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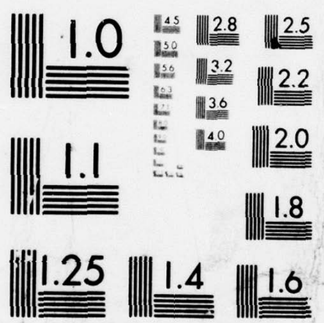

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

ON A SIMPLE RESOURCE-ALLOCATION GAME

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Group 92

TECHNICAL NOTE 1978-49

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LEXINGTON MASSACHUSETTS

I. Introduction

In his 1956 paper[1], Moser mentions a lottery puzzle posed by Cayley in 1875[2]. Moser discusses a formalized game in optimal stopping, closely related to Cayley's puzzle, and supplies a solution.

In Section II of this paper, we describe Moser's Game and repeat its solution, following, in the main, the lucid treatment of Howard[3]*. In Section III we introduce a generalization of the game -- an extension which we believe to be novel -- and give its solution. Section IV briefly discusses the applicability of the extended problem to matters involving the allocation of limited resources.

II. Moser's Game

The problem under discussion may be posed in the form of a game against Nature:

> A player with a marker observes repeated draws of a random sample from a known probability distribution on the real numbers. Upon observing the magnitude, x, of a draw, he may choose to tag (T) that draw with his marker: should he do so, the game is over and its value is x. Should he not

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* A discussion of Moser's Game, under a variety of names, is found in many treatises on Dynamic Programming or Optimal Stopping Theory.

tag ($\overline{T}$) that draw, it is discarded and another draw made and offered to him. At the beginning of the game he knows that he may observe as many as s draws if he chooses to do so.

What stopping strategy ought he to adopt to maximize the value of the game? I.e., how shall he decide when to tag a draw?

Howard's formulation of the solution (with notational changes) is followed below, with some simplification. Let:

$s \equiv$ No. of remaining choices or draws (or <u>stages</u>).

$t(s)$, or t, $\equiv$ Threshold for stopping at Stage s.
  I.e., if the draw at Stage s exceeds t, the player will tag it; else, he will pass it and go to Stage (s-1).

$v(s) \equiv$ The expected value of the game when the player is in Stage s and is using a general tagging strategy.

$v^*(s) \equiv$ The expected value of being in Stage s, given that the player utilizes an <u>optimal</u> tagging strategy.

The player will tag a draw if it is sufficiently large in magnitude, in the expectation that none of the limited number of future draws will be larger. Thus, a tagging strategy is simply a sequence of thresholds, one for each stage. The player compares the draw at a stage with the appropriate threshold and tags or not according as the draws exceeds or fall short of the threshold. Then the value of the game is the weighted sum of the values which are expected to result from tagging or from continuing:

$$v(s) = Pr(T|t) \langle Value|T,t\rangle + Pr(\overline{T}|t)v(s-1) \qquad (1)$$

Now, the probability of tagging is the probability that the draw, x, exceed the threshold:

$$Pr(T|t) = Pr(x \geqq t)$$

Of course:

$$Pr(\overline{T}|t) = 1 - Pr(x \geqq t)$$

The value of the halted game is the expected value of x, given that it exceeds the threshold:

$$\langle Value|T,t\rangle = E(x|x \geqq t)$$

Eq (1) may, therefore, be written:

$$v(s) = Pr(x \geqq t)\ E(x|x \geqq t) + \{1-Pr(x \geqq t)\}\ v(s-1) \qquad (2)$$

<u>Uniform Distribution</u>

In the special case of drawing from a distribution which is uniform over (0,1), we have:

$$Pr(x \geqq t) = (1-t)$$

$$E(x|x \geqq t) = \tfrac{1}{2}(1+t)$$

and Eq (2) becomes:

$$v(s) = \tfrac{1}{2}(1-t^2) + tv(s-1) \qquad (3)$$

If we assume that the player is using an optimal tagging strategy, we must have:

$$v^*(s) = \max_t \{\tfrac{1}{2}(1-t^2) + tv^*(s-1)\} \tag{4}$$

By differentiating the terms within brackets with respect to t(s), we find that the optimal threshold is given by:

$$t^*(s) = t^* = v^*(s-1)$$

I.e., the best threshold at a stage equals the value of the game at the subsequent stage. (Quite reasonable: if the current draw promises to be better than the value of the residual game, then tag the draw; otherwise, continue.) Substituting this into Eq(4):

$$\begin{aligned} v^*(s) &= \tfrac{1}{2}\{1 + t^{*2}(s)\} \\ &= \tfrac{1}{2}\{1 + v^{*2}(s-1)\} \end{aligned} \tag{5}$$

Eq(5) is a recursion formula for the value of the game at any stage in terms of the values at later stages. To initiate the recursion we require, as a given condition to the problem, the value $v^*(0)$ -- how much the game is worth if the player runs out of draws before applying his marker. The most interesting case is $v^*(0) = 0$, and we shall invariably assume this to hold. (This implies that the game has a null value if the player never tags a draw.)

