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A VIEW OF COMPLEMENTARY PIVOT THEORY (OR SOLVING EQUATIONS WITH--ETC(U)
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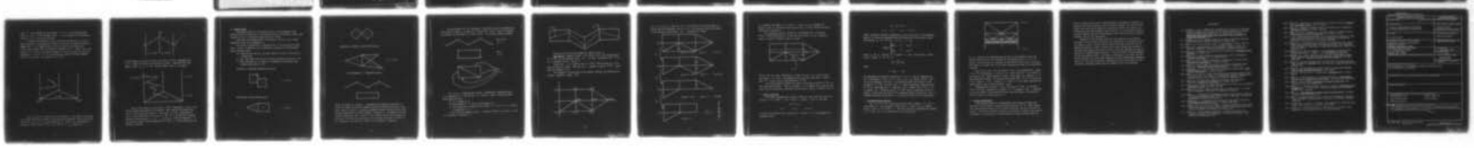

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A VIEW OF COMPLEMENTARY PIVOT THEORY
(or Solving Equations with Homotopies).

by

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B. Curtis Eaves

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A VIEW OF COMPLEMENTARY PIVOT THEORY
(or Solving Equations with Homotopies)

1. Introduction

Our purpose here is to give a brief, valid, and painless view of the equation solving computational method variously known as complementary pivot theory and/or fixed point methods. Our view begins in Section 2 with a bit of history and some of the successes of the method. In Section 3 the unique convergence proof of the method is elucidated with a riddle on ghosts. In Section 4 the general approach for solving equations with complementary pivot theory is encapsulated in the "Homotopy Principle". In Section 5 a simple example is used to illustrate both the convergence proof and the "Homotopy Principle". Rudiments of the general theory are stated and the "Main Theorem" is exhibited in an example in Section 6. Two representative complementary pivot algorithms are presented vis-a-vis the "Homotopy Principle" and "Main Theorem" in Section 7. Finally, in Section 8 the principal difficulty of the method is discussed and some of the studies for dealing with this difficulty are mentioned.

2. A Bit of History

If a point in time can be specified as the beginning of complementary pivot theory it is with the paper of Lemke and Howson [20]. In this paper a startling convergence proof was given for a finite algorithm for computing a Nash equilibrium of a bimatrix game. To understand their contribution, it was previously known that a Nash equilibrium existed via the Brouwer fixed point theorem and that exhaustive search offered a finite procedure for computing such an equilibrium. Furthermore, there is no theoretical proof that the Lemke-Howson algorithm has advantages over exhaustive search, and, in fact, one can construct examples where the Lemke-Howson algorithm is no better than exhaustive search. However, the point is, as a practical matter, the Lemke-Howson algorithm versus exhaustive search enables one to solve bimatrix games with characteristic size of, say, one thousand versus thirty. The situation is analogous to that of Dantzig's simplex method and linear programs.

In the paper [19] Lemke specified what is now known as "Lemke's Algorithm" and, thereby, showed that the convergence proof could be used for a much broader class of problems including quadratic programs.

The next steps came from Scarf in [26,27,28]. Using the convergence proof of Lemke and Howson he proved for the first time that a balanced game has a nonempty core and he described algorithms for computing a Brouwer fixed point and an equilibrium point for the general competitive equilibrium model. Once again, it has not been proved that the algorithm improved upon exhaustive search, but as a practical matter, problems could be solved that could not be solved before (note, for example, that Brouwer's theorem can be proved by repeated applications of Sperner's Lemma and that only finitely many simplexes need be examined at each iteration in order to find a complete simplex).

There are now at least two hundred papers in complementary pivot theory, and many very exciting developments have occurred. The convergence proof of Lemke and Howson is now understood to be intimately related to homotopy theory, a matter which is the crux of this paper. Many classical results have been given new "complementary pivot" proofs; to mention a few: Freidenfelds [9] and a connected set theorem of Browder, Kuhn [18] and the fundamental theorem of algebra, Garcia [11] and the last theorem of Poncairé, and Meyerson and Wright [23] and the Borsak-Ulman theorem.

A great deal of effort and ingenuity has been expended in making the complementary pivot algorithms more efficient; however, we shall not discuss these matters until the last section wherein we will describe the principal weakness of the complementary pivot algorithms.

Complementary pivot theory has been used to solve a number of specific problems; for instance, the following papers are concerned with complementary pivot theory applied to the solution of differential equations: Allgower and Jeppson [1,2], Wilmuth [34], Cottle [4], Netravali and Saigal [24], and Kaneko [14]. Katzenelson's [15] algorithm, and subsequent developments thereof, for electrical network problems also fits comfortably into the framework of complementary pivot theory as discussed in this paper.

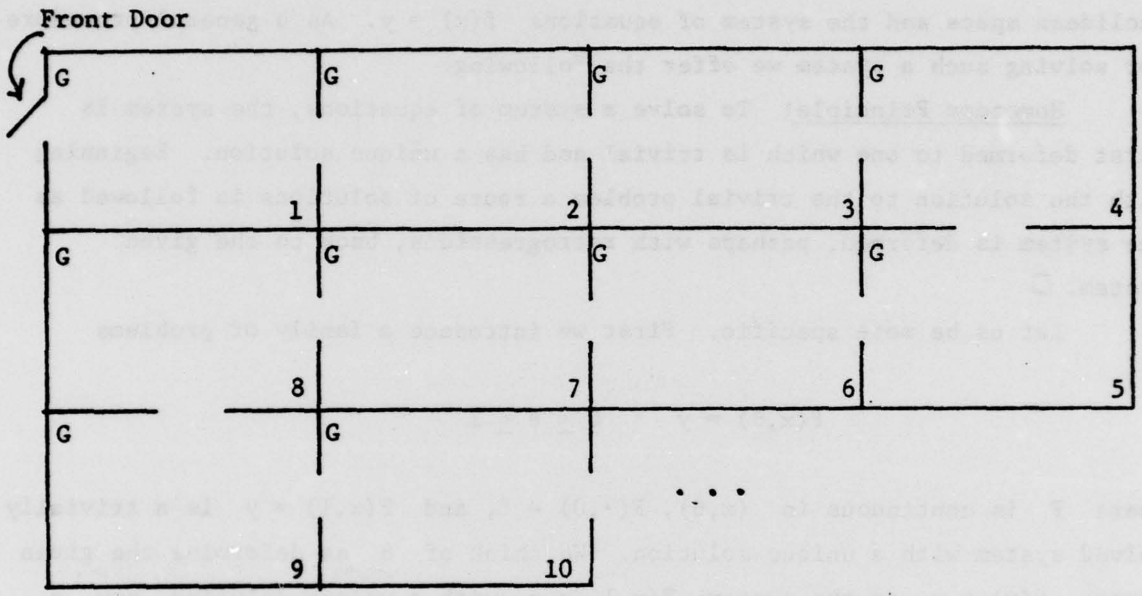
This paper is based upon Eaves and Scarf [8] and Eaves [6]. The reader might also want to consult Hirsch [12], Scarf [29] and Todd [31].

3. Convergence Proof

The following riddle and its solution illustrates the extraordinary convergence proof of Lemke and Howson.

Ghost Riddle: I eased through the front door of the allegedly haunted house. Just as a ghost appeared, the front door slammed shut behind me. He spoke, "You are now locked inside our house, but it is your fate that except for this room which has one open door, every other room with a ghost has two open doors." I thought, "Is there a room without a ghost?" □

The diagram below spills the beans.



