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TOLERANCE LIMITS AND VARIABLES SAMPLING PLANS: SOME POWER CALCU--ETC(U)
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*Applied Research in Statistics - Mathematics - Operations Research*

## TOLERANCE LIMITS AND VARIABLES SAMPLING PLANS: SOME POWER CALCULATIONS FOR THE NORMAL AND LOGNORMAL DISTRIBUTIONS

by

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## I. INTRODUCTION

For product evaluation, life testing or reliability studies, a one-sided tolerance limit or a variables sampling plan is often required. Typical statements that may require these techniques are:

(1) at least a desired proportion, $p_0$, of the product must meet or exceed a lower specification limit, L, with $\gamma_0$ confidence

or (2) at least a desired proportion, $p_0$, of the product must not exceed an upper specification limit, U, with $\gamma_0$ confidence.

If $p_0$ is taken to be 50% in a normal distribution, then the problem reduces to the familiar one of finding a one-sided confidence limit for $\mu$, or testing a statistical hypothesis concerning $\mu$. The problem under consideration here is the general problem where $p_0$ is not restricted to be 50%.

Consider, for example, the U.S. Navy's requirement to have an oil-water separator capable of producing an effluent with less than 15 parts per million (ppm) of oil in the effluent water 85% of the time with 95% confidence. In this case, $p_0 = 85\%$, $U = 15$, $\gamma_0 = 95\%$ and inferences related to the mean are not appropriate. As another example, suppose one gallon paint containers are filled by a machine. The manufacturer wishes to be 95% confident that at least 99% of the paint containers have one gallon or more in them. Here, $p_0 = 99\%$, $\gamma_0 = 95\%$ and $L = 1$.

These problems can be solved using either a parametric or a nonparametric procedure. In this report it has been assumed that a sample $x_1, \ldots, x_n$ is available from a population which follows a normal distribution with mean $\mu$ and variance $\sigma^2$ (both $\mu$ and $\sigma^2$ unknown). Since the lognormal distribution can be transformed into a normal distribution, the results of power and sample size calculations will also apply to the lognormal distribution.

One approach that could be used in this situation is the construction of a one-sided tolerance limit. For example, in the case of a specification limit U that must not be exceeded, an upper tolerance limit could be constructed. An upper one-sided tolerance limit, A, is a quantity that can be calculated from the sample. This quantity is determined by

$$\Pr[\Pr(X \leq A) \geq p_0] = \gamma_0. \tag{1.1}$$

In other words, after an observed value of A is calculated for a sample, there would be $\gamma_0$ confidence that $P(X \leq A)$ is at least $p_0$. The proportion $p_0$ is referred to as the coverage and $\gamma_0$ as the confidence coefficient.

In the oil-water separator example, attention would be on the probability that the amount of oil in the effluent water is less than or equal to A. The value of A would be determined such that this probability is at least 85% with 95% confidence. The obvious change can be made if a lower one-sided tolerance limit is required. For example, in the paint container illustration, attention would be on the probability that the amount of paint in the container is at least A. The value of A would be determined such that this probability is greater than or equal to 99% with 95% confidence. In the case of a lower tolerance limit, $p_0$ is often referred to as the reliability of the product.

Based on a random sample, A could be calculated and then compared with U or L. The product would be accepted if U > A for an upper one-sided tolerance limit and rejected otherwise. Similarly, for a lower one-sided tolerance limit, the product would be accepted if $A > L$ and rejected otherwise. If the product just meets the specification, there is a probability of $1 - \gamma_0$ of accepting it. For values of $\gamma_0$ that are generally used, this procedure places a large risk on a product being rejected if, in fact, the product just meets the specification.

Another approach to this problem is hypothesis testing by means of a one-sided variables sampling plan. This is a plan which guarantees that at least a proportion $p_0$ of the population is less than a given value U with a probability $\gamma_0$. Specifically, if $p = \Pr[X < U]$, then it is desired to test $H_0$: $p < p_0$ versus $H_1$: $p \geq p_0$. If a lower specification limit L is required, the problem can be described in similar statements where the hypotheses have been appropriately altered.

The tolerance limit approach and the variables sampling plan approach to the problem under consideration have been shown by Owens [6] to be equivalent. For brevity, the case of an upper specification limit will be assumed throughout the remainder of this report, although the results also apply to the case of a lower specification limit.

No matter how one approaches the problem, one must, implicitly or explicitly, be concerned with testing the null hypothesis $H_0$: $p < p_0$ against the alternative $H_1$: $p \geq p_0$.

