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INCOMPRESSIBLE FLOWS AS A SYSTEM OF CONSERVATION LAWS WITH A CO--ETC(U)
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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER APL/JHU-Reprint 2062	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) "Incompressible Flows as a System of Conservation Laws with a Constraint"		5. TYPE OF REPORT & PERIOD COVERED ~~Reprint~~ PAPER
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Joel C. W. Rogers		8. CONTRACT OR GRANT NUMBER(s) N00024-78-C-5384
9. PERFORMING ORGANIZATION NAME & ADDRESS The Johns Hopkins University Applied Physics Laboratory [illegible] Johns Hopkins Road [illegible] Laurel, MD 20810		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS ONR-NR-334-003
11. CONTROLLING OFFICE NAME & ADDRESS Office of Naval Research 800 North Quincy Street Arlington, VA 22217		12. REPORT DATE 11 May 1978
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS Naval Plant Representative, Laurel, MD Applied Physics Laboratory Johns Hopkins Road Laurel, MD 20810		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE None
16. DISTRIBUTION STATEMENT (of this Report) A DISTRIBUTION STATEMENT A Approved for public release; Distribution Unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) A		
18. SUPPLEMENTARY NOTES None		

DDC RECEIVED MAR 2 1979 D

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Water waves/incompressible flow/free boundary layer problem/Stefan problem/conservation laws/constraints/computational algorithms/Boltzmann equation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Some inadequacies of conventional formulations of inviscid hydrodynamics with regard to the solution of free boundary problems are analyzed. An alternative algorithmic formulation which produces the flow field incrementally in time is proposed. In this formulation paramountcy is given to the equations of mass and momentum conservation, which are solved subject to a constraint on the density. Energy decay is possible. The new formulation is examined with regard to its consistency with standard formulations when flow quantities are sufficiently smooth. Extensions to the treatment of stratified incompressible flows and transonic flows are discussed.

INCOMPRESSIBLE FLOWS AS A SYSTEM OF CONSERVATION LAWS WITH A CONSTRAINT

by

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A lecture presented at Institut de Recherche d'Informatique et d'Automatique in Rocquencourt, France, on May 11, 1978. This lecture is included in the publication Seminaires IRIA Analyse et Contrôle de Systèmes - 1978.

1. Introduction

The need for the reformulation of hydrodynamics presented here arose in connection with the study of the hydrodynamic free surface problem. As the problem is conventionally formulated, one looks for a solution $(u(x,t),P(x,t),\mathcal{R}(t))$ of the initial value problem for the equations

$$u_t + u\cdot\nabla u = -\frac{1}{\rho_0}\nabla P - g\vec{k}, \quad x\in\mathcal{R}(t), \quad 0 < t < T, \tag{1.1a}$$

$$\nabla\cdot u = 0, \quad x\in\mathcal{R}(t),\ 0 < t < T, \tag{1.1b}$$

where u is the velocity field and P is the pressure. ρ_0 is the density of water, g is the gravitational constant, and $\vec{k}$ is a unit vector in the z-direction. The initial conditions for (1.1) are

$$u(x,0) = u_0, \tag{1.2a}$$

$$\mathcal{R}(0) = \mathcal{R}_0. \tag{1.2b}$$

Boundary conditions are

$$P \to -\rho_0 g z \quad \text{as } z \to -\infty, \tag{1.3a}$$

$$u\cdot n = V_s\cdot n \quad \text{on } \partial\mathcal{R}_s(t), \tag{1.3b}$$

$$P = 0 \text{ on } \partial\mathcal{R}_f(t), \tag{1.3c}$$

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$$V \cdot n = u \cdot n \quad \text{on } \partial\mathcal{R}_f(t), \tag{1.3d}$$

where

$$\partial\mathcal{R} = \partial\mathcal{R}_f \cup \partial\mathcal{R}_s \quad , \quad \partial\mathcal{R}_f \cap \partial\mathcal{R}_s = \emptyset \quad , \tag{1.4}$$

$V \cdot n$ is the outward normal velocity of the flow region $\mathcal{R}(t)$, and $V_s \cdot n$ is prescribed on $\partial\mathcal{R}_s(t)$. $\partial\mathcal{R}_f$ is called the "free boundary". (For a more detailed discussion of the asymptotic conditions (1.3a), see chapter II of the report by Rogers (Ref. 6).)

For classical solutions of the differential equations (1.1), the velocity field $u(x,t)$ will possess derivatives with respect to x and t. When an initially differentiable velocity field u_0 evolves into a field $u(x,t)$ which is no longer differentiable, it becomes necessary to re-interpret the problem (1.1) - (1.4) in a suitably generalized manner. Such, in fact, is the case when a wave spills over and falls back on the surface. At the moment of impact with the surface, the velocity field at the point of impact is discontinuous.

Pursuing further the problem of a wave spilling over, or in general of the collision of two incompressible fluids with free surfaces, we may consider the idealized problem shown in Figure 1. Here we imagine that we have a cylinder in a region of space where there is no gravity, and that this cylinder contains two equal masses of liquid which are moving *without* friction with equal and opposite velocities along the axis. The free surface of each liquid mass consists of two components orthogonal to the cylinder axis. At the moment of collision we expect the condition (1.1b) on the velocity to be violated. Instead, if we denote distance along the cylinder axis by z, set the origin in the plane of collision, and denote the speed of each liquid mass before collision by U, we get at the moment of collision

$$\nabla \cdot u = -2\,U\delta(z) \quad . \tag{1.5}$$

Thus this example indicates that we will need to re-evaluate the condition (1.1b) in expressing the laws of hydrodynamics in a form suitable for treatment of the general free boundary problem.

A different sort of problem arises in connection with the delineation of the free boundary. When equations (1.1) - (1.4) have a classical solution, one finds from the requirement of consistency between (1.1a) and (1.1b) that

$$\Delta P = -\rho_0 \sum \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \tag{1.6}$$