Table 1 (based on Howard, but derived independently) shows $v^*(s)$ for the first 30 values of s. The value of the game approaches 1 asymptotically for increasing s, as one might expect from the nature of the game and the form of Eq(5).

TABLE 1

OPTIMUM VALUES OF MOSER'S GAME,

DRAWING FROM UNIFORM DISTRIBUTION (AFTER HOWARD[3])

S	VALUE
	0.5000
	0.6250
	0.6953
	0.7417
5	0.7751
	0.8004
	0.8203
	0.8364
	0.8498
10	0.8611
	0.8707
	0.8791
	0.8864
	0.8929
15	0.8986
	0.9037
	0.9084
	0.9126
	0.9164
20	0.9199
	0.9231
	0.9261
	0.9288
	0.9313
25	0.9337
	0.9359
	0.9379
	0.9399
	0.9417
30	0.9434

General Distribution

If, during the game, the drawings are made, not from a uniform distribution, but rather from one with a probability density, $f(\cdot)$, it can readily be shown that the recursion formula for $v^*(s)$ takes the form:

$$v^*(s) = \int_{t^*}^{\infty} z f(z)\, dz + v^*(s-1) \int_{-\infty}^{t^*} f(z)\, dz \qquad (6)$$

By differentiating the r.h.s. of Eq(6) with respect to t^* and equating the result to 0, we find that the optimal threshold again obeys the relation:

$$t^*(s) = v^*(s-1) \qquad (7)$$

Numerical problems may be encountered if the functions, $f(z)$ and $zf(z)$, are not integrable in closed form.

III. Generalization of Moser's Game

We generalize Moser's Game by giving the player, not one, but n markers. As before, he has s draws and may tag a draw or let it pass; if he tags it and if draws and markers remain, the game resumes. At the end, the value of the game is the sum of the tagged draws. We write:

$v_n(s) \equiv$ Expected value of a game which is in Stage s with n markers remaining.

$t_n(s)$, or $t \equiv$ Threshold used in Stage s with n markers.

Then we may write an equation analogous to Eq(1):

$$v_n(s) = Pr(T|t)\{\langle \text{Value } T,t,n\rangle + v_{n-1}(s-1)\}$$
$$+ \{1-Pr(T|t)\}v_n(s-1) \tag{8}$$

Eq(8) expresses the fact that the value, $v_n(s)$, is the weighted sum of the values of two exclusive alternatives:

i. Tagging the current draw, which accrues the value increment, $\langle \text{Value}|T,t,n\rangle$, and leaves the player with the value of the residual game, viz , $v_{n-1}(s-1)$

ii. Passing the current draw, which leaves the player with a game of value, $v_n(s-1)$

By manipulations similar to those in Section II, we find:

$$v^*_n(s) = \max_t \Big[Pr(T|t)\{\langle \text{Value } T,t,n\rangle + v^*_{n-1}(s-1)\}$$
$$+ \{1-Pr(T|t)\}v^*_n(s-1)\Big] \tag{9}$$

Uniform Distribution

Once again, matters simplify if the drawings arise from a uniform distribution over (0,1). Eq(9) becomes:

$$v^*_n(s) = \max_t\{\tfrac{1}{2}(1-t^2) + (1-t)v^*_{n-1}(s-1) + tv^*_n(s-1)\} \tag{10}$$

By differentiating within the brackets, the optimum threshold is found to be:

$$t^*_n(s) = v^*_n(s-1) - v^*_{n-1}(s-1) \tag{11}$$

whence:

$$v^*_n(s) = \tfrac{1}{2}\left[1 + \{v^*_n(s-1)-v^*_{n-1}(s-1)\}^2\right] + v^*_{n-1}(s-1) \tag{12}$$

This last equation may be more suggestively written as:

$$v^*_n(s) = \tfrac{1}{2}\{1 + t^{*2}_n(s)\} + v^*_{n-1}(s-1) \tag{13}$$

to show its parallelism with Eq(5).

Eqs(11) and (12) provide a recurrence relation from which $v^*_n(s)$ can be determined, knowing the values of the game for fewer markers and fewer draws. As before, initial conditions must be supplied, and we take $v^*_1(0) = v^*_2(0) = \ldots = v^*_n(0) = 0$, which assigns a null value to any game in which the player never tags a draw.