Assume we are standing in Room 1 with the front door closed. According to the riddle there is exactly one open door, so let us pass through the door into the adjoining room which we now call Room 2. In Room 2, assuming the presence of a ghost, there are two open doors available to us, one of which we just entered; so let us exit the other and enter Room 3. We continue in this fashion to Rooms 4, 5, etc. The essential property of this process is that no room is entered (i.e., numbered) more than once, which is to say, there is no cycling. A proof of this fact is available by assuming the contrary and examining the first room entered twice. Consequently, if there are only m rooms in the house, then the process must stop with m steps or less, and there is a room without a ghost. On the other hand, if the house is sufficiently haunted so as to have infinitely many rooms (George Dantzig supports this possibility) then either the process stops with a solution, that is, a room without a ghost, or it proceeds forever always entering new rooms.

In this isolated form the convergence proof appears uselessly simple. When we apply the convergence principle, the rooms will become pieces of linearity of some function and circumstances will not be quite so transparent.

4. Homotopy Principle

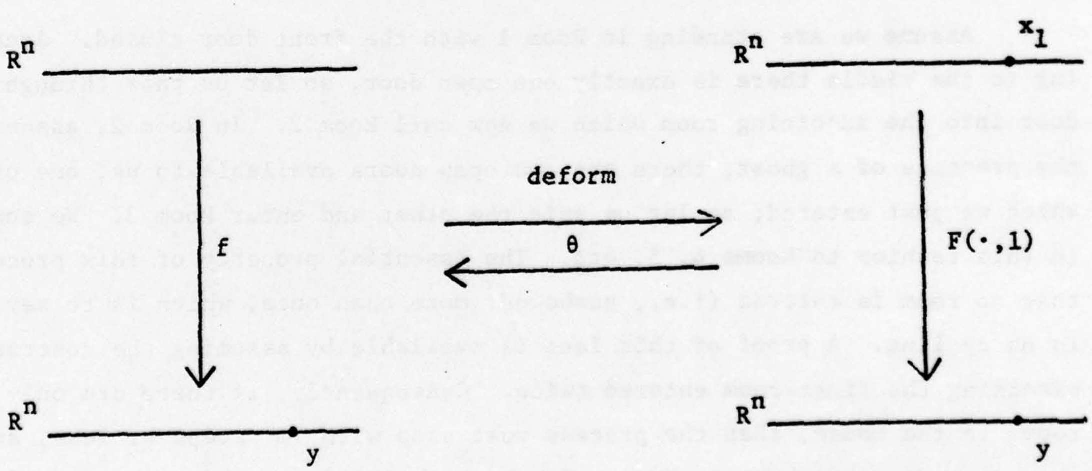
Now consider the continuous function $f : R^n \to R^n$ on n-dimensional Euclidean space and the system of equations $f(x) = y$. As a general procedure for solving such a system we offer the following.

Homotopy Principle: To solve a system of equations, the system is first deformed to one which is trivial and has a unique solution. Beginning with the solution to the trivial problem a route of solutions is followed as the system is deformed, perhaps with retrogressions, back to the given system. □

Let us be more specific. First we introduce a *family of problems*

$$F(x,\theta) = y \qquad 0 \leq \theta \leq 1$$

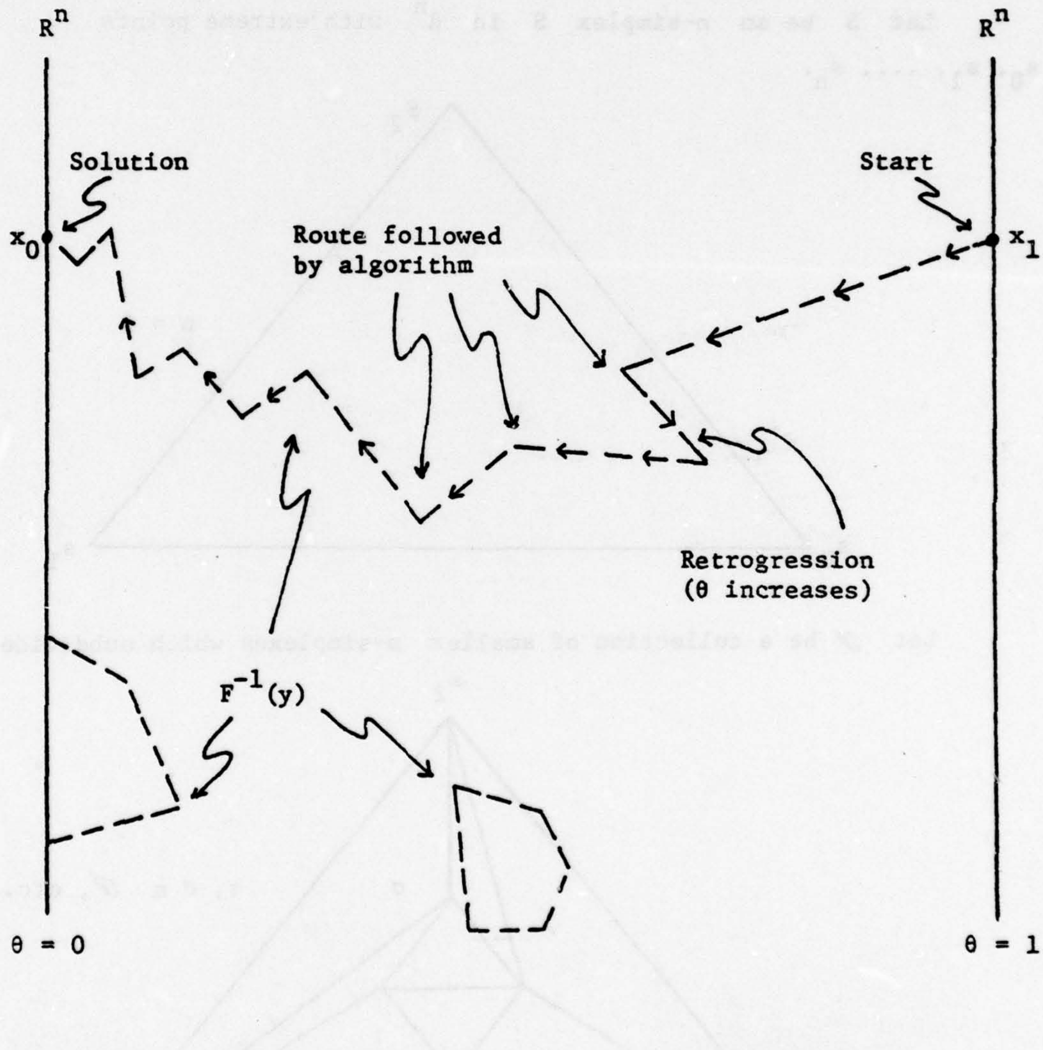
where F is continuous in (x,θ), $F(\cdot,0) = f$, and $F(x,1) = y$ is a trivially solved system with a unique solution. We think of θ *as deforming* the given system $f(x) = y$ to the system $F(x,1) = y$ with a unique solution, say x_1.