The need for consistency between (1.1a) and (1.3b) leads to a Neumann condition on the pressure at $\partial\mathcal{R}_s$. This, the asymptotic condition (1.3a), and the free surface condition (1.3c) combine with (1.6) to determine the pressure throughout $\mathcal{R}$. With P determined, (1.1a) can be solved for u, and (1.3d) then serves to determine the time evolution of $\mathcal{R}(t)$. In the case when $\partial\mathcal{R}_s = \emptyset$, we find that the problem (1.1) - (1.4) is invariant under the transformation

$$x' = \alpha x, \; t' = \alpha^{\frac{1}{2}}t \tag{1.7}$$

for any constant $\alpha > 0$. What this means is, for example, that if a flow starting from rest with a particular initial surface deformation results in the falling over and splashing down of a wave in time T_0, then the problem with similar initial conditions, except that the surface deformation is scaled relative to that in the first problem by a factor $\alpha < 1$, results in the falling over and splashing down of a wave in time $\alpha^{\frac{1}{2}}T_0$. Accordingly, we may generally expect that, for any flow, small perturbations on the flow will have the property that waves on sufficiently small scales will be continually falling over, and that the topology of such perturbed flows may not even be determinate. In the case of the breaking of a single wave, but even more so in this case of multiple breaking, the determination of the evolution of the free surface through an equation like (1.3d) becomes ambiguous. For, if we regard (1.3d) as an ordinary differential equation for the motion of a fluid element on the free surface, such an equation does not have a unique solution when the velocity field u is discontinuous at the free surface. In addition, it is open to question whether the portion of the free boundary contained in any unit ball will have bounded measure, or whether the points on the free boundary will be regular points for the Poisson equation (1.6). If they are not regular points, the meaning of the condition (1.3c) will need re-examination.

2. Role of Conservation Laws

Experience with other free boundary problems has shown that the equation which drives the motion of the free surface is often merely an expression of a fundamental conservation law. For example, in the Stefan problem, conservation of energy is paramount (Ref. 1) and in some model hyperbolic problems, the shock conditions are an expression of other conservation laws (Ref. 7). Recognition of this fact has made it possible to devise topology-independent algorithms to solve such free boundary problems. The use of such methods is especially desirable in the present problem where, as we have indicated, the surface may become so complicated that it may become impossible to follow its motion, not only in practice, but also in theory.

The question of what conservation laws are appropriate in order that our mathematical model adequately represent a physical situation is, of course, a problem of physics. In our relatively simple situation, the answer seems rather clear. The best-known conservation laws of Newtonian physics are those of mass, momentum, and energy. For the classical flow of an inviscid liquid one may derive energy conservation from the conservation laws for mass and momentum and generally energy conservation plays a subsidiary role, being derivable from the equations of motion in reversible physical situations and requiring reformulation in irreversible situations. In hydrodynamic theory the precedence of mass and momentum conservation over energy conservation is assumed, for example, in the derivation of the jump conditions for solutions of the nonlinear shallow water equations (Ref. 12). The energy which is lost is assumed somehow dissipated in other, irreversible processes, or in turbulence. This relation of energy loss to irreversibility is a natural complement to the connection of energy conservation with temporal homogeneity in Hamiltonian mechanics. (In dynamical systems which are richer in degrees of freedom than ours, the burden of irreversibility is shifted from the energy to the entropy.)

Regarding the conservation laws, one notes that (1.1a) is a statement of conservation of momentum (in the case $g = 0$). For a fluid whose elements do not undergo a density change as they move and whose velocity is uniformly differentiable from one point to another, so that the trajectories of different elements remain distinct, (1.1b) is a statement of mass conservation. When the density varies discontinuously in space, as it does at the water surface, the governing equation for the free surface, (1.3d), is also an expression of mass conservation. Thus, it would appear that at least one of the problems referred to above, how the free surface evolves in time, may be resolved by writing a law of mass conservation. When the density ρ and velocity u are differentiable, this takes the form

$$\rho t + \nabla \cdot (\rho u) = 0 \qquad (2.1)$$

As we have noted, classically one may think of (1.1b) as an "equation of state" in terms of which the pressure is determined. In the classical picture the equation of state is a constraint, and in the process of satisfying this constraint the momentum is altered by the term $-\nabla P$. We go one step further, and suggest that in the absence of a constraint the pressure vanishes, that is, the pressure arises only when the constraint cannot be satisfied without it. In accordance with the kinetic theory interpretation of pressure, we may identify the pressure with the transfer of momentum across the surface of a fluid element in the direction of its normal brought about by the action of the constraint. The boundary condition (1.3c) suggests that at the free boundary the constraint is automatically satisfied.

Let us inquire further what the nature of the constraint should be for the generalized flows of interest to us. To this end, we reconsider the situation depicted in Figure 1. We make two observations: First, the dynamics of the system should in no way be affected by the way we extend the velocity field to the region where $\rho = 0$ -- it is momentum and not velocity which is dynamically important. Second, there is no problem with a velocity distribution whose divergence approaches (1.5) at the moment of collision. The fundamental fact regarding the collision that we want to maintain is that the two bodies of fluid should not enter each other. That is, the fluid density should not exceed ρ_0. We write this as a one-sided constraint

$$\rho \leq \rho_0. \tag{2.2}$$

We will see that (2.2) is the appropriate generalization of the classical constraint (1.1b) and also of the boundary condition (1.3c). Its one-sided nature reminds us of variational inequalities.

3. Algorithmic Description of Hydrodynamics

One of our governing equations is the conservation of mass, (2.1). In accordance with the observations of the last section, in the absence of the constraint we will have the equations of momentum "conservation":

$$(\rho u)_t + \nabla \cdot (\rho uu) = - \rho g \vec{k} \tag{3.1}$$

Equations (3.1) and (2.1) form a set of hyperbolic conservation laws. For the actual hydrodynamic flow, we will solve them subject to the constraint (2.2).

Of course (2.1), (3.1), and (2.2) are generally inconsistent, and we have to make clear what we mean by "solving" (2.1) and (3.1) subject to (2.2). We will do this by giving an algorithm for approximating the solution of the evolutionary problem in which we start with initial data $\rho(0)$, $u(0)$ and try to find $\rho(t)$, $u(t)$ for $t > 0$. Our algorithm will be dependent on a parameter τ which we call the time step, and will generate from a pair of quantities ρ, ρu, with ρ satisfying (2.2), another pair of quantities $\bar{\rho}$, $\overline{\rho u}$, with $\bar{\rho}$ satisfying (2.2). We denote the result of this operation symbolically as

$$(\bar{\rho}, \overline{\rho u}) = \bar{S}(\tau) \quad (\rho, \rho u) \quad . \tag{3.2}$$

When we speak of "solution" of the problem, we mean that for $t > 0$ the operators

$$\left(\bar{S}\left(\frac{t}{n}\right)\right)^n \to S^*(t) \tag{3.3}$$

as $n \to \infty$. (3.3) is to be understood to hold in an appropriate function space. More will be said about this in our next lecture (Ref. 9). However, it behooves us to point out that we have not yet proven the crucial step (3.3), and thus we cannot speak of a "solution" of the problem in any rigorous mathematical sense. In this lecture we content ourselves with an indication that, when the flow quantities have sufficient regularity in space and time, our algorithm reduces to an approximate algorithm for solving the Euler equations, which may be expected to converge to the actual solution as $\tau \to 0$, under the same presuppositions regarding regularity. The next lecture will focus on the sorts of solutions we expect to emerge from the analysis of convergence, and the sense in which the inviscid hydrodynamic initial value problem may generally be regarded as well-posed. But the problems of convergence and regularity of the flows converged to are ongoing problems, presently uncompleted.