Under the normality assumption, A can be taken in the form of $\bar{x} + ks$, where $\bar{x}$ and $s^2$ are the sample unbiased estimators of $\mu$ and $\sigma^2$, respectively. To construct an upper one-sided tolerance limit, one must determine k such that

$$\Pr[\Pr(X \leq \bar{x} + ks) \geq p_0] = \gamma_0. \qquad (1.2)$$

For a given $p_0$ and $\gamma_0$, only the sample size is needed to determine the value of k and hence the tolerance limit. Values of k are given in many statistical tables, and approximation formulas are also available. See, for example, Natrella [5], Owens [6], Eisenhart, Hastay and Wallis [1].

Tolerance limits are usually used in estimation. That is, given $p_0$ and $\gamma_0$ it is desired to estimate $\mu + \Phi^{-1}(p_0)\sigma$ with $\gamma_0$ confidence, where $\Phi$ is the standard normal cumulative distribution function. However, an upper one-sided tolerance limit may also be used to test a hypothesis in the situation where a specification limit U has been imposed on the product. The procedure is to conclude that the specification has been met if $\bar{x} + ks \leq U$ and to conclude that it has not been met otherwise.

Guenther [3] linked certain one-sided tolerance limits to hypothesis testing. He showed that for the normal distribution, a test based on the tolerance limit $\bar{x} + ks$ is the uniformly most powerful invariant test.

## II. STATISTICAL FRAMEWORK

As was mentioned previously, the problem may be considered as a test of the hypothesis $H_0$: $p < p_0$ versus the alternative hypothesis $H_1$: $p \geq p_0$ where $p = Pr[X < U]$. Equation (1.2) is equivalent to

$$Pr[T(f,\delta) < k\sqrt{n} | \delta = \Phi^{-1}(p_0)\sqrt{n},\ f = n - 1] = \gamma_0 \qquad (2.1)$$

where $T(f,\delta)$ is a noncentral t random variable with f degrees of freedom and $\delta$ as the noncentrality parameter. (For a derivation of this equivalence, see Owens [6].)

In hypothesis testing, it is customary to speak in terms of the level of significance $\alpha = 1 - \gamma_0$ rather than the confidence. From the formulation of the problem in (2.1), it is clear that k is determined by $p_0$, n, and $\alpha$ (or $\gamma_0$); i.e., k is determined by $H_0$. Under the alternative hypothesis, only the noncentrality parameter $\delta$ is changed.

Consider now the power of the test. It is a function of four parameters and can be written as

$$G(p_0, \gamma_0, p, n) = Pr[T(f,\delta) \geq k\sqrt{n} | \delta = \Phi^{-1}(p)\sqrt{n},\ f = n - 1] \qquad (2.2)$$

Since the alternative hypothesis is a one-sided alternative, it is only necessary to consider the power for values of $p \geq p_0$.

Before the power can be calculated, the value of k must be determined. This can be done by choosing the k which minimizes

$$(Pr[T(f,\delta) < k\sqrt{n} | \delta = \Phi^{-1}(p_0)\sqrt{n},\ f = n - 1] - \gamma_0)^2.$$

With this value of k, the calculation of the power $G(p_0, \gamma_0, p, n)$ for various values of p is an evaluation of the probability that a noncentral t random variable is greater than or equal to $k\sqrt{n}$. Tables 1 through 8 present the values of $G(p_0, \gamma_0, p, n)$ for selected values of the parameters. All calculations were carried out in double precision using the IBM 370 computer.

Although the tables are given in terms of the confidence level $(\gamma_0)$ and the power (G), some experimenters may find the tables easier to interpret in terms of consumer's and producer's risks. The consumer's risk in this problem is the probability of rejecting $H_0$ and accepting the product when the specification limit has not been met. This is the level of significance of the test and is controlled at a maximum of $\alpha$. The producer's risk is the probability of rejecting the product when it does meet the standard. This can easily be calculated from the table, since this is the probability of a Type II error, and hence is equal to $1 - G(p_0, \gamma_0, p, n)$. For a given n and $\gamma_0$, as $p - p_0$ becomes larger, the producer's risk becomes smaller.

