu,0} + \bar{\delta}_{\nu,0} M_{\nu,11} + \bar{\delta}_{\xi 0} M_{\xi 11}; \\ M_{\xi 00} &= \gamma_{\xi 0} + \bar{\bar{\delta}}_{\nu,0} M_{\nu,11} + \bar{\bar{\delta}}_{\xi 0} M_{\xi 11}, \end{aligned}\right| \tag{45}$$

where

$$\left.\begin{aligned} \bar{\delta}_{\nu,0} &= -\frac{P_1^{c\xi} g_{01}^{c\nu} + P_1^{s\xi} g_{01}^{s\nu}}{g_{01}^{c\xi} g_{01}^{c\nu} + g_{01}^{s\xi} g_{01}^{s\nu}}; & \bar{\bar{\delta}}_{\nu,0} &= -\frac{P_1^{s\xi} g_{01}^{c\xi} - P_1^{c\xi} g_{01}^{s\xi}}{g_{01}^{c\xi} g_{01}^{c\nu} + g_{01}^{c\xi} g_{01}^{s\nu}}; \\ \bar{\delta}_{\xi 0} &= \frac{P_1^{s\nu} g_{01}^{c\nu} - P_1^{c\nu} g_{01}^{s\nu}}{g_{01}^{c\xi} g_{01}^{c\nu} + g_{01}^{s\xi} g_{01}^{s\nu}}; & \bar{\bar{\delta}}_{\xi 0} &= \frac{P_1^{c\nu} g_{01}^{c\xi} - P_1^{c\xi} g_{01}^{s\xi}}{g_{01}^{c\xi} g_{01}^{c\nu} + g_{01}^{s\xi} g_{01}^{s\nu}}; \\ \gamma_{\nu 0} &= \frac{g_{01}^{c\nu} F_0^y + g_{01}^{s\nu} F_0^x}{\text{[illegible]}}; & \gamma_{\xi 0} &= \frac{g_{01}^{c\xi} F_0^y - g_{01}^{s\xi} F_0^x}{\text{[illegible]}} \end{aligned}\right| \tag{46}$$

Then let us examine the system of equations:

$$\left.\begin{aligned} M_{\nu,00} P_{01}^{c\nu} + M_{\nu,11} g_1^{c\nu} + M_{\nu,22} P_2^{c\nu} - M_{\xi 00} P_{01}^{s\nu} - M_{\xi 11} g_1^{s\nu} - M_{\xi 22} P_2^{s\nu} &= F_1^x; \\ M_{\nu,00} P_{01}^{c\xi} + M_{\nu,11} g_1^{c\xi} + M_{\nu,22} P_2^{c\xi} + M_{\xi 00} P_{01}^{s\xi} + M_{\xi 11} g_1^{s\xi} + M_{\xi 22} P_2^{s\xi} &= F_1^y, \end{aligned}\right| \tag{47}$$

where

$$F_1^x = -(p^2+\omega^2)\left(\sum_{i=2}^{i=z} t_{1i} x_i + S_1 x_1\right); \quad t_{1i} = m_i\left(\frac{1}{N_{1i}} - \frac{1}{N_{0i}}\right); \quad S_1 = m_1\frac{1}{N_{01}};$$

$$F_1^y = -p^2\left(\sum_{i=2}^{i=z} t_{1i} x_i + S_1 x_1\right).$$

If we then substitute (45) in (47) and carry out grouping, then it is possible to obtain an expression analogous to expression (44);

$$\left.\begin{aligned} M_{\nu,11} g_1^{c\nu*} + M_{\nu,22} P_2^{c\nu} - M_{\xi 11} g_1^{s\nu*} - M_{\xi 22} P_2^{s\nu} &= F_1^{x*}; \\ M_{\nu,11} g_1^{c\xi*} + M_{\nu,22} P_2^{c\xi} + M_{\xi 11} g_1^{s\xi*} + M_{\xi 22} P_2^{s\xi} &= F_1^{y*}. \end{aligned}\right| \tag{48}$$

where

$$
\left.\begin{array}{l}
g_1^{c\xi *} = g_1^{c\xi} + P_{01}^{c\xi}\bar{\delta}_{\eta,0} - P_{01}^{s\eta}\bar{\bar{\delta}}_{\eta,0}; \quad g_1^{s\eta *} = g_1^{s\eta} - P_{01}^{c\xi}\bar{\delta}_{\xi 0} + P_{01}^{s\eta}\bar{\bar{\delta}}_{\xi 0}; \\
g_1^{s\xi *} = g_1^{s\eta} + P_{01}^{s\xi}\bar{\delta}_{\eta,0} + P_{01}^{c\eta}\bar{\bar{\delta}}_{\eta,0}; \quad g_1^{c\eta *} = g_1^{c\eta} + P_{01}^{s\xi}\bar{\delta}_{\xi 0} + P_{01}^{c\eta}\bar{\bar{\delta}}_{\xi 0}; \\
F_1^{x *} = F_1^{x} - P_{01}^{c\xi}\gamma_{\eta,0} + P_{01}^{s\eta}\gamma_{\xi 0}; \quad F_1^{y *} = F_1^{y} - P_{01}^{s\xi}\gamma_{\eta,0} - P_{01}^{c\eta}\gamma_{\xi 0}.
\end{array}\right| \qquad (49)
$$

These operations are sequentially repeated for each pair of equations. For the j-th pair they are written in the following manner:

$$
\left.\begin{array}{l}
g_j^{c\xi *} = g_j^{c\xi} + P_{ij}^{c\xi}\bar{\delta}_{\eta,i} - P_{ij}^{s\eta}\bar{\bar{\delta}}_{\eta,i}; \quad g_j^{s\eta *} = g_j^{s\eta} - P_{ij}^{c\xi}\bar{\delta}_{\xi i} - P_{ij}^{s\eta}\bar{\bar{\delta}}_{\xi i}; \\
g_j^{s\xi *} = g_j^{s\eta} + P_{ij}^{s\xi}\bar{\delta}_{\eta,i} + P_{ij}^{c\eta}\bar{\bar{\delta}}_{\eta,i}; \quad g_j^{c\eta *} = g_j^{c\eta} + P_{ij}^{s\xi}\bar{\delta}_{\xi i} + P_{ij}^{c\eta}\bar{\bar{\delta}}_{\xi i}; \\
F_j^{x *} = F_j^{x} - P_{ij}^{c\xi}\gamma_{\eta,i} + P_{ij}^{s\eta}\gamma_{\xi i}; \quad F_j^{y *} = F_j^{y} - P_{ij}^{s\xi}\gamma_{\eta,i} - P_{ij}^{c\eta}\gamma_{\xi i},
\end{array}\right\} \qquad (50)
$$

where

$$
\left.\begin{array}{l}
\bar{\delta}_{\eta,i} = -\dfrac{P_j^{c\xi} g_i^{c\eta *} + P_j^{s\xi} g_i^{s\eta *}}{g_i^{c\xi *} g_i^{c\eta *} + g_i^{s\xi *} g_i^{s\eta *}}; \quad \bar{\bar{\delta}}_{\eta,i} = -\dfrac{P_j^{s\xi} g_i^{c\xi *} - P_j^{c\xi} g_i^{s\xi *}}{g_i^{c\xi *} g_i^{c\eta *} + g_i^{s\xi *} g_i^{s\eta *}}; \\
\bar{\delta}_{\xi i} = \dfrac{P_j^{s\eta} g_i^{c\eta *} - P_j^{c\eta} g_i^{s\eta *}}{g_i^{c\xi *} g_i^{c\eta *} + g_i^{s\xi *} g_i^{s\eta *}}; \quad \bar{\bar{\delta}}_{\xi i} = -\dfrac{P_i^{c\eta} g_i^{c\xi *} + P_j^{s\eta} g_i^{s\xi *}}{q_i^{c\xi *} q_i^{c\eta *} + q_i^{s\xi *} q_i^{s\eta *}}; \\
\gamma_{\eta,i} = \dfrac{q_i^{c\eta *} F_i^{x *} + q_i^{s\eta *} F_i^{y *}}{q_i^{c\xi *} q_i^{c\eta *} + q_i^{s\xi *} q_i^{s\eta *}}; \quad \gamma_{\xi i} = \dfrac{q_i^{c\xi *} F_i^{y *} - q_i^{s\xi *} F_i^{x *}}{q_i^{c\xi *} q_i^{c\eta *} + q_i^{s\xi *} q_i^{s\eta *}}.
\end{array}\right| \qquad (51)
$$

The expressions for the flexural moments are:

$$
\left.\begin{array}{l}
M_{\eta i} = \gamma_{\eta i} + \bar{\delta}_{\eta,i} M_{\eta jj} - \bar{\bar{\delta}}_{\xi i} M_{\xi jj}; \quad M_{\xi i} = \gamma_{\xi i} + \bar{\delta}_{\eta,i} M_{\eta jj} + \bar{\bar{\delta}}_{\xi i} M_{\xi jj}; \\
[illegible] = m_i\left(\dfrac{1}{N_{i,i-1}} - \dfrac{1}{N_{i-1,i}}\right); \quad S_i = m_i\dfrac{1}{N_{i-1,i}}.
\end{array}\right| \qquad (52)
$$

As a result we will obtain equations connecting in pairs the two flexural moments in the plane of the greatest rigidity with the two flexural moments in the plane of the least rigidity right up to ...(illegible)...............