Our algorithm "solves" (2.1) and (3.1) subject to (2.2) in the following sense: The hyperbolic conservation laws (2.1) and (3.1) are "solved" for a time interval τ. Then the densities of mass and momentum are adjusted to satisfy (2.2), in a manner consistent with global conservation of mass and momentum. When we say that incompressible flows may be considered to evolve through a system of conservation laws in conjunction with a constraint, our statement is premised on the conjectured, but as yet unproven, existence of the limit of the algorithm as $\tau \to 0$. There is no doubt a certain lack of elegance in our approach through a family of solutions dependent on a parameter τ, but it is perhaps no worse than the situation which arises in making precise the solution of an initial value problem for an ordinary differential equation.

In what follows, we will attempt a reasonably complete description of the algorithm which is to render an approximation to the flow. However, what we present is by no means our first approach to the problem, and along the way mathematical simplifications have arisen which have removed the algorithm somewhat from its pristine physical orientation. For a more complete description of the physical considerations which led us to make some of our initial choices, we refer the reader to a more complete write-up (Ref. 10). What we present here is a mathematical object, which will rise to the status of theory or fall into disrepute according to its internal consistency. No doubt later versions will differ in detail, but we suspect that the main elements will remain intact.

To "solve" the conservation laws (2.1) and (3.1) for a time τ, we introduce a distribution function $F(x,v,t)$ satisfying the collisionless Boltzmann equation

$$F_t + v \cdot \nabla F - g \frac{\partial F}{\partial v_z} = 0 \quad , 0 < t < \tau \, , \tag{3.4a}$$

and initial conditions

$$F(x,v,0) = \rho(x) \;\; \delta\,(v-u(x)) \; . \tag{3.4b}$$

It is easiest to give boundary conditions for F in terms of the characteristics, whose equations, away from boundaries, are

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = - g\,\vec{k} \tag{3.5}$$

These are just the equations of classical particles moving under the influence of gravity without collisions. At a rigid boundary, we require that the equations of the characteristics describe the trajectories of particles reflecting specularly from the boundary. Thus, if a characteristic strikes a rigid boundary at time t_c and the velocity of the rigid boundary is V_s at the point and time where it is struck by the characteristic, we set

$$x(t_c^+) = x(t_c^-) \quad , \tag{3.6a}$$

$$v(t_c^+) = n \times (v(t_c^-) \times n) + (2V_s \cdot n - v(t_c^-) \cdot n)n, \tag{3.6b}$$

where n is the unit outward normal to the boundary at the point and time referred to.

Finally, we determine approximate solutions of (2.1) and (3.1) at time τ through

$$\tilde{\rho} = \int F(x,v,\tau)dv \quad , \tag{3.7a}$$

$$\widetilde{\rho u} = \int vF(x,v,\tau)dv \quad . \tag{3.7b}$$

In physical language, equations (3.7) state that all fluid elements at the same location after the passage of time τ have collided inelastically. We point out that this assumption that collisions are inelastic is not mandatory, but it seems like a simple and reasonable first approximation for the problems that interest us. Other assumptions are possible. The allowance of inelastic collisions permits the decay of energy, and an element of irreversibility enters into our algorithm, although the Euler equations themselves are formally reversible in time. We note that some additional assumptions regarding the nature of collisions have been needed to make the evolutionary problem determinate in the general case, and our treatment of the conservation laws has provided a set of such assumptions. For example, in the situation depicted in Figure 1, a number of possibilities after collision will be consistent with the requirements we have made heretofore. One possibility is for the two liquid masses to collide and them come to rest instantaneously, with all energy lost inelastically at the moment of impact. Another possibility is for them to collide totally elastically, bouncing off one another, with the flow totally reversible. There are also intermediate possibilities, with a loss of speed for all the fluid being one, and with some of the fluid being brought to rest and the remainder rebounding elastically being another. Although we are getting somewhat out of sequence, since we have not described how the algorithm treats the constraint condition (2.2) yet, we note that, according to the assumption of inelastic collisions made in (3.7), in the limit as $\tau \to 0$ for the case shown in Figure 1, we will get the first possibility listed above.

Classical flows in which the velocity is Lipschitz continuous in space uniformly in time will not permit the collision of fluid elements for τ sufficiently small, and thus this element of irreversibility will not enter. We will refer to the conservation laws (2.1) and (3.1) with $g = 0$ as the higher-dimensional form of Burgers' equation. We have noted elsewhere (Ref. 7) how in one dimension the proper solution of (2.1) and (3.1), as outlined in (3.4) and (3.7), differs from the solution of the formally equivalent conservation law

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad .$$

We will say nothing in this lecture about the convergence as $\tau \to 0$ of the solution of the conservation laws outlined in (3.4) and (3.7). The subject will be raised in the next lecture.

Assuming that we have adequately solved (2.1) and (3.1), let us see how we satisfy the constraint (2.2). If $\tilde{\rho} \le \rho_0$, the constraint has no effect, and we set $\bar{\rho} = \tilde{\rho}$, $\overline{\rho u} = \widetilde{\rho u}$. If $\tilde{\rho} > \rho_0$ somewhere, in violation of (2.2), we have to realize that during the time τ while mass was being convected according to (3.4), (3.5), (3.6), and (3.7a), other processes were also taking place. It may help to observe that $\tilde{\rho}$ is a linear functional of ρ, and that we may envisage $\tilde{\rho}$ as the accumulation of independently moving mass densities, or "streams". The other processes that took place in the time τ were of the following sort: Whenever there was an accumulation of mass yielding a density $> \rho_0$, the particles in the region of excess density were considered to be undergoing rapid elastic collisions which resulted in their spreading out from the region of density excess in an isotropic manner. As new density excesses arrived at a point from additional streams, they also, in addition to the excess particles which had not yet spread out from previous collisions and which therefore still contributed to the density excess at the point, underwent such elastic collisions with a resultant spreading out. Such collisions occurred with extreme rapidity, with the result that after the time τ all streams which had contributed to the density excess at a point had spread out and no excess was left.