Given System ($\theta = 0$) Trivial System ($\theta = 1$)

To obtain a solution to the given system $f(x) = y$ we follow the solution of $F^{-1}(y)$ beginning with $(x_1,1)$. Except for degenerate (rare) cases the component of $F^{-1}(y)$ that meets $(x_1,1)$ is a route, that is, a path.

Assuming F is piecewise linear on $R^n \times (0,1]$ the next schema illustrates the situation quite well.



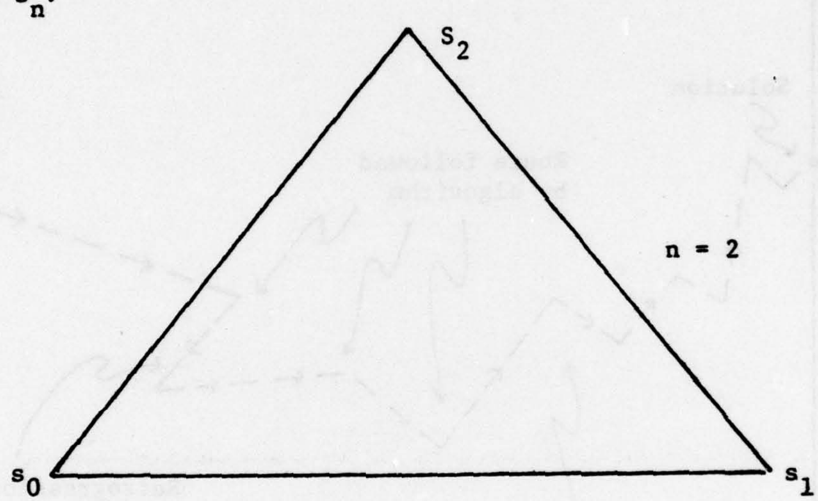
The algorithm begins with the point $(x_1,1)$ and follows the route of $F^{-1}(y)$. Under various conditions it can be shown that the route eventually leads to $R^n \times 0$ and thus yields a solution of the given problem.

This principle is illustrated in Section 5 and given theoretical credence in Section 6.

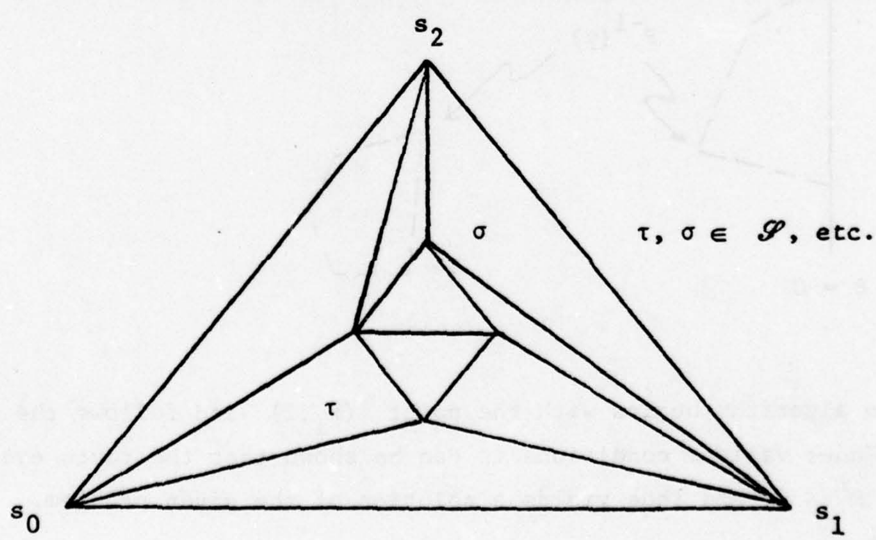
5. An Example

In this section we exhibit the complementary pivot convergence proof and the "Homotopy Principle" by solving a system of piecewise linear equations. This particular system of equations was chosen for its pedagogial value; later we examine merely continuous functions.

Let S be an n-simplex S in R^n with extreme points $s_0, s_1, \ldots, s_n$.



Let $\mathscr{S}$ be a collection of smaller n-simplexes which subdivides S.



By a vertex of $\mathcal{S}$ we mean a vertex of any element of $\mathcal{S}$.

Now let $f : S \to S$ be a continuous function on the simplex with the following three properties.

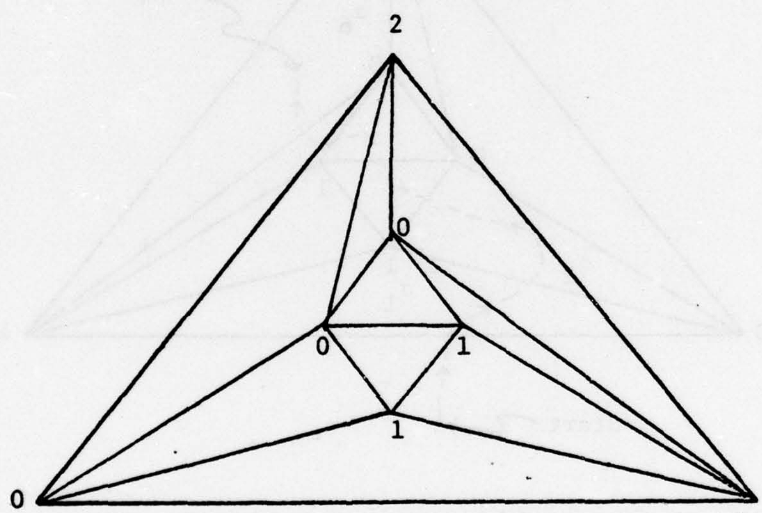
a) On each element of $\mathcal{S}$, f is linear, that is, affine.

b) On the boundary of S, f is the identity, that is,
$f(x) = x$ for x in ∂S.

c) Vertices of the subdivision are carried by f into extreme points of S, that is, $f(\text{vertices}) \subset \{s_0, \ldots, s_n\}$.

Let y be some interior point of S and we consider the system of equations $f(x) = y$. Toward solving this system label the vertices of $\mathcal{S}$ according to the extreme point to which it is mapped, that is, define the labeling function ℓ on the vertices by $\ell(v) = i$ if $f(v) = s_i$.

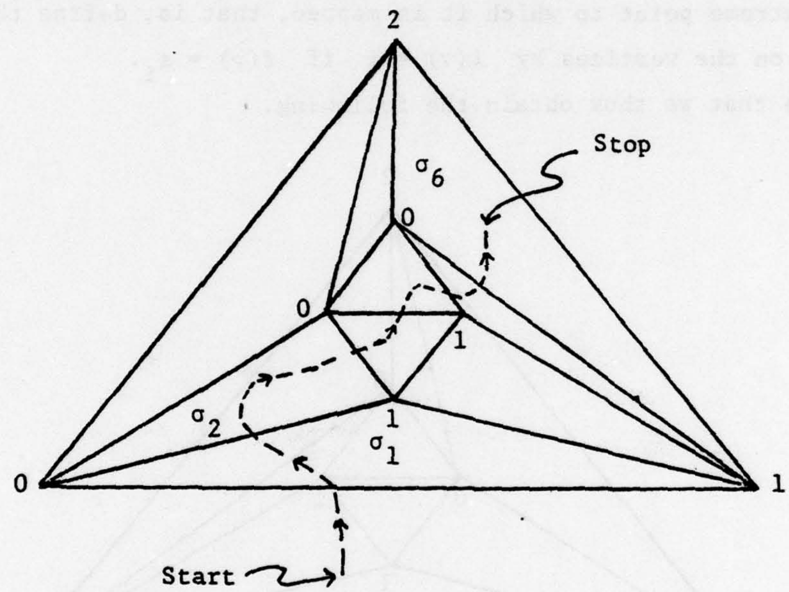
Assume that we thus obtain the following.



