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STABILITY RESULTS FOR A CLASS OF SYSTEMS WITH MULTIPLICATIVE STATE NOISE

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ABSTRACT

The stochastic stability of linear systems with non-Gaussian multiplicative (state dependent) noise is analyzed. The particular noise processes considered are a form of filtered Poisson processes. A technique is presented for investigating the p^{th} moment asymptotic stability of linear systems satisfying certain Lie-algebraic conditions. Several examples are given to illustrate the technique.

INTRODUCTION

The analysis of linear systems with multiplicative or state dependent noise (i.e. bilinear stochastic systems) has recently attracted a great deal of attention [1-14]. In particular, the stochastic stability of such systems for which the noise processes are Gaussian has been studied in some detail. In this paper the stochastic stability of a class of linear systems with non-Gaussian multiplicative noise is analyzed. The particular noise processes considered are a form of filtered Poisson processes.

Our technique for investigating stochastic stability is based upon a method used by Willsky, Marcus, and Martin [1]. In the following sections we describe a technique for investigating the stochastic stability of a class of systems. The technique is illustrated with examples.

PRELIMINARIES

We are interested in systems of the form

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^k A_i f_i(t) \right] x(t) , \quad (1)$$

where the A_i are known $n \times n$ matrices, $x(t)$ is an n dimensional vector, and $f_1(t), \dots, f_k(t)$ are noise processes. We assume that the noise processes are independent of the initial condition $x(0)$.

A Lie algebra of $n \times n$ matrices is a subspace of $n \times n$ matrices that is closed under the commutator product $[A, B] \triangleq AB - BA$. The Lie algebra generated by a set of $n \times n$ matrices is the smallest Lie algebra containing that set of matrices. A Lie algebra is said to be solvable if and only if there exists a nonsingular matrix P (possibly complex valued) such that PAP^{-1} is upper triangular for every A belonging to the Lie algebra (see [1]). Let \mathcal{L} denote the Lie algebra generated by A_0, A_1, \dots, A_k and assume that \mathcal{L} is solvable.

Let p be a positive integer and let $x_i(t)$ denote the i^{th} component of the vector $x(t)$. We say that the null solution of (1) is p^{th} moment asymptotically stable if, for all initial states $x(0)$ possessing the desired moments and independent of $f_1(t), \dots, f_k(t)$, we have

$$\lim_{t \rightarrow \infty} E \left\{ \left[x_1(t) \right]^{p_1} \left[x_2(t) \right]^{p_2} \dots \left[x_n(t) \right]^{p_n} \right\} = 0$$

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for any set of nonnegative integers p_i that sum to p . It is shown in [1] that, if \mathcal{L} is solvable, then the p^{th} moment asymptotic stability of (1) is equivalent to the first moment asymptotic stability of another system of the form (1), also with a solvable Lie algebra. Therefore, we can confine our discussion to the first moment asymptotic stability of (1), that is

$$\lim_{t \rightarrow \infty} E \{x_i(t)\} = 0, \quad i=1, \dots, n.$$

DEVELOPMENT

We write the solution to (1) in the form

$$x(t) = \phi_f(t,0) x(0),$$

where $\phi_f(t,0)$ is the transition matrix for (1), thought of as an explicit function of the processes $f_i(t)$. Since \mathcal{L} is assumed to be solvable, we know that there is a nonsingular matrix P such that

$$B_i = P A_i P^{-1}$$

is upper triangular for $i=0, 1, \dots, k$. The equation

$$\dot{Y}(t) = \left[B_0 + \sum_{i=1}^k B_i f_i(t) \right] Y(t), \quad Y(0) = I,$$

where $Y(t)$ is an $n \times n$ matrix, can be solved by straightforward calculations. Thus we have that

$$\phi_f(t,0) = P^{-1} Y(t) P,$$

and we can see that $\phi_f(t,0)$ involves nothing more complicated than exponentials of integrals of the $f_i(t)$, polynomials in the $f_i(t)$, and various combinations, products, and integrals of such quantities. This was the approach used in [1], where it was assumed that the $f_i(t)$ were mutually Gaussian random processes. Using the properties of the Gaussian distribution, it was then possible to evaluate

$$E \{ \phi_f(t,0) \}. \tag{2}$$

Since $E\{x(t)\} = E\{\phi_f(t,0)\} E\{x(0)\}$, we see that it is necessary to know the quantity in (2) in order to investigate the first moment asymptotic stability of (1).