Each group of collisions with attendant spreading out of mass was isotropic, and there were many such processes going on, until a sort of "steady state" was achieved. Now in fact the effect of an isotropic spreading out of a mass distribution $d\sigma(x)$ is to replace $d\sigma$ by

$$\left(\int P(x,x')d\sigma(x')\right)dx$$

where

$$\int P(x,x')dx = 1 \quad , \tag{3.8a}$$

$$\int P(x,x')(x-x')dx = 0 \quad , \tag{3.8b}$$

$$\int P(x,x')(x-x')_i(x-x')_j dx = q(x')\delta_{ij} \tag{3.8c}$$

Repeated application of minute ($q \to 0$) processes of this sort is equivalent to repeated application of Gaussian distributions (Ref. 4), and thus we may replace P by a Gaussian with small variance:

$$P(x,x') \to \frac{1}{(4\pi d\alpha(x'))^{N/2}} e^{-(x-x')^2/4d\alpha(x')} \tag{3.9}$$

where N is the dimensionality of the space. (3.9) holds in the interior of the fluid. Of course, at a rigid boundary any fluid which spreads out cannot pass through the boundary. Instead, there must be a reflection at the boundary, as there was in the case described by equations (3.6). So in general, we replace (3.9) by

$$P(x,x') \to e^{\Delta^+ d\alpha(x')} = S^+(d\alpha(x')) \tag{3.10}$$

where $S^+(d\alpha(x'))$ is the semigroup generated by the Laplace operator for the region exclusive of rigid bodies, with the requirement of zero normal gradient at the rigid boundaries.

In the hydrodynamic case only that part of the mass distribution corresponding to density exceeding ρ_0 spreads out, and thus the operator $S^+(d\alpha(x'))$ acts only on

$$f(\hat{\rho}(x'))dx'$$

where

$$f(\hat{\rho}) = \begin{cases} \hat{\rho} - \rho_0 & \hat{\rho} \geq \rho_0 \\ 0 & \hat{\rho} \leq \rho_0 \end{cases} \tag{3.11}$$

and $\hat{\rho}$ is a general mass density. What happens, then, is that an initial mass density $\hat{\rho}$ is replaced by

$$\hat{\rho}^{(1)} = \hat{\rho} - f(\hat{\rho}) + S^+(d\alpha_1(x')f(\hat{\rho}) \equiv F_1\hat{\rho} \quad , \tag{3.12a}$$

$\hat{\rho}^{(1)}$ is replaced by

$$\hat{\rho}^{(2)} = F_2\hat{\rho}^{(1)} = \hat{\rho}^{(1)} - f(\hat{\rho}^{(1)}) + S^+(d\alpha_2(x'))\, f(\hat{\rho}^{(1)}) \quad , \tag{3.12b}$$

and in general $\hat{\rho}^{(n)}$ is replaced by

$$\hat{\rho}^{(n+1)} = F_{n+1}\,\hat{\rho}^{(n)} = \hat{\rho}^{(n)} - f(\hat{\rho}^{(n)}) + S^+(d\alpha_{n+1}(x'))\; f\,(\hat{\rho}^{(n)}). \tag{3.12c}$$

After many such collisions $n \to \infty$ we achieve a steady state.

Referring to (3.10) and letting $d\alpha(x') \to 0$, we see that the steady state is the steady state ($\alpha \to \infty$) of the equation

$$\hat{\theta}_\alpha = \Delta^+ (\mu(x,\alpha) f (\hat{\theta})) , \tag{3.13a}$$

$$\hat{\theta} (\alpha = 0) = \hat{\rho} \tag{3.13b}$$

where

$$\mu(x,\alpha) = \frac{d\alpha(x,\alpha)}{d\alpha} , \quad d\alpha = \sup_x d\alpha(x,\alpha) , \tag{3.13c}$$

and the assumption that collisions take place wherever there is a mass density excess is reflected in the condition

$$\mu(x,\alpha) > 0 \tag{3.13d}$$

$\tilde{\rho}$ has been envisioned as the accumulation of a number of independently moving streams, and hence to find the new mass density after all the mass redistributions due to collisions have taken place, we should solve equation (3.13) with an initial density $\hat{\rho}$ and an inhomogeneous term on the right-hand side reflecting the addition of other contributions to $\tilde{\rho}$ as the parameter α runs from 0 to ∞. However, we have found (Ref. 8) that the steady state is independent of the order in which contributions to $\tilde{\rho}$ are inserted into the equation. A reduction in $\mu(x,\alpha)$ may be viewed as a change in the order in which contributions appear, and thus we note that the steady state is the same as that for the problem

$$\theta_\alpha = \Delta^+ f (\theta), \tag{3.14a}$$

$$\theta(\alpha = 0) = \tilde{\rho} . \tag{3.14b}$$

(3.14) is recognized as a one-phase Stefan problem. In terms of the solution of this problem, the new mass density is

$$\bar{\rho} = \lim_{\alpha \to \infty} \theta (\alpha). \tag{3.15}$$

(3.14) and (3.15) serve to determine the location of the hydrodynamic free boundary.

We turn next to the effect of the mass redistribution on the momentum density. Just as the elastically colliding particles carry a mass with them as they move, they also carry a velocity $\bar{u}$, which is as yet undetermined. But in addition, since velocity is nothing but rate of spatial displacement, the particles must have associated with them a momentum due to the fact of their redistribution. Since all these processes take place in a time τ, to lowest order in τ we may associate with a particle which has moved from x' to x the velocity $(x-x')/\tau$. Away from a rigid boundary, in the process represented by (3.12a), the momentum density $\hat{\rho}\hat{u}$ will be replaced by

$$(\hat{\rho}\hat{u})^{(1)} = \hat{\rho}\hat{u} - \bar{u}\, f(\hat{\rho}) + S(d\alpha_1(x'))(f(\hat{\rho})\bar{u})$$

$$+ \int \frac{1}{(4\pi d\alpha_1(x'))^{N/2}} \frac{x-x'}{\tau} e^{-(x-x')^2/4d\alpha_1(x')} f(\hat{\rho}(x'))dx'$$

$$= \hat{\rho}\hat{u} + (S(d\alpha_1(x'))-1)\,(\bar{u}\, f(\hat{\rho}))$$

$$- \frac{2}{\tau} \nabla\, (S(d\alpha_1(x'))\, f(\hat{\rho}(x'))d\alpha_1(x'))\ , \tag{3.16a}$$

where

$$S(d\alpha_1(x')) = e^{\Delta d\alpha_1(x')} \tag{3.16b}$$

The process represented by (3.12c) will result in the replacement of $(\hat{\rho}\hat{u})^{(n)}$ by

$$(\hat{\rho}\hat{u})^{(n+1)} = (\hat{\rho}\hat{u})^{(n)} + (S(d\alpha_{n+1}(x')) - 1)\,(\bar{u}\, f(\hat{\rho}^{(n)}))$$

$$- \frac{2}{\tau} \nabla(S(d\alpha_{n+1}(x'))f(\hat{\rho}^{(n)}(x'))d\alpha_{n+1}(x'))\ . \tag{3.16c}$$

Letting $n \to \infty$ and $d\alpha_1(x') \to 0$, we get, independent of the order in which the collisional processes associated with the mass redistribution occur,

$$\bar{\rho}\bar{u} = \tilde{\rho}\tilde{u} + \Delta(\bar{u}v) - \frac{2}{\tau}\nabla v \tag{3.17a}$$

where

$$v = \int_0^\infty f(\theta)d\alpha \tag{3.17b}$$

and θ satisfies (3.14). (3.17) is to be solved subject to the boundary condition (3.6) on $\bar{u}$ at the rigid boundary $\partial\mathcal{R}_s(t)$. (More precisely, the normal component $\bar{u}\cdot n$ satisfies (1.3b), and the derivative in the normal direction of the tangential component $\bar{u}\times n$ vanishes.)