By carrying out a similar operation for the z-th pair of equations ...

(illegible)...we will obtain

$$\left.\begin{array}{l} M_{x,z-1,z-1}\, g^{xx*}_{z-1} - M_{y,z-1,z-1}\, g^{yy*}_{z-1} = F^{y*}_{1}; \\ M_{x,z-1,z-1}\, g^{xy*}_{z-1} + M_{y,z-1,z-1}\, g^{yx*}_{z-1} = F^{x*}_{1}. \end{array}\right| \qquad (53)$$

since at the free end of the blade $M_{y,z-1,z} = M_{x,z-1,z} = 0$.

Employing the discussed algorithm, it is possible to write, that

$$M_{x,z-1,z-1} = \gamma_{x,z-1}, \quad M_{y,z-1,z-1} = \gamma_{y,z-1},$$

where $\gamma_{x,z-1}$ and $\gamma_{y,z-1}$ are also determined according to the formulas of (51).

Having obtained from (53) the values of moments $M_{x,z-1,z-1}$ and $M_{y,z-1,z-1}$, it is then possible in in reverse order, solving in turns the systems of two equations of the type of (52), to obtain the values of the flexural moments in all cross sections of the blade:

$$M_{x,ii} = \frac{(F_i^{x*} - M_{xjj} P_j^{xx} - M_{yjj} P_j^{xy})\, g_i^{yy*} + (F_i^{y*} - M_{xjj} P_j^{yx} - M_{yjj} P_j^{yy})\, g_i^{xy*}}{g_i^{xx*}\, g_i^{yy*} + g_i^{xy*}\, g_i^{yx*}}. \qquad (54)$$

There remains the task of determinign the values of β^y_z and β^x_z. It is possible to obtain tem from the solution of equations $(z+1)_x$ $(z+1)_y$. However, these values are not needed in the successive approximations for a standard blade. The exception is the case when the end of the blade is not free. In this case the penultimate pair of equations should be rearranged depending on these conditions, and accordingly the moments on the end of the blade and the components of the first derivative, of the elastic line, determined from system (43) will participate in the successive approximations. We will not dwell on this, since the corresponding operations are analogous to those untertaken with different boundary conditions in the root part of the blade; these operations are discussed below.

Another case is rigid attachment in the plane of rotation and hinged attachment in the vertical plane.

In comparison with the first case the boundary conditions vary somewhat, notably

$$\beta_0^{z} = 0; \quad M_z = 0;$$
$$\sum_{i=0}^{i=z} m_i U_{zi} = 0; \quad \sum_{i=0}^{i=z} m_i U_{yi} = 0.$$

In consequence of hinged attachment

$$M_{z00} \cos \varphi_{01} - M_{y00} \sin \varphi_{01} = 0. \tag{55}$$

Hence, the following order of the resolution of system (43) follows: in system (43) equation 1_y is replaced by equality (55), and all the sub sequence steps are retained unchanged in comparison with the case of total attachment. The expressions of (46) are also retained, it is necessary as soon as possible to place in them

$$g_0^{\zeta\gamma} = \cos \varphi_{01}; \quad g_0^{\xi\xi} = \sin \varphi_{01}; \quad P_1^{\xi\xi} = P_1^{\zeta\gamma} = F_0^{\gamma} = 0.$$

The third case is the hinged attachment in both planes; the hinges can be located at different distances from the zero point.

System (43) was obtained by assuming the equality of the derivative of the elastic line on the right end of the i-th section and on the left end of the (j-k)-th section. The hinge at the j-th point violates this condition and one of the equations of the (j+1)-th system (43) should be replaced by an equation which takes into account the presence of the hinge. If the axis of the hinge is located in the vertical plane the following condition should be fulfilled

$$[illegible] \sin \varphi_{jk} = 0. \tag{56}$$

If the axis of the hinge is located in the horizontal plane the following condition should be fulfilled

$$M_{\psi jk} \sin \varphi_{jk} + M_{\xi jk} \cos \varphi_{jk} = 0.$$

Thus, in the **sequential transition from the first** pair of equations to the last the (j+1)-th system should be solved (for example, in the case of a vertical hinge):

$$M_{\psi jj} g_j^{\xi\psi} + M_{\psi kk} P_k^{\xi\xi} + M_{\xi jj} g_j^{\zeta\psi} + M_{\xi kk} P_k^{\zeta\xi} = F_j^{\zeta};$$
$$M_{\psi jj} \cos \varphi_{jk} - M_{\xi jj} \sin \varphi_{jk} = 0.$$

With this order of the resolution of system (43) all the expressions of (51) are retained, it it is assumed that

$$g_j^{\xi\psi} = \cos \varphi_{jk}; \quad g_j^{\zeta\psi} = \sin \varphi_{jk}; \quad F_j^{\zeta} = P_k^{\xi\xi} = P_k^{\zeta\xi} = 0.$$

In the case of the positioning of a horizontal hinge at any i-th point the (i+1)th system is solved:

$$M_{\psi ii} g_i^{\zeta\psi} + M_{\psi jj} P_j^{\zeta\xi} - M_{\xi ii} g_i^{\xi\psi} - M_{\xi jj} P_j^{\xi\eta} = F_i^{\zeta};$$
$$\dot{M}_{\psi ii} \sin \varphi_{ij} + M_{\xi ii} \cos \varphi_{ij} = 0,$$

and there will accordingly be obtained in the expressions of (51)

$$g_i^{\zeta} = \cos \varphi_{01}; \quad g_i^{\xi\xi} = \sin \varphi_{01}; \quad P_j^{\zeta\xi} = P_j^{\xi\eta} = F_i^{\psi\zeta} = 0.$$

If hinges are combined at any point k (including at the zero point), then

$$M_{\psi kk} = 0; \quad M_{\xi kk} = 0.$$

For conserving the algorithm it is necessary in the expressions of (51) to assume

$$P_{k+1}^{\zeta\xi} = P_{k+1}^{\xi\xi} = P_{k+1}^{\zeta\eta} = P_{k+1}^{\xi\eta} = \dot{F}_{k+1}^{\psi\zeta} = F_{k+1}^{\zeta} = 0.$$

The appearance of hinges imposes the requirement for one more operation in solving system (43) - this is the determination of the component of the first derivative of the elastic line in the plane perpendicular to the axis of the hinge. The requirement can be fulfilled by solving the special equation for the section, the left boundary of which is combined with the hinge.

