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(6) EXPECTED VALUE ERRORS IN
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STRATEGIC MISSILE TARGETING.

THESIS

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EXPECTED VALUE ERRORS IN KILL PROBABILITY FORMULAS IN STRATEGIC MISSILE TARGETING

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

David K. Roberts
Lt USAF

Graduate Operations Research

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Preface

The purpose of this research was to examine the effects of incorrect uses of expected values in strategic missile targeting. This topic arose from some research for a classroom exercise, in which it was found that expected numbers of weapons are being incorrectly used in some published calculations for kill probability. I hope that this report will lead to a more cautious use of expected values in targeting calculations. This has certainly been a true learning experience for me.

I would like to thank my advisor, Dr. Keith Womer, for allowing me to pick this particular topic and for lending me some of his expertise. Special thanks goes to my wife Pat for her love and moral support at critical times. She typed the report, but, of course, I am solely responsible for any unnoticed errors.

David K. Roberts

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ABSTRACT

This report examines the incorrect uses of expected values in kill probability calculations that exist in some strategic targeting articles and models. Generally stated, the type of error is the incorrect use of expected numbers of weapons in probability calculations in place of numbers of weapons that are actually random variables. The most common example found was the use of the expected number of reliable missiles or warheads in kill probability calculations for silo targets.

In Chapter II, the results of a general literature search are given. These results are in the form of classifying various strategic targeting models. Three alternative classifications are offered -- two from the literature and one by the author. The author notes a definite lack of targeting literature covering missiles that both are unreliable and have multiple warhead capabilities. Of the targeting models that were examined, the author found a few articles that make the type of mistake studied in the report, but these mistakes are not common.

In Chapter III, a simple missile allocation problem is examined and a possible expected value error is discussed. For single-warhead missiles, the incorrect use of the expected number of reliable missiles in place of the random variable reliable missiles can lead to highly significant numerical

errors for some parameter values. The correct kill probability formula is given and several measures of discrepancy between the two formulas are given. The discrepancy measures are of two types -- the ratio of the missile allocations for the two methods, and the ratio of the respective probabilities of kill that result from the two allocations. The effects of changes in parameters such as accuracy and silo hardness, are proven mathematically. Numerical data on the discrepancy measures are tabled for various parameter values, and these data show that the error can be highly significant.

In Chapter IV, a specific example of an expected value error is discussed. A series of articles with incorrect kill probability calculations for MIRVed weapons is examined. The incorrect articles use the expected number of reliable warheads in place of the random number of reliable warheads. The author offers a corrected formula and numerically compares the differences based on data values for present and projected U.S. and Soviet arsenals. Then the author offers a general kill probability formula for silo targets; the formula is general enough to include mixed types of multiple-warhead missiles.

EXPECTED VALUE ERRORS IN KILL PROBABILITY FORMULAS IN STRATEGIC MISSILE TARGETING

I. Introduction

The primary objective of most strategic targeting and allocation problems can be simply stated. The goal is to gain the maximum amount of destructive power with the resources that are available. In other words, it is to find the "best" force mix of strategic weapons and the "best" targeting structure -- best in terms of military effectiveness. The problem may be couched in broader terms, such as "maximal deterrence", or "effective equivalence", but in many cases the objective can be simplified to one of destructive power.

Yet, even if the broad objective of a strategic allocation problem is clear, its precise definition may be difficult. Often, a particular strategic study or model will use a single Measure of Effectiveness (MOE) to define destructive power. With a group of possible enemy targets, three common examples of MOE's used are probability of destroying the targets, expected number of targets killed, or expected fractional damage of the targets. In computing each of these MOE's, several factors must be considered. Some common examples are the accuracy of the weapons systems, their reliability, and the number of weapons available.

Problem Statement

In some research for a class exercise involving a strategic allocation problem, errors were found in some published strategic studies used for the exercise. In its simplest and most general form, the mistake was in the use of an expected value of a random variable in the computation of a very common MOE -- probability of kill for a group of targets. The expected value of the random quantity, number of reliable missiles,was used to represent the variable that should have been left in its random form in the calculation.

The kill probability calculation is in general a function of the reliable number of missiles, and the use of the expected number of reliable missiles as a substitute in this case is inconsistent with the laws of probability concerning functions of random variables. As is more fully explained in Chapters III and IV of this thesis, these incorrect uses of the expected number of reliable missiles or warheads cannot be justified on the grounds that they lead to expected value formulas for kill porbability. The formulas discussed for kill probability are the types of functions of random variables (for example, the number of reliable missiles) in which the expected value of the function is in general not equal to the function of the expected value.

For example, if R represents the probability that a single-warhead missile performs without mechanical failure (is reliable), and if X represents the number launched of that type missile, then the number of reliable missiles per launch is a binomial random variable with sample size equal to X and binomial parameter equal to R. The expected number of reliable missiles per launch is then R times X, but kill probability formulas that are based on the number of reliable missiles should not, in general, substitute the expected value for the random variable. Other examples of this type of expected value error are analogous -- the use of the expected number of discriminated decoys and of the expected number of offensive missiles surviving the defenses, in place of their respective random variables.

Not only are these expected value errors conceptually incorrect, but they lead to serious numerical errors for some parameter values. In computing the number of single-warhead missiles that are necessary to destroy silo targets, the incorrect use of the expected number of missiles may lead to differences of more than 100% in terms of missiles allocated and effectiveness gained.

Chapter Topics

Following this chapter, the results of the literature search for this thesis are discussed. A general framework for targeting models is presented, with special considerations given to the assumptions and applicability of the models. The literature search determined that there is a definite lack of targeting models in the open literature that give kill probability formulas for missiles that both are unreliable and have multiple-warhead capabilities. The expected value errors of the type examined in this thesis were not found to be common in the open literature, but where found, they caused significant numerical errors. Some examples of expected value errors found are given in Chapter II, but are not examined in detail.

In Chapter III, a simple example of expected value error is examined in great detail. For single-warhead missiles, using, perhaps implicitly, the expected number of reliable missiles in calculations of kill probability against silo targets leads to serious numerical errors in terms of numbers of missiles allocated and in terms of kill probability gained. In Chapter III, an incorrect missile allocation method is given and proved to be incorrect. This allocation method tends to produce a consistent overestimation of kill probability for unreliable missiles and thus an underestimation of the number of missiles necessary to obtain any given probability of kill. These two results are proven mathematically along with the examination of effects of parameter changes. Sample missile allocations and resultant kill probabilities are calculated for various parameter values and show highly significant errors resulting from the incorrect use of the expected number of reliable missiles.

In Chapter IV of this thesis, a series of articles in published targeting literature is examined. The articles by Dr. Kosta Tsipis of the Stockholm International Peace Research Institute contain incorrect formulas for kill probability for multiple-warhead missiles. These formulas incorrectly use the expected number of reliable warheads from the same missile in place of the random variable of reliable warheads. Tsipis's formulas lead to overestimation of kill probability, as did the simpler example presented in Chapter III. A correction of Tsipis's formula is given for multiple-warhead missiles against silo targets. Comparative calculations of kill probability for multiple-warhead missiles in present and projected U.S. and Soviet arsenals are tabled. These tabled values show significant numerical errors that arise from Tsipis's formula.

Finally, in Chapter IV, a very general formula for kill probability for multiple-warhead missiles against point (silo) targets is presented. This formula is flexible enough to include cases where warheads from a single missile may be sent more than one target. The formula is also easily adaptable to linear programming for optimal missile allocation.

Objectives and Scope

The objectives of this report are simple. The first one is to show conceptually that a particular type of error exists in kill probability calculations, at least in some instances. Second is to determine how widespread these types of expected value errors are in the open literature on targeting. When expected value errors are found in published formulas, the third objective is to determine the numerical errors caused by the erroneous formulas. Special emphasis will be focussed on the most recent studies -- later than 1970.

The scope of this report, due to the size and importance of the broad field of strategic allocation, is very limited in nature. Only strategic allocation problems will be considered, although many types of calculations are common to both the strategic and tactical fields. Instead of studying both missile and bomber allocation, this report will discuss only missile allocation because it is relatively simpler in terms of the number of factors to be considered, while the error can be illustrated in both types of allocation. This analysis is not intended to be a comprehensive methodological review of strategic allocation -- it is simply an analysis of one type of calculation error and its applicability. A more general topic related to this error is the use of expected values of random variables in intermediate stages of multistage calculations. This related topic will be discussed in the report, but only as a peripheral issue. Only the military effectiveness aspects of strategic allocation problems will be discussed, ignoring the costing that is the focus of some allocation problems. This particular report is solely a mathematical and analytical work, taking as given data on nuclear effects, guidance systems, targeting, and operational considerations.

II. Results of the Literature Search

A wide variety of open literature on missile targeting has been written since 1970 -- variety in both applicability and complexity. The purpose of this report is not to present a general survey of targeting literature, but a limited background is necessary for further chapters. Two general surveys of unclassified material are available that are very useful.

A general analytical survey was presented by A. Ross Eckler and Stefan A. Burr in a book sponsored and published by the Military Operation Research Society (MORS) in 1972 [Eckler, 1972]. This very useful reference provides a very solid mathematical background for targeting analysis. Eckler and Burr seem to organize mathematical targeting models into two types of classifications. The first is to classify a given model by its Criterion of Effectiveness, which depends partly on the assumption and applicability of the model. The second classification is by the mathematical solution technique used by the model.

A general survey on missile allocation was presented by Samuel Matlin in 1970 in Operations Research [Matlin, 1970: 334-373]. Matlin's article uses two types of classification. It classifies the submodels involved -- the targeting aspects that every model must address. It also classifies the models themselves into four types based on applicability.

Following summaries of the Eckler/Burr and Matlin classifications, an eight part characterization of missile targeting literature will be presented.

Methodology

The methodology for this thesis was a simple one. The research was accomplished in two distinct parts. One phase or category of research was a literature search to determine where expected value errors occurred and their prevalence. The second category of research included analyses of published articles or models where these types of errors occurred. These analyses involved the correction of erroneous formulas with special emphases on the numerical errors caused by the incorrect formulas. Two analyses of erroneous kill probability formulas are presented in Chapters III and IV of this thesis.

The literature search for the thesis was concentrated solely in the open published literature -- no classified material was examined. Only strategic articles and studies were reviewed -- tactical analysis was ignored. Most of the articles were centered around kill probability calculations for point targets like silos, but some of the models included area targets like cities and industrial complexes. Primarily, targeting models were studied that included unreliable missiles -- missiles that have positive probabilities of failure at some stage of operation.

The articles themselves were mainly found through the government reports indexes, the Defense Documentation Center, and the Science Citation Index. Once the Tsipis series of articles, which are analyzed in Chapter IV of this thesis, were found, much of the literature search was centered around material related to those articles.

Once found, all the targeting models were analyzed with particular emphases on each targeting model's assumptions and probability calculations. If the articles or studies were found to have erroneous probability formulas, the correct formulas for these calculations were derived for the same assumptions. Where possible, extensive sets of probability values were calculated to determine the numerical seriousness of any errors. In Chapters III and IV of this thesis, two analyses of expected value errors are presented. In Chapter III, kill probability formulas for single-warhead missiles against silo targets are examined; in Chapter IV, kill probability formulas for multiple-warhead missiles are studied.

Mathematical Models Survey From Eckler and Burr

The MORS work by Eckler and Burr provides a good foundation for analysis of targeting. It begins with general formulas for kill probabilities under a large variety of assumptions and then discusses specific mathematical models and their applicability. The book admittedly has defensive missile optimization as a frame of reference and the models discussed are generally designed to be used without computer solutions (although not exclusively so).

Classification By Criterion of Effectiveness. Eckler and Burr provide a classification of targeting models by their effectiveness criteria. For example, if a model's Measure of Effectiveness (MOE) were percentage of targets destroyed, then a sample criterion would be to pick the feasible strategy that would maximize the percentage of targets destroyed. Criteria of effectiveness for targeting models are largely determined by the model's frame of reference (offense or defense) and by the types of targets. Criteria of effectiveness given by Eckler and Burr are [Eckler, 1972: 3-6]

1. Maximize (minimize) the expected number of targets destroyed, for offense (defense).
2. Maximize (minimize) the expected value of the targets destroyed, for offense (defense).
3. Maximize (minimize) a uniform probability of kill across a group of targets, for offense (defense).
4. Minimizing the expected surviving number of preferentially ranked targets, ranked by offense, for offense.
5. Maximize the expected surviving number of preferentially ranked targets, ranked by defense, for defense.
6. Maximize the probability that no targets are destroyed, for defense.
7. Maximize the probability that all targets are destroyed, for offense.

<u>Classification By Mathematical Solution Technique</u>. A second classification scheme used by Eckler and Burr is by the mathematical technique used to solve for the optimal

missile allocation. Some targeting models require the use of more than one technique from the following [Eckler, 1972: 8-11]:

1. Assumption of continuity for functions of integral-valued numbers of missiles.
2. Standard elementary optimization -- optimize a single objective function with calculus and algebra.
3. Langrangian multiplier optimization.
4. Linear Programming.
5. Game theory.
6. Dynamic Programming.
7. Direct optimization -- for a given proposed solution, show that the solution satisfies optimality conditions and is unique.
8. Monte Carlo method.
9. Search techniques.

Eckler and Burr proposed that game theory be used more frequently, since "game theory can be defined as the mathematical theory of conflict." [Eckler, 1972:11]

Missile-Allocation Literature Survey From Matlin

Samuel Matlin produced a general survey of targeting literature as an annotated bibliography [Matlin, 1970:334-373]. Although somewhat outdated at this time, Matlin's survey outlines a broad approach of classifying missile-allocation models. Primarily from an offensive frame of reference, Matlin provides short abstracts of thirty-nine articles. Matlin explicitly defines the applicability of missile-allocation models;

> The problem considered is: given an existing weapon force and a set of targets, what is the optimal allocation of weapons to target? This is not to be confused with the force-mix problem, which asks: what weapon mix should be developed, under constraints of time and money, to maximize the damage to the enemy? Target allocation is actually a submodel of the latter problem . . . [Matlin, 1970:334]

Matlin has two classification schemes for missile-allocation models.

Matlin lists five types of submodels that are universal to allocation models. Each model should address the issues of [Matlin, 1970:337-346]:

1. The weapon system.
2. The target complex.
3. The engagement.
4. The damage submodel (measures of damage).
5. The solution algorithm.

Matlin classifies the surveyed targeting models into four types [Matlin, 1970:346-357]. The first two are the allocation of single weapon types, and, the allocation of multiple weapon types. The third is a group of game theory models -- models that involve active strategies by both attacker and defender. Matlin's fourth model classification is a group of special-feature models that do not fit into the three previous classifications. Some "special features" listed by Matlin are:

1. Targeting in the small -- targeting at a small group

of "local" area targets.

2. Force structuring.
3. Indirect values -- some intrinsically valueless targets, such as command and control centers, that contribute to the values of other targets.
4. Defense orientation.
5. Special optimization criteria -- other than expected value killed or survived.

An Alternative Characterization of Targeting Models

Based on the literature search for this thesis, an alternative characterization of models may be proposed that more closely suits the analysis for this report. It is a general method for characterizing missile models by examining the aspects of targeting problems. Strategic targeting models generally seem to address certain common issues and the assumptions underlying them. The characterization includes only the analytical parts of strategic targeting and not the deterrent or psychological aspects. The following characterization borrows some of the information from the classification schemes of Eckler and Burr and of Matlin. Each missile targeting model should explicitly address at least the following issues:

1. User of the model, or the frame of reference (offense or defense).
2. Types of targets.
3. Measures of Effectiveness (MOE's).

4. Missile types.

5. Independence of events.

6. Quantity of information.

7. Costs -- inclusion or omission.

8. Solution techniques for missile allocation.

The types and values of targets must explicitly be stated. Targets can roughly be categorized into two types. Force targets are enemy missile silos and military installations that could threaten the attacker. Value targets are population and industrial centers, and non-threatening military installations. In general, these value targets may be assigned different values by the attacker (and by the defender), either in numerical or priority scales. Besides the type of targets, if the model user is the attacker, he must learn about the defenses at the targets, if any.