It should also be emphasized that if the specifications are exactly met, $(p = p_0)$, and the test is conducted at $\alpha = 1 - \gamma_0$ level of significance, then the producer's risk is $\gamma_0$. This means that the performance must indeed be better than the stated requirement to have much of a chance of being accepted. For example, if 80% confidence is required, then the producer's risk is 80% if his product meets, but is not better than, the stated requirement. In this case, the probability is 80% that the product will be rejected, even though it does meet the standard.

| Sample Size | k | True Value of $P(X \leq U)$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0.800 | 0.850 | 0.900 | 0.920 | 0.940 | 0.960 | 0.980 |
| 10 | 1.4741 | 0.100 | 0.198 | 0.378 | 0.482 | 0.609 | 0.755 | 0.906 |
| 20 | 1.2404 | 0.100 | 0.267 | 0.579 | 0.729 | 0.865 | 0.959 | 0.997 |
| 30 | 1.1540 | 0.100 | 0.326 | 0.721 | 0.865 | 0.958 | 0.994 | 1.000 |
| 40 | 1.1061 | 0.100 | 0.380 | 0.818 | 0.935 | 0.988 | 0.999 | 1.000 |
| 50 | 1.0747 | 0.100 | 0.428 | 0.883 | 0.970 | 0.997 | 1.000 | 1.000 |
| 60 | 1.0522 | 0.100 | 0.473 | 0.926 | 0.986 | 0.999 | 1.000 | 1.000 |
| 70 | 1.0351 | 0.100 | 0.515 | 0.954 | 0.994 | 1.000 | 1.000 | 1.000 |
| 80 | 1.0215 | 0.100 | 0.553 | 0.971 | 0.997 | 1.000 | 1.000 | 1.000 |
| 90 | 1.0103 | 0.100 | 0.589 | 0.982 | 0.999 | 1.000 | 1.000 | 1.000 |
| 100 | 1.0010 | 0.100 | 0.622 | 0.989 | 1.000 | 1.000 | 1.000 | 1.000 |
| 110 | 0.9930 | 0.100 | 0.653 | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 |
| 120 | 0.9861 | 0.100 | 0.681 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 |
| 130 | 0.9801 | 0.100 | 0.708 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| 140 | 0.9747 | 0.100 | 0.732 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| 150 | 0.9700 | 0.100 | 0.754 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| 160 | 0.9656 | 0.100 | 0.775 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 170 | 0.9617 | 0.100 | 0.794 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 180 | 0.9582 | 0.100 | 0.812 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 190 | 0.9549 | 0.100 | 0.828 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 200 | 0.9519 | 0.100 | 0.843 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

**Table 1:** Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .80$ and $\gamma_0 = .90$

| Sample Size | k | True Value of $P(X \le U)$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0.800 | 0.850 | 0.900 | 0.920 | 0.940 | 0.960 | 0.980 |
| 10 | 1.7032 | 0.050 | 0.108 | 0.233 | 0.316 | 0.428 | 0.581 | 0.786 |
| 20 | 1.3711 | 0.050 | 0.157 | 0.414 | 0.569 | 0.741 | 0.896 | 0.986 |
| 30 | 1.2528 | 0.050 | 0.201 | 0.567 | 0.748 | 0.898 | 0.980 | 0.999 |
| 40 | 1.1882 | 0.050 | 0.244 | 0.688 | 0.861 | 0.964 | 0.997 | 1.000 |
| 50 | 1.1463 | 0.050 | 0.285 | 0.781 | 0.926 | 0.988 | 1.000 | 1.000 |
| 60 | 1.1163 | 0.050 | 0.325 | 0.849 | 0.962 | 0.996 | 1.000 | 1.000 |
| 70 | 1.0936 | 0.050 | 0.364 | 0.897 | 0.981 | 0.999 | 1.000 | 1.000 |
| 80 | 1.0756 | 0.050 | 0.401 | 0.931 | 0.991 | 1.000 | 1.000 | 1.000 |
| 90 | 1.0609 | 0.050 | 0.436 | 0.955 | 0.996 | 1.000 | 1.000 | 1.000 |
| 100 | 1.0486 | 0.050 | 0.471 | 0.970 | 0.998 | 1.000 | 1.000 | 1.000 |
| 110 | 1.0381 | 0.050 | 0.503 | 0.981 | 0.999 | 1.000 | 1.000 | 1.000 |
| 120 | 1.0291 | 0.050 | 0.534 | 0.988 | 1.000 | 1.000 | 1.000 | 1.000 |
| 130 | 1.0212 | 0.050 | 0.564 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 |
| 140 | 1.0141 | 0.050 | 0.592 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 |
| 150 | 1.0079 | 0.050 | 0.619 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 |
| 160 | 1.0022 | 0.050 | 0.644 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| 170 | 0.9971 | 0.050 | 0.668 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| 180 | 0.9925 | 0.050 | 0.691 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 |
| 190 | 0.9882 | 0.050 | 0.712 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 200 | 0.9843 | 0.050 | 0.733 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 2: Probability of Exhibiting $P(X \le U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{\bar{x}} + ks$ is Calculated for $p_0 = .80$ and $\gamma_0 = .95$