In accordance with (15) and (16) we have

$$b_{ij}(U_{yij} - U_{yii}) = d_{ij}^{\eta} M_{\xi ij} + l_{ij}^{\eta} M_{\xi ii} + \beta_{ii}^{\eta};$$
$$b_{ij}(U_{\xi ij} - U_{\xi ii}) = d_{ij}^{\xi} M_{\eta ij} + l_{ij}^{\xi} M_{\eta ii} + \beta_{ii}^{\xi}.$$

Having multiplied the first equation by the negative value of $\sin\varphi_{ij}$, and the second by cos φ_{ij}, let us add them, employing (28). We will obtain

$$b_{ij}(U_{xj} - U_{xi}) = M_{\eta ii} l_{ij}^{\xi} \cos\varphi_{ij} + M_{\eta jj}(d_{ij}^{\xi}\cos\Delta\varphi_j \cos\varphi_{ij} - \sin\Delta\varphi_j \sin\varphi_{ij} d_{ij}^{\eta}) -$$
$$- M_{\xi ii} l_{ij}^{\eta}\sin\varphi_{ij} - M_{\xi jj}(d_{ij}^{\xi}\sin\Delta\varphi_j\cos\varphi_{ij} + d_{ij}^{\eta}\cos\Delta\varphi_j\sin\varphi_{ij}) + \beta_{xi}. \qquad (57)$$

From (7) we will obtain

$$b_{ij}(U_{[illegible]} - U_{[illegible]}) = f_{ij} M_{yij} - f_{ij} M_{yii} - \frac{Q_{xij}}{N_{ij}}$$

or

$$b_{ij}(U_{[illegible]} - U_{[illegible]}) = f_{ij}(-M_{\xi ij}\sin\varphi_{ij} + M_{\eta ij}\cos\varphi_{ij}) -$$
$$- [illegible] \frac{Q_{[illegible]}}{N_{ij}}$$

After the transformations taking into account the substitutions of (41) let us finally determine

$$\beta_{xi} = [illegible] - M_{\xi ij} P_j^{[illegible]} + \frac{Q_{[illegible]}}{N_{ij}}. \qquad (58a)$$

Carrying out analogous operations it is possible to find the expressions for determining β_y ;

$$\beta_{yi} = - M_{\eta ii} g_{ij}^{\xi} - M_{\eta jj} P_j^{\xi} - M_{\xi ii} g_{ij}^{\eta} - M_{\xi jj} P_j^{\eta} + \frac{Q_{yij}}{N_{ij}}. \qquad (58b)$$

It is easy to see that equations (58a) and (58b) do not differ in structure from analogous equations for the root section of the blade (42b). For determining components β_{x0} and β_{y0} it is sufficient to use the expressions

$$\beta_{x0} = - M_{\eta 00} g_{01}^{\xi} - M_{\eta 11} P_1^{\xi} + M_{\xi 00} g_{01}^{\eta} + M_{\xi 11} P_1^{\eta} + \frac{Q_{x01}}{N_{01}};$$
$$\beta_{y0} = - M_{\eta 00} g_{01}^{\xi} - M_{\eta 11} P_1^{\xi} - M_{\xi 00} g_{01}^{\eta} - M_{\xi 11} P_1^{\eta} + \frac{Q_{y01}}{N_{01}}.$$

If a combined hinge occurs somewhere, then

$$
\begin{aligned}
\beta_{xi} &= M_{\eta jj} P_j^{\xi i} + M_{\xi jj} P_j^{\eta i} + \frac{Q_{xij}}{N_{ij}}; \\
\beta_{yi} &= -M_{\eta jj} P_j^{\xi i} - M_{\xi jj} P_j^{\eta i} + \frac{Q_{yij}}{N_{ij}}.
\end{aligned}
\qquad (59)
$$

Determining the Displacements

System (40) is used for determining the displacements; this system in essence represents two unconnected systems, one for the plane xOz, the other for plane yOz. Taking the boundary conditions into consideration

$$\sum_{i=0}^{i=z} m_i U_{xi} = 0; \quad \sum_{i=0}^{i=z} m_i U_{yi} = 0.$$

For this the displacements at each point are represented in the form of a binomial

$$\tilde{U}_{xi} = \tilde{U}_{x0} + \tilde{\tilde{U}}_{xi}; \quad U_{yi} = \tilde{U}_{y0} + \tilde{\tilde{U}}_{yi}; \qquad (60)$$

where

$$\tilde{\tilde{U}}_{x0} = \tilde{\tilde{U}}_{y0} = 0.$$

In accordance with this the determination of the displacements breaks down into two steps.

In the general case, in the first step the displacements $\tilde{\tilde{U}}_{xi}$ and $\tilde{\tilde{U}}_{yi}$ are determined from simple reccurrence formulas:

$$
\begin{aligned}
\tilde{\tilde{U}}_{xk} = \frac{1}{b_{jk}} (M_{\eta ii} d_{ij}^{\xi i} + M_{\eta ji} l_j^{\xi i} + M_{\eta kk} d_k^{\xi i} - M_{\xi ii} d_{ij}^{\eta i} - \\
- M_{\xi ji} l_j^{\eta i} - M_{\xi kk} d_k^{\eta i} - b_{ij} \tilde{\tilde{U}}_{xi} - a_j \tilde{\tilde{U}}_{xj});
\end{aligned}
\qquad (61a)
$$

$$
\begin{aligned}
\tilde{\tilde{U}}_{yk} = \frac{1}{b_{jk}} (M_{\xi ii} d_{ij}^{\xi i} + M_{\xi ji} l_j^{\xi i} - M_{\xi kk} d_k^{\xi i} + M_{\eta ii} d_{ij}^{\eta i} - \\
- M_{\eta ji} l_j^{\eta i} - M_{\eta kk} d_k^{\eta i} - b_{ij} \tilde{\tilde{U}}_{yi} - a_j \tilde{\tilde{U}}_{yj})
\end{aligned}
\qquad (61b)
$$

Then U and U are determined

$$\tilde{U}_{x0} = -\frac{\sum_{i=0} m_i \tilde{U}}{\sum m_i}; \quad \tilde{U}_{y0} = -\frac{\sum_{i=0} m_i \tilde{U}}{\sum_{i=0} m_i}. \tag{62}$$

With the presence of hinges in accordance with (57) the formulas vary only for the point following after a hinge.

If a vertical hinge is located at point j, then

$$\tilde{U}_{xk} = \frac{1}{b_{jk}}(M_{yjj} l_{jk}^{c} + M_{ykk} d_k^{c} - M_{zjj} l_{jk}^{c} - M_{zkk} d_k^{c} + b_{jk} U_{xj} + \beta_{xj}). \tag{63a}$$

If a horizontal hinge is located at point j, then

$$\tilde{U}_{yk} = \frac{1}{b_{jk}}(M_{yjj} l_{jk} + M_{ykk} d^{c} + M_{zjj} l_{jk}^{c} + M_{zkk} d_k^{c} + b_{jk} U_{yj} + \beta_{yj}). \tag{63b}$$

The subsequent process occurs according to the same formulas (61a) and (61b).

In the case of the placing of a hinge at the zero point formulas (63a) and (63b) change accordingly. In the case of the placing of both hinges at the zero pointthe formulas are simplified

$$\tilde{\tilde{U}}_{x1} = \frac{1}{B_{01}}(M_{y11} d_1^{c} - M_{z11} d_1^{c} + \beta_{x0});$$
$$\tilde{\tilde{U}}_{y1} = \frac{1}{B_{01}}(M_{y11} d_1^{c} + M_{z11} d_1^{c} + \beta_{y0}).$$

Certain Results of Calculations

The possibilities of the method are not examined in the present section in detail, however, the figures presented below make it possible to give it a certain evaluation.

The most important characteristics of the method are the required value of accuracy for obtaining a satisfactory solution and the calculation time. It appears that the method discussed above requires calculations with relatively low accuracy, and the number of approximations needed is such, that for the purposes of calculation practically any electronic computer can be used. The results of calculations presented below were obtained on a M-20 computer.

All the calculations were carried out with respect to a hypothetical blade, the prototype of which was the blade of the tail rotor of the Mi-6 helicopter. The diameter of the rotor was taken equal to 6.8 m and its rate of rotation n = 680 r/min. The other characteristics of the blade varied depending on the purposes of the calculation. These characteristics are shown in Fig. 4. Also show here are how these data were represented for calculation on the computer.

Fig. 5 shows the number of approximations n and the time τ, required for accomplishing these approximations, depending on the accuracy , which is defined as the greatest difference between the values of the deformations at any point of the blade in two successive approximations. It is possible to see, that the setting angle of the blade with respect to the plane of rotation somewhat affects the number of approximations.

Fig. 6 shows the change in the frequency of the natural oscillations of the blade depending on the accuracy of the solution.

It turns out that beginning with an accuracy of the value of the frequency ...(rest of sentence and next paragraph is illegible)....

This characteristic becomes especially important when the actual frequencies of the natural oscillations are close to each other. In this case the values of the natural frequencies obtained by calculation appear to be sensitive to the accuracy of the definition of functions employed in the calculation.

This circumstance compels the calculators to somehow modify the calculational methods, to increase the accuracy of the definition of functions and the accuracy in the intermediate operations. And this in its turn is frequently limited by the possibilities of computers or in any case significantly increases the calculation time.

A certain amount of uncertainty in the reliability of the obtained results does not favor the successful application of these methods.