In this paper we will assume that the noise processes $f_1(t), \dots, f_k(t)$ are independent random processes and that each is a type of filtered Poisson process. Specifically, we assume that a noise process has the following form:

$$f(t) = \begin{cases} 0, & N_t = 0 \\ \sum_{i=1}^{N_t} U_i h(t, \tau_i), & N_t \geq 1 \end{cases} \tag{3}$$

where N_t is a Poisson counting process with intensity function $\lambda(t) \geq 0$,

τ_i is the i^{th} occurrence time, and the amplitudes U_i are independent and identically distributed and independent of N_t . It will be assumed that $\lambda(t)$ is integrable over bounded sets and positive on a set of positive Lebesgue measure. Also, it is assumed that U_1 possesses a moment generating function. Finally, it is assumed that the impulse response, or weighting function, $h(t, \tau)$ is causal; that is, $h(t, \tau) = 0$ for $t < \tau$. Noise processes of this general type are a popular model for many physical phenomena.

The characteristic functional of the random process $f(t)$ defined in (3) is given by the following [15]:

$$C_f(g) \triangleq E \left\{ \exp \left[j \int_0^t f(s) dg(s) \right] \right\} = \exp \left(\int_0^t \lambda(\tau) \left[\phi \left(\int_{\tau}^t h(s, \tau) dg(s) \right) - 1 \right] d\tau \right), \quad (4)$$

where $\phi(a) = E\{\exp(jaU_1)\}$ is the characteristic function of U_1 . By the proper choice of the function $g(\cdot)$, we can use the characteristic functional in (4) to evaluate (2). We now illustrate the method with examples.

Example 1: Let $n = k = 2$, and let

$$A_0 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} .$$

In this case we can easily check that \mathcal{L} is solvable and that

$$P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

upper triangularizes the system. Letting $y = Px$, we have

$$\dot{y}(t) = \begin{bmatrix} -1 + f_1(t) & -1 + f_1(t) \\ 0 & -1 + f_2(t) \end{bmatrix} y(t) .$$

Thus we find that

$$y(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ 0 & a_{22}(t) \end{bmatrix} y(0) ,$$

where

$$a_{ii}(t) = \exp \left[-t + \int_0^t f_i(\tau) d\tau \right] , \quad i = 1, 2$$

and

$$a_{12}(t) = \int_0^t \exp \left[-t + \int_s^t f_1(\tau) d\tau + \int_0^s f_2(\tau) d\tau \right] (-1 + f_1(s)) ds .$$

Assume that both of the noise processes have the form of (3) and that for each of the noise processes the corresponding amplitudes $\{U_i\}$ have a Gaussian distribution with zero mean and unit variance. Let

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$h_1(t, \tau) = \exp(\tau - t) u(t - \tau)$ and $h_2(t, \tau) = u(t - \tau) - u(t - \tau - 1)$, where $u(\cdot)$ is the unit step function.

Using (4) we find that

$$E \{a_{ii}(t)\} = \exp \left(-t + \int_0^t \lambda_i(\tau) \left[M \left(\int_\tau^t h_i(s, \tau) ds \right) - 1 \right] d\tau \right),$$

where $M(a) = \exp(a^2/2)$. Since

$$M \left(\int_\tau^t h_i(s, \tau) ds \right) \leq \sqrt{e} \quad \text{for all } t, \tau,$$

we have that

$$E \{a_{ii}(t)\} \leq \exp \left[-t + (\sqrt{e} - 1) \int_0^t \lambda_i(\tau) d\tau \right].$$

Therefore, if

$$\int_0^\infty \lambda_i(\tau) d\tau < \infty \quad (5)$$

or if

$$\lambda_i(\tau) \leq \frac{\alpha}{(\sqrt{e} - 1)} \quad \text{for some } \alpha < 1, \quad (6)$$

then we see that

$$\lim_{t \rightarrow \infty} E \{a_{ii}(t)\} = 0.$$

Now consider the off-diagonal term:

$$\begin{aligned} a_{12}(t) = & - \int_0^t \exp \left[-t + \int_s^t f_1(\tau) d\tau + \int_0^s f_2(\tau) d\tau \right] ds \\ & + \int_0^t f_1(s) \exp \left[-t + \int_s^t f_1(\tau) d\tau + \int_0^s f_2(\tau) d\tau \right] ds. \end{aligned} \quad (7)$$