It may seem that there is some mystery associated with our determination of a velocity field in terms of the displacement of moving particles in a time interval, as opposed to its determination through higher order time derivatives of the displacement, namely, the acceleration. However, the determination here really has grown out of a consistency argument, and we can give an example from elementary mechanics to illustrate our point.

Consider the situation shown in Figure 2. We have a particle moving on the surface of a rigid body under the influence of gravity. The particle may move on the surface or above it, but may not enter the rigid body. Thus we have a one-sided constraint on the motion of the particle, similar in some respects to (2.2). One may devise an algorithm to determine the motion of the particle as follows: Given the particle position and velocity (r,u) at a given time, we let the particle follow the familiar parabolic path appropriate to motion in a gravitational field without any constraints. This carries the particle to $\tilde{P}$ after a time τ, when it has velocity $\tilde{u}$. If $\tilde{P}$ lies on or above the rigid body, no correction is necessary, and we can set $\bar{P} = \tilde{P}$, $\bar{u} = \tilde{u}$ for the new position and velocity of the constrained motion. If $\tilde{P}$ lies inside the rigid body, we satisfy the constraint by moving the particle back to the nearest point $\bar{P}$ on the body's surface. Then, to be consistent with the fact that velocity is rate of spatial displacement, we have to add to $\tilde{u}$ the displacement from $\tilde{P}$ to $\bar{P}$ divided by τ. This addition to $\tilde{u}$ is known in mechanics as the normal force exerted on the particle by the body. At a point P_f on the body the particle may leave the surface. P_f may be thought of as a "free boundary".

We regard the region of flow where $0 < \rho < \rho_0$ to be a "spray". This is more a mathematical artifice than a physically complete representation of an actual spray. A more detailed description of some of the physical assumptions made in our characterization of the fluid in the region where $0 < \rho < \rho_0$ as a spray is given elsewhere (Ref. 10). The possibility of the development of a spray in the non-classical formulation of hydrodynamics is analogous to the possibility of "slush" formation in the non-classical formulation of the Stefan problem (Ref. 1). Indeed, as seen in (3.11) and (3.14), there is a clear correspondence between the enthalpy and latent heat in the one-phase Stefan problem, on the one hand, and the mass and liquid density in hydrodynamics, on the other. Similar interpretations may likewise be given to "spray" and "slush". In the latter case, slush occupying a region $\mathcal{R}$ of positive measure may be conceived as a mixture of minute volumes of two phases of a substance, such that the volume of each phase has a positive measure in each subset of positive measure in $\mathcal{R}$. In the former case, we think of spray occupying a region $\mathcal{R}$ of positive measure as consisting of minute volumes of liquid ($\rho = \rho_0$) and vacuum ($\rho = 0$), with the volume of each phase in each subset of $\mathcal{R}$ of positive measure having positive measure. As yet we do not have any examples of flows in which we can show rigorously that sprays must develop in order for a solution of the equations to exist. Nevertheless, as we pointed out in the Introduction, it is by no means clear that the hydrodynamic free boundary can always be sharply defined, and we leave open the possibility of the formation of a diffuse free boundary.

In concluding this section dealing with an algorithmic representation of a generalized hydrodynamics, we remark that numerical results based on the algorithm have been obtained, and are currently being prepared for publication (Ref. 11). The numerical treatment of the hyperbolic conservation laws follows the path laid down in (3.4) - (3.7). The steady state of the one-phase Stefan problem (3.14) is found using the algorithm (3.12) for the case $d\alpha_1(x') = d\alpha$ (Ref. 1). The linear elliptic problem (3.17) is solved by finding the steady state of a parabolic equation, which is in turn solved through a variation on an algorithm applicable to a class of hyperbolic and parabolic problems (Ref. 2).

4. Consistency of the Algorithm

To demonstrate consistency of the reformulated hydrodynamics with classical hydrodynamics in the regime where the latter is meaningful, we examine alternate forms of our equations when the velocity field is differentiable. Consistency will be demonstrated if we show that the lowest order terms in τ are identical in both formulations. The consistency of (3.4) - (3.7) with the hyperbolic conservation laws (2.1) and (3.1) is quite straightforward. As we have pointed out, when the velocities are differentiable no collisions of fluid elements will occur for τ sufficiently small, and (3.4) - (3.7) will then solve (2.1) and (3.1) exactly in the interior of the flow region. As regards boundary conditions, it follows from (3.6) that the average, over a small time interval, of the component of momentum normal to a rigid boundary must approach that component of velocity of the boundary times the average over the same time interval of the density, as the distance to the boundary approaches zero. This is consistent with (1.3b).

Let us then focus attention on the second half of the algorithm, which deals with the ramifications of the constraint (2.2). If we can show that (3.17) goes over, to first order in τ, to the equation

$$\bar{\rho}\bar{u} = \tilde{\rho}\tilde{u} - \tau\nabla P \qquad (4.1)$$

where P satisfies (1.6), we will have shown that our algorithm reverts to a split-step scheme for solving the classical (1.1). Comparing (4.1) with (3.17), we see that the obvious correspondence to make is that

$$\frac{2}{\tau^2} v \rightarrow P \qquad (4.2)$$

(3.17b), (3.14), and (3.15) lead to

$$\Delta^+ v = \bar{\rho} - \tilde{\rho} \qquad (4.3)$$

Suppose at a given time we have a density $\rho \leq \rho_0$ and a velocity u satisfying (1.1b). In the interior of the liquid, $\rho = \rho_0$. Now, assume the velocity is differentiable, so that (2.1) and (3.1) imply

$$u_t + u \cdot \nabla u = -g\vec{k} \qquad (4.4)$$

Integrating (4.4) over the time interval τ to get $\tilde{u}$, we find to first order in τ

$$\nabla \cdot \tilde{u} = -\tau \sum \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \quad . \tag{4.5}$$