Physical attributes of all possible weapons, such as accuracy, reliability, size, power, and range, must be considered. But beyond the parameter descriptions of missiles, their functions and operations are usually clearly stated. Some missile operation questions that may affect strategy are whether or not the missiles are offensive or defensive, land-based or sea based, single-warhead or multiple-warhead. An important additional description is the number of distinct stages of operation -- booster launch, booster flight, re-entry vehicle (RV) separation, RV flight, and RV detonation.

Related to the description of operation stages is a clear delineation of which events in a strategic engagement are independent. For example, for some strategic targeting problems, availability of one group of missiles may affect the availability of others, or missiles launches may not be independent of one another, or successful detonation of one warhead may hamper the successful guidance of following re-entry vehicles (Fratricide). Clear descriptions of events like these and the assumptions that support them help lead to a clear understanding of a targeting problem and can have a marked influence on probability calculations.

Quality of information may be the prime determinant of strategy and engagement outcomes. There are at least three types of information that apply to strategic conflict. First is the quality of information about the physical parameters of the arsenals of both sides. A strategist wants to understand how well-defined his estimates of, say accuracy and reliability, are for both himself and his opponent. A second type of information is prior information about the strategy of the opponent, if two-sided stragety is considered. Third, the availability of intermediate information in a strategic conflict is important. The re-assignment of missiles to different targets to replace unreliable missiles or to destroy yet unharmed targets is only possible if intermediate information is available. Massive strikes at pre-assigned targets, or one-wave strikes, ignore the value of such information.

Very few missile targeting models consider the dollar costs of missiles. Most are missile-allocation models like those surveyed by Matlin. Consideration of costs as a criterion by a model is a type of force-structuring.

The list of mathematical solution techniques from Eckler and Burr (repeated above) is a comprehensive one. Solution techniques can be loosely classified as analytical (for example, Linear Programming and Lagrangian optimization) or as based on simulation (for example, game theory and Monte Carlo theory).

Limited Literature Discussion Of Missile Allocation For Unreliable Missiles With Multiple Warheads

In the literature search that was centered on kill probabilities for point targets, very little information was found on kill probabilities for missiles that are unreliable and have multiple-warhead capability. If it is true that most future United States Missiles will have at least Multiple Independently-targeted Re-entry Vehicles (MIRV's), and since physical missiles always have a positive probability of failure, there definitely needs to be more allocation techniques in the open (unclassified and available) literature for these types of missiles.

The only strategic model that was examined that is used by the military as a planning tool is the Arsenal Exchange Model (AEM). The mathematical formulation for the AEM (seventh revision) was produced in 1973 Bosovich, 1973 . The chapter on damage functions -- probabilities of kill -- do not contain any general formulas for kill probabilities for MIRVed weapons, although unreliability is included [Bosovich, 1973:IV-B-1--IV-B-30]. The prototype arsenal examples [Bosovich, 1973:20-21] used throughout the model consist of single-warhead missiles.

Dr. Kosta Tsipis of the Stockholm International Peace Research Institute offers a kill probability formula for unreliable, MIRVed missiles but, as is discussed in Chapter IV of this report, Tsipis's equations for this type of missile are wrong. On the thirty-nine annotated references given by Matlin, only one [Morgan, 1968] considers multiple-warhead missiles in its damage calculations. But the same article, according to Matlin, does not directly address the issue of reliability.

For the single-wave type of missile attack against hard targets that is emphasized in this thesis, *no formula for* kill probability was found that is similar to any of Eqs. (4.9), (4.10), and (4.20), which are correct. General formulas like those seem to be difficult to find in the open literature.

Prevalence of the Type of Errors Discussed In This Report

Strategic targeting models and studies were researched to determine the prevalence of a type of error found in some models. Loosely stated, the error consists of using expected values of weapons in probability calculations when the numbers of weapons are actually random variables. For example, the number of reliable weapons in any attack wave is actually a random variable related to the missile's reliability and to the number launched. Similarly, the number of warheads that penetrate enemy area defenses is actually a random variable related to the number of reliably performing missiles and the probability of penetration for any given missile. Substituting the expected values of either of these two random variables in place of the random variables itself may lead to serious errors in probability of kill calculations. For example, the fact that seventy percent of missiles launched are expected to perform reliably is not, in general, a sufficient reason for assuming that seventy percent of the missiles will perform reliably each launch. In Chapter III, a simple but general example of this type of error is examined. For non-MIRVed missiles, using the expected value of reliable missiles in kill probability calculations leads to serious errors in some problems. In Chapter IV, a more complicated, but specific example from a series of published articles on targeting is examined. For MIRVed missiles, using the expected number of reliable re-entry vehicles can lead to serious errors in probabilities of kill.

A simple error of the type discussed in Chapter III of this paper can be found in a Systems Analysis book published by the Industrial College of the Armed Forces [Snyder, 1967: 61-84]. The methodology for this case study advises computing a necessary missile allocation in two stages -- first, to compute the number of reliable missiles necessary for a given kill probability, and, from this result, compute the necessary number to be launched from the formula relating the number of missiles launched to the expected number reliable. This approach is incorrect and is explained in Chapter III.

The Arsenal Exchange Model (AEM) seems to make an error of this type in one part of its calculation of kill probability against defended targets [Bosovich, 1973:IV-B-12--IV-B-15]. In a complicated attack with decoys and defense, AEM defined d as the number of decoys per warhead, p_D as the probability that a given warhead is discriminated by the enemy. Then UFØ, defined as the number of undiscriminated objects per warhead, is computed by AEM to $UFØ = 1 + (1 - p_D)d$. In reality, the number of undiscriminated objects per warjead is a random variable with expected value $E(UFØ) = 1 + (1 - p_D)d$. AEM uses its calculation of UFØ to calculate other quantities and probabilities throughout the model. The use of this expected value is of the type discussed above and seems to be conceptually incorrect. Due to the complexity of the AEM, it was not determined whether this conceptual mistake leads to serious numerical errors in the results of the AEM. This could possibly be a simplifying assumption in the AEM, except that other similar random variables are included as random variables and not as expected values.

In the literature researched for the thesis, it was found that the incorrect use of expected numbers of missiles or warheads is not a common error. Early research did show some examples of incorrect usages, and these are explained above. But the majority of the open articles examined did not make these types of expected value errors.

III. Conceptual Discussion of a Particular Error

One of the most common Measures of Effectiveness (MOE's) used in strategic targeting calculations is the computed probability of kill, or Probability of Destruction (PD), of a selected group of targets. For strategic targets classified as point targets, PD is often easy to compute. Also, the use of PD is often applicable across broad classes of targets and thus provides a common MOE for the evaluation of alternative allocations of weapons. One possible allocation plan is to accept some common level of PD for a group of targets while minimizing the use of resources necessary to gain that PD level. For instance, decision-makers may conclude that it is desirable to have a probability of 0.8 of destroying a particular group of fifty Soviet silos. Then it might be necessary to find out how many Minuteman III missiles, for instance, would need to be targeted at the silos.

Problem Definition

A specific example of the type of errors discussed in this study will be examined in this chapter. This type of strategic problem, while exceedingly simple, is very common in targeting literature and amply illustrates how an error of this type can be made. It was the discovery of incorrect treatments of a similar strategic problem in some published targeting articles that caused this question to be researched in more detail.

Objectives and Assumptions. This targeting problem can be stated as follows. Suppose that a particular group of targets are to be destroyed by an allocation of one type of missile. The targets can be considered point targets, such as enemy missile silos. Suppose that the targets are to have an equal probability of being destroyed, and call this common level of PD the Desired Probability of Destruction (DPD) for the targets. The objective of this hypothetical problem is to find the minimum number of missiles per target that are necessary to obtain this DPD. The targets are considered equal-valued and therefore the same number of missiles will be sent to each target. For this simple problem, assume that there is an unlimited number of identical missiles and that they do not have multiple-warhead capability. These missiles are to be independently launched and their detonation can be timed such that no fratricide will occur.

Definition of Reliability Parameter R. An important parameter to consider for this problem is the mechanical reliability of the missiles. Reliability as used here is meant to include all aspects of missile system operation except for accuracy. Important components of total missile reliability are launch reliability, stage or flight reliability, and detonation reliability. This total system Reliability (R) can be considered to represent the overall fraction of missiles that will function reliably from countdown through detonation (excluding accuracy). Or equivalently, this parameter R represents the probability that any particular missile will perform reliably. Defined in this way, this parameter R can be used to find, for a group of missiles targeted, the expected number of missiles that would be successful through detonation if the order to launch were given. That is, the expected value, in a probabilistic sense, of the number of reliable missiles can easily be found -- this expected value is the product of reliability R times the total number of missiles targeted, or called into action.

Definition of Parameter P. One other parameter that is important in evaluating missile system performance is the single-shot terminal probability of killing the target. This probability, P, is a conditional probability, and is conditioned upon the missile system's effectiveness through detonation. If the missile's warhead is assumed to detonate upon ground impact, the calculation of P depends on three factors -- the accuracy of the missile, the yield (Y) or megatonnage of the warhead, and the hardness of the target. The accuracy of a missile is usually expressed in Circular Error Probable (CEP), defined as the distance such that a circle of radius CEP would be expected to contain one-half the landings of all the missiles targeted for the center of the circle. In the case of silo hard targets, the hardness (H) of a target is defined as the minimum blast overpressure that would be required to render the contents of the silo ineffective. The hardness of a silo is usually measured in pounds per square inch (psi). It is not the purpose of this thesis to examine nuclear blast effects, but one way to calculate the conditional terminal probability of kill P is given in [Tsipis, 1974:37-38]. If yield Y is expressed in megatons, CEP in nautical miles, and H in pounds per square inch, then:

$$P = 1 - e^{-1/2 \frac{Y^{2/3}}{(CEP)^2 (0.19 - 0.23\, H^{1/2} + 0.068\, H)^{2/3}}} \qquad (3.1)$$

Summary List of Problem Aspects. With the assumption that enemy defenses are incapable of preventing missile attack, the facts and assumptions of this allocation problems are as follows:

1. Equal-valued strategic point targets (silo)
2. Identical single-warhead missiles
3. Well-known parameters of missiles and targets
4. Independently launched missiles and exclusion of fratricide
5. No enemy defenses
6. Goal of finding minimum number of missiles, given DPD

First Solution Method

One possible approach to the targeting allocation problem outlined above is found in some of the open literature on targeting. This method, which shall be called Approach One or Method One, for lack of a better name, can be summarized as follows. By isolating on a single target, the number of missiles that are necessary to gain some level of Desired Probability of Destruction (DPD) is denoted by X_1. The solution X_1 is the minimum number of missiles targeted to each silo. This solution approach begins by solving for the number of reliable missiles needed, and then proceeds to find out how many missiles actually need to be targeted at each silo.

Two Step Approach. The effective, or reliable, number of missiles that are targeted to a silo is defined as the number of missiles that actually land and detonate; denote this general quantity by N. Then, if P is known, it is straightfoward to calculate the Probability of Destruction (PD) from any value of N:

$$PD = 1 - (1 - P)^N \tag{3.2}$$

For this problem, PD is desired to be DPD. So, the solution for any given DPD is:

$$N_1 = \frac{\ln(1 - DPD)}{\ln(1 - P)} \tag{3.3}$$

Thus, N_1 reliable missiles must be targeted at each silo to obtain DPD. Since R represents the average fraction of missles that are reliable, an expected value approach might lead one to compute the allocation X_1 from the parameter R and from N_1, the necessary number of effective weapons.

That is,

$$X_1 = \frac{N_1}{R} \tag{3.4}$$

From this technique, X_1 missiles need to be targeted at each silo to obtain a probability equal to DPD of killing the silo.

The solution technique outlined in Eqs. (3.2)-(3.4) can be synthesized into an equivalent expression for PD for any general amount of missiles X:

$$PD = 1 - (1 - P)^{RX} \tag{3.5*}$$

Then, by setting PD equal to the given DPD, the solution for X_1 is equivalent to that of Eqs. (3.2)-(3.4):

$$X_1 = \frac{\ln(1 - DPD)}{R \ln(1 - P)} \tag{3.6*}$$

The logic that seems to underlie the two-step approach of Eqs. (3.2)-(3.4) is based on what can be called "expected value grounds". One can easily compute the minimum number of reliable missiles necessary, N_1, and for any amount of targeted missiles X, one can easily compute the expected value of those that will be reliable. Therefore, it should follow that these two computations can be combined because they are individually correct. The implicit assumptions in this reasoning are that the expected value of PD is all that is needed, and that Eqs. (3.2)-(3.4) are based on an expected value calculation of PD.

Fallacy of First Method. In truth, the number of reliable missiles needed, N_1, can be computed from Eq. (3.3). And the expected number of reliable missiles, E(N), if X missiles are targeted, is:

$$E(N) = RX \tag{3.7}$$

So individually, the two calculations are correct, but their combination is not. Eq. (3.5) is equivalent to:

$$PD = 1 - (1 - P)^{E(N)} \tag{3.8*}$$

This calculation is invalid, even if based on expected value terms. The expected value of PD is not given by Eq. (3.8). In this case, the expected value of N, equal to RX, was used in the calculation of the expected value of PD. From the laws of probability, for any function f and random variable Y, E(f(Y)) is in general not equal to f(E(Y)). It is clear that N, the number of reliable missiles, is in reality a binomial random variable with parameter equal to the reliability R and sample size equal to the number of missiles launched X.

Therefore, Eqs. (3.5), (3.6), and (3.8) are incorrect combinations of individually correct calculations. These three equations have been labelled with asterisks to prevent confusion. Thus this solution technique, embodied in Eqs. (3.2)-(3.4), is an incorrect way to approach this simple strategic problem.

Second Solution Method

In the preceding paragraphs, a possible approach to solving a particular strategic targeting problem has been shown, and that this approach is invalid. It is probably easier to show that an alternative approach is correct, and to highlight the differences between the two methods.

Using the same notation and terminology as before, R was defined as the probability that a given missile performs reliably through detonation. The conditional probability P was defined as the probability that a given missile will destroy the target, conditioned on successful detonation. Then it is clear that the single-shot unconditional probability of destroying the target is RP. Then, if X missiles are targeted at the silo, this multiple shot unconditional probability of killing the target should be:

$$PD = 1 - (1 - RP)^{X} \tag{3.9}$$

If PD is set to the desired level, DPD, then the necessary number of missiles computed under this method, X_2, is found from:

$$X_2 = \frac{\ln(1 - DPD)}{\ln(1 - RP)} \tag{3.10}$$

If the system parameters P and R are well-defined, then the solution from this second method, given in Eq. (3.10), is the accurate one.

First Discrepancy Measure X_1/X_2

There is clearly a conceptual mathematical difference between the two methods, as shown by contrasting the two solutions given in Eqs. (3.6) and (3.10). Given that the error committed in Eq. (3.6) is a fairly common one, then an important consideration is the determination of the empirical properties of the discrepancy.

Formula For X_1/X_2 Ratio. One possible numerical measure of the discrepancy between the two methods is the simple ratio of their two solutions X_1 and X_2. Referring to Eqs. (3.6) and (3.10), this ratio is:

$$\frac{X_1}{X_2} = \frac{\ln(1 - RP)}{R \ln(1 - P)} \tag{3.11}$$

Note that the value of this ratio is independent of the value of DPD. It can be shown that if the value of the reliability R is less than one, as it is in all realistic cases, then this ratio X_1/X_2 is always less than one. That is, the first method's solution of X_1 missiles is always less than the second method's solution of X_2 missiles. Thus, under the first (incorrect) method, military planners would not get the level of protection, in terms of Probability of Destruction, that was sought. The Desired Probability of Destruction (DPD) would not be gained.