We call a simplex σ of $\mathcal{S}$ completely labeled only if all labels $0, 1, \ldots, n$ are present on its vertices. In the figure above there is exactly one such, namely the upper right one.

Let us observe that solving $f(x) = y$ is equivalent, modulo solving a system of linear equations, to finding a completely labeled simplex. If a simplex τ of $\mathcal{S}$ has only labels $\{0, \ldots, n\} \sim \{i\}$ then in view of the linearity of f on τ, f would map the entirety of τ into the face S spanned by $\{s_0, \ldots, s_n\} \sim \{s_i\}$. Consequently, no point of τ would hit the interior point y. On the other hand, if a simplex τ of $\mathcal{S}$ is completely labeled then τ is mapped onto S, and here some point of τ hits y. So for the moment we focus on the task of finding a completely labeled simplex.

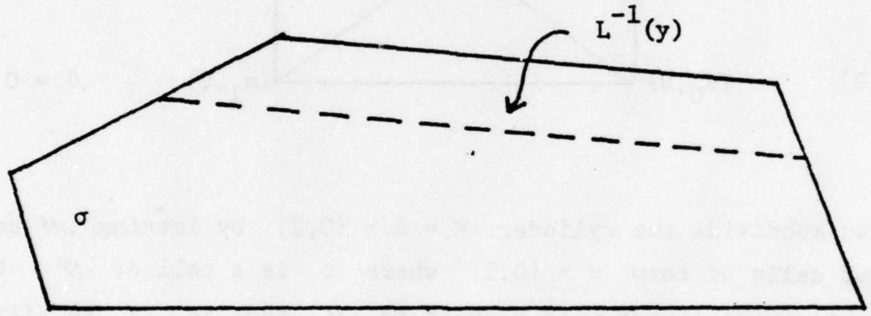
To execute the task of finding a completely labeled simplex we shall employ the "Ghost Riddle" as follows to obtain "Cohen's Algorithm", see [3]. We regard as a room a simplex of $\mathcal{S}$ and as an open door a face of a simplex of $\mathcal{S}$ with labels 0 and 1 (if $n > 2$ a face of a simplex of $\mathcal{S}$ with labels $(0, \ldots, n - 1)$). By passing through open doors the path followed is indicated below and terminates with a completely labeled simplex.



Prima facia this procedure works so smoothly that it seems rigged. But suppose that the n-simplex σ of $\mathcal{S}$ has vertices $(t_0, \ldots, t_{n-1})$ with labels $(0, \ldots, n - 1)$. If the remaining vertex t_n has labels n, then one has a completely labeled simplex. But, if the remaining label is i for some $0 < i < n - 1$, then there is exactly one other face of σ, namely, that spanned by $\{t_0, \ldots, t_n\} \sim \{t_i\}$, that has labels $\{0, \ldots, n - 1\}$. Thus, if a room has at least one open door, then either it is the target (i.e., a completely labeled simplex) or it has exactly two open doors. Since only one door passes from outside the simplex S to the inside and since there are only finitely many simplexes, the procedure must terminate with a completely labeled simplex.

Our aim in the above exercise was principally to exhibit the convergence proof in action. Next we exhibit use of the "Homotopy Principle" and show that it yields the algorithm just given, "Cohen's Algorithm", for solving $f(x) = y$.

Before applying the "Homotopy Principle" to solve $f(x) = y$ let us recall an elementary fact from linear algebra. Let $L : R^{n+1} \to R^n$ be a linear map of rank n from $(n+1)$-space to n-space. If y is any point in the range R^n, then $L^{-1}(y)$ is a line. Taking matters one step further let σ be an $(n+1)$-cell, that is, a closed polyhedral convex set of dimension $n + 1$. Let $L : \sigma \to R^n$ be a linear map where $L(\sigma)$ has dimension n. Then for most values of y in R^n either $L^{-1}(y)$ is empty or is a chord of σ

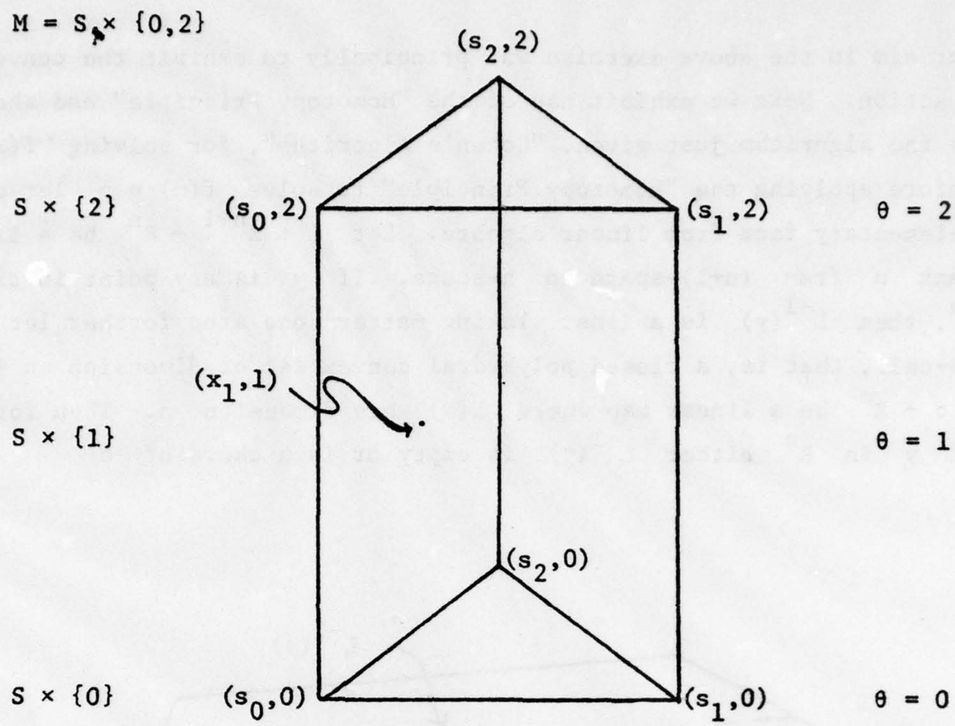


whose endpoints lie interior to n-faces of σ. There are a few values of y where $L^{-1}(y)$ meets an $(n-1)$-face of σ; these y's we call degenerate (or critical) but for convenience we shall always assume that our y is regular, that is, not degenerate. There are measures for dealing with degenerate y's but the treatment for them will not be discussed in this paper, (see Eaves [6]).

Given the system $f(x) = y$ we introduce a family of problems $F(x,0) = y$. Let x_1 be any point in interior of the face of S spanned by s_0 and s_1 (for $n > 2$ by $s_0, s_1, \ldots, s_{n-1}$) and define the homotopy, that is, function, F by

$$F(x,\theta) = f(x) + (y-x_1)\,\theta$$

for $0 \le \theta \le 2$. So F carries points of the cylinder $M = S \times [0,2]$ into R^n.