The magnitude of the expectation of the first summand in (7) can be upper bounded by

$$\int_0^t \exp \left[-t + (\sqrt{e} - 1) \int_s^t \lambda_1(\tau) d\tau + (\sqrt{e} - 1) \int_0^s \lambda_2(\tau) d\tau \right] ds. \quad (8)$$

Thus we see that if λ_1 and λ_2 each satisfy either (5) or (6) then (8) goes to zero as t approaches infinity.

To evaluate the expectation of the second summand in (7) we proceed

as follows. Using (4) we find that

$$E \left\{ \exp \left[r f_1(s) + \int_s^t f_1(\tau) d\tau \right] \right\} = \exp \left(\int_s^t \lambda_1(\tau) \left[M(r+1 - e^{\tau-t}) - 1 \right] d\tau \right).$$

Thus,

$$\begin{aligned} E \left\{ f_1(s) \exp \left[\int_s^t f_1(\tau) d\tau \right] \right\} &= \frac{d}{dr} E \left\{ \exp \left[r f_1(s) + \int_s^t f_1(\tau) d\tau \right] \right\} \Big|_{r=0} \\ &= \exp \left[\int_s^t \lambda_1(q) \left[M(1 - e^{q-t}) - 1 \right] dq \right] \int_s^t \lambda_1(\tau) (1 - e^{\tau-t}) M(1 - e^{\tau-t}) d\tau. \end{aligned}$$

Therefore,

$$\left| E \left\{ f_1(s) \exp \left[\int_s^t f_1(\tau) d\tau \right] \right\} \right| \leq \sqrt{e} \int_s^t \lambda_1(\tau) d\tau \exp \left[\int_s^t \lambda_1(q) (\sqrt{e} - 1) dq \right]. \quad (9)$$

If λ_1 satisfies (5), then (9) can be upper bounded by a constant. If λ_1 satisfies (6), then (9) can be upper bounded by $3(t-s) \exp[\alpha(t-s)]$.

Now consider

$$E \left\{ \exp \left[\int_0^s f_2(\tau) d\tau \right] \right\}.$$

This can be upper bounded by

$$\exp \left[(\sqrt{e} - 1) \int_0^s \lambda_2(\tau) d\tau \right].$$

If λ_2 satisfies (5) or (6), this can be upper bounded by a constant times $\exp(\alpha s)$. Therefore, if λ_1 and λ_2 each satisfy either (5) or (6), then $E\{a_{12}(t)\}$ goes to zero as t approaches infinity, and we conclude that the system of Example 1 is asymptotically stable.

Example 2: Consider the following scalar equation:

$$\dot{x}(t) = \alpha x(t) + \beta x(t) f(t) \quad (10)$$

where $\beta \neq 0$. The solution is given by

$$x(t) = x(0) \exp \left[\alpha t + \beta \int_0^t f(\tau) d\tau \right].$$

Assume that $f(t)$ is a compound Poisson process; that is, in (3), $h(t, \tau) = u(t - \tau)$. Then, using (4), we have that

$$E \left\{ \exp \left[\beta \int_0^t f(\tau) d\tau \right] \right\} = \exp \left(\int_0^t \lambda(\tau) \left(M[\beta(t-\tau)] - 1 \right) d\tau \right).$$

where M is the moment generating function of U_1 . It is straightforward to show that if $P\{\beta U_1 > 0\} > 0$ then there exists a $\gamma > 0$ such that for $t > \tau$, $M[\beta(t-\tau)] \geq \exp[\gamma(t-\tau)]$. Assuming that $P\{\beta U_1 > 0\} > 0$, we have that

$$E \left\{ \exp \left[\beta \int_0^t f(\tau) d\tau \right] \right\} \geq \exp \left[\int_0^t \lambda(\tau) \left(\exp[\gamma(t-\tau)] - 1 \right) d\tau \right].$$

Notice that

$$\frac{d}{dt} \left(\int_0^t \lambda(\tau) \left(\exp[\gamma(t-\tau)] - 1 \right) d\tau \right) = \gamma e^{\gamma t} \int_0^t \lambda(\tau) e^{-\gamma\tau} d\tau. \quad (11)$$

Since $\lambda(\cdot)$ is positive on a set of positive Lebesgue measure, we see that (11) approaches infinity as t approaches infinity. This yields the following result.