Proof That X_1/X_2 Is Less Than One. The proof that the ratio X_1/X_2 is less than one is based on treating the ratio as a function of three variables -- DPD, P, and R. From Eq. (3.11), it is easily seen that if the reliability R is equal to 1, then the ratio X_1/X_2 is equal to one for all values of P and DPD. And:

$$\frac{\partial}{\partial R}\left(\frac{X_1}{X_2}\right) = \frac{\partial}{\partial R}\left[\frac{\ln(1 - RP)}{R \ln(1 - P)}\right]$$

$$= \frac{1}{\ln(1 - P)}\left[\ln(1 - RP) \cdot \frac{-1}{R^2} + \frac{1}{R} \cdot \frac{(-P)}{(1 - RP)}\right]$$

$$= \frac{1}{\ln(1 - P)}\left[\frac{-(1 - RP)\ln(1 - RP) - (PR)}{R^2 (1 - RP)}\right]$$

$$= \frac{-1}{R^2 (1 - RP) \ln(1 - P)}\left[(1 - RP)\ln(1 - RP) + RP\right] \tag{3.12}$$

To derive the sign of $\frac{\partial}{\partial R}\left(\frac{X_1}{X_2}\right)$, examination of the last term in Eq. (3.12) shows that the sign is the same as that of the quantity in the square brackets, $(1 - RP)\ln(1 - RP) + RP$. This latter quantity is zero if R is zero. Taking the partial derivative of this quantity with respect to R:

$$\frac{\partial}{\partial R}\left[(1 - RP)\ln(1 - RP) + RP\right]$$

$$= \left[(1 - RP)\frac{(-P)}{(1 - RP)} + (-P)\ln(1 - RP) + P\right]$$

$$= -P \ln(1 - RP) \tag{3.13}$$

This partial derivative is positive for all P and R both greater than zero and less than one, that is, for all of the possible ranges of P and R. Since the bracketed quantity in the last term of Eq. (3.12) is zero for R equal to zero and its partial derivative with respect to R is positive for all R greater than zero, then the quantity is positive for R greater than zero. Therefore, the partial derivative of the discrepancy ratio X_1/X_2 with respect to R is positive. And since, for R equal to one, the ratio X_1/X_2 is equal to one, then X_1/X_2 is less than one for all R lying in the open interval (0.0, 1.0). That is,

$$R \in (0.0,\ 1.0)$$

$$X_1/X_2 = 1 \text{ for } R = 1$$

$$\frac{\partial}{\partial R}\left(\frac{X_1}{X_2}\right) > 0 \text{ for } R < 1$$

taken together imply:

$$X_1/X_2 < 1 \tag{3.14}$$

Therefore, in all realistic cases where the probability of successful detonation is less than unity, the minimum number of missiles computed under the first (incorrect) method is less than the actual missiles needed. So for this type of problem, the first method is always biased towards underprotection, or lower probability of destroying the targets. A graph of the discrepancy ratio X_1/X_2 versus reliability R illustrates the effects of changes in R. Such a graph is shown in Fig. (1), with P equal to 0.7.

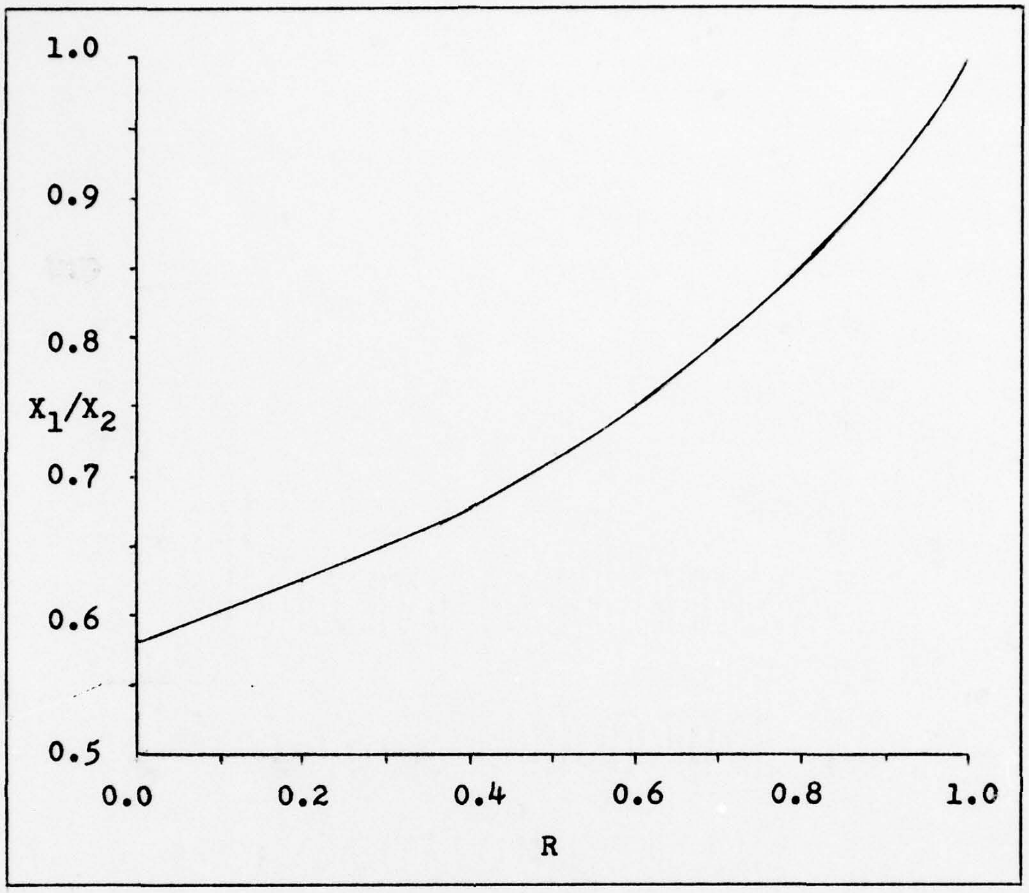

Figure 1. Discrepancy Ratio X_1/X_2 Versus Reliability R With DPD = 0.8, P = 0.7

Second Discrepancy Measure PD_1/PD_2

Possibly a more important question than the discrepancy in necessary missiles is the resultant discrepancy in the Probability of Destruction (PD) due to the first (incorrect) method. It is clear that if the number of missiles calculated from the first method is less than the actual number of missiles needed, then the actual PD gained from the first method is lower than that being sought.

Formula For PD_1/PD_2 Ratio. A more useful discrepancy measure would be the calculation of the ratio of the PD gained from the first method to the PD gained from the second. Defining PD_1 as the actual PD gained from using X_1 missiles, PD_1 can be calculated by substituting X_1, from Eq. (3.6), for X in Eq. (3.9):

$$PD_1 = 1 - (1 - RP)^{\frac{\ln(1 - DPD)}{R \ln(1 - P)}} \tag{3.15}$$

This would be the actual probability of destroying the targets if X_1 missiles were targeted at each one. Since the probability of destruction obtained from the second method's solution X_2 is precisely DPD, then the ratio of PD_1 to PD_2 is:

$$\frac{PD_1}{PD_2} = \frac{1}{DPD} \left(1 - (1 - RP)^{\frac{\ln(1 - DPD)}{R \ln(1 - P)}} \right) \tag{3.16}$$

An alternative formulation for PD_1/PD_2 that is useful for some analysis is:

$$\frac{PD_1}{PD_2} = \frac{1}{DPD} \left(1 - (1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right) \tag{3.17}$$

This last expression is equivalent to Eq. (3.16) since, for any general a,b,c:

$$a^{b \ln(c)} = c^{b \ln(a)} \tag{3.18}$$

Simple Proof that PD_1/PD_2 Is Less Than One. This discrepancy calculation PD_1/PD_2 can be viewed as a measure of error caused by using the first method. This is a useful measure because it is the ratio of the Measures of Effectiveness that are the actual outputs of the two methods. Since the true expression for PD given in Eq. (3.9) is an increasing function of missiles X, and X_1 is less than X_2, then it follows that PD_1 is less than PD_2 for all parameter values DPD, P and R between zero and one.

Mathematical Properties of PD_1/PD_2

In analyzing the discrepancy between these two solution methods for this simple strategic problem, it is useful to examine the mathematical properties of the discrepancy measure PD_1/PD_2 and its empirical qualities. For this measure, the mathematical properties are well-defined and the numerical errors caused by the first method can be significant, depending upon the values of the three parameters -- P, R, and DPD. By examining the effects of changes in the individual parameters, "worst-case" and "best-case" situations can be found.

Since, among various strategic problems of this type, the parameters DPD, P, and R can assume wide ranges of values between zero and one, the discrepancy measure PD_1/PD_2 can be considered a function of three variables. By examining the partial derivatives of PD_1/PD_2 with respect to the three parameters DPD, P, and R, one can determine at which parameter values the error is serious.

Proof That PD_1/PD_2 Increases With R. Repeating the alternative expression for PD_1/PD_2 that is given in Eq. (3.17):

$$\frac{PD_1}{PD_2} = \frac{1}{DPD} \left(1 - (1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right) \tag{3.19}$$

From this expression, it is easily seen that if the reliability R assumes the value one, the ratio PD_1/PD_2 is equal to one also because the power to which the quantity (1 - DPD) is being raised is unity if R is equal to one. That is, there is no difference in the two methods, regardless of the values of the other two parameters DPD and P, if R is one -- there is no error involved in using the first method in this case. So the error only arises when R is less than one, which agrees with the earlier result of identity between X_1 and X_2 only if R is one.

From Eq. (3.19), the partial derivative of PD_1/PD_2 with respect to R is:

$$\frac{\partial}{\partial R}\left(\frac{PD_1}{PD_2}\right) = \frac{-1}{DPD}\left[(1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}}\right] \cdot \left[\ln(1 - DPD)\right]$$

$$\cdot \frac{\partial}{\partial R}\left[\frac{\ln(1 - RP)}{R \ln(1 - P)}\right] =$$

$$= \frac{-\ln(1 - DPD)}{DPD \ln(1 - P)}\left[(1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}}\right] \cdot$$

$$\cdot \left[\ln(1 - RP) - \frac{1}{R^2} + \frac{1}{R}\frac{(-P)}{(1 - RP)}\right] =$$

$$= \frac{-\ln(1 - DPD)}{DPD \ln(1 - P)} \left[(1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right] \cdot$$

$$\cdot \left[\frac{-1}{(1 - RP) \quad R^2} \right] \left[(1 - RP) \ln(1 - RP) + RP \right] \quad (3.20)$$

Careful examination of the last term of Eq. (3.20) shows that, with all parameters DPD, P, and R greater than zero and less than one, $\frac{\partial}{\partial R}\left(\frac{PD_1}{PD_2}\right)$ has the same sign as the last quantity in the last term: $\left[(1 - RP) \ln(1 - RP) + RP \right]$. But the quantity $\left[(1 - RP) \ln(1 - RP) + RP \right]$ was shown to be positive for all realistically possible parameter values when the ratio X_1/X_2 was analyzed (Eq. (3.13)). Therefore, the partial derivative of PD_1/PD_2 with respect to R is positive. That is, as the reliability decreases from a maximum value of one, the discrepancy ratio PD_1/PD_2 decreases, or equivalently, the error is greater. As R decreases, the underprotection, which arises from the first method's miscalculation of the necessary missiles, becomes worse. As R approaches one, the error is less significant. These effects of changes in reliability R are shown in Fig. (2), which is a plot of PD_1/PD_2 versus R for the parameter values DPD = 0.8, P = 0.7.

Proof That PD_1/PD_2 Decreases With P. Similar derivations can give the partial derivative of PD_1/PD_2 with respect to P. From Eq. (3.19):

$$\frac{\partial}{\partial P}\left(\frac{PD_1}{PD_2}\right) =$$

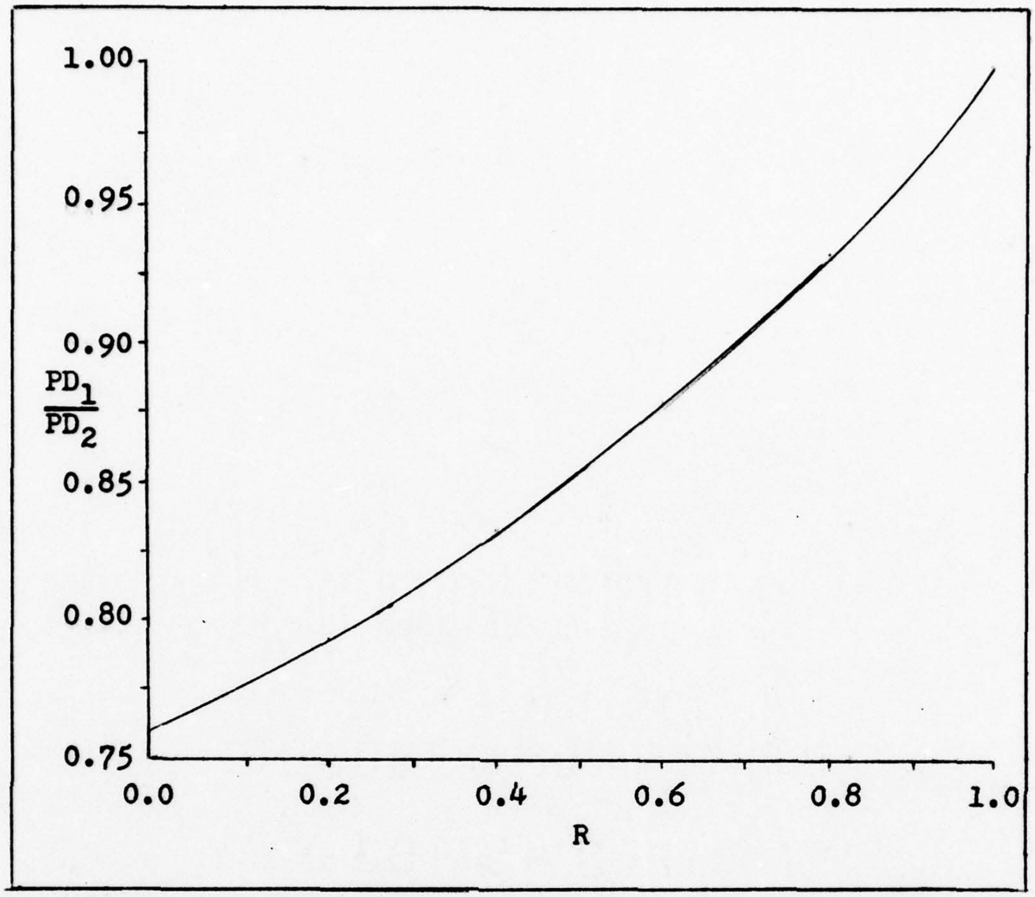

Figure 2. Discrepancy Ratio PD_1/PD_2 Versus Reliability R With DPD = 0.8, P = 0.7

$$= -\frac{1}{DPD}\left[(1 - DPD)^{\frac{\ln(1 - RP)}{R\ln(1 - P)}}\right]\left[\ln(1 - DPD)\right] \cdot \frac{\partial}{\partial P}\left[\frac{\ln(1 - RP)}{R\ln(1 - P)}\right] \tag{3.21}$$

From this expression, the sign of $\frac{\partial}{\partial P}\left(\frac{PD_1}{PD_2}\right)$ is the same as the sign of the last quantity: $\frac{\partial}{\partial P}\ \frac{\ln(1 - RP)}{R\ln(1 - P)}$. And:

$$\frac{\partial}{\partial P}\left[\frac{\ln(1 - RP)}{R\ln(1 - P)}\right] = \frac{1}{R}\left[\ln(1 - RP)\frac{(-1)}{\left[\ln(1 - P)\right]^2}\cdot\right.$$

$$\left.\cdot\frac{(-1)}{(1 - P)} + \frac{1}{\ln(1 - P)}\cdot\frac{(-R)}{(1 - RP)}\right]$$

$$= \frac{1}{R}\left[\frac{\ln(1 - RP)}{\left[\ln(1 - P)\right]^2(1 - P)} + \frac{(-R)}{\ln(1 - P)(1 - RP)}\right]$$

$$= \frac{1}{R}\left[\frac{(1 - RP)\ln(1 - RP) + (-R)(1 - P)\ln(1 - P)}{\left[\ln(1 - P)\right]^2(1 - P)(1 - RP)}\right] \tag{3.22}$$

Since the denominator of the last term of Eq. (3.22) is uniformly positive for R and P in the range from zero to one, the sign of the whole quantity in Eq. (3.22), and thus that of $\frac{\partial}{\partial P}\left(\frac{PD_1}{PD_2}\right)$, is the same as that of the numerator: $(1 - RP)\ln(1 - RP) + (-R)(1 - P)\ln(1 - P)$. *If P were zero*, this last quantity would be zero also. And the partial derivative of this numerator with respect to P is

$$\frac{\partial}{\partial P}\left[(1 - RP)\ln(1 - RP) + (-R)(1 - P)\ln(1 - P)\right]$$

$$= \ln(1 - RP)(-R) + R\ln(1 - P)$$

$$= R\left[-\ln(1 - RP) + \ln(1 - P)\right]$$

$$= R\ln\left[\frac{(1 - P)}{(1 - RP)}\right] \tag{3.23}$$

Since $RP < P < 1$, then the last term of Eq. (3.23) is negative. Therefore, since the numerator of the last term of Eq. (3.22) is zero for P equal to zero and uniformly decreases with P for P greater than zero, then this numerator is negative for all values of positive DPD, P, and R. Then each term of Eq. (3.22) is negative, which implies that

$\frac{\partial}{\partial P}\left(\frac{PD_1}{PD_2}\right)$ is negative also.