We can subdivide the cylinder $M = S \times \{0,2\}$ by letting $\mathcal{M}$ be the collection of cells of form $\sigma \times \{0,2\}$ where σ is a cell of $\mathcal{S}$. Now observe that F is piecewise linear with respect to $\mathcal{M}$, that is, F is affine on each cell of $\mathcal{M}$.

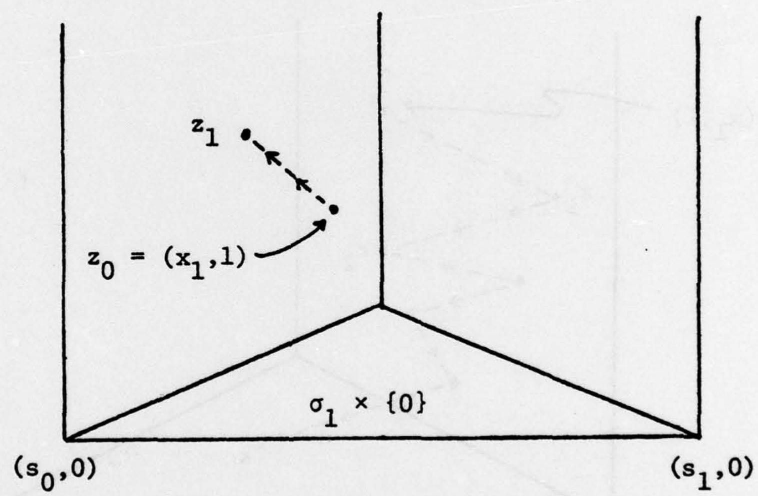
At $\theta = 0$ the system $F(x,\theta) = y$ is the system $f(x) = y$. For $\theta > 0$ the system of particular interest is

$$
* \left\{ \begin{array}{l} F(x,\theta) = y \\ (x,\theta) \in \partial M \\ \theta > 0 \end{array} \right.
$$

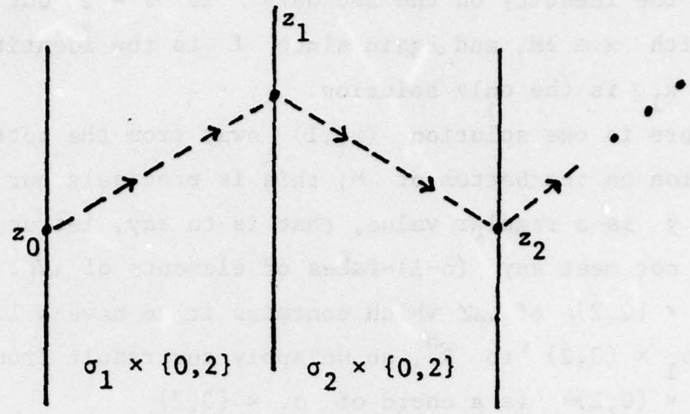
The second condition requires that (x,θ) is in the boundary of M, that is, in the top, bottom or some side of M. We argue that the system $*$ has exactly one solution, namely, $(x_1,1)$.

If $\theta > 1$ then $y - \theta(y - x_1)$ is not in S and clearly $f(x) = y - \theta(y - x_1)$ can have no solution. If $0 < \theta < 1$ then $y - \theta(y - x_1)$ is interior to S and $f(x) = y - \theta(y - x_1)$ with $x \in \partial S$ can have no solution, since f is the identity on the boundary. If $\theta = 1$ our system becomes $f(x) = x_1$ with $x \in \partial M$, and again since f is the identity on the boundary, clearly $x = x_1$ is the only solution.

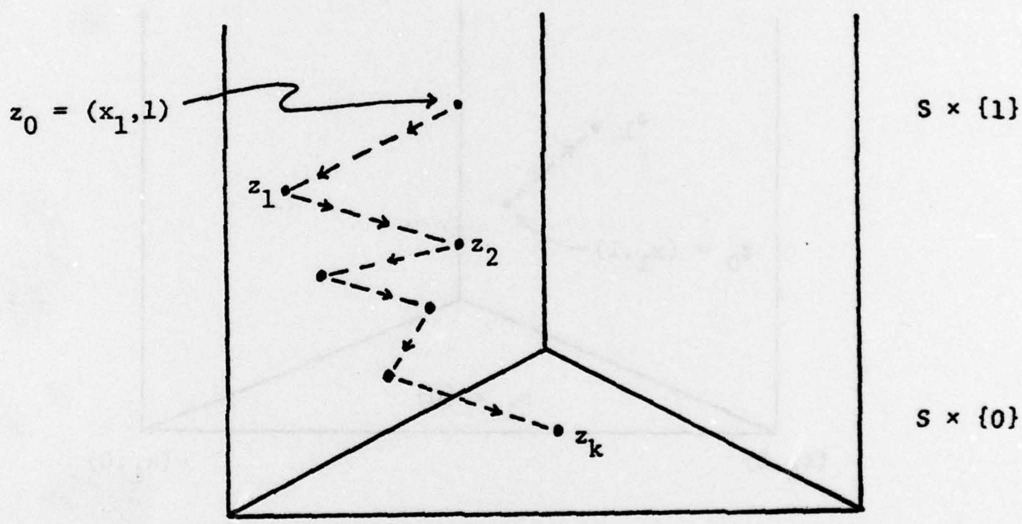
So there is one solution $(x_1,1)$ away from the bottom of M and we seek a solution on the bottom of M; this is precisely our desired situation. Assume that y is a regular value, that is to say, let us assume that $F^{-1}(y)$ does not meet any $(n-1)$-faces of elements of $\mathcal{M}$. Given $(x_1,1)$ and the cell $\sigma_1 \times \{0,2\}$ of $\mathcal{M}$ which contains it we have a linear map from the $(n+1)$-cell $\sigma_1 \times \{0,2\}$ to R^n, so we apply our result from linear algebra. $F^{-1}(y) \cap (\sigma_1 \times \{0,2\})$ is a chord of $\sigma_1 \times \{0,2\}$



Let $z_0 = (x_1,1)$ be one end of the chord and z_1 the other; calculating z_1 is just a matter of solving a linear system of equations. Next we go to the $(n+1)$-cell $\sigma_2 \times \{0,2\}$ of $\mathcal{M}$ that contains z_1 but not z_0 and repeat the procedure to get a chord $F^{-1}(y) \cap (\sigma_2 \times \{0,2\})$ of $\sigma_2 \times \{0,2\}$, etc.



In this manner we continue to follow the route of $F^{-1}(y)$ beginning with $(x_1,1)$. *Observe that this route can have no forks.* Eventually the route yields a point z_k in $S \times \{0\}$ and the system $f(x) = y$ is solved.



What is the relation of the route of $F^{-1}(y)$ beginning at $(x_1,1)$ and "Cohen's Algorithm" applied to the problem? Well, they are in essence identical once the smoke has cleared. If the route of $F^{-1}(y)$ beginning with $(x_1,1)$ is projected down to the base $S \times \{10\}$ of M we see that it passes through the same sequence σ_1, σ_2, ... of rooms as "Cohen's Algorithm"! In this sense we regard "Cohen's Algorithm" for $f(x) = y$ as that yielded by the "Homotopy Principle".

6. General Theory

The general theory is to be sketched here. As the example of the previous section contains most of the ideas involved in the general theory, the conceptual step to the material presented here is small.

Cells are our building blocks. We define an m-cell to be a closed polyhedral convex set of dimension m.