Theorem: Assume that $f(t)$ is a compound Poisson process whose amplitudes U_i possess a moment generating function and that the intensity function $\lambda(\cdot)$ is positive on a set of positive Lebesgue measure. Also, assume that $P\{\beta U_1 > 0\} > 0$. Then (10) is not asymptotically stable.

Notice that if the support of the distribution of U_1 includes both positive and negative numbers, then $P\{\beta U_1 > 0\} > 0$.

Example 3: Now consider the following n^{th} order system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t) f(t) \quad (12)$$

where A_0 and A_1 are upper triangular $n \times n$ matrices and $x(t)$ is an n dimensional vector. Notice that the i^{th} term of the diagonal of the transition matrix $\phi_f(t, 0)$ will be of the form

$$\exp \left[\alpha_i t + \beta_i \int_0^t f(\tau) d\tau \right], \quad (13)$$

where α_i and β_i are the i^{th} terms of the diagonals of A_0 and A_1 , respectively. If the initial state $x(0)$ is a constant vector with a one in the i^{th} position and zeros elsewhere, then the i^{th} component of $x(t)$ will be given by (13). This results in the following corollary.

Corollary: If at least one diagonal element β of A_1 is nonzero and if $f(t)$ satisfies the conditions of the Theorem, then (12) is not asymptotically stable.

Example 4: Consider the system (10) and let the noise process be given by (3) where

$$h(t, \tau) = \exp[-\gamma(t-\tau)] u(t-\tau), \quad \gamma > 0.$$

Let $M(\cdot)$ be the moment generating function of U_1 . We find that

$$E \left\{ \exp \left[\beta \int_0^t f(\tau) d\tau \right] \right\} = \exp \left[\int_0^t \lambda(\tau) \left(M \left[\frac{\beta}{\gamma} \left(1 - \exp[-\gamma(t-\tau)] \right) \right] - 1 \right) d\tau \right].$$

Assuming that $\lambda(\cdot)$ is integrable, we see that this quantity can be easily upper bounded. In this case we see that (10) is asymptotically stable if $\alpha < 0$.

Example 5: Consider the scalar equation (10). Let $f(t)$ be given by (3) and assume that h is a function of the difference of its arguments, say

$$h(t, \tau) = \tilde{h}(t-\tau).$$

Assume that U_1 possesses the moment generating function $M(\cdot)$. Also, assume that

$$\int_0^\infty |\tilde{h}(t)| dt < \infty. \quad (14)$$

Then from (4) we see that

$$E \left\{ \exp \left[\beta \int_0^t f(\tau) d\tau \right] \right\} = \exp \left[\int_0^t \lambda(\tau) \left(M \left[\beta \int_0^{t-\tau} \tilde{h}(s) ds \right] - 1 \right) d\tau \right].$$

From (14) we conclude that

$$M \left[\beta \int_0^{t-\tau} \tilde{h}(s) ds \right] \leq M_0 < \infty.$$

Assume that the intensity function $\lambda(\cdot)$ is such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda(\tau) d\tau = 0.$$

Then we have that

$$\left| E \{ x(t) \} \right| \leq \left| E \{ x(0) \} \right| \exp \left[\alpha t + M_0 \int_0^t \lambda(\tau) d\tau \right] .$$

Thus we see that if $\alpha < 0$, the system is asymptotically stable.

CONCLUSIONS

We have presented a method for analyzing the p^{th} moment asymptotic stability of a class of linear systems with multiplicative state noise. The class of systems considered were required to satisfy a certain Lie algebraic condition. The noise processes were taken to be a form of filtered Poisson noise. The utility of the method was illustrated with several examples. We note that these methods can be extended in a straightforward fashion to include noise processes having the form considered in this paper plus an additive independent Gaussian component.

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