This analysis shows that as the missile's single-shot conditional probability P of killing the target is increased, the ratio of probabilities of destruction PD_1/PD_2 gets smaller. That is, as P increases, using the first method solution results in a true probability of destruction that is farther from the goal of DPD. This is the opposite effect of that which occurred for changes in the reliability R. For P close to zero, the error caused by using the first solution would be less significant. A representative graph of PD_1/PD_2 versus P is given in Fig. (3), with parameter values DPD = 0.8, R = 0.5. The value of PD_1/PD_2 is undefined for both P equal to zero and P equal to one.

Proof That PD_1/PD_2 Increases With DPD. Since the values of the Desired Probability of Destruction DPD may vary for different problems, it is useful to know the effects on the ratio PD_1/PD_2 caused by changes in DPD. Referring to Eq. (3.19), the partial derivative of PD_1/PD_2 with respect to DPD is:

$$\frac{\partial}{\partial (DPD)}\left(\frac{PD_1}{PD_2}\right) = -\frac{1}{(DPD)^2} - \frac{1}{DPD}\left[\frac{\ln(1 - RP)}{R\ln(1 - P)} \cdot (1 - DPD)\left[\frac{\ln(1 - RP)}{R\ln(1 - P)} - 1\right] \cdot (-1)\right] -$$

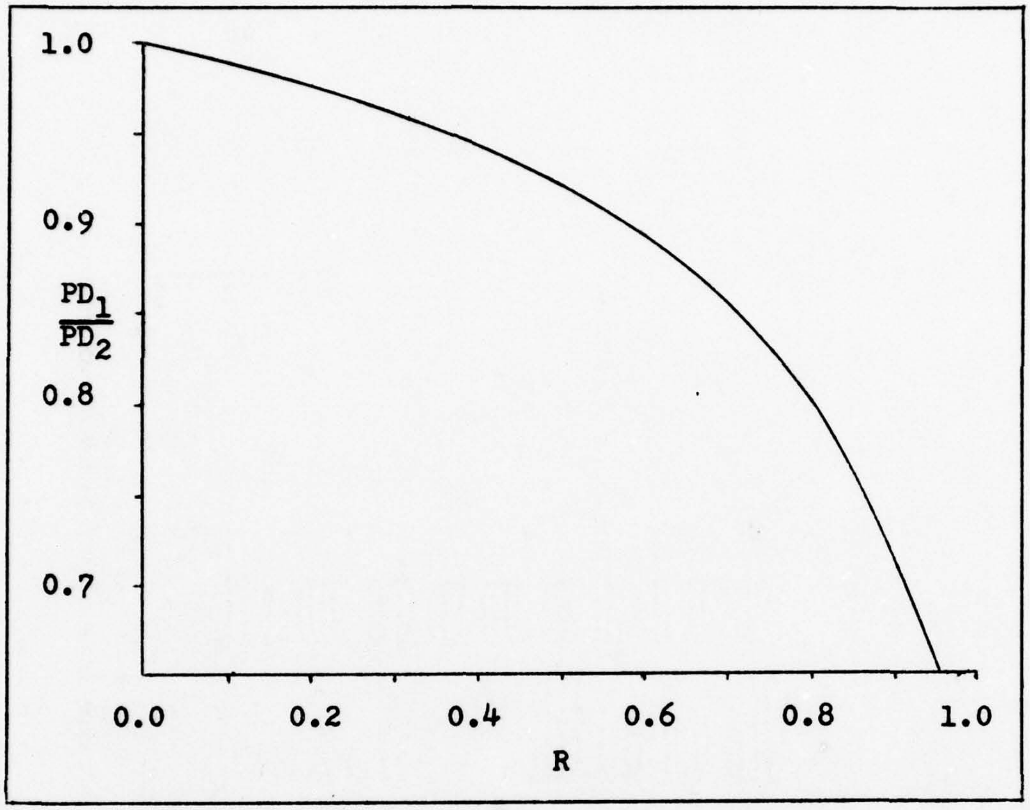

Figure 3. Discrepancy Ratio PD_1/PD_2 Versus P With DPD = 0.8, R = 0.5

$$
\begin{aligned}
&- \left[\frac{(-1)}{(DPD)^2} (1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right] \\
&= \frac{1}{(DPD)^2} \left\{ \left[\frac{DPD \ln(1 - RP)}{R \ln(1 - P)} \right] \cdot \right. \\
&\quad \cdot (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] + \\
&\quad \left. + \left[(1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right] - 1 \right\} \qquad (3.24)
\end{aligned}
$$

Examination of the last term of Eq. (3.24) shows that

$\frac{\partial}{\partial (DPD)} \left(\frac{PD_1}{PD_2} \right)$ has the same sign as the quantity in the braces in the last term. This quantity is equal to zero for DPD equal to zero. And its partial derivative with respect to DPD is:

$$\frac{\partial}{\partial (DPD)} \left\{ \frac{DPD \ln(1 - RP)}{R \ln(1 - P)} (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] + \right.$$
$$\left. + \left[(1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right] - 1 \right\}$$
$$= \frac{\ln(1 - RP)}{R \ln(1 - P)} \left\{ DPD \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] \cdot \right.$$
$$\cdot (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 2 \right] (-1) +$$
$$\left. + (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] \right\} +$$
$$+ \frac{\ln(1 - RP)}{R \ln(1 - P)} (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] (-1) - 0$$
$$= \frac{\ln(1 - RP)}{R \ln(1 - P)} \left\{ DPD \cdot \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] \right.$$
$$\left. (1 - DPD) \left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 2 \right] \cdot (-1) \right\} \tag{3.25}$$

By referring to the formula for the missile ratio X_1/X_2 given in Eq. (3.11):

$$\left[\frac{\ln(1 - RP)}{R \ln(1 - P)} - 1 \right] = \left[\frac{X_1}{X_2} - 1 \right] < 0 \tag{3.26}$$

Therefore, the last term in Eq. (3.25) is positive for all values of DPD, P, and R greater than zero and less than one. Then the last term of Eq. (3.24) is positive, that is,

$\frac{\partial}{\partial(DPD)}\left(\frac{PD_1}{PD_2}\right)$ is positive for the realistic values of DPD, P, and R.

The effect of increasing the chosen value of the Desired Probability of Destruction is the same as that of increasing R -- it results in increasing the discrepancy ratio PD_1/PD_2. That is, as DPD is increased, the error caused by the first method is less significant, if the values of the other two parameters P and R remain constant. Conversely, if DPD is decreased, say for solution sensitivity analysis, the ratio PD_1/PD_2 decreases also, or equivalently, the error becomes more significant. Fig. (4) shows the relationship between PD_1/PD_2 and DPD for constant P = 0.7, R = 0.5.

Among practical problems, the values of the three parameters may vary widely. It is then useful to outline general situations when the first method causes significant errors. In the preceding discussion, the partial derivatives of PD_1/PD_2 with respect to R, P, and DPD were proven to be positive, negative, and positive, respectively. Then, the first method's error is most significant in problems with low R, high P, and low DPD values. That is, the ratio of probabilities of kill is lowest in this situation. Conversely, the error is least significant in problems with high R, low P, and high DPD values. These two situations identify

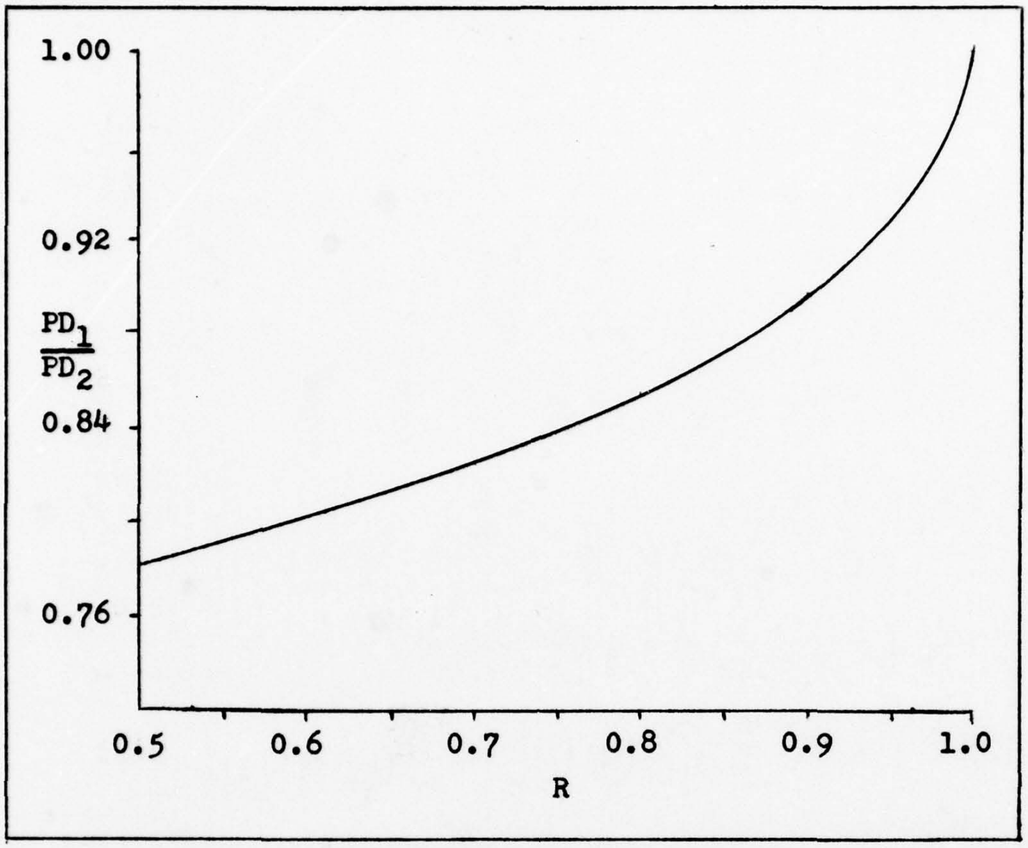

Figure 4. Discrepancy Ratio PD_1/PD_2 Versus DPD With P = 0.7, R = 0.5

"worst-case" and "best-case" conditions.

Effects of Changes of Other Parameters. Two values, R and P, actually depend on other missile system parameters. The reliability R is a single composite figure that depends on the various components of mechanical reliability -- for example, launch reliability and detonation reliability. If any of these individual component reliabilities decrease, the overall reliability R decreases, and so does the discrepancy ratio PD_1/PD_2. The error caused by the first solution method becomes more significant for any decrease in an individual component reliability.

The single-shot conditional probability of kill P, as computed in Eq. (3.1), depends upon three other values -- missile megatonnage Y, missile accuracy CEP, and target hardness H. As Y increases, P obviously increases, causing the ratio PD_1/PD_2 to decrease. As the accuracy measure CEP increases (less accuracy), P decreases, and PD_1/PD_2 increases. As H increases, P decreases, and PD_1/PD_2 increases. That is, the first method's error is most significant for high Y, low CEP, and low H values.

Limit of PD_1/PD_2 As R Approaches Zero. One final mathematical result will be presented for this simple three parameter problem. It is possible to derive a lower limit for PD_1/PD_2 for any given problem. Referring to Eq. (3.17), the limit of PD_1/PD_2 as R approaches zero is given by:

$$\lim_{R \to 0} \left(\frac{PD_1}{PD_2} \right) = \lim_{R \to 0} \left[\frac{1}{DPD} \left(1 - (1 - DPD)^{\frac{\ln(1 - RP)}{R \ln(1 - P)}} \right) \right] =$$

$$= \frac{1}{DPD} \left\{ 1 - (1 - DPD)^{\frac{1}{\ln(1 - P)} \lim_{R \to 0} \left[\frac{\ln(1 - RP)}{R} \right]} \right\} \tag{3.27}$$

The limit of the ratio $[\ln(1 - RP)] / R$ as R approaches zero would be an undefined quantity -- zero divided by zero. From L'Hospital's Rule for limits of ratios:

$$\lim_{R \to 0} \left[\frac{\ln(1 - RP)}{R} \right] = \lim_{R \to 0} \left[\frac{-P(1 - RP)^{-1}}{1} \right]$$

$$= \lim_{R \to 0} \left(\frac{-P}{(1 - RP)} \right) = -P \tag{3.28}$$

Then:

$$\lim_{R \to 0} \left(\frac{PD_1}{PD_2} \right) = \frac{1}{DPD} \left[1 - (1 - DPD)^{\frac{-P}{\ln(1 - P)}} \right] \qquad (3.29)$$

This limit is not valuable as a realistic value for PD_1/PD_2 because R never assumes the value zero. But the limit does provide a lower bound for the discrepancy ratio PD_1/PD_2 for any given problem with known P and DPD.

Empirical Results

The mathematical properties of two alternative solution methods for a simple strategic allocation problem have been discussed in this chapter. It would also be useful to examine their empirical properties to determine the numerical significance of errors caused by the first method. For practical problems, the Desired Probability of Destruction (DPD) for point targets is usually high -- greater than 80% or 0.8. The parameters reliability R and single-shot conditional probability of kill P may assume any values, depending on the individual missile system considered.

Graphs of PD_1/PD_2 Versus R. One way to examine the numerical errors caused by the first method is to graph the discrepancy ratio PD_1/PD_2 for realistic values of the three parameters DPD, P and R. Such a graph is given in Fig. (5).