Table 2 lists values of $v^*_n(s)$ for n from 2 to 9, and for the first 30 values of s. The corresponding thresholds are shown as well. The asymptotic value of $v^*_n(s)$ is, of course, n, the approach to the limit becoming slower with increasing n. For the first n values of a game in which there are s choices, the incremental value of the game is ½ per additional stage; this reflects the fact that, whenever the player has as many markers left as choices, he tags each draw regardless of its magnitude, and gains an average of ½ in value each time. Only when there are more draws than markers left in a game does the player have an effective choice of tagging or passing.

General Distribution

For a general density function, f(•), for the draws, the optimal threshold is still given by Eq(11), and the recursion relation for the value of a game is:

$$v^*_n(s) = \int_{t^*}^{\infty} zf(z)\, dz + v^*_{n-1}(s-1) \int_{t^*}^{\infty} f(z)\, dz + v^*_n(s-1) \int_{-\infty}^{t^*} f(z)\, dz \qquad (14)$$

The numerical problems which might be encountered are similar to those for the unextended game.

IV. Discussion

Moser's Game and its generalization are encountered in some applications where a limited number of resources ('markers') are to be allocated against selected realizations of a quantity of recurring opportunities ('draws'). The particular application of interest to the writer, which drew his attention to this game, was this:

> A defense commander controls a finite number (n) of weapons. These are to be expended on an individual basis against selected members of an attacking force (of which there are $s \geq n$ units). The attackers come into range one after the other and the commander must decide, from an observed score on each unit -- higher scores being correlated with more dangerous attackers --, which n units to intercept and which, perforce, to pass.

When not all attacking units can be engaged, and when the attackers must be dealt with on a one-by-one basis, the

TABLE 2

VALUES OF GENERALIZED MOSER'S GAME DRAWING FROM UNIFORM DISTRIBUTION

	N = 2			N = 3	
S	VALUE	THRESHOLD	S	VALUE	THRESHOLD
	0.5000	0.0		0.5000	0.0
	1.0000	0.0		1.0000	0.0
	1.1953	0.3750		1.5000	0.0
	1.3203	0.5000		1.7417	0.3047
5	1.4091	0.5786	5	1.9091	0.4214
	1.4761	0.6340		2.0341	0.5000
	1.5287	0.6757		2.1318	0.5580
	1.5712	0.7084		2.2105	0.6031
	1.6064	0.7347		2.2756	0.6393
10	1.6360	0.7565	10	2.3303	0.6692
	1.6613	0.7749		2.3770	0.6943
	1.6833	0.7906		2.4174	0.7157
	1.7024	0.8042		2.4528	0.7342
	1.7194	0.8160		2.4839	0.7503
15	1.7344	0.8265	15	2.5116	0.7646
	1.7479	0.8358		2.5365	0.7772
	1.7600	0.8441		2.5588	0.7886
	1.7710	0.8517		2.5791	0.7988
	1.7810	0.8585		2.5975	0.8080
20	1.7902	0.8647	20	2.6143	0.8164
	1.7986	0.8703		2.6298	0.8241
	1.8064	0.8755		2.6440	0.8312
	1.8135	0.8803		2.6572	0.8377
	1.8202	0.8847		2.6694	0.8437
25	1.8263	0.8888	25	2.6808	0.8493
	1.8321	0.8927		2.6914	0.8545
	1.8375	0.8962		2.7013	0.8593
	1.8425	0.8996		2.7106	0.8638
	1.8473	0.9027		2.7193	0.8680
30	1.8517	0.9056	30	2.7275	0.8720

TABLE 2

(Continued)

	N = 4			N = 5	
S	VALUE	THRESHOLD	S	VALUE	THRESHOLD
	0.5000	0.0		0.5000	0.0
	1.0000	0.0		1.0000	0.0
	1.5000	0.0		1.5000	0.0
	2.0000	0.0		2.0000	0.0
5	2.2751	0.2593	5	2.5000	0.0
	2.4761	0.3660		2.8004	0.2249
	2.6318	0.4420		3.0287	0.3243
	2.7568	0.5000		3.2105	0.3969
	2.8597	0.5462		3.3597	0.4538
10	2.9462	0.5842	10	3.4847	0.5000
	3.0200	0.6159		3.5912	0.5385
	3.0837	0.6429		3.6831	0.5712
	3.1394	0.6663		3.7633	0.5994
	3.1885	0.6866		3.8340	0.6240
15	3.2321	0.7046	15	3.8969	0.6456
	3.2712	0.7205		3.9531	0.6647
	3.3064	0.7347		4.0037	0.6819
	3.3382	0.7476		4.0495	0.6973
	3.3672	0.7592		4.0912	0.7112
20	3.3937	0.7698	20	4.1293	0.7239
	3.4181	0.7794		4.1642	0.7355
	3.4405	0.7883		4.1965	0.7462
	3.4612	0.7964		4.2262	0.7560
	3.4804	0.8040		4.2538	0.7650
25	3.4983	0.8110	25	4.2795	0.7734
	3.5149	0.8175		4.3034	0.7812
	3.5305	0.8235		4.3258	0.7885
	3.5451	0.8292		4.3467	0.7953
	3.5588	0.8345		4.3664	0.8017
30	3.5717	0.8395	30	4.3849	0.8076