The equation of mass conservation,

$$\rho_t + u \cdot \nabla\rho = -\rho\nabla \cdot u \ ,$$

gives for $\tilde{\rho}$ in the interior of the region $\rho = \rho_0$, to second order in τ

$$\begin{aligned} \tilde{\rho} &= \rho_0 - \rho_0 \frac{\tau}{2} (\nabla \cdot u + \nabla \cdot \tilde{u}) \\ &= \rho_0 + \rho_0 \frac{\tau^2}{2} \sum \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \quad . \end{aligned} \tag{4.6}$$

For points interior to the liquid region at the given time ($\rho = \rho_0$) and also the liquid region a time step later ($\bar{\rho} = \rho_0$), it follows from (4.3) that

$$\Delta v = -\frac{\rho_0 \tau^2}{2} \sum \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \ ,$$

in agreement with (1.6) and the correspondence (4.2). We recall that (1.6) is just the condition to make $\nabla \cdot \bar{u} = 0$ to lowest order in τ.

With respect to boundary conditions, if we have a classical flow with a sharp free boundary, $\tilde{\rho}$ will fall rapidly from the expression (4.6) to 0 at the free boundary. In the interior of the liquid, as $\tau \to 0$, we find from (4.6) and (4.3) that $|\Delta v|$ is small compared to ρ_0, whereas outside the free boundary we will have approximately $\Delta v = \rho_0$. Thus, although one derives from (3.17b) and (4.3) that $v = \nabla v = 0$ at the free boundary, there will be a sort of "boundary layer" there in which ∇v changes from 0 to a finite value. Since by (4.6) $\tilde{\rho}$ will differ from ρ_0 by $0(\tau^2)$ in the interior of the liquid, mass conservation will require that this "boundary layer" have thickness $0(\tau^2)$ if the free boundary has bounded curvatures. From (4.3), ∇v will change by $0(\tau^2)$ over this boundary layer and v will change by $0(\tau^4)$. Just inside the boundary layer $\left(2/\tau^2\right)\nabla v$ will assume a value which does not necessarily vanish as $\tau \to 0$, but $\left(2/\tau^2\right) v$ will $\to 0$ as $\tau \to 0$. Hence to lowest order in τ the boundary condition (1.3c) will be redeemed.

Something similar occurs at the rigid boundaries. On account of (3.14), we get $n \cdot \nabla v = 0$ at rigid boundaries. However, (3.4)-(3.7) predict that, over a "boundary layer" with thickness $0(g\tau^2)$, $\tilde{\rho} - \rho_0$ will be $0(\rho_0)$. Across this boundary layer $2/\tau^2 \ n \cdot \nabla v$ will jump from 0 to a value $0(\rho_0 g)$. Interpreting the asymptotic condition (1.3a) as an approximation to the case where the fluid is bounded below by a portion of a rigid plane situated at a large nagative value of z, we easily confirm the validity with respect to this condition of the correspondence (4.2). Similar results obtain at other rigid boundaries, but we note that agreement between the different formulations is built in by requiring solutions of (3.17) to have the derivative in the normal direction of their tangential component vanish, and to have the normal component satisfy (1.3b).

With the correspondence (4.2), the term $\Delta(\bar{u}v)$ on the right of (3.17a) is seen to be $O(\tau^2)$, and thus to have no effect on the consistency of our formulation with the classical equations (1.1) - (1.4). Reference to (3.16) and the discussion preceding (3.16) shows that the extra term represents the fact that the elastically colliding particles in our picture carry the mean velocity $\bar{u}$ with them. Even though this term may be removed from (3.17a) without changing the consistency of the equations, we have retained it, since without it the equations would lack Galileian invariance.

In the Introduction we raised the question of whether points on the free boundary $\partial\mathcal{Q}_f$ would generally be regular points for the Poisson equation (1.6). This was one of the reasons for our search for a formulation of the problem which did not entail the solution of a partial differential equation in the liquid region subject to boundary data on $\partial\mathcal{Q}_f$. In our reformulation of the problem, the one-phase Stefan problem (3.14) and (3.17b) take the place of (1.6) and (1.3c). We have indicated that (3.14) is solved in practice by using the algorithm (3.12) with $d\alpha_1(x') = d\alpha$. The result of such an algorithm has been proven to converge to a solution of the problem (3.14) *for any given* $\alpha = \alpha_0$ *as* $d\alpha \to 0$ (Ref. 2), and it is not hard to extend this to a proof of convergence to the steady state solution. Our remarks in this section indicate that, inside a boundary layer of thickness $O(\tau^2)$ near the boundary, we may expect $(2/\tau^2)v$ and $\nabla(2/\tau^2)v$ to converge to P and ∇P, respectively.

Accordingly, it is of some interest to inquire to what extent, when the boundary layer becomes infinitely thin and the steady state Stefan problem reverts to a linear elliptic boundary value problem, the boundary conditions demanded by (1.3c) at the free boundary are actually attained by the result of our algorithm in the limit $d\alpha \to 0$. Note that this limit of an infinitely thin boundary layer can also be achieved by letting $\rho_0 \to \infty$ in (3.11), (3.14), (3.17b), and (4.3). An error bound (Ref. 10) indicates that, for smooth boundaries $\partial\mathcal{Q}_f$, the steady state $G(d\alpha)$ given by the algorithm (3.12) with $d\alpha_1(x') = d\alpha$ and (3.17b) in the limit $\rho_0 \to \infty$ has an L^∞ error, when N = 3 of

$$||G(d\alpha)-G(x,x_0)||_{L^\infty} = O\left(\frac{\sqrt{d\alpha}}{d^2}\left(\ln\left(\frac{d}{\sqrt{d\alpha}}\right)\right)^{\frac{1}{2}}\right),\ d = \mathrm{dist}(x_0,\ \partial\mathcal{Q}_f), \qquad (4.7)$$

for the computation of $G(x,x_0)$ given by

$$\Delta G = -\delta(x-x_0)\ ,\quad x_0 \in \mathcal{Q}_f,$$

$$G|_{\partial\mathcal{Q}_f} = 0.$$

(A recent paper (Ref. 3) in which the same algorithm is described appears to give an error $O(\sqrt{d\alpha})$, which is better than the one given above. I have not tracked down the discrepancy in the estimates.*) Our analysis (Ref. 10) also shows, for more general regions $\mathcal{R}_f$, that the steady state $G(d\alpha)$ converges to a limit

$$G(0) = \inf_{d\alpha} G(d\alpha) \tag{4.8}$$

as $d\alpha \to 0$, and that $G(0) \to 0$ as one approaches a regular point of $\partial(\mathcal{R}_f \cup \partial\mathcal{R}_f)$ from the interior of $\mathcal{R}_f$. Similar results would apply to the limit as $d\alpha \to 0$ of the steady state for $\rho_0 \to \infty$ computed by (3.12) with $d\alpha_1(x') = d\alpha$, (3.17b), and (4.2), that is, $P = 0$ at all regular points of $\partial(\mathcal{R}_f \ \partial\mathcal{R}_f)$. From a physical point of view, this may be more reasonable than requiring $P = 0$ at all regular points of $\partial\mathcal{R}_f$, as in (1.3c).