In this figure, PD_1/PD_2 is plotted versus R for eight combinations of values of DPD and P. These curves illustrate the mathematical properties of the ratio PD_1/PD_2 that were

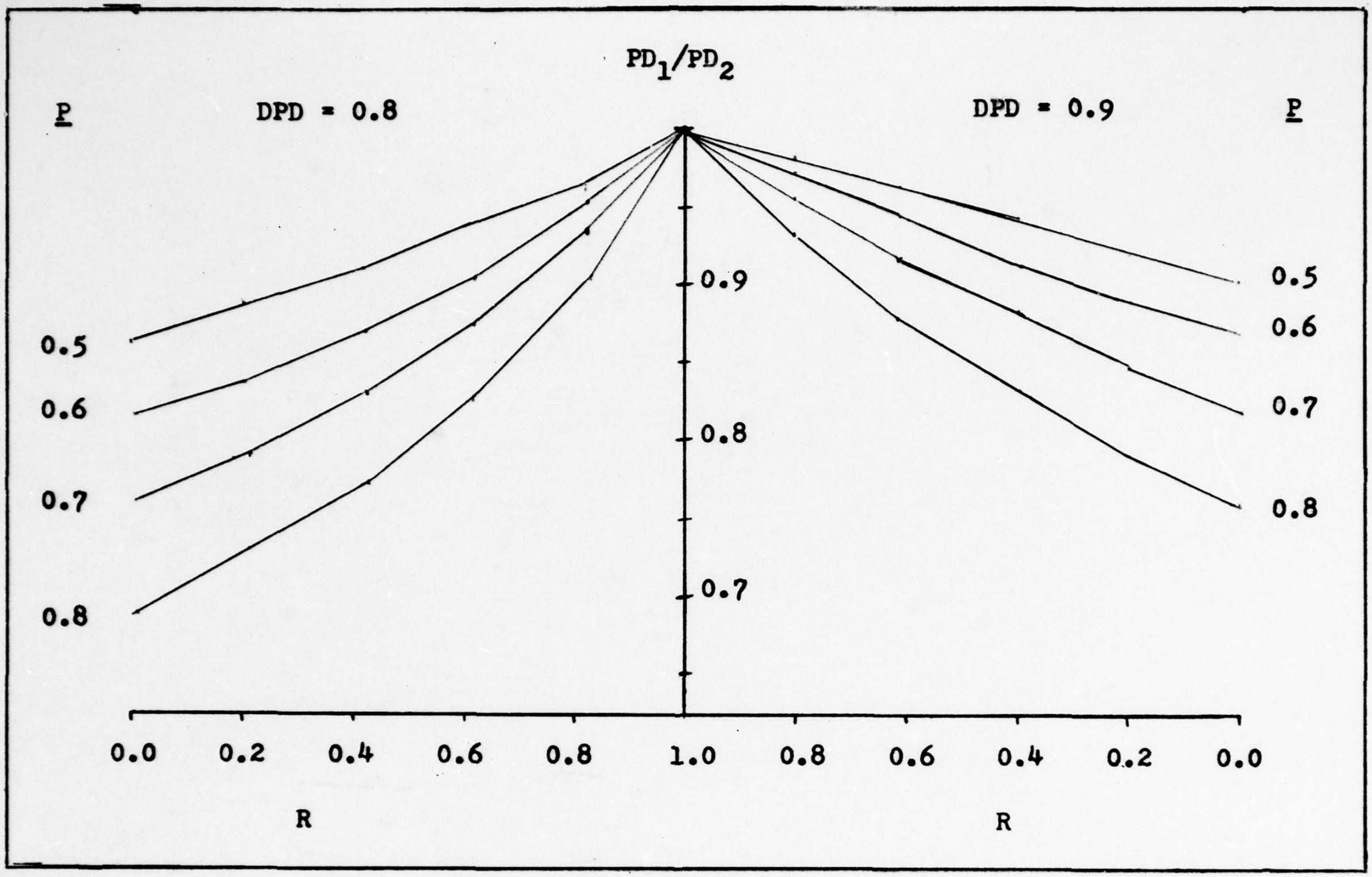

Figure 5. Comparative Plots of Discrepancy Ratio PD_1/PD_2 Versus Reliability R

discussed earlier. For any given combination of values of DPD and P both between zero and one, the ratio increases with R, from a lower limit given by Eq. (3.29) for R equal to zero, to a high of one for R equal to one. Comparisons of the curves show that the ratio uniformly decreases with P for any one combination of values of DPD and R. And the ratio increases as DPD is increased from 0.8 to 0.9, for any given combination of R and P.

Besides the effects of changes in parameter values, overall examination of Fig. (5) gives information about the magnitude of the error for realistic problems. As the ratio PD_1/PD_2 decreases, the first method generally leads to solutions that achieve smaller percentages of the method's goal -- level of probability of destruction. For instance, for the realistic situation where the parameters R, P, and DPD are 0.5, 0.7, and 0.8, respectively, the ratio PD_1/PD_2 is less than 0.86. That is, the first method achieves only 86% of its goal. Examination of Fig. (5) shows even more alarming discrepancies for some of the parameter values.

Definitions of Rounded Measures $\overline{X}_1/\overline{X}_2$, $\overline{PD}_1/\overline{PD}_2$, and $\overline{PD}_1$/DPD. The discrepancy ratios X_1/X_2 and PD_1/PD_2 previously derived are actually based on the computed numbers of missiles necessary. In general, these computed amounts X_1 and X_2 will not be integers. Of course, fractions of missiles cannot be targeted for silos. The number of missiles must, in real problems, be rounded up to the nearest integer to assure a Probability of Destruction greater than or equal to the desired level DPD. For small numbers of missiles, this rounding causes relatively large percentage changes in X, and consequently in PD.

By representing the upward roundings of X_1 and X_2 as $\bar{X}_1$ and $\bar{X}_2$, respectively, the two discrepancy ratios can be re-computed. By separately inserting $\bar{X}_1$ and $\bar{X}_2$ into the true PD calculation given in Eq. (3.9), the resulting ratio can be called $\overline{PD}_1/\overline{PD}_2$. For small values of X_1 and X_2, the two ratios $\bar{X}_1/\bar{X}_2$ and $\overline{PD}_1/\overline{PD}_2$ can be significantly different from the original ratios X_1/X_2 and PD_1/PD_2. An example is shown in Fig. (6). The two curves are individual plots of PD_1/PD_2 and $\overline{PD}_1/\overline{PD}_2$, both versus R. For both plots, DPD is equal to 0.8 and P is equal to 0.7. While the PD_1/PD_2 plot exhibits the same smooth, convex curve characterized in Fig. (5), the $\overline{PD}_1/\overline{PD}_2$ plot exhibits step-like behavior. This is reasonable since the roundings of X_1 and X_2 produce different effects, depending on how close X_1 and X_2 are to the rounded $\bar{X}_1$ and $\bar{X}_2$. In fact, the $\overline{PD}_1/\overline{PD}_2$ function is discontinuous at the step points. As R increases, the difference between PD_1/PD_2 and $\overline{PD}_1/\overline{PD}_2$ generally increases, because the numbers of missiles X_1 and X_2 decrease with R, causing more percentage difference due to rounding. The effects of changes of parameters DPD, P, and R on $\overline{PD}_1/\overline{PD}_2$ are generally the same as the effects derived earlier for PD_1/PD_2. For instance, $\overline{PD}_1/\overline{PD}_2$ generally increases with R, though not strictly increasing (due to the step characteristics).

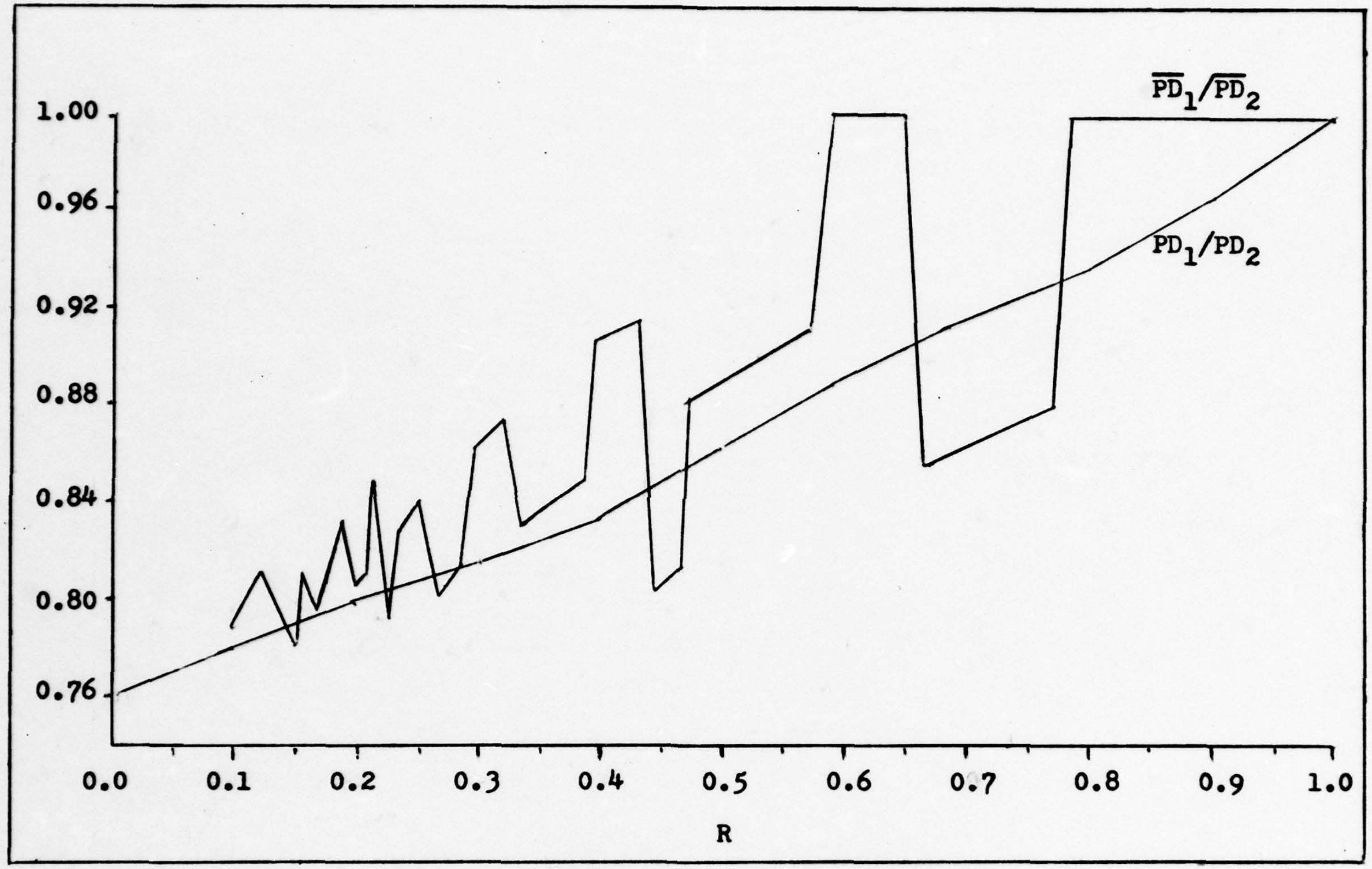

Figure 6. Comparison of Ratios PD_1/PD_2 and $\overline{PD}_1/\overline{PD}_2$, Both Versus R, DPD=0.8, P=0.7

A third discrepancy measure involving rounded figures, and possibly the most useful, is the ratio $\overline{PD}_1$/ DPD. This measure is the actual ratio of probability of destruction obtained by using the rounded $\overline{X}_1$ number of missiles to the desired probability of kill. This ratio is a true evaluation of how close the first method comes to reaching its goal of DPD.

Numerical Comparisons of Rounded and Non-rounded Measures. To further illustrate the effects of rounding the numbers of missiles X_1 and X_2, Tables (I)-(VIII) give a comprehensive comparison of calculations for various values of parameters DPD, P, and R. Each table corresponds to a single curve from Fig. (5). Table (I), for instance, gives a listing of calculations based on values of DPD and P equal to 0.8 and 0.5, respectively, with R varying from 0.1 to 0.9 in 0.1 increments. These eight tables fully illustrate the numerical differences between the two solution methods. In some cases the rounded, or practical, values of the ratios $\overline{X}_1/\overline{X}_2$ and $\overline{PD}_1/\overline{PD}_2$ are significantly less than one.

Examination of Tables (I)-(VIII) show that for a few combinations of values of parameters DPD, P, and R, the value of $\overline{PD}_1$ is actually greater than the value of DPD. Thus the discrepancy measure $\overline{PD}_1$/DPD is greater than one for these cases. For example, if DPD = 0.8, P = 0.7, R = 0.6, then $\overline{PD}_1$ = 0.805 or $\overline{PD}_1$/DPD = 1.006. In this and other similar cases, the first method of solution obtains a level of PD that is greater than its goal of DPD, but this method

TABLE I

Comparative Calculations Between Two Methods, With DPD = 0.8, P = 0.5

DPD	P	R	X_1	X_2	X_1/X_2	$\overline{X}_1$	$\overline{X}_2$	$\overline{X}_1/\overline{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.8	0.5	0.1	23.2	31.4	0.740	24	32	0.750	0.696	0.870	0.708	0.806	0.878
0.8	0.5	0.2	11.6	15.3	0.760	12	16	0.750	0.706	0.882	0.718	0.815	0.881
0.8	0.5	0.3	7.7	9.9	0.782	8	10	0.800	0.716	0.895	0.728	0.803	0.906
0.8	0.5	0.4	5.8	7.2	0.805	6	8	0.750	0.726	0.908	0.738	0.832	0.887
0.8	0.5	0.5	4.6	5.6	0.830	5	6	0.833	0.737	0.921	0.763	0.867	0.928
0.8	0.5	0.6	3.9	4.5	0.858	4	5	0.800	0.748	0.936	0.760	0.832	0.913
0.8	0.5	0.7	3.3	3.7	0.888	4	4	1.000	0.760	0.951	0.821	0.821	1.000
0.8	0.5	0.8	2.9	3.2	0.921	3	4	0.750	0.773	0.966	0.784	0.870	0.901
0.8	0.5	0.9	2.6	2.7	0.958	3	3	1.000	0.786	0.983	0.834	0.834	1.000

TABLE II

Comparative Calculations Between Two Methods, With DPD = 0.8, P = 0.6

DPD	P	R	X_1	X_2	X_1/X_2	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_1/\bar{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.8	0.6	0.1	17.6	26.0	0.675	18	26	0.692	0.663	0.828	0.672	0.800	0.840
0.8	0.6	0.2	8.8	12.6	0.698	9	13	0.692	0.675	0.843	0.684	0.810	0.844
0.8	0.6	0.3	5.9	8.1	0.722	6	9	0.667	0.687	0.859	0.696	0.832	0.836
0.8	0.6	0.4	4.4	5.9	0.749	5	6	0.833	0.700	0.875	0.746	0.807	0.925
0.8	0.6	0.5	3.5	4.5	0.779	4	5	0.800	0.714	0.893	0.760	0.832	0.913
0.8	0.6	0.6	2.9	3.6	0.812	3	4	0.750	0.729	0.912	0.738	0.832	0.887
0.8	0.6	0.7	2.5	3.0	0.849	3	3	1.000	0.745	0.931	0.805	0.805	1.000
0.8	0.6	0.8	2.2	2.5	0.892	3	3	1.000	0.762	0.953	0.859	0.859	1.000
0.8	0.6	0.9	2.0	2.1	0.942	2	3	0.667	0.780	0.975	0.788	0.903	0.873

TABLE III

Comparative Calculations Between Two Methods, With DPD = 0.8, P = 0.7

DPD	P	R	X_1	X_2	X_1/X_2	$\overline{X}_1$	$\overline{X}_2$	$\overline{X}_1/\overline{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.8	0.7	0.1	13.4	22.2	0.603	14	23	0.609	0.621	0.776	0.638	0.812	0.786
0.8	0.7	0.2	6.7	10.7	0.626	7	11	0.636	0.635	0.794	0.652	0.810	0.805
0.8	0.7	0.3	4.5	6.8	0.653	5	7	0.714	0.650	0.813	0.692	0.808	0.857
0.8	0.7	0.4	3.3	4.9	0.682	4	5	0.800	0.666	0.833	0.731	0.807	0.907
0.8	0.7	0.5	2.7	3.7	0.716	3	4	0.750	0.684	0.855	0.725	0.821	0.883
0.8	0.7	0.6	2.2	3.0	0.754	3	3	1.000	0.703	0.879	0.805	0.805	1.000
0.8	0.7	0.7	1.9	2.4	0.799	2	3	0.667	0.724	0.904	0.740	0.867	0.853
0.8	0.7	0.8	1.7	2.0	0.852	2	2	1.000	0.746	0.933	0.806	0.806	1.000
0.8	0.7	0.9	1.5	1.6	0.918	2	2	1.000	0.772	0.969	0.863	0.863	1.000