TABLE 2

(Continued)

	N = 6			N = 7	
S	VALUE	THRESHOLD	S	VALUE	THRESHOLD
	0.5000	0.0		0.5000	0.0
	1.0000	0.0		1.0000	0.0
	1.5000	0.0		1.5000	0.0
	2.0000	0.0		2.0000	0.0
5	2.5000	0.0	5	2.5000	0.0
	3.0000	0.0		3.0000	0.0
	3.3203	0.1996		3.5000	0.0
	3.5712	0.2916		3.8364	0.1797
	3.7756	0.3607		4.1064	0.2653
10	3.9462	0.4158	10	4.3303	0.3308
	4.0912	0.4615		4.5199	0.3841
	4.2162	0.5000		4.6831	0.4288
	4.3252	0.5331		4.8252	0.4669
	4.4212	0.5619		4.9502	0.5000
15	4.5064	0.5871	15	5.0611	0.5290
	4.5826	0.6096		5.1603	0.5547
	4.6512	0.6296		5.2495	0.5776
	4.7133	0.6476		5.3302	0.5982
	4.7698	0.6639		5.4036	0.6168
20	4.8215	0.6787	20	5.4707	0.6337
	4.8688	0.6922		5.5322	0.6492
	4.9125	0.7046		5.5889	0.6634
	4.9528	0.7160		5.6412	0.6764
	4.9902	0.7266		5.6898	0.6894
25	5.0249	0.7363	25	5.7349	0.6996
	5.0573	0.7454		5.7769	0.7099
	5.0876	0.7539		5.8163	0.7196
	5.1160	0.7618		5.8531	0.7286
	5.1426	0.7692		5.8876	0.7371
30	5.1677	0.7762	30	5.9201	0.7450

TABLE 2

(Continued)

	N = 8			N = 9	
S	VALUE	THRESHOLD	S	VALUE	THRESHOLD
	0.5000	0.0		0.5000	0.0
	1.0000	0.0		1.0000	0.0
	1.5000	0.0		1.5000	0.0
	2.0000	0.0		2.0000	0.0
5	2.5000	0.0	5	2.5000	0.0
	3.0000	0.0		3.0000	0.0
	3.5000	0.0		3.5000	0.0
	4.0000	0.0		4.0000	0.0
	4.3498	0.1636		4.5000	0.0
10	4.6360	0.2435	10	4.8611	0.1502
	4.8770	0.3057		5.1613	0.2251
	5.0837	0.3571		5.4174	0.2843
	5.2633	0.4006		5.6394	0.3337
	5.4212	0.4381		5.8340	0.3760
15	5.5611	0.4710	15	6.0064	0.4129
	5.6861	0.5000		6.1603	0.4453
	5.7985	0.5259		6.2985	0.4741
	5.9002	0.5491		6.4235	0.5000
	5.9926	0.5700		6.5371	0.5233
20	6.0771	0.5891	20	6.6409	0.5445
	6.1545	0.6064		6.7360	0.5638
	6.2258	0.6223		6.8236	0.5815
	6.2917	0.6370		6.9045	0.5977
	6.3528	0.6505		6.9795	0.6128
25	6.4096	0.6630	25	7.0492	0.6267
	6.4625	0.6747		7.1141	0.6396
	6.5119	0.6855		7.1748	0.6516
	6.5582	0.6957		7.2316	0.6629
	6.6017	0.7052		7.2850	0.6734
30	6.6426	0.7141	30	7.3351	0.6833

sequential decisions faced by any commander take a form similar to that of Moser's Game, although realistic considerations produce considerable complexities. Nevertheless, an understanding of the generalized Moser Game is useful as a basis for deducing an optimal engagement strategy.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A player is offered a finite set of numbers, one after another, each drawn at random from a known distribution. The player may choose a number when offered (not knowing the future choices) and thereby stop the game, whose value becomes the magnitude of the chosen number. The strategy by which the player decides how to choose a number to maximize the value of the game, is presented, and the game is generalized to the case in which it continues until the player has made n choices (the value of the game then being the sum of the n numbers chosen).

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