| Sample Size | k | True Value of $P(X \leq U)$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.850 | 0.900 | 0.920 | 0.940 | 0.960 | 0.980 |
| 10 | 1.7332 | 0.100 | 0.218 | 0.297 | 0.407 | 0.559 | 0.767 |
| 20 | 1.4710 | 0.100 | 0.304 | 0.447 | 0.628 | 0.821 | 0.966 |
| 30 | 1.3754 | 0.100 | 0.377 | 0.568 | 0.774 | 0.934 | 0.996 |
| 40 | 1.3227 | 0.100 | 0.443 | 0.664 | 0.866 | 0.977 | 1.000 |
| 50 | 1.2884 | 0.100 | 0.502 | 0.741 | 0.922 | 0.992 | 1.000 |
| 60 | 1.2638 | 0.100 | 0.555 | 0.802 | 0.955 | 0.997 | 1.000 |
| 70 | 1.2451 | 0.100 | 0.603 | 0.849 | 0.975 | 0.999 | 1.000 |
| 80 | 1.2303 | 0.100 | 0.647 | 0.886 | 0.986 | 1.000 | 1.000 |
| 90 | 1.2131 | 0.100 | 0.686 | 0.914 | 0.992 | 1.000 | 1.000 |
| 100 | 1.2030 | 0.100 | 0.721 | 0.935 | 0.996 | 1.000 | 1.000 |
| 110 | 1.1994 | 0.100 | 0.752 | 0.952 | 0.998 | 1.000 | 1.000 |
| 120 | 1.1919 | 0.100 | 0.780 | 0.964 | 0.999 | 1.000 | 1.000 |
| 130 | 1.1854 | 0.100 | 0.806 | 0.973 | 0.999 | 1.000 | 1.000 |
| 140 | 1.1796 | 0.100 | 0.828 | 0.980 | 1.000 | 1.000 | 1.000 |
| 150 | 1.1744 | 0.100 | 0.848 | 0.985 | 1.000 | 1.000 | 1.000 |
| 160 | 1.1697 | 0.100 | 0.866 | 0.989 | 1.000 | 1.000 | 1.000 |
| 170 | 1.1655 | 0.100 | 0.882 | 0.992 | 1.000 | 1.000 | 1.000 |
| 180 | 1.1616 | 0.100 | 0.896 | 0.994 | 1.000 | 1.000 | 1.000 |
| 190 | 1.1581 | 0.100 | 0.909 | 0.996 | 1.000 | 1.000 | 1.000 |
| 200 | 1.1548 | 0.100 | 0.920 | 0.997 | 1.000 | 1.000 | 1.000 |

Table 3: Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .85$ and $\gamma_0 = .90$

| Sample Size | k | True Value of $P(X \leq U)$ 0.850 | 0.900 | 0.920 | 0.940 | 0.960 | 0.980 |
|---|---|---|---|---|---|---|---|
| 10 | 1.9881 | 0.050 | 0.120 | 0.174 | 0.254 | 0.381 | 0.594 |
| 20 | 1.6145 | 0.050 | 0.182 | 0.295 | 0.460 | 0.680 | 0.909 |
| 30 | 1.4833 | 0.050 | 0.241 | 0.406 | 0.627 | 0.854 | 0.985 |
| 40 | 1.4121 | 0.050 | 0.296 | 0.507 | 0.752 | 0.938 | 0.998 |
| 50 | 1.3660 | 0.050 | 0.349 | 0.595 | 0.840 | 0.976 | 1.000 |
| 60 | 1.3332 | 0.050 | 0.400 | 0.670 | 0.899 | 0.991 | 1.000 |
| 70 | 1.3084 | 0.050 | 0.448 | 0.734 | 0.938 | 0.997 | 1.000 |
| 80 | 1.2888 | 0.050 | 0.494 | 0.787 | 0.962 | 0.999 | 1.000 |
| 90 | 1.2728 | 0.050 | 0.537 | 0.831 | 0.977 | 1.000 | 1.000 |
| 100 | 1.2594 | 0.050 | 0.577 | 0.867 | 0.987 | 1.000 | 1.000 |
| 110 | 1.2480 | 0.050 | 0.614 | 0.895 | 0.992 | 1.000 | 1.000 |
| 120 | 1.2382 | 0.050 | 0.649 | 0.918 | 0.995 | 1.000 | 1.000 |
| 130 | 1.2296 | 0.050 | 0.681 | 0.937 | 0.997 | 1.000 | 1.000 |
| 140 | 1.2220 | 0.050 | 0.711 | 0.951 | 0.999 | 1.000 | 1.000 |
| 150 | 1.2152 | 0.050 | 0.738 | 0.962 | 0.999 | 1.000 | 1.000 |
| 160 | 1.2091 | 0.050 | 0.763 | 0.971 | 1.000 | 1.000 | 1.000 |
| 170 | 1.2036 | 0.050 | 0.786 | 0.978 | 1.000 | 1.000 | 1.000 |
| 180 | 1.1985 | 0.050 | 0.807 | 0.983 | 1.000 | 1.000 | 1.000 |
| 190 | 1.1939 | 0.050 | 0.827 | 0.987 | 1.000 | 1.000 | 1.000 |
| 200 | 1.1897 | 0.050 | 0.844 | 0.990 | 1.000 | 1.000 | 1.000 |