While not carrying out a systematic investigation of the possibilities of the method examined above, let us illustrate them by a comparison of it with the Bubnov-Galerkin Method in the form in which it was employed in solving analogous problems, for example, in work [1].

Fig. 7 shows the dependence of the frequency of the natural oscillations of a blade fastened to a hub with the aid of one horizontal hinge on the angle of inclination of the plane of the greatest rigidity to the plane of rotation.

[daN·cm^2][daN/m]

——— actual dependence
----dependence, represented in the calculation

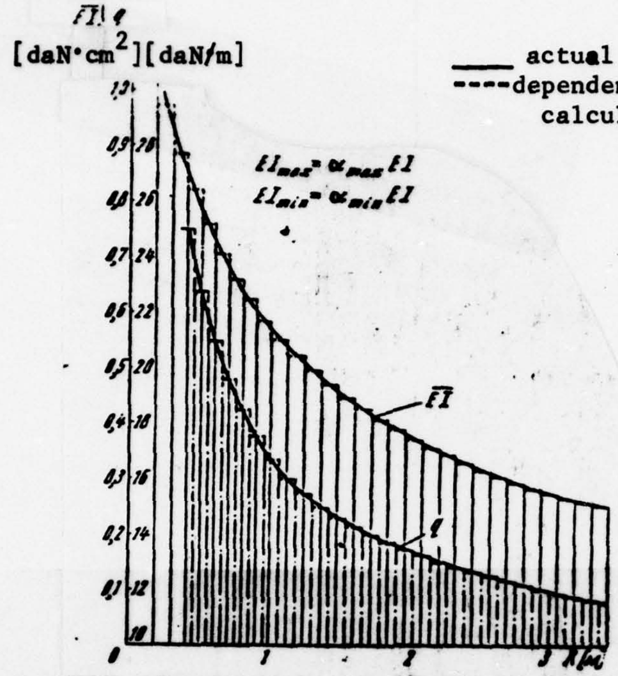

Fig.. 4.

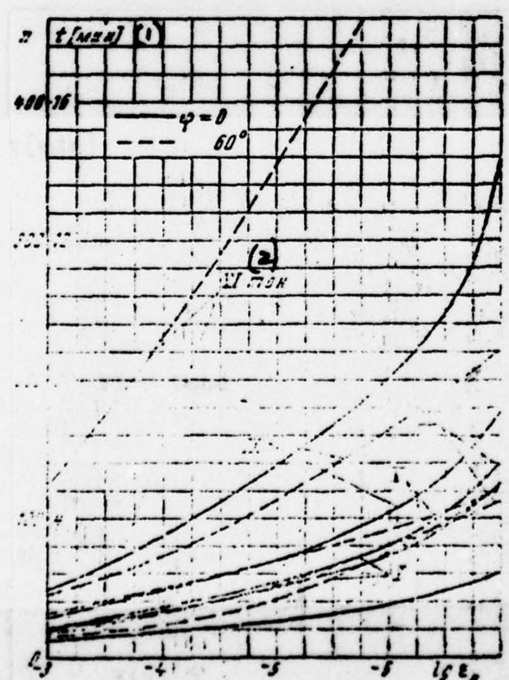

Fig. 5. Key: 1 - t[min]. 2 - VI tone.

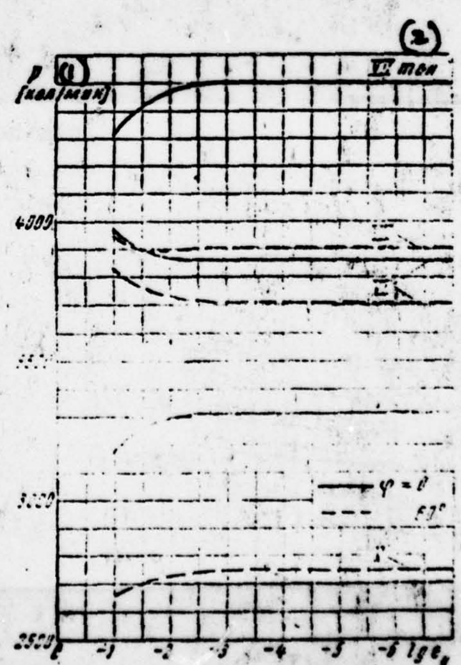

Fig. 6. Key: 1 - [osc/min]. 2 - VII tone

—— Calculation employing the Bubnov-Galerkin Method
---- Calculation by the Finite Element Method with Iterations

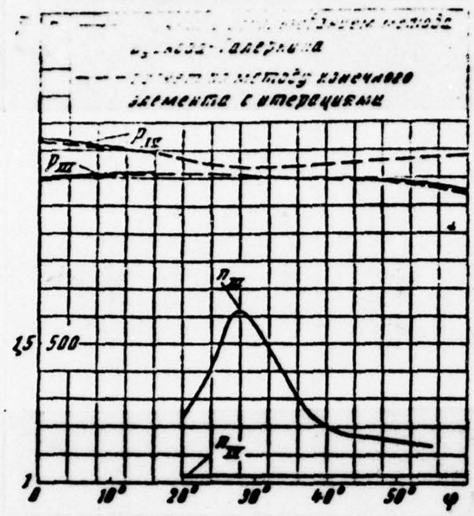

Fig. 7

[osc/min]

—— Calculation employing the Bubnov-Galerkin Method
----- Calculation by the Finite Element Method with Iterations

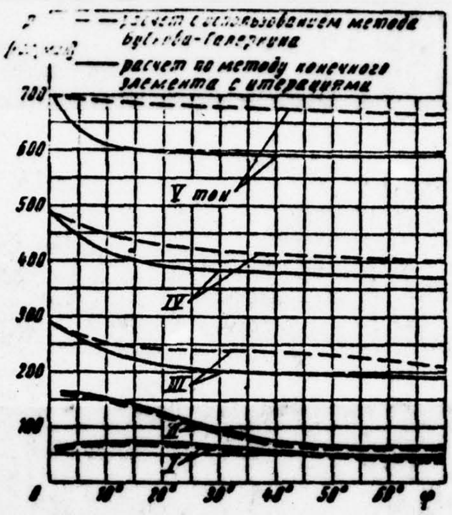

Fig. 8

to the plane of rotation. The flexural moments in the plane of the greatest and least rigidity were selected in such a manner, so that the frequencies of the natural oscillations of the blade with regard to the their and fourth tones were as close as possible to each other.

The graph shows that the Finite Element Method with Iterations makes it possible to determine the frequencies of the natural oscillations with medium accuracy (lg ε_1 = — 7) over the entire range of the setting angle values in question. In this case the number of approximations at the sites of the greatest convergence of frequencies, although it increases, it remains moderate.

The methodology, which takes advantage of the Bubnov-Galerkin Method does not make it possible to find frequencies over a broad range of setting angle values (14^{o}–44^{o}). Moreover, it is necessary to emphasize that in a whole series of cases...(remainder of sentence is illegible)....... It is also necessary to consider that the Bubnov-Galerkin Method is favored by its simplicity and by the fact that it makes it possible to employ ready developed operation elements, from which a future methodology can be compiled. For example, let us cite the results of the calculation of the dependence of the frequencies of a plane untwisted blade attached to a hub by a horizontal hinge on its setting angle.

The dependences obtained employing the Bubnov-Galerkin Method and the method of successive approximations (Fig. 8) very satisfactorily agree for the first two tones of the natural oscillations of the tail rotor of a helicopter. For higher tones their divergence attains 20%.

The calculations cited for blades with a different combination of parameters make it possible to more accurately define certain assumptions and simplifications established in blade design practice.

Thus, it is usually assumed, that the frequencies and the forms of the natural oscillations of blades with a typical construction (for a helicopter rotor) of the attachment of the blades to the hub (horizontal and vertical hinge) very slightly depend on the design twistedness of the blade and the setting angle relative to the plane of rotation. It is evident from Fig. 9 that both for a twisted as well as for a plane blade the frequency of the natural oscillations with preferential deformations in the plane of rotation rather noticeably depends on the setting angle

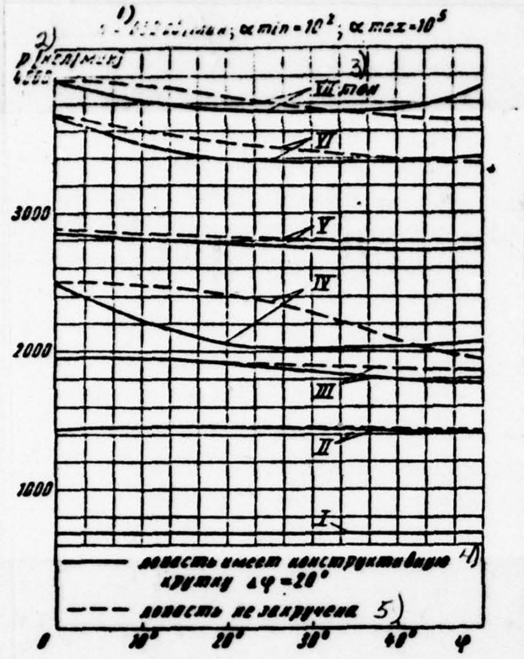

Fig. 9.