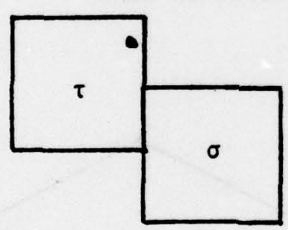
Let $\mathcal{M}$ be a collection of m-cells and let M be the union of these cells. $(M, \mathcal{M})$ is defined to be a subdivided m-manifold if the following three conditions hold.

a) Given any two cells of $\mathcal{M}$ either they do not meet or they meet in a common face.

b) Any (m-1)-face of a cell of $\mathcal{M}$ lies in at most two cells of $\mathcal{M}$.

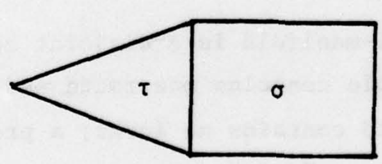
c) Given any point of M there is a neighborhood which meets only finitely many cells of $\mathcal{M}$.

Condition a) prohibits construction such as



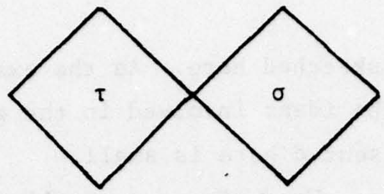
$\tau, \sigma \in \mathcal{M}$

and requires constructions such as

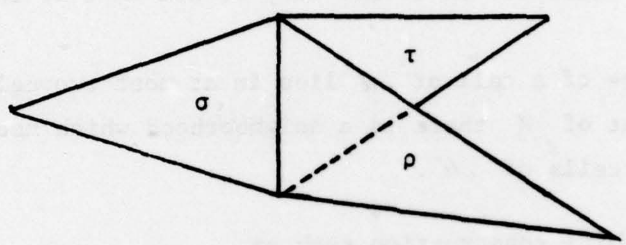


$\tau, \sigma \in \mathcal{M}$

or

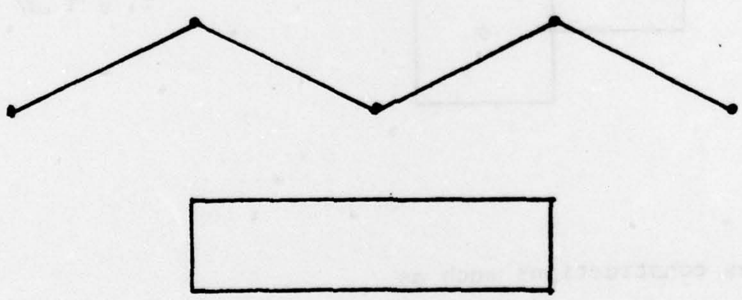


Condition b) prohibits construction such as



$\sigma, \tau, \rho \in \mathcal{M}$

As an example of a 1-manifold we have

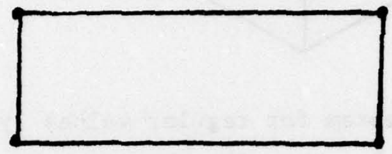


where $\mathcal{M}$ contains 8 1-cells. A 1-manifold is a disjoint collection of routes and loops; the previous example contains one route and one loop. Note that a route or loop of a 1-manifold contains no forks; a proof of this point requires essentially the "Ghost Argument" of Section 3. As examples of a subdivided 2-manifold we have $(S,\mathcal{S})$ of the previous section of the surface of cube where $\mathcal{M}$ is the set containing the top, bottom, and sides. We call M an m-manifold if for some $\mathcal{M}$, $(M,\mathcal{M})$ is a subdivided m-manifold.

By the boundary ∂M of a subdivided m-manifold we mean the union of all $(m-1)$-faces of cells of $\mathcal{M}$ that lie in exactly one m-cell. Thus, for example, the boundary of the surface of a cube is empty. For further examples consider

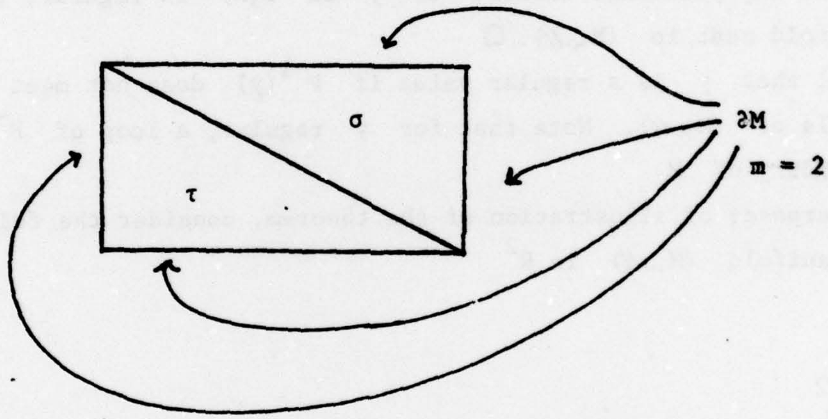
$\partial M = \{a,b\}$

$m = 1$



$\partial M = \phi$

$m = 1$



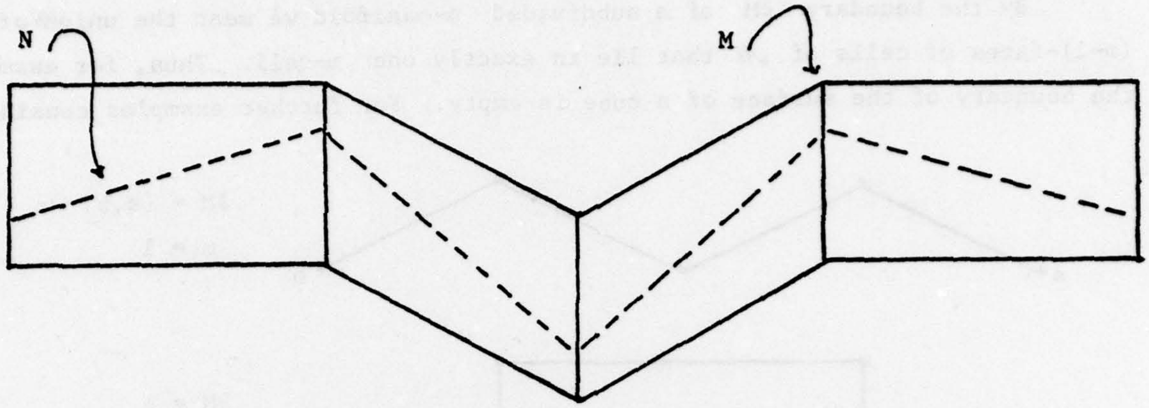
Let N be a 1-manifold and $(M,\mathcal{M})$ a subdivided m-manifold, where N is contained in M. We say that N is neat in $(M,\mathcal{M})$ if the following three conditions hold.

a) N is closed in M.

b) The boundary of N lies in the boundary of M.

c) The collections of nonempty sets of form $N \cap \sigma$ with σ in $\mathcal{M}$ forms a subdivision of N.

In the following diagram we show a 1-manifold (which is a route) N neat in a subdivided 2-manifold M.