Table 4: Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .85$ and $\gamma_0 = .95$

| Sample Size | k | True Value of $P(X \leq U)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.900 | 0.910 | 0.920 | 0.930 | 0.940 | 0.950 | 0.960 | 0.970 | 0.980 | 0.990 |
| 10 | 2.0659 | 0.100 | 0.121 | 0.146 | 0.178 | 0.218 | 0.269 | 0.334 | 0.421 | 0.541 | 0.715 |
| 20 | 1.7654 | 0.100 | 0.133 | 0.176 | 0.233 | 0.306 | 0.399 | 0.515 | 0.654 | 0.806 | 0.945 |
| 30 | 1.6573 | 0.100 | 0.143 | 0.202 | 0.280 | 0.382 | 0.507 | 0.652 | 0.800 | 0.925 | 0.991 |
| 40 | 1.5981 | 0.100 | 0.152 | 0.224 | 0.323 | 0.449 | 0.597 | 0.753 | 0.889 | 0.973 | 0.999 |
| 50 | 1.5597 | 0.100 | 0.159 | 0.245 | 0.363 | 0.509 | 0.673 | 0.827 | 0.939 | 0.991 | 1.000 |
| 60 | 1.5322 | 0.100 | 0.167 | 0.265 | 0.400 | 0.563 | 0.735 | 0.880 | 0.968 | 0.997 | 1.000 |
| 70 | 1.5114 | 0.100 | 0.174 | 0.284 | 0.435 | 0.612 | 0.787 | 0.918 | 0.983 | 0.999 | 1.000 |
| 80 | 1.4949 | 0.100 | 0.180 | 0.303 | 0.468 | 0.656 | 0.829 | 0.944 | 0.991 | 1.000 | 1.000 |
| 90 | 1.4815 | 0.100 | 0.187 | 0.320 | 0.499 | 0.696 | 0.864 | 0.962 | 0.996 | 1.000 | 1.000 |
| 100 | 1.4702 | 0.100 | 0.193 | 0.337 | 0.526 | 0.731 | 0.891 | 0.975 | 0.998 | 1.000 | 1.000 |
| 110 | 1.4607 | 0.100 | 0.199 | 0.354 | 0.556 | 0.762 | 0.914 | 0.983 | 0.999 | 1.000 | 1.000 |
| 120 | 1.4524 | 0.100 | 0.204 | 0.370 | 0.582 | 0.790 | 0.932 | 0.989 | 0.999 | 1.000 | 1.000 |
| 130 | 1.4452 | 0.100 | 0.210 | 0.385 | 0.607 | 0.815 | 0.946 | 0.993 | 1.000 | 1.000 | 1.000 |
| 140 | 1.4388 | 0.100 | 0.216 | 0.400 | 0.631 | 0.838 | 0.958 | 0.995 | 1.000 | 1.000 | 1.000 |
| 150 | 1.4330 | 0.100 | 0.221 | 0.415 | 0.653 | 0.857 | 0.967 | 0.997 | 1.000 | 1.000 | 1.000 |
| 160 | 1.4279 | 0.100 | 0.226 | 0.430 | 0.674 | 0.875 | 0.974 | 0.998 | 1.000 | 1.000 | 1.000 |
| 170 | 1.4232 | 0.100 | 0.231 | 0.444 | 0.694 | 0.890 | 0.980 | 0.999 | 1.000 | 1.000 | 1.000 |
| 180 | 1.4190 | 0.100 | 0.237 | 0.457 | 0.713 | 0.904 | 0.984 | 0.999 | 1.000 | 1.000 | 1.000 |
| 190 | 1.4150 | 0.100 | 0.242 | 0.471 | 0.730 | 0.916 | 0.988 | 0.999 | 1.000 | 1.000 | 1.000 |
| 200 | 1.4115 | 0.100 | 0.246 | 0.484 | 0.747 | 0.927 | 0.990 | 1.000 | 1.000 | 1.000 | 1.000 |