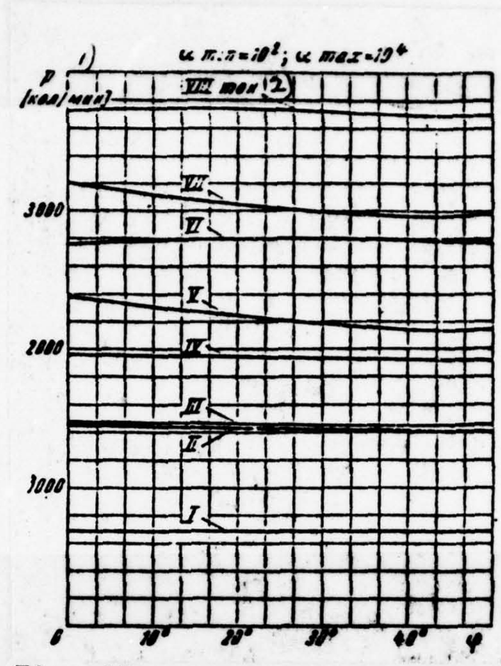

Fig. 10.

Fig. 9. Key: 1) r/min; 2) [osc/min]; 3) tone; 4) blade has design twist; 5) blade is not twisted.

Fig. 10. Key: 1) [osc/min]; 2) tone.

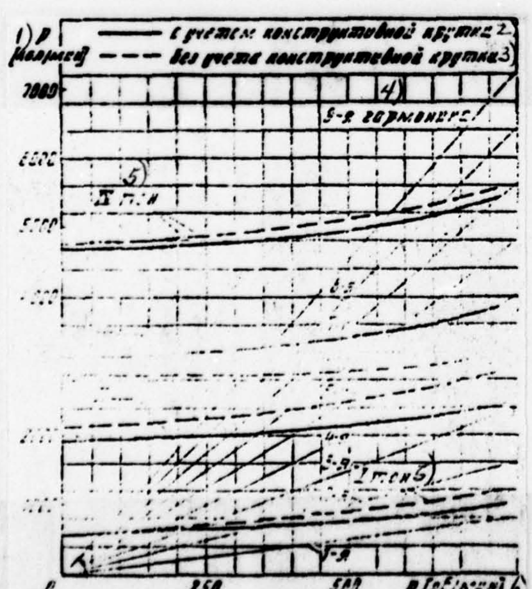

Fig. 11. Key: 1) [osc/min]; 2) taking the design twist into account; 3) without taking design twist into account; 4) 9-th harmonic; 5) tone; 6) [r/min].

within the usual limits for helicopter blades. It is obvious that this sensitivity to setting angle depends on blade characteristics, in particular, on the relationship of the rigidities in the plane of the least and greatest rigidity. The results, shown in Fig. 10, of the design of a blade with the greatest rigidity reduced by 10 times as compared with the blade, represented in Fig. 9, illustrate this fact.

In conclusion, let us illustrate the advisability of taking blade twist into consideration for calculating the frequencies and the forms of the natural oscillations with respect to the blades of one of the aircraft propellers.

Fig. 11 shows the resonance diagram for such blades, obtained by taking into account twist in the cetrifugal force field, and also presented here is a resonance diagram which does not consider the design twist of the blade, i.e., because this is done in design practice. One can easily be convinced of the advisability of calcualting by a methodology, which takes blade twist into consideration.

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8. ...(Illegible)...

Manuscript submitted 2 Dec. 1971

Flutter Analysis of a Helicopter Rotor in Flight

by V. V. Nazarov

Summary

Methods of flutter analysis of a helicopter rotor in flight have been developed; these methods make it possible to consider blade elasticity, the effect of air compressibility and the pliability of the cyclic pitch control system. A description of the methods, a block diagram of the program and certain results of flutter analysis of rotors are presented.

Conventional Designations

N - the number of rotor blades;

ω - rotational speed of the rotor [1/s];

$\xi\ (r_1)$ - moment of control rod relative to the feathering (flapping) hinge;

$\varkappa = \frac{r_1}{\xi}$ - gear ratio of the blade-flapping control;

$k_x,\ k_y,\ k_z$ - coefficients of the total rigidity of the of the elements of the check respectively of the pitch and lateral control and the collective pitch control from the control stick to the cyclic pitch control

$M_\mu = \frac{W}{a}$ - dimensionless flying speed;

EJ - rigidity of blade in flexure;

b - blade chord;

GT - rigidity of blade in torsion;

$\varphi_x,\ \varphi_y$ - angles of inclination of the cyclic pitch control respectively during deformation of the cables of the pitch and lateral control;

φ_j -complete angle of turn of the transverse cross section of the j-th blade due to the kinematics of the blade-flapping control and the elastic deformations of the blade;

- ...(Illegible)...

- ...(Illegible)...

$\bar{x}_0 = \frac{x_0}{b}$ - relative position of the axis of the feathering hinge.

- coefficient of total rigidity of elements of cables from the blade to cyclic pitch control;

$\frac{r}{R}$ - relative distance from transverse cross section of the blade to the rotor axis;

- azimuth of the control rod of the j-th blade of the rotor;

W - flying speed of the helicopter [m/s];

j - rotor blade number (j=0,1, ...,N-1);

$M_{\circ} = \frac{\omega R}{a}$ - circular Mach number of rotor;

ψ_j - azimuth of the j-th blade;

The broken line designates the derivative with respect to the radius, the dotted line designates the derivative with respect to time.

Introduction

The purpose of this work is the creation of a method of analyzing a rotor for flutter in flight, which makes it possible to consider both the elasticity of the rotor blades, as well as the pliability of the control system, kinematically connecting the motion of all the rotor blades.

This refinement of the analytical scheme, in comparison with the analytical methods [3] developed earlier by other authors and in comparison with the analytical method discussed in [7] was evoked by the following circumstances. First, the experiments with dynamically similar models of rotors of large diameter showed, that the effective centering of the blades (see [3]) for such rotors is not a comprehensive characteristic of distributed antiflutter balancing. The balancer, which changes the effective centering by one and the same value, is mounted at different cross sections of the blade; it cahnges the critical number of revolutions of rotor flutter differently. This can be explained only by the effect of blade elasticity, which, consequently, should be taken into account in flutter analysis. Secondly, the pliability of the control system cables, which is usually taken into account by introducing an isolated blade of effective rigidity into the calculation, cannot be taken into account in the case, when the lateral control of the helicopter is accomplished not by means of the cyclic pitch control, which is characteristic for helicopters of transverse arrangement. In this case the rigidity of the attachment of the cyclic pitch control with respect to the lateral inclinations is extremely great, and even in the hovering mode the analysis of rotor flutter does not agree with the calculation of the isolated blade [6]. Moreover, as is shown in [7], the calculational set up , which makes it possible to calculate rotors of high-speed helicopters and rotary-wing aircraft of different arrangements in different modes, for example, in the mode of large , it is necessary to include in it the calculation of the effect of the compressibility of air on the aerodynamic characteristics of the airfoils and to make it possible to take into account their periodic change under flight conditions more accurately, than by simple averaging. This becomes necessary in calculations with large numbers. Moreover, the purpose of this work was the achievement of a method with all its variations in a single algorithm and in a single program for a computer, of sufficiently fast-response for carrying out parametric investigations.

Assumptions. General Equations

In the investigation of flutter it is necessary to take the following factors into consideration as a totality:

- the elasticity of the blades and the rotor control system,

- the high subsonic tip velocities of the blades,

- the large nonlinear forces, close in character to the forces of ...(Word illegible)... dry friction arising during oscillations of the blades in the feathering hinge, loaded with centrifugal forces,

- the nonsteady-state nature of the flow around the blades, when disturbances remain for a long time close to the spinning disk.