TABLE IV

Comparative Calculations Between Two Methods, With DPD = 0.8, P = 0.8

DPD	P	R	X_1	X_2	X_1/X_2	$\overline{X}_1$	$\overline{X}_2$	$\overline{X}_1/\overline{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.8	0.8	0.1	10.0	19.3	0.518	10	20	0.500	0.566	0.707	0.566	0.811	0.697
0.8	0.8	0.2	5.0	9.2	0.542	5	10	0.500	0.582	0.727	0.582	0.825	0.705
0.8	0.8	0.3	3.3	5.9	0.568	4	6	0.667	0.599	0.749	0.666	0.807	0.825
0.8	0.8	0.4	2.5	4.2	0.599	3	5	0.600	0.619	0.773	0.686	0.855	0.802
0.8	0.8	0.5	2.0	3.2	0.635	2	4	0.500	0.640	0.800	0.640	0.870	0.735
0.8	0.8	0.6	1.7	2.5	0.677	2	3	0.667	0.664	0.830	0.730	0.859	0.849
0.8	0.8	0.7	1.4	2.0	0.729	2	2	1.000	0.691	0.863	0.806	0.806	1.000
0.8	0.8	0.8	1.3	1.6	0.793	2	2	1.000	0.721	0.901	0.870	0.870	1.000
0.8	0.8	0.9	1.1	1.3	0.879	2	2	1.000	0.757	0.946	0.922	0.922	1.000

TABLE V

Comparative Calculations Between Two Methods, With DPD = 0.9, P = 0.5

DPD	P	R	X_1	X_2	X_1/X_2	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_1/\bar{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.9	0.5	0.1	33.2	44.9	0.740	34	45	0.756	0.818	0.909	0.825	0.901	0.916
0.9	0.5	0.2	16.6	21.9	0.760	17	22	0.773	0.826	0.918	0.822	0.902	0.924
0.9	0.5	0.3	11.1	14.2	0.782	12	15	0.800	0.835	0.927	0.858	0.913	0.940
0.9	0.5	0.4	8.3	10.3	0.805	9	11	0.818	0.843	0.937	0.866	0.914	0.947
0.9	0.5	0.5	6.6	8.0	0.830	7	8	0.875	0.852	0.947	0.867	0.900	0.963
0.9	0.5	0.6	5.5	6.5	0.858	6	7	0.857	0.861	0.957	0.882	0.918	0.962
0.9	0.5	0.7	4.8	5.4	0.888	5	6	0.833	0.871	0.967	0.884	0.925	0.956
0.9	0.5	0.8	4.2	4.5	0.921	5	5	1.000	0.880	0.978	0.922	0.922	1.000
0.9	0.5	0.9	3.7	3.9	0.958	4	4	1.000	0.890	0.989	0.908	0.908	1.000

TABLE VI

Comparative Calculations Between Two Methods, With DPD = 0.9, P = 0.6

DPD	P	R	X_1	X_2	X_1/X_2	$\overline{X}_1$	$\overline{X}_2$	$\overline{X}_1/\overline{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.9	0.6	0.1	25.1	37.2	0.675	26	38	0.684	0.789	0.876	0.800	0.905	0.884
0.9	0.6	0.2	12.6	18.0	0.698	13	18	0.722	0.799	0.888	0.810	0.900	0.900
0.9	0.6	0.3	8.4	11.6	0.722	9	12	0.750	0.810	0.900	0.832	0.908	0.917
0.9	0.6	0.4	6.3	8.4	0.749	7	9	0.778	0.822	0.913	0.854	0.915	0.932
0.9	0.6	0.5	5.0	6.5	0.779	6	7	0.857	0.833	0.926	0.882	0.918	0.962
0.9	0.6	0.6	4.2	5.2	0.812	5	6	0.833	0.846	0.940	0.893	0.931	0.958
0.9	0.6	0.7	3.6	4.2	0.849	4	5	0.800	0.859	0.954	0.887	0.934	0.949
0.9	0.6	0.8	3.1	3.5	0.892	4	4	1.000	0.872	0.969	0.927	0.927	1.000
0.9	0.6	0.9	2.8	3.0	0.942	3	3	1.000	0.886	0.984	0.903	0.903	1.000

TABLE VII

Comparative Calculations Between Two Methods, With DPD = 0.9, P = 0.7

DPD	P	R	X_1	X_2	X_1/X_2	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_1/\bar{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.9	0.7	0.1	19.1	31.7	0.603	20	38	0.526	0.750	0.834	0.766	0.937	0.818
0.9	0.7	0.2	9.6	15.3	0.626	10	16	0.625	0.764	0.848	0.779	0.910	0.855
0.9	0.7	0.3	6.4	9.8	0.653	7	10	0.700	0.777	0.864	0.808	0.905	0.892
0.9	0.7	0.4	4.8	7.0	0.682	5	8	0.625	0.792	0.880	0.807	0.928	0.869
0.9	0.7	0.5	3.8	5.4	0.716	4	6	0.667	0.808	0.897	0.821	0.925	0.889
0.9	0.7	0.6	3.2	4.2	0.754	4	5	0.800	0.824	0.915	0.887	0.934	0.949
0.9	0.7	0.7	2.7	3.4	0.799	3	4	0.750	0.841	0.935	0.867	0.932	0.930
0.9	0.7	0.8	2.4	2.8	0.852	3	3	1.000	0.860	0.955	0.915	0.915	1.000
0.9	0.7	0.9	2.1	2.3	0.918	3	3	1.000	0.879	0.977	0.949	0.949	1.000

TABLE VIII

Comparative Calculations Between Two Methods, With DPD = 0.9, P = 0.8

DPD	P	R	X_1	X_2	X_1/X_2	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_1/\bar{X}_2$	PD_1	PD_1/PD_2	$\overline{PD}_1$	$\overline{PD}_2$	$\overline{PD}_1/\overline{PD}_2$
0.9	0.8	0.1	14.3	27.6	0.518	15	28	0.536	0.697	0.774	0.714	0.903	0.790
0.9	0.8	0.2	7.2	13.2	0.542	8	14	0.571	0.713	0.792	0.752	0.913	0.824
0.9	0.8	0.3	4.8	8.4	0.568	5	9	0.556	0.730	0.811	0.746	0.915	0.815
0.9	0.8	0.4	3.6	6.0	0.599	4	6	0.667	0.748	0.831	0.786	0.901	0.872
0.9	0.8	0.5	2.9	4.5	0.635	3	5	0.600	0.768	0.853	0.784	0.922	0.850
0.9	0.8	0.6	2.4	3.5	0.677	3	4	0.750	0.790	0.877	0.859	0.927	0.927
0.9	0.8	0.7	2.0	2.8	0.729	3	3	1.000	0.813	0.904	0.915	0.915	1.000
0.9	0.8	0.8	1.8	2.3	0.793	2	3	0.667	0.839	0.932	0.870	0.953	0.913
0.9	0.8	0.9	1.6	1.8	0.879	2	2	1.000	0.868	0.964	0.922	0.922	1.000

never obtains a level of PD greater than that gained by the second method.

Table of Lower Limits for PD_1/PD_2. As a final numerical result for this problem, the lower limits of PD_1/PD_2 for given combinations of DPD and P are given in Table (IX). These values represent the limit of PD_1/PD_2 as R approaches zero for constant DPD and P. The formula for each limit is given in Eq. (3.29). The eight lower limits in Table (IX) correspond to the eight curves given in Fig. (5). These lower limits are not realistic values of PD_1/PD_2, of course, since R never assumes the value zero. But the limit does give a lower bound for PD_1/PD_2 for any strategic problem of this

Table IX

Lower Limit of PD_1/PD_2 As R Approaches Zero

DPD	P	$\lim_{R \to 0} \frac{PD_1}{PD_2}$
0.8	0.5	0.859
0.8	0.6	0.814
0.8	0.7	0.760
0.8	0.8	0.688
0.9	0.5	0.900
0.9	0.6	0.865
0.9	0.7	0.820
0.9	0.8	0.757

type.

General Results and Conclusions

Some attempt has been made in this chapter to examine the differences between two alternative solution techniques for a particularly simple strategic allocation problem. The problem involves targeting groups of a single type of missile at a set of point targets to assure a given probability of destruction. This simple, but common, problem was chosen because it highlights the sometimes overuse of what can be called "expected-value mentality." This strategic problem was simple enough to point out mathematical and empirical discrepancies between the two methods and still have reasonably proveable results. Other more complicated strategic problems have been empirically examined (for example, targeting various types of missiles with multiple-warhead capability). The erroneous results of expected value thinking given in this chapter can in some cases be generalized to these more complicated situations, and several numerical examples tend to re-inforce this generalization (for example, see Chapter IV).

For this simple problem, the relevant facts and derived results are easily summarized. A list of points defining the problem is:

1. The targets are equal-valued point targets (silos).
2. There is only one type of missile -- without multiple-warhead capability.

3. The operating characteristics of the missile are well-known.

4. The parameters P and R for the missile system are defined as in pp. (21)-(23) of this report.

5. The objective is to find the minimum number X of missiles per target necessary to obtain a Desired Probability of Destruction of the targets.

The results discussed in this chapter are:

1. For two alternative solution methods, the ratio X_1/X_2 was derived and proven to be less than one for all realistic values of R.

2. For the two methods, the ratio of the two (non-rounded) actual probabilities of destruction, PD_1/PD_2, was derived and proven to be less than one.

3. The ratio PD_1/PD_2 uniformly increases with R, decreases with P, and increases with DPD.

4. Low R, high P, and low DPD values cause the first method's error to be most significant (worst case).

5. High R, low P, and high DPD values cause the error to be negligible (best case).

6. The lower bounds for PD_1/PD_2 were derived and tabled for various values of DPD and P.

7. The ratios of rounded results, $\bar{X}_1/\bar{X}_2$ and $\overline{PD}_1/\overline{PD}_2$, produced empirical results similar, but not exactly so, to their non-rounded counterparts.

8. The first method can in a few situations produce $\overline{PD}_1$, based on the rounded $\bar{X}_1$, to be greater than DPD.

9. The numerical errors caused by the first method can be numerically significant for a wide variety of parameter values.

IV. An Example of a Serious Error In Published Literature

In this chapter, the analysis contained in a series of articles by noted atomic physicist Kosta Tsipis will be examined. Doctor Tsipis was, when the articles were written, the Senior Researcher at the Stockholm International Peace Research Institute (SIPRI) and a Research Associate at the Center for International Studies at the Massachusetts Institute of Technology. Three articles by Tsipis will be discussed, along with three articles by other authors that used or referred to his analysis. The first article, and primary reference for this chapter, appeared in the October/November, 1974, issue of Technology Review [Tsipis, 1974:34-47].

The discussion will be centered on an erroneous set of equations designed to compute probabilities of kill in counter-silo problems for unreliable missiles that have multiple-warhead capabilities. For multiple-warhead missiles, the errors in computing kill probabilities also lead to invalidation of Tsipis's primary measure of merit for missiles -- his "K" measure which depends on the missile warhead's accuracy and its megatonnage. The errors lead to calculations of probability of silo kill that are higher than the correct results. As the mechanical reliability of the missile decreases, the errors in kill probability become more significant. Overestimation of kill probability for single missiles in turn leads to underestimation of the numbers of missiles required to gain a given level of kill probability.

In this chapter, Tsipis's analysis is traced, and then alternative formulas will be presented that are corrections for the same assumptions. Then comparative probabilities of kill will be computed, using mainly his data, to draw attention to numerical errors caused by his formulas. The errors do not arise either for perfectly reliable weapons or for single-warhead missiles.

Missile Pre-Allocation Problem

The applicability of this analysis is limited in scope. Tsipis examines both counterpopulation and counterforce deterrence, but the discussion here is limited only to counter-force targeting -- the objective of the problem is to destroy or cripple the enemy's missile force to limit his second-strike capability. This strategic allocation problem is generally the same as that discussed in Chapter III, except that the missiles have multiple-warhead capability.

Assumptions. For this problem, the attacker is trying to obtain the capability of destroying the defender's land missile force. The silos that house the missile force can reasonably be considered to be point targets, with what is commonly called a "cookie-cutter" damage function. A "cookie-cutter" damage function is one in which the target is destroyed if the attack weapon detonates within some given distance of the target, and the target is unharmed if the weapon detonates any further from the target. That is, there is no possibility of fractional damage. This critical distance from the target (silo) is usually called the lethal radius of the target.

The attacker is restricted to launching his available missiles simultaneously, thus ignoring the possible benefits of sequential targeting. He is forced to analyze his optimal targeting strategy before launch. In this problem, enemy defenses are ignored. Tsipis chooses to ignore the bombers presently in U.S. and Soviet arsenals in his analysis, as will this chapter. The possibility of warhead fratricide is ignored, assuming that detonations can be timed to reduce the possibility of one warhead detonation interfering with another.

Definitions of Variables. Besides the numbers of missiles used to target at silos, there are four basic variables in this problem. Where possible, Tsipis's notation will be used. Three basic variables relate to the weapons -- missile accuracy, missile launcher reliability, and warhead yield. Missile accuracy is measured as in Chapter III -- Circular Error Probable (CEP), in nautical miles. Tsipis only discusses missile launcher reliability, denoted by ρ , ignoring re-entry vehicle reliability for multiple-warhead missiles. This reliability is the probability of the missile launcher performing reliably until its release of the re-entry vehicles, for multiple-warhead missiles. The warhead yield Y is usually measured in megatons. A fourth basic variable is related only to the target. The silo hardness H is defined as the minimum blast overpressure required to render the contents of the silo ineffective; H is usually measured in pounds per square inch (psi).

Calculation of Single-Shot Terminal Probability of Kill, P

An intermediate value will be useful for further analysis. The single-shot terminal probability of kill, P, was defined in Chapter III. This probability P is the conditional probability of a single warhead destroying the target and is conditioned on two things -- first, the warhead and its missile delivery system perform reliably through detonation, and, second, the target has not been destroyed by a previous warhead. While the second condition may seem trivial, its realization is important in the formal derivations of later formulas. For silo point targets, P is then the probability that the warhead lands closer than the target's lethal radius, given the two conditions.

Tsipis himself offers a calculation for P, which he calls P_k, in his Eq. (11) [Tsipis, 1974:38]. If the warhead landing error is assumed to follow an independent two-dimensional Gaussian (normal) probability distribution centered at the target, as is commonly assumed, and if the possibility of missing the target in a given direction is generally the same as for any other direction, then P can be calculated from this simplification of Tsipis's Eq. (11):

$$P = P_k = 1 - e^{-1/2\left[\frac{Y^{2/3}}{(CEP)^2\,(0.19 - 0.23\,H^{1/2} + 0.068\,H)^{2/3}}\right]} \tag{4.1}$$

There is some question as to whether this equation is a valid formula for P. In a published discussion of this formula, John Walsh of the Office of the Director of Defense Research and Engineering, DOD, criticizes the formula and offers an alternative one [Walsh, 1975:1117-1118]:

$$P = P_k = 1 - e^{\left[\frac{-4.17\ Y^{2/3}}{(CEP)^2\ H^{2/3}}\right]} \qquad (4.2)$$

Walsh contends that Tsipis made an error in the use of the bivariate Gaussian distribution, while Tsipis believes that his own use of $(CEP)^2$ as an estimate of the variance of the distribution for a given weapon produces at most five percent error [Tsipis, 1975:1119]. Tsipis does seem to have made an error in the use of the distribution, contradicting other sources [Eckler, 1972:17]. On the other hand, Walsh's formula seems to use a one-term approximation of Tsipis's formula relating lethal radius r_s to yield Y and silo hardness H:

$$r_s = \frac{Y^{1/3}}{(0.19 - 0.23\ H^{1/2} + 0.068\ H)^{1/3}} \qquad (4.3)$$

Walsh's formula just uses the one-term approximation $(0.068^{1/3} \cdot H^{1/3})$. In any case, most of the debate between the two seems to center around the nuclear effects on silos. It is not the purpose of this paper to discuss a proper calculation of P, since it depends on assumptions about nuclear effects. The errors that are examined below are in the use of P to calculate probability of kill, not in the calculation of P. For that reason, Tsipis's formula for P given in Eq. (4.1) will be used for the rest of this chapter, so that any numerical differences that arise will be due to other differences.