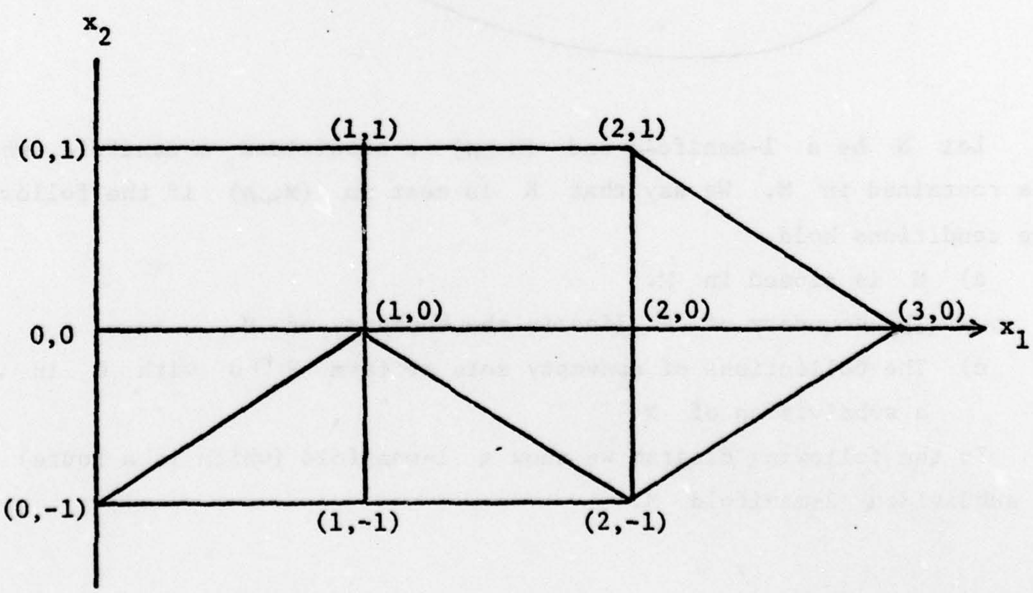


Now we can state the main theorem for regular values y.

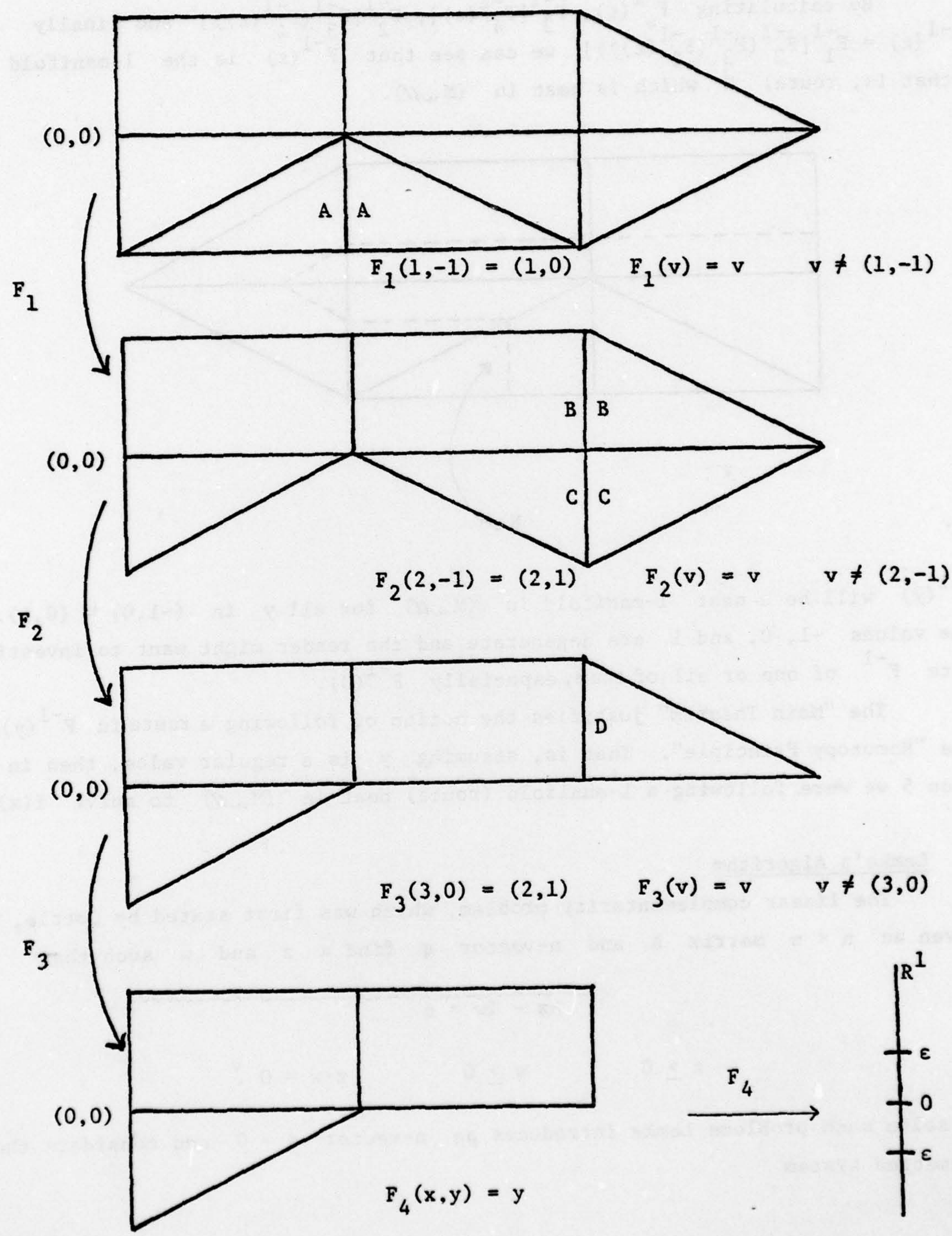
Main Theorem (Regular Values): Let $(M,\mathcal{M})$ be an $(n+1)$-manifold and $F : M \to R^n$ be $\mathcal{M}$ piecewise linear. If y in $F(M)$ is regular, then $F^{-1}(y)$ is a 1-manifold neat in $(M,\mathcal{M})$. □

Recall that y is a regular value if $F^{-1}(y)$ does not meet any $(n-1)$-faces of cells of $(M,\mathcal{M})$. Note that for y regular, a loop of $F^{-1}(y)$ cannot meet the boundary of M.

For purposes of illustration of the theorem, consider the following subdivided 2-manifold $(M,\mathcal{M})$ in R^2

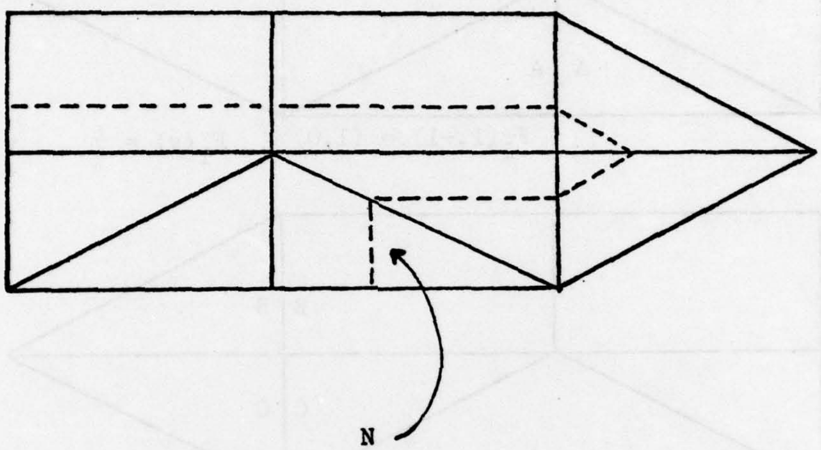


Let $F = F_1 F_2 F_3 F_4$ where the F_i's are defined in the schema below together with the requirement that the F_i's are piecewise linear with respect to the indicated subdivisions. Let v represent any vertex.