The following assumptions are introduced when developing a method for analyzing rotor flutter.

1) the hypothesis of plane transverse cross sections is assumed valid, in accordance with which the blade is replaced by a beam with variable elasto-mas linear characteristics, to which is fastened nondeformed transverse plate-sections;

2) the blade is considered untwisted; we disregard its static flexural and torsional deformations;

3) the mass of the blade is assumed distributed in the plane of the chord;

4) before the appearance of the oscillations the plane of the blade chord is assumed perpendicular to the rotor axis;

5) the flapping and drag hinges are assumed to be located at one stationary point, at which the boundary conditions at the blade root are assigned;

6) the hypothesis of stationariness is assumed valid in determining the aerodynamic effects on an oscillating blade;

7) the assumption,approximately valid for the majority of blade designs is introduced; this assumption states that before the appearance of oscillations the axis of rigidity of the blade is a straight line, coinciding with the axis of the feathering hinge.

In the assumptions enumerated, the differential equations of the movement of a blade element of unit length can be written in the following form (see [2], [3])

$$(EIu'')'' + m\ddot{u} - m\sigma\ddot{\varphi} - \omega^2\left[u'\int_r^R mr\,dr\right]' + \omega^2\left[(\varphi r)'\int_r^R m\sigma\,dr\right]' + P'_r = 0$$

$$-(GT\varphi')' + I_m\ddot{\varphi} - m\sigma\ddot{u} + \omega^2 r\left[u'\int_r^R m\sigma\,dr\right]' + \omega^2 I\varphi + M'_r = 0, \qquad (1)$$

where P'_r and M'_r are a linear aerodynamic force and moment, acting on an airfoil, oscillating in a plane-parallel flow.

The expressions for P'_r and M'_r, derived on the basis of the hypothesis of stationariness [1] have the form

$$
\left.
\begin{aligned}
P'_r &= -\frac{1}{2}\rho b V^2 c_y^\alpha\left[\varphi + \left(\frac{3}{4} - x_0\right)\frac{b\dot{\varphi}}{V} + \frac{\dot{u}}{V} - \frac{V_t u'}{V}\right];\\
M'_r &= \frac{1}{2}\rho b^2 V^2 c_y^\alpha (\bar{x}_0 - \bar{x}_\phi)\left[\varphi + \right.\\
&\left. + \left(\frac{3}{4} - \bar{x}_0 - \frac{\pi \bar{c}_{m0}}{8c_y^\alpha(\bar{x}_0 - \bar{x}_\phi)}\right)\frac{b\dot{u}}{V} - \frac{\dot{u}}{V} - \frac{V_t u'}{V}\right],
\end{aligned}
\right\}
\qquad (2)
$$

where

$$V = \omega r + W\cos\chi;\quad V_t = -W\sin\chi;\quad \bar{c}_{m0} = \frac{c_{m0}}{(c_{m0})_{M=0}};\quad \bar{X}_\phi = -\frac{\partial m_z}{\partial c_y}.$$

The differential equations of (1) are used for investigating the oscillations of blades during flutter, which arise and develop near steady-state flywheel motion.

The aerodynamic coefficients of the profiles of the transverse cross sections of a blade c_y^α, $\bar{x}_\phi$ and c_{m0}, which enter into the equations of (1), essentially depend on the magnitude of the local Mach number, i.e., on the normal velocity component of flow V, incident on the blade. Since the velocity of the incident flow V varies periodically with time, then the aerodynamic coefficients are periodic functions of time and at each given moment they are different for different cross sections of the blade. Thus, the equations of (1) represent a system of differential equations in partial derivatives with variable coefficients, which are periodic functions of time. In forming the boundary conditions for equation system (1) a number of assumptions is introduced, which schematize the rotor control system. In this case, the usual type of rotor is examined with individual hinge attachment of each blade to the hub and with cyclic pitch control. It is assumed, that all the elements of the control cables are weightless and the elasticity of the control cable elements is taken into consideration by introducing appropriate elastic elements. The friction forces (see [3]) are taken into account only in the feathering hinge, since the magnitude of the moment of these forces is of approximately of the same order as the magnitude of the moment of the external forces acting on the blade. It is possible to show [5], that with these assumptions the boundary conditions in the root of each blade take the form (see [5])

$$\left.\begin{aligned} &(u_j)_{r=0} = 0; \\ &(EIu''_j)_{r=0} = \bar{k}x\left[(\varphi_j + u'_j x) - \frac{1}{\xi}\zeta_j\right]_{r=0}; \\ &(GT\varphi'_j)_{r=0} = \bar{k}\left[(\varphi_j + u'_j x) - \frac{1}{\xi}\zeta_j\right]_{r=0}, \end{aligned}\right\} \qquad (3)$$

where the magnitude of ζ_j expresses the connection between the rotor blades through the cyclic pitch control and is equal to the displacement along the vertical of the lower point of attachmant of the control rod of the j-th blade of the cyclic pitch control.

It is possible to show [5], after having compiled the equations of equilibrium of the cyclic pitch control under the effect of all the elastic forces, that

$$\frac{1}{\xi}\zeta = \frac{1}{N}\sum_{n=0}^{N-1}(\varphi_n + u'_n x)_{r=0}(G_z + 2G_x \sin\psi_j \sin\psi_n + 2G_y \cos\psi_j \cos\psi_n), \qquad (4)$$

where

$$G_z = \frac{\bar{k}}{\bar{k} + \bar{k}_z}; \quad G_x = \frac{\bar{k}}{\bar{k} + \bar{k}_x}; \quad G_y = \frac{\bar{k}}{\bar{k} + \bar{k}_y};$$

$$\bar{k} = k\xi^2; \quad \bar{k}_x = \frac{2}{N}\frac{1}{1 + x^2}k_x; \quad \bar{k}_y = \frac{2}{N}\frac{1}{1 + x^2}k_y; \quad \bar{k}_z = \frac{1}{N}\xi^2 k_z.$$

It is shown in work [7], how the equation of flutter by the Bubnov-Galerkin Method reduces to a system of usual equations with periodic coefficients. Taking the pliability of the cyclic pitch control into account the equation of flutter for the j-th blade takes the form

$$C\ddot{\bar{x}}_j + \omega D(t)\dot{\bar{x}}_j + \omega^2 [B(t) + C^*] \bar{x}_j +$$
$$+ A \left[\bar{x}_j - \frac{1}{N} \sum_{n=0}^{N-1} (G_z + 2G_x \sin\psi_j \sin\psi_n + 2G_y \cos\psi_j \cos\psi_n) \bar{x}_n \right] = 0$$
$$(j = 0, 1, 2, \ldots;\ N-1); \tag{5}$$

here x_j is the vector of the generalized coorditates of blade motion;

G is the matrix of the inertial coefficients;

A is the matrix of rigidity;

C is the matrix of the centrifugal inertial coefficients;

D(t) is the matrix of aerodynamic damping;

B(t) is the matrix of aerodynamic rigidity.

If k_x, k_y, k_z are infinitely great, which corresponds to the rigid fastening of the cyclic pitch control, then $\vdots = 0$ and system (5) develops into a flutter equation system for an isolated blade.

In those cases, when the coefficients of rigidity of the cables of pitch and lateral control differ greatly from each other or the elastic attachment of the nonrotating part of the cyclic pitch control is such, that the center of rigidity of the attachment is located relatively far from the axis of the rotor, it is necessary to solve the flutter problem of the rotor as a whole.

In those cases, when the oscillations of a rotor with flutter have the aspect of normal forms (for example, in a hovering mode with identical rigidities k_x and k_y), it is possible to reduce a rotor flutter problem as a whole to an examination of the oscillations of one isolated blade with different values of equivalent rigidity of the control system, which correspond to different normal forms of rotor oscillations.

Taking into Account Blade Elasticity and the Pliability of the Control Cables of Cyclic Pitch Control

The method of analyzing the flutter in an isolated blade was discussed in work [7], in which employing the Bubnov-Galerkin Method the separate inherent forms of flexural and torsional oscillations of a nonrotating blade as an absolutely rigid body were taken as natural forms of flexural and torsional oscillations. A calculational method was proposed in this work, which made it possible to take blade elasticity into account, which was attained by introducing additional flexural forms of oscillations, obtained beforehand from frequency calculation. In this case, in order that it would be possible to use the separate natural forms of the oscillations of a beam a substitution of the variables of the following form is introduced

$$\varphi(r, t) = \theta(r, t) - \varkappa u'_{r=0}, \qquad (6)$$

which made it possible to separate the boundary conditions in the blade root.

$$(EIu'')'_{r=0} = 0,$$

In this case, it is assumed that

since the moment of the external forces relative to the flapping hinge is much greater than the moment of reaction in the blade setting.