Tsipis's Formulas For Multiple-Shot P_k = PD

Tsipis gives equations for calculating the unconditional probability of killing a target, P_k, which was called PD in Chapter III, for two separate cases. First is the case when a group of warheads, all from different missiles, are launched at a single target. Second is the launching of a group of warheads, all on the same missile launcher, at the single target. It is Tsipis's calculations for the second case, which are repeated in other articles, that seem to be in error. Both cases will be outlined below.

Calculation of P_k For N_1 Warheads From N_1 Different Missile Launchers. If N_1 warheads on N_1 individual, but identical, missiles, each with reliability ρ , are sent to the same target, then the calculation of P from Eq. (4.1) can be used to compute the multiple-shot probability of kill. From Tsipis's Eq. (21), [Tsipis, 1974:39] ;

$$P_k(\rho, N_1) = PD = 1 - (1 - \rho P)^{N_1} \quad (4.4)$$

or equivalently, substituting the formula for P given in Eq. (4.1) gives:

$P_k(\rho , N_1) =$

$= PD =$

$$= 1 - \left[1 - \rho \left\{ 1 - e^{-\left[\frac{Y^{2/3}}{2(CEP)^2(0.19 - 0.23\ H^{1/2} + H)^{2/3}}\right]} \right\}\right]^{N_1} \qquad (4.5)$$

This last equation is in terms of the four basic variables -- accuracy CEP, yield Y, reliability ρ, and silo hardness H.

<u>Calculation of P_k For N_2 Warheads From Same Missile Launcher</u>. For missiles with multiple-warhead capability, say one that has Multiple Independently-targeted Re-entry Vehicles (MIRV) it seems reasonable that a general formula for multiple-shot kill probability would have to include cases where some warheads from the same missile are launched at the same target. If N_2 warheads from the same missile launcher that has reliability ρ are sent to a single target, then Tsipis's Eq. (22), [Tsipis, 1974:39], gives the following formula for kill probability:

$P_k(\rho , N_2) =$

$= PD =$

$$= 1 - e^{-\left[\frac{\rho\ Y^{2/3}\ N_2}{2(CEP)^2(0.19 - 0.23\ H^{1/2} + 0.068\ H)^{2/3}}\right]} \qquad (4.6)^*$$

In terms of the single-shot terminal probability of kill P, this equivalent to:

$$P_k(\rho, N_2) = PD = 1 - (1 - P)^{\rho N_2} \qquad (4.7)^*$$

A quote from the Technology Review article [Tsipis, 1974:39] clearly defines when Tsipis feels that Eqs. (4.6) and (4.7) can be used; "If n warheads aimed against a silo are carried by the same missile of reliability ρ, the kill probability becomes ..."

Fallacy of Eqs. (4.6) and (4.7). There is a simple way to show that Tsipis's Eq. (22) for kill probability, which corresponds to Eqs. (4.6) and (4.7) above, seems to be incorrect. By examining Eq. (4.6) it is seen that if the equation were valid, the kill probability could be made arbitrarily close to one just by loading enough warheads N_2 on the launcher. This point is even made clearer by referring to Eq. (4.7), since P is always less than one in an imperfect world. But if the single missile launcher system that delivers the warheads has at most a probability of ρ of getting to the target area, then the kill probability is at most equal to ρ. That is, the probability of destroying the target with a single missile with multiple warheads cannot be any higher than the reliability of the missile system delivering those warheads.

If Z is defined as the number of reliable warheads that reach the target area, a random variable, then the expected value of Z, E(Z), for this situation is just the missile reliability ρ times the number of warheads per launcher, N_2. That is, Eq. (22) of Tsipis and Eq. (4.7) above are equivalent to:

$$P_k(\rho, N_2) = PD = 1 - (1 - P)^{E(Z)} \quad (4.8)^*$$

This is an incorrect use of an expected value in a probability calculation -- the type of error that was previously discussed in Chapter III. If ρ is, for example, equal to 0.7, then on the long-term average, 70% of the total missiles launched, and thus 70% of the total warheads launched, would perform reliably. But for any given missile launch, one of two events can occur. Either the missile delivery system performs reliably and all of the N_2 warheads are delivered to the target area, or, the system fails and none of the warheads are delivered. The first event has probability of $\rho = 0.7$ and the second has probability of $1 - \rho = 0.3$. Eqs. (4.6), (4.7), and (4.8) would be valid only if 70% of the warheads were certain to be delivered per launch. These latter three equations are invalid and have been labelled with asterisks to prevent confusion.

Correct Formulas For P_k = PD For N_2 Warheads From Same Missile

If the terminal probability of kill P can be effectively computed, then a corrected alternative formula for computing the probability of killing a target with N_2 warheads from the same missile is easily obtained. Two forms of this formula will be presented.

If the reliability ρ is the reliability of the missile, and if the re-entry vehicles (warheads) are assumed to perform reliably (as Tsipis does implicitly in his Eq. (22)), then the probability of N_2 warheads from a single missile destroying the target is:

$$P_k(\rho, N_2) = PD = \rho \cdot (1 - (1 - P)^{N_2}) \tag{4.9}$$

Or, using Eq. (4.1), substituting for P gives $P_k(\rho, N_2)$ in terms of the four basic variables:

$$P_k(\rho, N_2) =$$

$$= PD =$$

$$= \rho \left\{ 1 - e^{-1/2\left[\frac{N_2 Y^{2/3}}{(CEP)^2(0.19 - 0.23\ H^{1/2} + 0.068\ H)^{2/3}}\right]} \right\} \tag{4.10}$$

Examination of either of these last two formulas shows that as the number of warheads per missile N_2 increases, the probability of kill asymptotically approaches ρ, not one, as occurred for Tsipis's equation. That is, the probability of kill from a single missile can be no higher than that missile's reliability, regardless of how many warheads are loaded on the missile. Neither Eq. (4.9) nor (4.10) involve more difficult calculations than their incorrect counterparts, Eqs. (4.7) and (4.6).

The validity of Eq. (4.9), and thus of Eq. (4.10), can be easily verified. The terminal probability of kill P is just the conditional probability of destroying the target for a single warhead, conditioned on the reliable performance of the missile. If the N_2 re-entry vehicle warheads are assumed to perform independently after their release from the missile launcher, then the conditional probability that the target survives all the warheads is just $(1 - P)^{N_2}$. That is, the conditional probability of destroying the target is $1 - (1 - P)^{N_2}$. Since this last quantity is conditioned on reliable missile performance, which has probability equal to the reliability ρ, then the multiplication of $1 - (1 - P)^{N_2}$ by ρ gives the unconditional probability of killing the target with N_2 warheads from the same missile of reliability ρ. This multiplication gives the result in Eq. (4.9). A simple diagram, Fig. (7), illustrates the derivation of Eq. (4.9).

Comparative Calculations Of P_k = PD For Different Data

If Tsipis's Eq. (22), repeated in Eqs. (4.6) and (4.7), is wrong, then it is important to highlight any numerical errors that arise from his formula for kill probability for missiles with multiple-warhead capabilities. Table X is a set of P_k calculations comparing Tsipis's formula and Eq. (4.10) for various U.S. and Soviet missiles, both present and proposed. Only missiles with multiple-warhead capabilities are considered because it is only for these missiles that Tsipis's

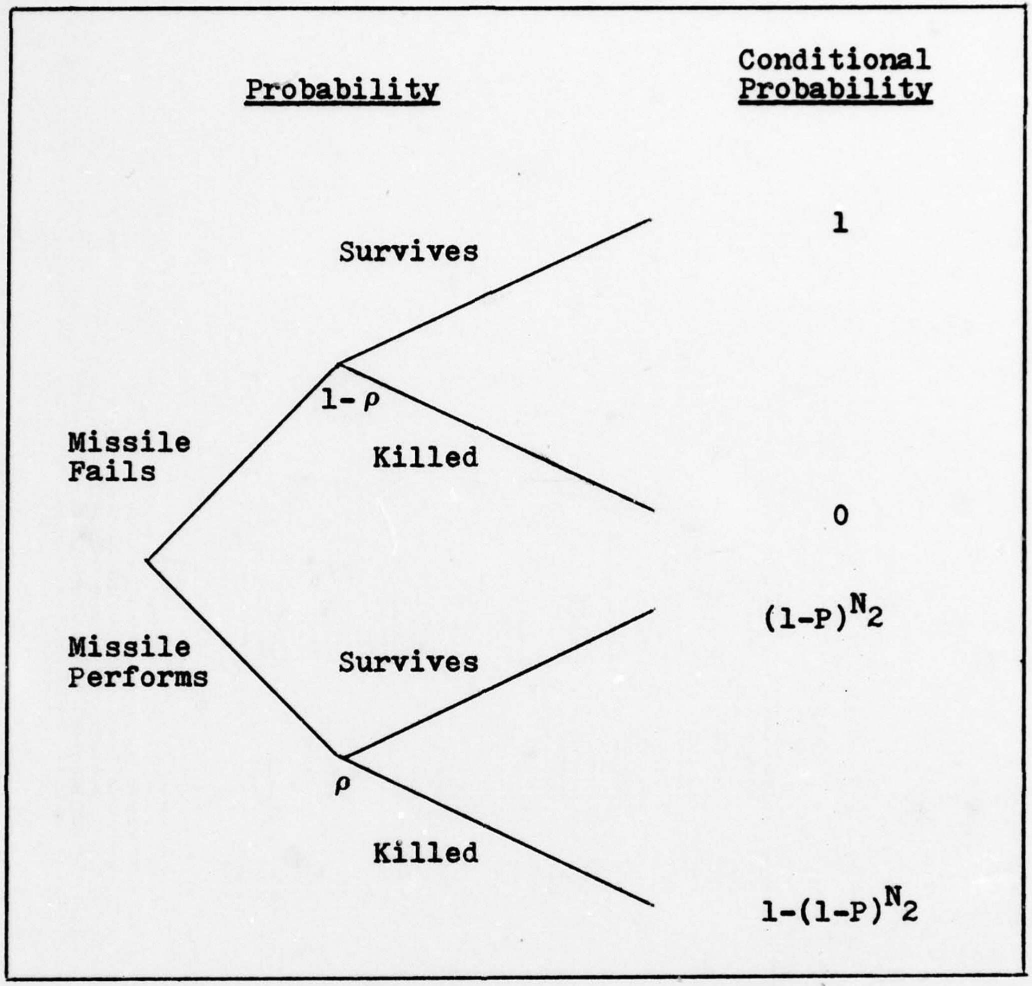

Figure 7. Event Probability Diagram For N_2 Warheads From The Same Missile Of Reliability ρ

P_k formulas are incorrect. Table X lists the compared probabilities of kill if only one missile of the listed type were sent to the corresponding target.

If m denotes the number of re-entry vehicles per launcher (RV/L), then N_2 = m and the tabled probabilities can be calculated from the four basic variables (CEP, Y, H, and ρ) and

Table X

Comparison of P_k Calculated From Eq. (4.10) (P_{k2}) and Tsipis Eq. (22) (P_{k1})

For Single Missiles From U.S. and Soviet MIRVed Missile Force, = 0.8

	Silo Hardness H (psi)							
	100		300		500		1000	
Present[1]:	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}
US Min. III, Y=0.16 mt CEP=0.2 nm, m=3 warheads	0.957	0.785	0.743	0.654	0.608	0.552	0.435	0.408
US Poseidon, Y=0.05 mt CEP=0.3 nm, m=10 warheads	0.884	0.746	0.604	0.549	0.472	0.440	0.323	0.308
US Polaris, Y=0.2 mt CEP=0.5 nm, m=3 warheads	0.443	0.415	0.223	0.216	0.159	0.156	0.101	0.099
Future[2]:								
US Min. III, Y=0.32 mt CEP=0.11 nm, m=3 warheads	1.000	0.800	0.999	0.800	0.993	0.798	0.950	0.781
US Poseidon, Y=0.1 mt CEP=0.2 nm, m=10 warheads	1.000	0.800	0.963	0.787	0.898	0.754	0.751	0.660
SovietSS-17, Y=0.2 CEP=0.5 nm, m=6 warheads	0.690	0.615	0.396	0.374	0.294	0.282	0.191	0.186
SovietSS-18, Y=1.0 mt CEP=0.5 nm, m=5 warheads	0.942	0.777	0.707	0.628	0.571	0.522	0.404	0.381

Table X (cont.)

	Silo Hardness H (psi)							
	100		300		500		1000	
Early 1980's[3]:	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}
US Min. III, Y=0.35 mt CEP=0.10 nm, m=3 warheads	1.000	0.800	1.000	0.800	0.998	0.800	0.979	0.793
US Trident I, Y=0.08 mt CEP=0.18, m=10 warheads	1.000	0.800	0.970	0.790	0.912	0.761	0.772	0.674
US Trident II, Y=0.08 mt CEP=0.18 nm, m=20 warheads	1.000	0.800	0.999	0.800	0.992	0.798	0.948	0.780
Soviet SS-18, Y=1.0 mt CEP=0.3 nm, m=6 warheads	1.000	0.800	0.983	0.795	0.941	0.776	0.821	0.707
Soviet SS-19, Y=0.2 mt CEP=0.3 nm, m=6 warheads	0.961	0.786	0.754	0.661	0.619	0.561	0.445	0.417

Table X (cont.)

	Silo Hardness H (psi)							
	100		300		500		1000	
Late 1980's[4]:	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}	P_{k1}	P_{k2}
US Minuteman III, Y=0.25 mt CEP=0.02 nm, m=3 warheads	1.000	0.800	1.000	0.800	1.000	0.800	1.000	0.800
US Trident I, Y=0.06 mt CEP=0.02 nm, m=10 warheads	1.000	0.800	1.000	0.800	1.000	0.800	1.000	0.800
US Trident II, Y=0.06 mt CEP=0.02 nm, m=20 warheads	1.000	0.800	1.000	0.800	1.000	0.800	1.000	0.800
Soviet SS-18, Y=2.0 mt CEP=0.1 nm, m=6 warheads	1.000	0.800	1.000	0.800	1.000	0.800	1.000	0.800
Soviet SS-19, Y=0.4 mt CEP=0.1 nm, m=6 warheads	1.000	0.800	1.000	0.800	1.000	0.800	1.000	0.800

Data Source:

1. Tsipis, Technology Review, [Tsipis, 1974]
2. Tsipis, Technology Review, [Tsipis, 1974]
3. Leggett, Armed Forces Journal International, [Leggett,1975]
4. Leggett, Armed Forces Journal International, [Leggett,1975]

m. The weapons system characteristics come from Tsipis's Technology Review article [Tsipis, 1974:40-43] and from Representative Robert L. Leggett's, (D-CA), article appearing in the Armed Forces Journal International [Leggett, 1975:30-32]. Part of this weapons data is repeated by Tsipis in another article [Tsipis, 1975:394-395]. The silo hardness H data appears to be in doubt, so all the possible silo hardness mentioned by these sources have been used -- 100 psi, 300 psi, 500 psi, and 1000 psi.

Tsipis does not produce tables for P_k using his Eq. (22), or Eq. (4.6) above, but actually calculates the necessary numbers of warheads to gain some given level of P_k, using another equation (Eq. (4.5) above) for perfectly reliable missiles (ρ = 1.0). Table X given here uses an arbitrary missile reliability of ρ = 0.8. The discrepancy between the two P_k figures from Tsipis's equation and Eq. (4.10) would depend on the value of ρ . If missile reliability ρ is lower than 0.8, as it may possibly be for older weapons, then the discrepancy would even be larger. For instance, for the present Minuteman III against a silo with hardness equal to 100 psi, if ρ is 0.8, then the P_k from Eq. (4.6) is 0.957 and the P_k from Eq. (4.10) is 0.785. But if ρ were equal to 0.7, then the two P_k values would have a much larger discrepancy -- 0.937 to 0.686. Even more significant discrepancies would occur for lower ρ values. In general, Tsipis's Eq. (22) leads to an overestimation of P_k for missiles with multiple warheads. This in turn would lead to an understatement of the number of those missiles necessary to obtain some given probability of kill. This is the same result as that discussed in Chapter III for the simpler type of error.