5. Other Versions

We have noted that there is some arbitrariness in the algorithm presented in Section 3 regarding the presence of higher order terms in τ, as there has to be for anything short of an exact solution for that time interval. What terms are added or dropped is largely a matter of taste. For example, the term $\Delta(\bar{u}v)$ was left in (3.17a) to guarantee Galileian invariance.

Eugene Isaacson has called our attention to a general approach for solving equations subject to a constraint (Ref. 5). In this approach one would add a perturbation to the equations so that the constraints might be satisfied, and then try to minimize the perturbation in some appropriate sense. For example, if one had the constraint (1.1b) on a velocity field, one could add a vector field to the right-hand side of the hyperbolic conservation laws (3.1). It is well known that the L^2 minimum vector field with given divergence and suitable homogeneous boundary conditions is a gradient, and one might in this way arrive at (1.1a). In our case, the only appropriate extension of this to the steady state one-phase Stefan problem we are aware of appears to be that, if λ is a vector field satisfying

$$\nabla \cdot \lambda = \rho_0 - \tilde{\rho}\,, \qquad x \in \mathcal{D}\,, \tag{5.1a}$$

$$\lambda \cdot n\big|_{\partial\mathcal{D}} = 0 \tag{5.1b}$$

and $\tilde{\rho}$ satisfies

$$\tilde{\rho} \geq \rho_0, \qquad x \in \mathcal{D}_0\,, \tag{5.2a}$$

$$\tilde{\rho} = 0\,, \qquad x \notin \mathcal{D}_0\,, \tag{5.2b}$$

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* I am indebted to Bertrand Mercier for informing me of this work and providing me with a preprint.

then the minimum of $\int_{\mathcal{D}} \lambda^2 dx$ over all λ satisfying (5.1) and over all $\mathcal{D} \supset \mathcal{D}_0$ is achieved for

$$\lambda = \nabla v \tag{5.3}$$

with v given by (3.17b) and (3.14) with Δ^+ replaced by Δ for $\mathbb{R}^N$. We have not followed this line of thinking in searching for a generalized hydrodynamics, because we have felt more secure proceeding on grounds with a more direct connection to physics.

Another item which we have not yet discussed relates to the degree to which the various stages of our algorithm conserve energy. It is obvious from (3.4) - (3.7) that energy cannot increase in the first half of our split-step scheme:

$$\int (\tfrac{1}{2}\tilde{\rho}\tilde{u}^2 + \tilde{\rho} gz)\, dx \leq \int (\tfrac{1}{2}\rho u^2 + \rho gz) dx \tag{5.4}$$

For the second half, the result is not so simple. From (3.17) and (4.3) we may derive

$$\bar{\rho}\bar{u}^2 - \tilde{\rho}\tilde{u}^2 = -\tilde{\rho}(\bar{u}-\tilde{u})^2 + \frac{4}{\tau} v \nabla \cdot \bar{u} - 2v(\nabla\bar{u})^2$$

$$+ \Delta(v\bar{u}^2) - \frac{4}{\tau} \nabla \cdot (v\bar{u}) . \tag{5.5}$$

The only term which can increase the energy here is $\frac{4}{\tau} v \nabla . \bar{u}$. To first order in τ, we have seen that an initially divergenceless velocity field remains so as long as the flow is a classical one, and hence to first order there is no energy change. Even if we had the possibility of a velocity field with a divergence, we would generally expect to lowest order in τ that $\tilde{\rho} < \rho_0$ where $\nabla \cdot u > 0$ and $\tilde{\rho} > \rho_0$ where $\nabla \cdot u < 0$. Then to lowest order we would expect $v = 0$ where $\nabla \cdot u > 0$ and $v > 0$ where $\nabla \cdot u < 0$. To lowest order in τ, we could replace $\nabla \cdot \bar{u}$ by $\nabla \cdot u$ and thus conclude that $\frac{4}{\tau} v \nabla \cdot \bar{u}$ would tend to reduce the energy.

Our general feeling is this: It may be acceptable for energy to be created in the second half of a time step as a compensation for too much energy dissipated in the inelastic collisions of the first half, but it is not physically acceptable for energy to be gained overall. On the other hand, a slight energy increase which is a manifestation of the time discretization in the algorithm as opposed to a sign of instability may not be disastrous. As it is, we can give an example of a flow for which the algorithm of Section 3 will predict a net increase of energy over a time step (Ref. 10). Needless to say, such a flow does not exhibit any great degree of regularity over the time step, and its treatment by (3.4) - (3.7) over the first half of the time step is questionable.

We may add a term to the right-hand side of (3.17a) which preserves momentum conservation, Galileian invariance, and consistency to lowest order in τ For example, we may write

$$\overline{\rho u}_i = \widetilde{\widetilde{\rho u}}_i + \Delta(\bar{u}_i v) - \frac{2}{\tau}\frac{\partial v}{\partial x_i} + \sum \frac{\partial}{\partial x_j} G_{ij} \ . \tag{5.6}$$

In place of (5.5) we get

$$\overline{\overline{\rho u}}^2 - \widetilde{\widetilde{\rho u}}^2 = -\tilde{\rho}(\bar{u} - \tilde{u})^2 + \frac{4}{\tau} v \nabla \cdot \bar{u} - 2v\ (\nabla \bar{u})^2$$
$$+ \Delta(v\bar{u})^2 - \frac{4}{\tau}\nabla \cdot (v\bar{u}) - 2\sum \frac{\partial \bar{u}_i}{\partial x_j} G_{ij} + 2\sum \frac{\partial}{\partial x_j}(\bar{u}_i G_{ij}). \tag{5.7}$$

A more detailed investigation (Ref. 10) suggests that is is possible to choose G_{ij} so as to preserve Galileian invariance and consistency, and to make

$$\int \overline{\overline{\rho u}}^2 dx \le \int \widetilde{\widetilde{\rho u}}^2 dx$$

in all cases. However, the price may be to replace (3.17a) by a nonlinear equation for $\bar{u}$, whose solution may have to be obtained iteratively. We have gone to some lengths to discuss to what extent energy conservation or nonconservation is an essential part of our theory because of our belief, elaborated on more fully in the next lecture, that the status of energy conservation for the limiting flow obtained as $\tau \to 0$ has a deeper connection to important qualitative properties of the flow (Ref. 9). As we have already indicated, (3.17a) is likely subject to further emendation, and it may be that there is no uniquely simple and acceptable formula for $\overline{\rho u}$, in contrast to equations (3.14) and (3.15) for $\bar{\rho}$.