**Table 5:** Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .90$ and $\gamma_0 = .90$

| Sample Size | k | True Value of $P(X \leq U)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.900 | 0.910 | 0.920 | 0.930 | 0.940 | 0.950 | 0.960 | 0.970 | 0.980 | 0.990 |
| 10 | 2.3549 | 0.050 | 0.062 | 0.076 | 0.095 | 0.120 | 0.154 | 0.199 | 0.264 | 0.364 | 0.534 |
| 20 | 1.9262 | 0.050 | 0.069 | 0.096 | 0.133 | 0.184 | 0.255 | 0.352 | 0.485 | 0.660 | 0.866 |
| 30 | 1.7775 | 0.050 | 0.075 | 0.113 | 0.167 | 0.243 | 0.349 | 0.489 | 0.660 | 0.838 | 0.970 |
| 40 | 1.6974 | 0.050 | 0.081 | 0.128 | 0.199 | 0.301 | 0.437 | 0.606 | 0.784 | 0.929 | 0.994 |
| 50 | 1.6458 | 0.050 | 0.086 | 0.143 | 0.231 | 0.355 | 0.517 | 0.701 | 0.868 | 0.971 | 0.999 |
| 60 | 1.6091 | 0.050 | 0.091 | 0.158 | 0.261 | 0.407 | 0.589 | 0.777 | 0.921 | 0.989 | 1.000 |
| 70 | 1.5814 | 0.050 | 0.095 | 0.172 | 0.291 | 0.457 | 0.652 | 0.835 | 0.954 | 0.996 | 1.000 |
| 80 | 1.5596 | 0.050 | 0.100 | 0.186 | 0.320 | 0.503 | 0.708 | 0.880 | 0.974 | 0.998 | 1.000 |
| 90 | 1.5418 | 0.050 | 0.104 | 0.199 | 0.349 | 0.547 | 0.756 | 0.913 | 0.985 | 0.999 | 1.000 |
| 100 | 1.5269 | 0.050 | 0.108 | 0.213 | 0.377 | 0.588 | 0.797 | 0.938 | 0.992 | 1.000 | 1.000 |
| 110 | 1.5143 | 0.050 | 0.112 | 0.226 | 0.403 | 0.626 | 0.832 | 0.956 | 0.996 | 1.000 | 1.000 |
| 120 | 1.5034 | 0.050 | 0.116 | 0.239 | 0.430 | 0.661 | 0.861 | 0.969 | 0.998 | 1.000 | 1.000 |
| 130 | 1.4939 | 0.050 | 0.120 | 0.252 | 0.455 | 0.693 | 0.886 | 0.978 | 0.999 | 1.000 | 1.000 |
| 140 | 1.4855 | 0.050 | 0.124 | 0.265 | 0.480 | 0.723 | 0.907 | 0.985 | 0.999 | 1.000 | 1.000 |
| 150 | 1.4730 | 0.050 | 0.127 | 0.277 | 0.503 | 0.750 | 0.924 | 0.990 | 1.000 | 1.000 | 1.000 |
| 160 | 1.4712 | 0.050 | 0.131 | 0.290 | 0.526 | 0.775 | 0.938 | 0.993 | 1.000 | 1.000 | 1.000 |
| 170 | 1.4651 | 0.050 | 0.135 | 0.302 | 0.549 | 0.798 | 0.950 | 0.995 | 1.000 | 1.000 | 1.000 |
| 180 | 1.4595 | 0.050 | 0.138 | 0.314 | 0.570 | 0.819 | 0.959 | 0.997 | 1.000 | 1.000 | 1.000 |
| 190 | 1.4544 | 0.050 | 0.142 | 0.326 | 0.591 | 0.838 | 0.967 | 0.998 | 1.000 | 1.000 | 1.000 |
| 200 | 1.4497 | 0.050 | 0.146 | 0.338 | 0.610 | 0.855 | 0.974 | 0.998 | 1.000 | 1.000 | 1.000 |