In analyzing the flutter of an elastic blade employing the Bubnov-Galerkin Method the zero and the first natural form of the flexural and the first natural form of the torsional oscillations of a nonrotating beam in a vacuum are taken. The friction in the feathering hinge both for an absolutely rigid, as well as for an elastic blade is taken into consideration in the form of additions to the forces of aerodynamic damping.

As was shown in a number of works (see, for example, [5,6]), the pliability of the rotor cyclic pitch control, arising due to the flexibility of the control cables, can have a considerable effect on rotor flutter, and also the peculiarities, as the different rigidity of the circuits of the pitch and lateral control, can even introduce qualitative changes in the flutter pattern. An attempt at analyzing rotor flutter in the flight mode taking into consideration the pliability of the control cables of the cyclic pitch control leads to great computational difficulties, mainly because in this case it is necessary to investigate a system of differential equations with periodic coefficients of a higher order than occurs for an isolated blade.

The lowering of the order of this system of equations by the method, which is employed in investigating ground resonance, generally speaking, is not possible. Work [5] substantiated the application of this calculational scheme by analyzing the accurate solutions of flutter equations for certain particular cases.

According to this work, in equation system (5) the susbtitution of of variables of the following form can be carried out

$$\bar{x}_j = \bar{y}_0 + \bar{y}_1 \cos \chi_j + \bar{y}_2 \sin \chi_j. \tag{7}$$

After this substitution an equivalent system of differential equations is obtained relative to the functions of $\bar{y}$, which contains periodic coefficients of the form sin KNt (K is a whole number and greater than unity). Thus, the averaging of coefficients with this method leads to smaller errors, than with the method of averaging discussed in [7]. Thus, substituting (7) in (5) and orthogonalizing the discrepancy to 1, $\cos \chi_j$, $\sin \chi_j$, in a set of values of discrete variable j, we will obtain an equation system of n times higher order than for an isolated blade [n is the number of terms in representation (7)].

It is possible to show [5], that as a result of transformation only periodic coefficients of the type sinKNt, cosKNt can remain in the new equation system, where K is an integer, the average of which does not affect the boundary of flutter, since the frequency of the oscillations during flutter is close to the frequency of the revolutions. The main difficulties in the transformation is the process of orthogonalizing to functions (7). It was proposed in work [5] to represent the periodic coefficients $D(\chi_j)$ and $B(\chi_j)$ by trigonometric polynomials. With this approach it is possible to obtain a new equation system in closed form only for an incompressible gas. The consideration of compressibility makes such a method of calculating analytical expressions for coefficients of matrices $D(\chi_j)$ and $B(\chi_j)$ unacceptably cumbersome.

However, in carrying out the indicated calculational method on an electronic computer it is proceed in the following manner. Let us substitute in (5)

$$\bar{x}_j = \begin{bmatrix} \bar{y}_0(\chi) \\ \bar{y}_1(\chi)\cos\psi_j \\ \bar{y}_2(\chi)\sin\psi_j \\ \vdots \end{bmatrix} = \begin{pmatrix} 1 \\ \cos\psi_j \\ \sin\psi_j \\ \vdots \end{pmatrix}^T \bar{y};$$

$$\bar{x}'_j = \begin{pmatrix} 1 \\ \cos\psi_j \\ \sin\psi_j \\ \vdots \end{pmatrix}^T \bar{y}' + \begin{pmatrix} 0 \\ -\sin\psi_j \\ \cos\psi_j \\ \vdots \end{pmatrix}^T \bar{y};$$

$$\bar{x}''_j = \begin{pmatrix} 1 \\ \cos\psi_j \\ \sin\psi_j \end{pmatrix}^T \bar{y}'' + \begin{pmatrix} 0 \\ -2\sin\psi_j \\ 2\cos\psi_j \\ \vdots \end{pmatrix}^T \bar{y}' + \begin{pmatrix} 0 \\ -\cos\psi_j \\ -\sin\psi_j \end{pmatrix}^T \bar{y}. \tag{8}$$

Let us orthogonalize the obtained discrepancy

$$A\begin{pmatrix} 1 \\ [illegible] \end{pmatrix}^T [illegible] \left[D\begin{pmatrix} 1 \\ \cos\psi_j \\ \vdots \end{pmatrix}^T + C\begin{pmatrix} 0 \\ -2\sin\psi_j \\ [illegible] \end{pmatrix}\right]^T \bar{y}' -$$

$$-\left[(B - C^*)\begin{pmatrix} \cos\psi_j \\ \sin\psi_j \\ \vdots \end{pmatrix} + D\begin{pmatrix} -\sin\psi_j \\ \cos\psi_j \end{pmatrix} + C\begin{pmatrix} -\cos\psi_j \\ -\sin\psi_j \end{pmatrix}\right.$$

$$\left. + \frac{1}{\omega^2} A\begin{bmatrix} (1 - G_z) \\ (1 - G_y)\cos\psi_j \\ (1 - G_x)\sin\psi_j \\ \vdots \end{bmatrix}\right] \bar{y} = 0 \tag{9}$$

to 1, cos ψ_i , sin ψ_i , ... in the set of values of discrete variables ψ_j . We will obtain a system of m x n equations [n is the order of the matrices A, B, C, D, and m is the number of terms in substitution (8)] with constant coefficients.

Thus, in analyzing an elastic blade for flutter taking cyclic pitch control into consideration it is necessary to investigate the stability of the solutions of a system of 18 differential equations of the first order with constant coefficients. If in equation (9) we set $G_z = 0$, $G_x = 0$, $G_y = 0$, then the connection between the blades through the cyclic pitch control disappears and system (9) now represents a system of differential equations of an isolated blade, but which takes into consideration a greater number of harmonics of oscillations, than in the simple averaging of the periodic coefficients in equation (5).

The question of the stability of the solutions of such approximate equations reduces to an analysis of the roots of a characteristic equation of this system. The order of the system of such equations, and consequently, also the accuracy of the calculation of the boundary of flutter depends on how many forms of natural oscillations we take for the isolated blade and how many terms are taken in substitution (7).

An analogous procedure of approximate investigation of the stability of a system of differential equations with periodic coefficients was used in work [9] in investigating the ground resonance of a helicopter rotor.

The effect on the stability of the motion and the flutter of a helicopter rotor of such factors, as blade elasticity and compressibility of the flow incident on the blade was investigated in this work, in contrast to the works indicated above. The maximum permissible order in such a method of the investigation of stability, achieved on an BESM-3 computer, is 12, i.e., nm $\leq$ 12, where n is the number of natural forms of oscillations of an isolated blade (m = 2 or 3), and m is the number of terms in substitution (7).

Results of the Analysis

Fig. 1-7 show the results of the flutter analysis of an isolated blade and a rotor in accordance with a program compiled for the BESM-3 computer.

The main parameter, which determines the margin of safety from flutter, is the margin with respect to parameter $\bar{x}_{\text{ц.т}}$. The variation in the effective centering of the blade to the assigned was attained by loading the trailing and leading edge with a load corresponding in magnitude. Fig. 1 shows the results of the flutter analysis of a dynamically similar model of a helicopter rotor blade of large diameter. The blade, having considerable elasticity in the plane of the flap or stroke was tested for flutter with different antiflutter balancers, which were located on the leading edge at different sites over the radius.

It was ascertained in the experiment, that the variation in revolutions depends not only on the increase in the effective centering, but also on the site at which the antiflutter balancer is located. The flutter analysis of the model, carried out taking three degrees of freedom into account, confirmed this dependence. The margin with respect to aerodynamic centering for an elastic blade is equal to $\sum_i c_i \nu_i$ (where c_i is the increase in effective centering for a semi-rigid blade in the i-th cross section;

is a coefficient, which depends on $\bar{r}_i$ and is approximately equal to $\bar{r}_i^{-2}$). For a semi-rigid blade in accordance with the calculation, the critical revolutions depend only on the the magnitude of the antiflutter balancer. Fig. 2 gives the results of the calculation of the critical revolutions for an elastic blade depending on flying speed. It is evident, that with an increase in M number from zero to 0.25, the critical revolutions of the blade fall off by approximately 10%. Fig. 3 shows the change in the decrements for elastic and rigid blades during transit of the flutter zone boundary. As is evident from the graphs, the increase in the oscillations in a blade, having considerable elasticity, during transit of the flutter zone boundary, is an order greater, than for an analogous rigid blade.