6. Stratified Flow and Transonic Flow

Incompressible flows with a non-constant density are amenable to a treatment like that offered here for the constant-density case. In this case we introduce, in addition to ρ and u, a new dependent variable $\mathcal{J}$ which represents the volume fraction of space filled at each point. Our hyperbolic conservation laws consist of

$$(\mathcal{J}\rho)_t + \nabla \cdot (\mathcal{J}\rho u) = 0 \ , \tag{6.1}$$

$$(\mathcal{J}\rho u)_t + \nabla \cdot (\mathcal{J}\rho uu) = -\mathcal{J}\rho g\vec{k} \tag{6.2}$$

and a conservation law for volume:

$$\mathcal{J}_t + \nabla \cdot (\mathcal{J}u) = 0 \tag{6.3}$$

In place of (2.2) we have the fundamental constraint

$$\mathcal{J} \le 1 \tag{6.4}$$

As before, we proceed for the first half time step from $(\vartheta, \rho, \rho u)$ to $(\tilde{\vartheta}, \tilde{\rho}, \widetilde{\rho u})$ by "solving" (6.1), (6.2), and (6.3) for a time τ. To satisfy the constraint (6.4) we solve

$$\theta_{\alpha}^{*} = \Delta f^{*}(\theta^{*}) \tag{6.5a}$$

with

$$\theta^{*}(\alpha=0) = \tilde{\vartheta} . \tag{6.5b}$$

Here

$$f^{*}(\theta^{*}) = \begin{cases} \theta^{*} - 1 & \theta^{*} \geq 1 \\ 0 & \theta^{*} \leq 1 \end{cases} \tag{6.6}$$

$\bar{\vartheta}$ is given by

$$\bar{\vartheta} = \lim_{\alpha \to \infty} \theta^{*}(\alpha) . \tag{6.7}$$

We define

$$v^{*} = \int_{0}^{\infty} f^{*}(\theta^{*}) d\alpha \tag{6.8}$$

and find $\bar{\rho}$ from

$$\bar{\vartheta}\bar{\rho} = \tilde{\vartheta}\tilde{\rho} + \Delta(\bar{\rho} v^{*}) \tag{6.9}$$

In place of (3.17a) we have

$$\bar{\vartheta}\bar{\rho}\bar{u} = \tilde{\vartheta}\tilde{\rho}\tilde{u} - \frac{2}{\tau}\nabla(\bar{\rho} v^{*}) + \Delta(\bar{\rho}\bar{u} v^{*}) \tag{6.10}$$

The algorithm in Section 3 is then a special case of (6.1) - (6.10). Note that in this section ρ refers to an intrinsic fluid property, whereas earlier in this paper ρ refers to mass density. To get the algorithm of Section 3 from (6.1) - (6.10), take the case where $\rho = \tilde{\rho} = \bar{\rho} = \rho_0$ in (6.1) - (6.10), and then replace $\vartheta\rho_0$, $\tilde{\vartheta}\rho_0$, and $\bar{\vartheta}\rho_0$ wherever they occur in (6.1) - (6.10) by ρ, $\tilde{\rho}$, and $\bar{\rho}$, respectively.

The equation of state (2.2) may be considered to be a special case of the more general equation of state for a barotropic flow:

$$P = P(\rho) . \tag{6.11}$$

Since we have solved (2.2) by solving a one-phase Stefan problem (3.14), one may ask if (6.11) can also be obtained through the solution of a nonlinear parabolic equation. One might even wonder if, for a polytropic fluid, the analog to the one-phase Stefan problem is the equation for flow in a porous medium. The answer to this seems quite clearly to be "No." We shall indicate some analogies which may carry over to the transonic case, but we do not think these have any practical computational value.

Let

$$\tilde{P}^{-1}(\xi) \equiv \sup\{x | P(x) = \xi\} \tag{6.12}$$

In the hydrodynamic case, $\tilde{P}^{-1}(\xi) = \rho_0 \forall \xi \geq 0$. Then an analog to (3.14) may be written as

$$\theta_\alpha = \Delta f(\theta(\alpha), v(\alpha)) \tag{6.13a}$$

$$\theta(\alpha = 0) = \tilde{\rho} \tag{6.13b}$$

where

$$f(\theta,v) = \max\left(\theta - \tilde{P}^{-1}\left(\frac{2}{\tau^2} v\right), 0\right) \tag{6.14}$$

and

$$v(\alpha) = \int_0^\alpha f(\theta(\alpha'), v(\alpha'))d\alpha' \tag{6.15}$$

In the cases of greatest interest, $\tilde{P}^{-1}(0) = 0$ and $\frac{d\tilde{P}^{-1}(\xi)}{d\xi}$ is bounded and non-negative. In such cases we can show that (Ref. 10) θ and v achieve limits $\bar{\rho}$ and $\bar{v}$ as $\alpha \to \infty$, that

$$\Delta\bar{v} = \bar{\rho} - \tilde{\rho} = \tilde{P}^{-1}\left(\frac{2}{\tau^2}\bar{v}\right) - \tilde{\rho} \tag{6.16}$$

and that $\bar{v}$ and $\bar{\rho}$ depend monotonically on $\tilde{\rho}$. Also, if $\tilde{\rho}$ has compact support, so does $\bar{v}$ when we have a polytronic fluid

$$P(\rho) = A\rho^\gamma \tag{6.17}$$

with $\gamma > 1$.

Acknowledgement

This work has been supported by the Office of Naval Research under Task No. NR 334-003.

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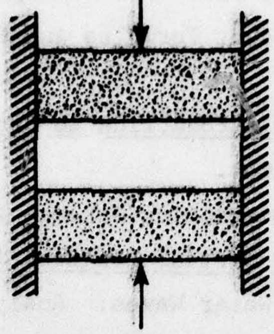

Fig. 1

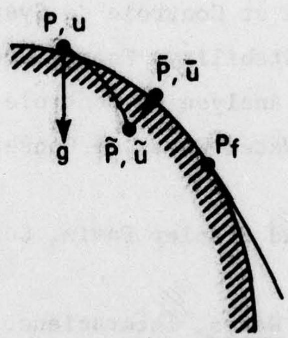

Fig. 2