Discussion of Tsipis's Measure of Lethality, K

Dr. Tsipis proposes a general measure to evaluate the counterforce strength of a strategic missile. This measure, a parameter called K, is calculated from two of the four basic variables of this counterforce problem, accuracy CEP and yield Y:

$$K \equiv \frac{Y^{2/3}}{(CEP)^2} \tag{4.17}$$

Most of Tsipis's equations for kill probability are actually in terms of this lethality measure K, rather than CEP and Y.

For the case with perfectly reliable missiles ($\rho = 1.0$), Tsipis's Eq. (21), repeated here as Eq. (4.5), reduces to:

$$P_k(\rho, N_1) = 1 - e^{\left[\frac{-Y^{2/3}N_1}{2(CEP)^2(0.19 - 0.23\,H^{1/2} + 0.068H)^{2/3}}\right]}$$

$$= 1 - e^{\left[\frac{-K\,N_1}{2(0.19 - 0.23\,H^{1/2} + 0.068\,H)^{2/3}}\right]} \tag{4.18}$$

This is his Eq. (14), [Tsipis, 1974:38] . Then for any desired probability of kill, called DPD in Chapter III, the necessary value of KN_1 can be solved for:

$$KN_1 = -2(0.19 - 0.23\,H^{1/2} + 0.068\,H)^{2/3} \cdot \ln(1 - DPD) \tag{4.19}$$

This is equivalent to his Eq. (17), [Tsipis, 1974:38]. Reverting to Tsipis's notation n for the number of warheads, then for any desired kill probability and each silo type, the value of K • n necessary to destroy one silo of that tyee can be computed. Then the number of warheads necessary can be computed using the warhead's K value. For instance, a Poseidon warhead (Y = 0.05 mt, CEP = 0.3 nm) has a K value of 1.5. And a silo with H = 300 psi requires a K • n value approximately equal to 30 to destroy it with probability 0.9. Thus it would take about 20 perfectly reliable Poseidon warheads to destroy a 300 psi silo with probability 90%.

The usefulness of the K measure of lethality for the unreliable case seems questionable, even using Tsipis's equations. For unreliable missiles, he offers his Eqs. (21) and (22) or Eqs. (4.5) and (4.6), for computing kill probability. But unlike his Eq. (14), the necessary value of K • n, for a given silo and desired kill probability, cannot easily be solved for in closed form from Eq. (4.5), that is, from his Eq. (21). In Eq. (4.10), the correct formula for kill probability for multiple-warhead missiles, the solution for K • n would involve reliability ρ and would not be the same as Tsipis's solution for K • n given in Eq. (4.19). In any case, for unreliable missiles, the necessary K • n value per silo cannot be computed from Eq. (4.19), since Eq. (4.19) applies only to perfectly reliable missiles. That K • n value cannot be used to compute the number of warheads necessary for unreliable missiles.

Published Discussion of Tsipis Analysis

There has been substantial comment on and use of Tsipis's analysis in published articles subsequent to his 1974 Technology Review article, but no corrections of his Eq. (22), repeated in two forms in Eqs. (4.6) and (4.7), were found. Tsipis himself repeats the analysis in an article in the 7 February issue of Science, [Tsipis, 1975a:393-397], except that he does not offer data projected into the future as he did in Technology Review. As mentioned earlier, there was a discussion between Tsipis and John Walsh on the proper calculation of the terminal probability of kill P, published in Science [Walsh, 1975:1118-1119], but Tsipis's Eq. (22) is not mentioned. Congressman Leggett [Leggett, 1975:30-32] uses Tsipis's K measure and more recent data to justify his Congressional debate that the U.S. has a marked advantage over the Soviet Union in countersilo capability. Several other articles by Tsipis, while of a less analytical nature than the 1974 Technology Review article, use the K measure as a valid general measure of merit for strategic missiles.

In a paper for the Center for Naval Analysis (CNA), Michael L. Squires [Squires, 1976] draws comparisons between Tsipis's K measure of merit and the results of a computer simulation model. The ninth version of the Arsenal Exchange Model (AEM) was used to compute three Measures of Effectiveness (MOE's) for U.S. and Soviet countersilo attacks. The three MOE's were the number of U.S. (Soviet) ICBM's surviving, the number of U.S. (Soviet) warheads surviving, and the number of U.S. (Soviet) equivalent megatons surviving.

Squires uses Tsipis's data for U.S. and Soviet silo hardness, and Congressman Leggett's data on weapons characteristics. While Tsipis concludes that the United States maintains (and will continue to maintain through the 1980's) a strategic arsenal vastly superior to that of the Soviet Union, based on total K values, Squires concludes that with given future projections, The Soviet Union will be essentially equivalent in offensive capability, based on the model's three MOE's, through the 1980's. Squires criticizes Tsipis's K measure as an oversimplification, but does not offer to explain why the Tsipis measure is invalid. Squires seems to base his belief about the invalidity of the K measure only on the fact that the measure produces different results from the AEM simulation runs. His reasoning, of course, implicitly assumes the correctness of the Arsenal Exchange Model. Squires does not explicitly refute any of Tsipis's probability calculations.

General Formula For Kill Probability For Multiple-Warhead Arsenals

In this chapter, formulas for kill probabilities have been given for single types of single-warhead and multiple-warhead missiles with reliability ρ. It would be useful to give a general formula for kill probability for different types of missiles launched at a single target. This formula assumes that a correct terminal probability of kill can be calculated, with P defined as before. This terminal probability of kill P is a conditional probability that is related both to the type of missile and the type of target.

If an attacker's missile arsenal has I different types of missiles, these missiles can be subscripted $i = 1$ to I. Then, for a given target type, let P_i be the terminal probability of kill for that missile type i. Let ρ_i be the missile delivery system reliability as before, and let R_i be the re-entry vehicle (warhead) reliability for missile system i. This is necessary because there are actually at least two phases of operation for a multiple warhead system -- missile system operation to re-entry vehicle separation, and re-entry vehicle operation after separation. Let m_i be the number of re-entry vehicles carried per missile for type i, and define a full warhead group to be a group of m_i warheads from a single missile. If full warhead groups can be divided and targeted to more than one target, then a general formula for kill probability needs to include cases where, for a single target, less than m_i warheads from a single missile can be used to destroy the single target. For example, for the present U.S. Minuteman III, the number of warheads per missile is $m_i = 3$. Then a general targeting strategy would allow the number of warheads sent to the target from a single missile to be less than 3, with the missile's remaining warheads allocated to other targets. Then a general formula for kill probability for multiple-warhead missiles would be:

$$P_k = 1 - \prod_{i=1}^{I} \prod_{j=i}^{m_i} \left\{ 1 - \rho_i \left[1 - (1 - R_i P_i)^j \right] \right\}^{X_{ij}} \quad (4.20)$$

The only unknown variables in Eq. (4.20) are the X_{ij}, which are the number of warhead groups from missile type i that have j warheads in them. When silo targets are clustered close together geographically, it may be more efficient for the attacker to split his MIRVed warhead groups between targets, at least in some targeting strategies. That is, in practical targeting allocations, the attacker may get a higher target destruction by splitting up his multiple-warhead weapons, if the constraints of the allocation problem permit. For example, the present U.S. Minuteman III has the number of warheads per missile $m_i = 3$. If possible, it is more efficient to send two warheads from one missile and one warhead from a separate missile, than it is to send three warheads from the same Minuteman III. If $\rho_i = 0.7$, $R_i = 0.9$, and $P_i = 0.7$, then sending two warheads from one missile and one warhead from another results in a kill probability of $P_k = 0.779$. This can be computed directly from Eq. (4.20) with $I = 1$, $X_{i1} = 1$, $X_{i2} = 1$, $X_{i3} = 0$. But if three warheads were sent from one single Minuteman III missile, then the resulting kill probability would be only $P_k = 0.665$. This P_k value is from Eq. (4.20) with $I = 1$, $X_{i1} = 0$, $X_{i2} = 0$, $X_{i3} = 1$. Of course, this targeting strategy of splitting the warhead groups would only be possible if some of the targets were close enough to allow different warheads from the same missile to be targeted to separate targets.

The formula for kill probability given in Eq. (4.20) is general enough to include this strategy of splitting the warhead groups to achieve optimal targeting. For a given target, then, the total number of warheads sent to each target would be the total sum of the products of the number of warheads per split warhead group, j, times the number of those groups, X_{ij}. That is, for a given target, the total number of warheads sent to target would be the double sum $\sum_{i=1}^{I} \sum_{j=1}^{m_i} j \cdot X_{ij}$. The formula for kill probability given in Eq. (4.20) is a general method for computing kill probability, assuming that all the parameters, especially the P_i, are known. Eq. (4.20) is just a direct extension of Eq. (4.9), with N_2 in Eq. (4.9) assuming the values of j in Eq. (4.20). For single-warhead missiles, like the present U.S. Titan, $m_i = 1$. Individual missiles are assumed to operate independently, and a group of warheads from a single missile is assumed to operate independently of warhead groups from other missiles. The use of Eq. (4.20) ignores the problem of fratricide, or equivalently, presumes that warhead detonations can be properly sequenced to reduce fratricide.

One useful feature of Eq. (4.20) is that it can easily be adapted for use in a linear (or integer) program to obtain optimal allocations of missiles. This is easily done because, for a given target, the logarithm of the survival probability P_s, that is, of $1 - P_k$, is linear in the unknown variables X_{ij}:

$$\ln(P_s) = \ln(1 - P_k) =$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{m_i} X_{ij} \cdot \ln\left\{1 - \rho_i \left[1 - (1 - R_i P_i)^j\right]\right\} , \tag{4.21}$$

Assuming that all the system parameters are well-known (ρ_i, R_i, P_i), then Eqs. (4.20) and (4.21) provide an extremely useful tool for optimally allocating multiple-warhead missiles.

Eq. (4.20) highlights a general point about probability calculations of this type. Events for multiple-warhead missiles may not be independent. The title Multiple Independently-Targeted Re-entry Vehicle may be a misnomer, because events for re-entry vehicles (warheads) from the same missile are not necessarily independent. When probabilities of multistage events like the successful operation of a MIRVed missile are expressed in conditional terms, then events may not be independent. For instance, the probability that a target survives one warhead from an unreliable missile is easily calculated:

$$P_s(1) = 1 - P_k(1) = 1 - \rho_i R_i P_i \tag{4.22}$$

But the probability that the target survives two warheads from the same missile is not equal to the product of the two individual probabilities because the events of targets surviving the individual warheads are not independent. That is:

$$P_s(2) \neq [P_s(1)]^2 = [1 - \rho_i R_i P_i]^2 \tag{4.23}$$

And for the above case of $\sum_{j=1}^{m_i} j \cdot X_{ij}$ warheads, the probability that the target survives the $\sum_{j=1}^{m_i} j \cdot X_{ij}$ warheads from missiles of type i is not equal to $1 - \rho_i R_i P_i$ raised to the $\sum_{j=1}^{m_i} j \cdot X_{ij}$ power.

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V. Conclusions

In this report, a specific type of error in strategic targeting was examined. In probability calculations, some weapons quantities are actually random variables and their expected values should not be used in place of the random quantities themselves. Mistakes of this type were examined in this report.

As a foundation for analysis, a general discussion of strategic targeting and missile allocation was presented in Chapter II. Two classifications of strategic targeting models found in published articles were presented. Then a classification of targeting models was proposed that was useful for the analysis for this report. The three classifications were generally based on model applicability, assumptions, and solution technique.

Two general results from the literature search were presented. One was that, although literature on targeting abounds and is available, very few available articles discussed kill probability calculations for missiles that both were unreliable and possessed multiple-warhead capabilities. Second, the type of errors discussed in this report were not common in the literature examined, although some errors existed.

In Chapter III, a simple missile allocation problem was examined and possible errors for the problem were discussed. For a problem of allocating identical single-warhead missiles among a group of identical silo targets in order to achieve some common desired kill probability, two methods of solution were examined. The first method involved the incorrect use of the expected number of reliable weapons in a calculation of kill probability. A second method for allocating the missiles was given that corrects the first. The first method was shown to generally underestimate the number of missiles necessary to gain the desired probability of kill, and therefore, to overestimate the resulting true probability of kill. Several measures of discrepancy between the two methods were proposed, two of them based on the non-rounded numbers of missiles and three based on rounded missile allocations. Indicated by the two discrepancy measures that were based on non-rounded allocations, the significance of errors caused by the first method was shown to be:

1. greater as missile reliability decreased
2. greater as warhead yield in megatons increased
3. greater as accuracy increased (lower circular error probable)
4. greater as target hardness decreased
5. greater as the desired kill probability decreased

The discrepancy measures based on rounded missile allocations cannot be said to strictly follow these five above trends because of discontinuities, but the trends are generally the same for both types of discrepancy measures, depending on the parameter values. Different missile allocations and discrepancy measures were evaluated for various values of the parameters missile reliability, warhead yield, missile accuracy, target hardness, and desired kill probability, and these data were tabled. The numerical results show that, at least for some parameter values, the numerical errors caused by the first method are significant when compared to the first method.

In Chapter IV, a specific method from published targeting literature was examined. A series of articles by Dr. Kosta Tsipis contain a formula for kill probability that is incorrect. For unreliable missiles with MIRV capability, Dr. Tsipis offers a formula for kill probability that involves an incorrect use of the expected number of reliable missile launchers. An alternative formula for the same kill probability calculation was given in Chapter IV. Dr. Tsipis's calculation of kill probability for a single missile approaches the value of one as the number of warheads increases, while the corrected calculation of kill probability can never be higher than the missile reliability (if only one missile is launched to the target). This result invalidates Tsipis's conclusion that strategic arsenal strength depends on the number of warheads, and not on the number of missile launchers. This difference between the two formulas can lead to large numerical discrepancies. For data on present and projected U.S. and Soviet arsenals that were obtained from Tsipis and other sources, comparative calculations from the two formulas were tabled and show significant differences.

A general formula for the probability of killing a silo target was presented. The formula is general enough to include mixed types of multiple-warhead missiles at a single target.

The general conclusion of this report is that anyone interested in strategic targeting analysis should be cautious about the use of expected weapons numbers in kill probability calculations. In some cases, incorrect used of expected value assumptions may lead to serious numerical errors that are quite separate from the resulting conceptual errors. The use of expected values based on simplifying assumptions should be cautiously made.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report examines the incorrect uses of expected values in kill probability calculations that exist in some strategic targeting articles and models. Generally stated, the type of error is the incorrect use of expected numbers of weapons in probability calculations in place of numbers of weapons that are actually random variables. The most common example found was the use of the expected number of reliable missiles or over

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warheads in kill probability calculations for silo targets.

In Chapter II, the results of a general literature search are given. These results are in the form of classifying various strategic targeting models. Three alternative classifications are offered -- two from the literature and one by the author. The author notes a definite lack of targeting literature covering missiles that both are unreliable and have multiple warhead capabilities. Of the targeting models that were examined, the author found a few articles that make the type of mistake studied in the report, but these mistakes are not common.

In Chapter III, a simple missile allocation problem is examined and a possible expected value error is discussed. For single-warhead missiles, the incorrect use of the expected number of reliable missiles in place of the random variable reliable missiles can lead to highly significant numerical errors for some parameter values. The correct kill probability formula is given and several measures of discrepancy between the two formulas are given. The discrepancy measures are of two types -- the ratio of the missile allocations for the two methods, and the ratio of the respective probabilities of kill that result from the two allocations. The effects of changes in parameters such as accuracy and silo hardness, are proven mathematically. Numerical data on the discrepancy measures are tabled for various parameter values, and these data show that the error can be highly significant.

In Chapter IV, a specific example of an expected value error is discussed. A series of articles with incorrect kill probability calculations for MIRVed weapons is examined. The incorrect articles use the expected number of reliable warheads in place of the random number of reliable warheads. The author offers a corrected formula and numerically compares the differences based on data values for present and projected U.S. and Soviet arsenals. Then the author offers a general kill probability formula for silo targets; the formula is general enough to include mixed types of multiple-warhead missiles.