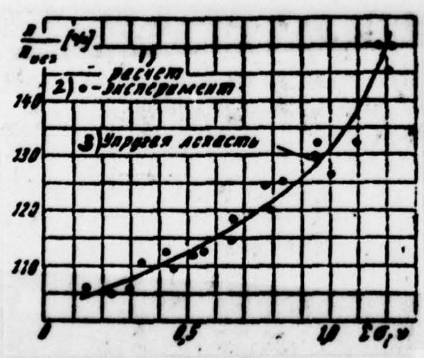

Fig. 1.

Key: (1) Calculation; (2) Experiment; (3) Elastic blade.

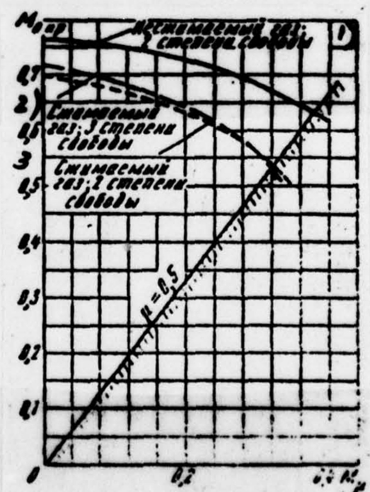

Fig. 2

Key: (1) Incompressible gas; (2) 2 degrees of freedom; (3) Compressible gas; (4) 3 degrees of freedom; (5) Compressible gas; (6) 2 degrees of freedom.

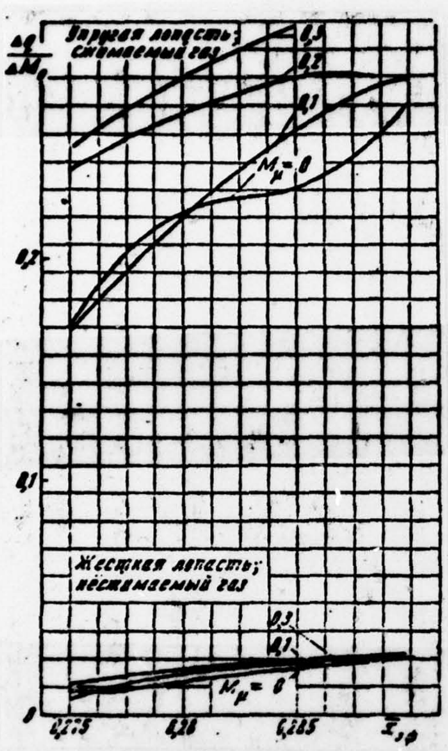

Fig. 3
Key: (1) Elastic blade; (2) Compressible gas; (3) Rigid blade; (4) Incompressible gas.

Fig. 4 shows the results of the flutter analyses obtained by different approximate methods, which take into account the different number of harmonics of periodic coefficients and gives a comparison of the calculations with an accurate solution. It is evident from the results of the calculation, that "simple" averaging of the periodic coefficients* makes it possible with sufficient accuracy for practical purposes to estimate the flutter boundary to a flying speed of $M_\mu \leq 0.25$. "Three-term" averaging after the introduction of substitution (7) makes it possible to rather accurately carry out an estimation of critical centering to a flight mode of $\mu = 0.7 - 0.8$.

It is also eveident from the figures shown, that the critical centering, obtained by taking the periodically changing coefficients into con- consideration, decreases more rapidly than the critical centering obtained during calculation with averaging of the periodic coefficients. For each averaging method there is a maximum value of the number μ, for which we will employ this calculational method. The greater the number of terms

* 121 is substitution of $\bar{x} = \bar{\bar{y}}_o$ (simple averaging), 122 is substitution of $\bar{x} = \bar{y}_o + \bar{y}_1 \cos \chi$, 123 is the substitution of $\bar{x} = \bar{y}_o + \bar{y}_1 \cos \chi + \bar{y}_2 \sin \chi$.

in substitution (7), then for the greater it is possible to carry out the calculation. However, beginning with 0.9 it is practically impossible to carry out the calculation with the number of terms in substitution (7) m > 4, since here computer error begins to be expressed. The same pattern is also observed in analyzing rotor flutter.

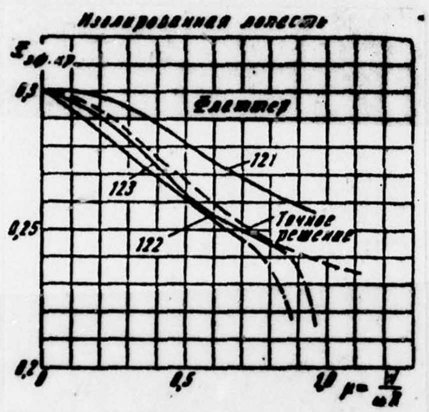

Fig. 4

Key: (1) Isolated blade; (2) Flutter; (3) Accurate solution.

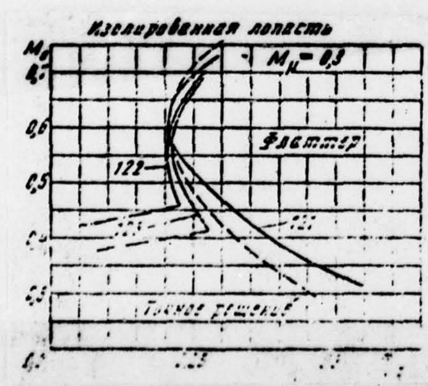

Fig. 5

Key: (1) Isolated blade; (2) Flutter; (3) Accurate solution

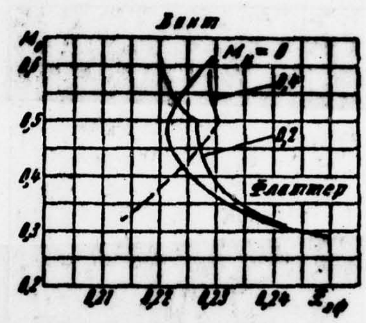

Fig. 6

Key: (1) Rotor; (2) Flutter.

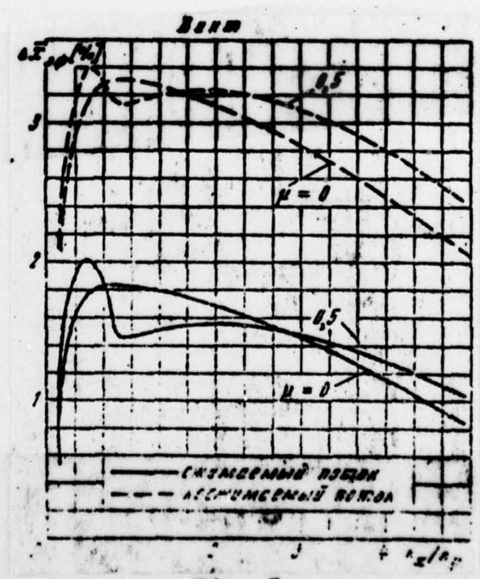

Fig. 7

Key: (1) Rotor; (2) Compressible flow; (3) Incompressible flow.

The practical range of numbers , in which it is possible to carry out helicopter rotor flutter analysis is – 0 – 0.8.

Fig. 7 presents the results of the flutter analysis of a helicopter rotor, which has different control cable rigidity. The caaculation was carried out observing the condition that the total rigidity of the circuits of pitch and lateral control remained constant. Changes in the margin with respect to parameter x_{14} were obtained for compressible and incompressible gases. It is evident from the figure, that calculation under the assumption of incompressibility qualitatively reflects the pattern of the variation in margins depending on the relationship of the rigidities of the control cables for different flight speeds. A sharp change in the increase in margin in the region $k_x/k_y = 1$ for flight speeds $M_\mu \neq 0$ is accompanied by a variation in flutter frequency

Conclusions

1. For a blade, having considerable elasticity in the flapping plane, the effectiveness of an antiflutter balancer depends not only on its magnitude, but also on the site along the span, at which it is located.

2. The oscillations of an elastic blade during transit of the flutter bboundary increase by an order more rapidly, than for an absolutely rigid blade.

3. The method of simple averaging of the periodic coefficients of flutter equations when $\mu > 0.25$ leads to an exaggeration of the actual margins with respect to the centering, which ensures safety from flutter in flight.

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