Table 6: Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .90$ and $\gamma_0 = .95$

| Sample Size | k | True Value of $P(X \leq U)$ | | | | |
|---|---|---|---|---|---|---|
| | | 0.950 | 0.960 | 0.970 | 0.980 | 0.990 |
| 10 | 2.5688 | 0.100 | 0.133 | 0.182 | 0.261 | 0.411 |
| 20 | 2.2092 | 0.100 | 0.154 | 0.242 | 0.389 | 0.642 |
| 30 | 2.0802 | 0.100 | 0.171 | 0.293 | 0.495 | 0.791 |
| 40 | 2.0107 | 0.100 | 0.187 | 0.340 | 0.585 | 0.882 |
| 50 | 1.9657 | 0.100 | 0.201 | 0.383 | 0.660 | 0.935 |
| 60 | 1.9337 | 0.100 | 0.214 | 0.423 | 0.723 | 0.965 |
| 70 | 1.9094 | 0.100 | 0.227 | 0.461 | 0.775 | 0.981 |
| 80 | 1.8903 | 0.100 | 0.239 | 0.496 | 0.818 | 0.990 |
| 90 | 1.8747 | 0.100 | 0.251 | 0.529 | 0.854 | 0.995 |
| 100 | 1.8616 | 0.100 | 0.263 | 0.561 | 0.883 | 0.997 |
| 110 | 1.8506 | 0.100 | 0.274 | 0.590 | 0.906 | 0.999 |
| 120 | 1.8410 | 0.100 | 0.285 | 0.617 | 0.925 | 0.999 |
| 130 | 1.8326 | 0.100 | 0.295 | 0.643 | 0.940 | 1.000 |
| 140 | 1.8252 | 0.100 | 0.305 | 0.668 | 0.953 | 1.000 |
| 150 | 1.8196 | 0.100 | 0.316 | 0.690 | 0.963 | 1.000 |
| 160 | 1.8126 | 0.100 | 0.325 | 0.712 | 0.970 | 1.000 |
| 170 | 1.8072 | 0.100 | 0.335 | 0.732 | 0.977 | 1.000 |
| 180 | 1.8023 | 0.100 | 0.345 | 0.750 | 0.982 | 1.000 |
| 190 | 1.7978 | 0.100 | 0.354 | 0.768 | 0.986 | 1.000 |
| 200 | 1.7937 | 0.100 | 0.363 | 0.784 | 0.989 | 1.000 |

Table 7: Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .95$ and $\gamma_0 = .90$

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| Sample Size | k | True Value of $P(X \leq U)$ | | | | |
|---|---|---|---|---|---|---|
| | | 0.950 | 0.960 | 0.970 | 0.980 | 0.990 |
| 10 | 2.9115 | 0.050 | 0.069 | 0.098 | 0.149 | 0.256 |
| 20 | 2.3965 | 0.050 | 0.082 | 0.138 | 0.246 | 0.473 |
| 30 | 2.2203 | 0.050 | 0.093 | 0.176 | 0.338 | 0.647 |
| 40 | 2.1259 | 0.050 | 0.103 | 0.212 | 0.424 | 0.774 |
| 50 | 2.0654 | 0.050 | 0.113 | 0.247 | 0.503 | 0.860 |
| 60 | 2.0226 | 0.050 | 0.122 | 0.281 | 0.574 | 0.915 |
| 70 | 1.9903 | 0.050 | 0.131 | 0.314 | 0.637 | 0.950 |
| 80 | 1.9648 | 0.050 | 0.140 | 0.346 | 0.693 | 0.971 |
| 90 | 1.9442 | 0.050 | 0.148 | 0.377 | 0.741 | 0.984 |
| 100 | 1.9269 | 0.050 | 0.156 | 0.407 | 0.783 | 0.991 |
| 110 | 1.9123 | 0.050 | 0.165 | 0.436 | 0.819 | 0.995 |
| 120 | 1.8997 | 0.050 | 0.173 | 0.465 | 0.849 | 0.997 |
| 130 | 1.8886 | 0.050 | 0.181 | 0.492 | 0.875 | 0.999 |
| 140 | 1.8739 | 0.050 | 0.189 | 0.518 | 0.897 | 0.999 |
| 150 | 1.8702 | 0.050 | 0.196 | 0.543 | 0.915 | 1.000 |
| 160 | 1.8624 | 0.050 | 0.204 | 0.568 | 0.931 | 1.000 |
| 170 | 1.8554 | 0.050 | 0.212 | 0.591 | 0.943 | 1.000 |
| 180 | 1.8489 | 0.050 | 0.220 | 0.613 | 0.954 | 1.000 |
| 190 | 1.8430 | 0.050 | 0.227 | 0.635 | 0.963 | 1.000 |
| 200 | 1.8376 | 0.050 | 0.235 | 0.655 | 0.970 | 1.000 |