F_1 collapses the region A, F_2 flips C over to B, F_3 collapses the region D, and F_4 projects to the vertical axis. So F_i^{-1} of any set can be discerned by inspection.

By calculating $F_4^{-1}(\varepsilon)$, $F_3^{-1}[F_4^{-1}(\varepsilon)]$, $F_2^{-1}[F_3^{-1}(F_4^{-1}(\varepsilon))]$ and finally $F^{-1}(\varepsilon) = F_1^{-1}[F_2^{-1}(F_3^{-1}(F_2^{-1}(\varepsilon)))]$ we can see that $F^{-1}(\varepsilon)$ is the 1-manifold (that is, route) N which is neat in $(M,\mathcal{M})$.



$F^{-1}(y)$ will be a neat 1-manifold in $(M,\mathcal{M})$ for all y in $(-1,0) \cup (0,1)$. The values -1, 0, and 1 are degenerate and the reader might want to investigate F^{-1} of one or all of them, especially $F^{-1}(0)$.

The "Main Theorem" justifies the notion of following a route in $F^{-1}(y)$ in the "Homotopy Principle". That is, assuming y is a regular value, then in Section 5 we were following a 1-manifold (route) neat in $(M,\mathcal{M})$ to solve $f(x) = y$.

7. Lemke's Algorithm

The linear complementarity problem, which was first stated by Cottle, is: Given an $n \times n$ matrix A and n-vector q find a z and w such that

$$Az - Iw = q$$

$$z \geq 0 \qquad w \geq 0 \qquad z \cdot w = 0 .$$

To solve such problems Lemke introduces an n-vector $d > 0$ and considers the augmented system

$$Az - Iw + d\theta = q$$

$$z \geq 0 \qquad w \geq 0 \qquad \theta \geq 0 \qquad z \cdot w = 0$$

Lemke's algorithm proceeds by generating a path of solutions to the augmented system. To explain his algorithm from the perspective of the general theory first define $f_i : R^1 \to R^n$ for $i = 1, 2, \ldots, n$ by

$$f_i(x_i) = \begin{cases} A_i x_i & \text{if} \quad x_i \geq 0 \\ I_i x_i & \text{if} \quad x_i \leq 0 \end{cases}$$

where A_i and I_i are the i^{th} columns of A and I, the identity, respectively. Define $f : R^n \to R^n$ by

$$f(x) = \sum_i^n f_i(x_i)$$

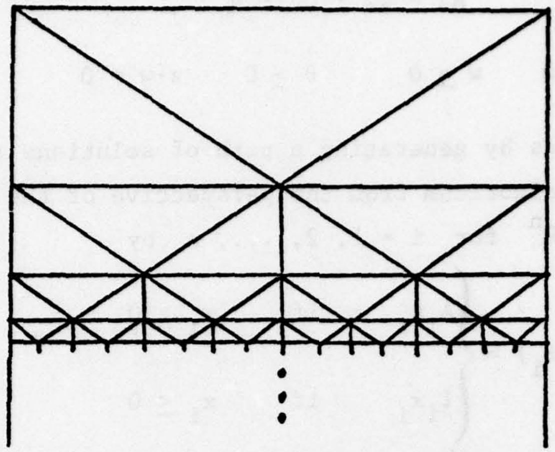
where

$$x = (x_1, \ldots, x_n) \; .$$

The complementary problem is equivalent to $f(x) = q$ and the augmented system is equivalent to $f(x) + \theta d = q$ with $\theta \geq 0$. So, define $F(x,\theta) = f(x) + \theta d$ as the homotopy; note that F is piecewise linear with respect to the orthants of $R^n \times R^1_+$. To solve $f(x) = q$ Lemke's algorithm follows the path of $F^{-1}(q)$ beginning with θ large and $x \leq 0$. Observe that $F^{-1}(q)$, if q is regular, is a 1-manifold neat in $R^n \times R^1_+$ which is subdivided by its orthants. To show that the route of the algorithm yields a solution to $f(x) = q$ requires conditions on the matrix A, an issue that will not be treated here.

8. The Eaves-Saigal Algorithm

Let $g : S \to \mathring{S}$ be a continuous function from a simplex S in R^n to its interior $\mathring{S}$. Let us compute a fixed point $x = g(x)$ of g, or equivalently, a zero of $f(x) \overset{\Delta}{=} g(x) - x$. The first step is to subdivide the cylinder $M = S \times (0,1]$ with $\mathcal{M}$ as indicated in the picture below.



S × {1} should not be subdivided and the size of the simplexes of the subdivision should tend to zero uniformly as the simplexes near S × {0}. We pause to note here that in a computer the subdivision exists only in the sense that there is a formula that can be used to generate portions of the subdivision as needed.

Let $F(x,\theta) = f(x)$ for all vertices (x,θ) of the subdivision. Define $F : M \to R^n$ by extending F to all of M in such a way that F is affine on the cells of $\mathcal{M}$; this extension is unique. Once again, in a computer F is only generated as needed. We now have the property that $F(\cdot,1)$ is linear and $F(\cdot,t)$ tends to f as t tends to 0. $F(x,1) = 0$ is a linear system and is easily shown to have a unique solution $(x_1,1)$.

Beginning with the point $(x_1,1)$ the route of $F^{-1}(0)$ is followed; $F^{-1}(0)$ is a neat 1-manifold in $(M,\mathcal{M})$ if 0 is regular. The x component of the route tends (as the subdivision gets finer and finer) to a solution of $f(x) = 0$, that is, a fixed point of g.

8. Internal Developments

The principle weakness of complementary pivot theory is simply that there are too many cells to traverse along the route of $F^{-1}(y)$. Many studies have improved the situation; let us mention those that seem to be the most important. The "restart" method of Merrill [22] permits, in effect, many cells to be skipped. In the presence of differentiability Saigal [25] has shown that

Merrill's method can be used to obtain quadratic convergence; in addition, it becomes clear from his analysis that one should formulate the system of equations ro be solved so that they are as smooth as possible; the routes to be followed are then less inclined to turn radically. In the absence of special structure Todd has shown that the simplexes (and cells) should be as round as possible, that is, not long slivers.

Kojima [17], and recently Todd [31], have shown how to use special structure of the function as linearity and separability to drastically reduce the number of cells. Van der Laan and Talman [33] have revived an idea of Shapley [30] which, in effect, enables one to move through some fraction of the cells more quickly; for lack of a better term this technique is often referred to a "variable dimension method". Garcia and Gould [10] have proposed an idea with similar affect. The works of Kellogg, Li, and Yorke [16] and Hirsch and Smale [13] avoid cells altogether by using differential homotopies, see Li [21], but, then, following a differential path is more difficult than following a piecewise linear route; the tradeoff is not yet understood.

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An elementary overview is offered of the complementary pivot and fixed point method of solving equations. The "homotopy principle" is described and then exhibited and justified with examples.

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