Table 8: Probability of Exhibiting $P(X \leq U) > p_0$ for a Normal Distribution When the One-sided Tolerance Limit $\bar{x} + ks$ is Calculated for $p_0 = .95$ and $\gamma_0 = .95$

## III. APPLICATIONS

These results can be used in a variety of situations. In general, the tables will be useful to any experimenter in a situation in which the test statistic follows a noncentral t distribution under both the null and the alternative hypothesis, with the difference in the distribution being only a change in the noncentrality parameter $\delta$. Specifically, the problems that have been considered by the authors are related to tolerance limits and variables sampling plans. The following examples illustrate some of these situations.

Example 1: Suppose a consumer, for example, the U.S. Navy, requires an oil-water separator system capable of separation such that not more than 20 parts per million (ppm) of oil is contained in the water effluent 85% of the time with 95% confidence. Previous experience indicates that the distribution of the amount of oil in the effluent water follows a lognormal distribution. A separator system is put on test, 170 samples are collected and analyzed. The natural logarithms of the observations were found, and the mean and standard deviations of the transformed scores were 1.7542 and 0.8262 respectively, With $k = 1.2036$ taken from Table 4 with $p_0 = .85$ and $\gamma_0 = .95$, it is found that $\bar{x} + ks = 2.7487$ which is less than ln(20). It is therefore concluded that the system does meet the requirement.

Example 2: If, in example 1, the requirement had been that the effluent contain not more than 15 ppm of oil 85% of the time with 95%

confidence, the system would have been judged unsatisfactory since 2.7487 > ln(15). After examining the tables, the manufacturer may in return have argued that his risk in this situation was too high. If his system meets, but does not exceed the requirement, he faces a 95% chance of having his system rejected. In fact, his system must produce less than 15 ppm of oil 90% of the time in order to have a probability of 76% of being accepted.

Example 3: Suppose a lower 95% tolerance limit is required for $p_0 = 85\%$. It is, of course, desirable to have this quantity close to the point it is estimating, $\mu + \Phi^{-1}(1 - p_0)\sigma$. A method of measuring this closeness and determining sample size was presented by Faulkenberry and Weeks [2]. Using this method, a $p > p_0$ and a small $\varepsilon > 0$ would be chosen. Then the sample size would be determined to satisfy

$$\Pr[\Pr(X > \bar{x} + ks) \geq p] \leq \varepsilon.$$

In other words, the probability of a Type II error is no larger than $\varepsilon$ at p.

If $p = 96\%$ and $\varepsilon = 5\%$ are chosen, the power at 96% must be at least 95%. In Table 4, it is seen that the required sample size is between 40 and 50. Therefore, a sample size 50 is adequate. (Exact calculation can be performed using the techniques of the previous section.)

It should be noted that the choice of p need not be made independently of n. A more useful procedure would be for the experimenter to consult the tables and choose n by examining the effect that an increased sample size has on the power (or Type II error). In this way, the sample size could be determined with the knowledge of how much power the sample size "buys" as compared to a different sample size.

Another example was suggested by Guenther [4]. He considered several hypothesis testing situations in which the statistic is distributed as a (central) t random variable under the null hypothesis, and as a noncentral t under the alternative hypothesis. One of the situations considered is where inferences are being made about the mean of a normal distribution. He determined the sample size by specifying the power when the difference between the mean under the null hypothesis and the mean under the alternative is expressed in quantiles. This result has been extended in this report to include the case where the inference is to be made about any given quantile, not necessarily the mean, of a normal distribution.

## IV. SUMMARY

In this technical report, tables for the power of a one-sided statistical hypothesis test are presented for the situation in which the statistic is distributed as a noncentral t under both the null and alternative hypothesis. Specifically, power and sample size problems are considered for a one-sided tolerance limit and for a variables sampling plan. Examples are also included to illustrate how these tables can be used in hypothesis testing situations and for determining sample size requirements in some estimation procedures.

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In this technical report, tables for the power of a one-sided statistical hypothesis test are presented for the situation in which the statistic is distributed as a noncentral t under both the null and alternative hypothesis. Specifically, power and sample size problems are considered for a one-sided tolerance limit and for a variables sampling plan. Examples are also included to illustrate how these tables can be used in hypothesis testing situations and for determining sample size requirements in some estimation procedures.

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